Bergman kernels and symplectic reduction

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Abstract. We generalize several recent results concerning the asymptotic expansions of Bergman kernels to the framework of geometric quantization and establish an asymptotic symplectic identification property. More precisely, we study the asymptotic expansion of the G-invariant Bergman kernel of the spin^c Dirac operator associated with high tensor powers of a positive line bundle on a symplectic manifold. We also develop a way to compute the coefficients of the expansion, and compute the first few of them, especially, we obtain the scalar curvature of the reduction space from the G-invariant Bergman kernel on the total space. These results generalize the corresponding results in the non-equivariant setting, which has played a crucial role in the recent work of Donaldson on stability of projective manifolds, to the geometric quantization setting. As another kind of application, we generalize some Toeplitz operator type properties in semi-classical analysis to the framework of geometric quantization. The method we use is inspired by Local Index Theory, especially by the analytic localization techniques developed by Bismut and Lebeau.

Contents

0. Introduction

The study of the Bergman kernel is a classical subject in the theory of several complex variables, where usually it concerns the kernel function of the projection operator to an infinite dimensional Hilbert space. The recent interest of the analogue of this concept in complex geometry mainly started in the paper of Tian [39], which was in turn inspired by a question of Yau. Here, the projection is onto a finite dimensional space instead.

After [39], the Bergman kernel has been studied extensively in [35], [42], [14], [23], establishing the diagonal asymptotic expansion for high powers of an ample line bundle. Moreover, the coefficients in the asymptotic expansion encode geometric information of the underlying complex projective manifolds. This asymptotic expansion plays a crucial role in the recent work of Donaldson [18], where the existence of Kähler metrics with constant scalar curvature is shown to be closely related to Chow-Mumford stability.

In [17], [26], [27], Dai, Liu, Ma and Marinescu studied the *full off-diagonal* asymptotic expansion of the (generalized) Bergman kernel of the spin^c Dirac operator and the renormalized Bochner–Laplacian associated to a positive line bundle on a compact symplectic manifold. As a by product, they gave a new proof of the results mentioned in the previous paragraph. They find also various applications therein, especially as pointed out in [27], the *full off-diagonal* asymptotic expansion implies Toeplitz operator type properties. This approach is inspired by the Local Index Theory, especially by the analytic localization techniques of Bismut-Lebeau [7, §11]. We refer to the above papers and the recent book [28] for detail informations of the Bergman kernel on compact symplectic manifolds.

In this paper, we generalize some of the results in [17], [26] and [27] to the framework of geometric quantization, by studying the asymptotic expansion of the G-invariant

Bergman kernel for high powers of an ample line bundle on symplectic manifolds admitting a Hamiltonian group action.

To start with, let (X,ω) be a compact symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle L over X endowed with a Hermitian connection ∇^L with the property that

$$\frac{\sqrt{-1}}{2\pi}R^L=\omega, \tag{0.1}$$

where $R^L=(\nabla^L)^2$ is the curvature of (L,∇^L).

Let (E,h^E) be a Hermitian vector bundle on X equipped with a Hermitian connection ∇^E and R^E denotes the associated curvature.

Let g^{TX} be a Riemannian metric on X. Let $\mathbf{J}:TX\to TX$ be the skew–adjoint linear map which satisfies the relation

$$\omega(u,v)=g^{TX}(\mathbf{J}u,v) \tag{0.2}$$

for $u,v\in TX$.

Let J be an almost complex structure such that

$$g^{TX}(Ju,Jv)=g^{TX}(u,v),\quad \omega(Ju,Jv)=\omega(u,v) \tag{0.3}$$

and that $\omega(\cdot,J\cdot)$ defines a metric on TX. Then J commutes with $\mathbf{J}$ and $J=\mathbf{J}(-\mathbf{J}^2)^{-1/2}$ (cf. (2.7)).

Let ∇^{TX} be the Levi-Civita connection on (TX,g^{TX}) with curvature R^{TX} and scalar curvature r^X. The connection ∇^{TX} induces a natural connection $\nabla^{\det}$ on $\det(T^{(1,0)}X)$ with curvature $R^{\det}$, and the Clifford connection ∇^{Cliff} on the Clifford module $\Lambda(T^{*(0,1)}X)$ with curvature R^{Cliff} (cf. Section 2.2).

The spin^c Dirac operator D_p acts on $\Omega^{0,\bullet}(X,L^p\otimes E)=\bigoplus_{q=0}^n\Omega^{0,q}(X,L^p\otimes E)$, the direct sum of spaces of $(0,q)$–forms with values in $L^p\otimes E$. We denote by D_p^+ the restriction of D_p on $\Omega^{0,\mathrm{even}}(X,L^p\otimes E)$. The index of D_p^+ is defined by

$$\mathrm{Ind}(D_p^+)=\mathrm{Ker}\,D_p^+-\mathrm{Coker}\,D_p^+. \tag{0.4}$$

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and $\dim G=n_0$. Suppose that G acts on X and its action on X lifts on L and E. Moreover, we assume the G-action preserves the above connections and metrics on TX,L,E and J. Then $\mathrm{Ind}(D_p^+)$ is a virtual representation of G. Denote by $(\mathrm{Ker}\,D_p)^G$, $\mathrm{Ind}(D_p^+)^G$ the G-trivial components of $\mathrm{Ker}\,D_p$, $\mathrm{Ind}(D_p^+)$ respectively.

The action of G on L induces naturally a moment map $\mu:X\to\mathfrak{g}^*$ (cf. (2.14)). We assume that $0\in\mathfrak{g}^*$ is a regular value of μ.

Set $P=\mu^{-1}(0)$. Then the Marsden-Weinstein symplectic reduction $(X_G=P/G,\omega_{X_G})$ is a symplectic orbifold (X_G is smooth if G acts freely on P).

Moreover, (L,∇^L), (E,∇^E) descend to (L_G,∇^{L_G}), (E_G,∇^{E_G}) over X_G so that the corresponding curvature condition $\frac{\sqrt{-1}}{2\pi}R^{L_G}=\omega_G$ holds (cf. [20]). The G-invariant almost complex structure J also descends to an almost complex structure J_G on TX_G, and h^L,h^E,g^{TX} descend to h^{L_G}, h^{E_G}, g^{TX_G} respectively.

One can construct the corresponding spin^c Dirac operator $D_{G,p}$ on X_G.

The geometric quantization conjecture of Guillemin-Sternberg [20] can be stated as follows: when E is the trivial bundle $\mathbb{C}$ on X, for any $p > 0$,

$$\dim\left(\mathrm{Ind}(D_p^+)^G\right) = \dim\left(\mathrm{Ind}(D_{G,p}^+)\right). \tag{0.5}$$

When G is abelian, this conjecture was proved by Meinrenken [31] and Vergne [41]. The remaining nonabelian case was proved by Meinrenken [32] using the symplectic cut techniques of Lerman, and by Tian and Zhang [40] using analytic localization techniques.

More generally, by a result of Tian and Zhang [40, Theorem 0.2], for any general vector bundle E as above, there exists $p_0 > 0$ such that for any $p \geq p_0$, (0.5) still holds.

On the other hand, by [25, Theorem 2.5] (cf. (2.13)), which is a direct consequence of the Lichnerowicz formula for D_p, for p large enough, $\mathrm{Coker}\, D_p^+$ is null (cf. also [10], [13]). Thus there exists $p_0 > 0$ such that for any $p \geq p_0$,

(0.6)

$$\begin{aligned}\dim(\mathrm{Ker}\, D_p)^G &= \dim(\mathrm{Ker}\, D_{G,p}) = \dim\left(\mathrm{Ind}(D_{G,p}^+)\right) = \int_{X_G} \mathrm{Td}(TX_G)\,\mathrm{ch}(L_G^p \otimes E_G) \\ &= \mathrm{rk}(E)\int_{X_G} \frac{(p\,c_1(L_G))^{n-n_0}}{(n-n_0)!} + \int_{X_G}\Big(c_1(E_G) + \frac{\mathrm{rk}(E)}{2}c_1(TX_G)\Big)\frac{(p\,c_1(L_G))^{n-n_0-1}}{(n-n_0-1)!} \\ &\qquad + \mathscr{O}(p^{n-n_0-2}),\end{aligned}$$

where $\mathrm{ch}(\cdot), c_1(\cdot), \mathrm{Td}(\cdot)$ are the Chern character, the first Chern class and the Todd class of the corresponding complex vector bundles (TX_G is a complex vector bundle with complex structure J_G).

Set $E_p := \Lambda(T^{*(0,1)}X)\otimes L^p \otimes E$. Let $\langle\ \ \rangle$ be the L^2-scalar product on $\Omega^{0,\bullet}(X, L^p\otimes E) = \mathscr{C}^\infty(X, E_p)$ induced by g^{TX}, h^L, h^E as in (1.19).

Let P_p^G be the orthogonal projection from $(\Omega^{0,\bullet}(X, L^p \otimes E), \langle\ \ \rangle)$ on $(\mathrm{Ker}\, D_p)^G$. The G-invariant Bergman kernel is $P_p^G(x, x')$ $(x, x' \in X)$, the smooth kernel of P_p^G with respect to the Riemannian volume form $dv_X(x')$.

Let pr_1 and pr_2 be the projections from $X \times X$ onto the first and second factor X respectively. Then $P_p^G(x, x')$ is a smooth section of $\mathrm{pr}_1^*(E_p) \otimes \mathrm{pr}_2^*(E_p^*)$ on $X \times X$. In particular, $P_p^G(x, x) \in \mathrm{End}(E_p)_x = \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_x$.

The G-invariant Bergman kernel $P_p^G(x, x')$ is an analytic version of $(\mathrm{Ker}\, D_p)^G$. In view of (0.6), it is natural to expect that the kernel $P_p^G(x, x')$ should be closely related to the corresponding Bergman kernel on the symplectic reduction X_G. The purpose of this paper is to study the asymptotic expansion of the G-invariant Bergman kernel $P_p^G(x, x')$ as $p \to \infty$, and we will relate it to the asymptotic expansion of the Bergman kernel on the symplectic reduction X_G.

Let $d^X(x, x')$ be the Riemannian distance between x, $x' \in X$.

In Section 2.4, we prove the following result which allows us to reduce our problem as a problem near $P = \mu^{-1}(0)$.

Theorem 0.1. *For any open G-neighborhood U of P in X, $\varepsilon_0 > 0$, $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ (depend on U, ε_0) such that for $p \geq 1$, $x, x' \in X, d^X(Gx, x') \geq \varepsilon_0$ or*

$x, x' \in X \setminus U$,

$$|P_p^G(x,x')|_{\mathscr{C}^m} \leq C_{l,m} p^{-l}, \tag{0.7}$$

where $\mathscr{C}^m$ *is the* $\mathscr{C}^m$*-norm induced by* ∇^L, ∇^E, ∇^{TX}, h^L, h^E *and* g^{TX}.

Assume for simplicity that G acts freely on P.

Let U be an open G-neighborhood of $\mu^{-1}(0)$ such that G acts freely on U.

For any G-equivariant vector bundle (F, ∇^F) on U, we denote by F_B the bundle on $U/G = B$ induced naturally by G-invariant sections of F on U. The connection ∇^F induces canonically a connection ∇^{F_B} on F_B. Let R^{F_B} be its curvature. Let

$$\mu^F(K) = \nabla^F_{K^X} - L_K \in \mathrm{End}(F) \tag{0.8}$$

for $K \in \mathfrak{g}$ and K^X the corresponding vector field on U.

Note that $P_p^G \in (\mathscr{C}^\infty(U \times U, \mathrm{pr}_1^* E_p \otimes \mathrm{pr}_2^* E_p^*))^{G \times G}$, thus we can view $P_p^G(x,x')$ (x, $x' \in U$) as a smooth section of $\mathrm{pr}_1^*(E_p)_B \otimes \mathrm{pr}_2^*(E_p^*)_B$ on $B \times B$.

Let g^{TB} be the Riemannian metric on $U/G = B$ induced by g^{TX}. Let ∇^{TB} be the Levi-Civita connection on (TB, g^{TB}) with curvature R^{TB}. Let N_G be the normal bundle to X_G in B. We identify N_G with the orthogonal complement of TX_G in $(TB|_{X_G}, g^{TB})$.

Let g^{TX_G}, g^{N_G} be the metrics on TX_G, N_G induced by g^{TB} respectively.

Let P^{TX_G}, P^{N_G} be the orthogonal projections from $TB|_{X_G}$ on TX_G, N_G respectively. Set

$$\begin{aligned} &\nabla^{N_G} = P^{N_G}(\nabla^{TB}|_{X_G})P^{N_G}, \quad \nabla^{TX_G} = P^{TX_G}(\nabla^{TB}|_{X_G})P^{TX_G}, \\ &{}^0\nabla^{TB} = \nabla^{TX_G} \oplus \nabla^{N_G}, \quad A = \nabla^{TB}|_{X_G} - {}^0\nabla^{TB}. \end{aligned} \tag{0.9}$$

Then ∇^{N_G}, ${}^0\nabla^{TB}$ are Euclidean connections on N_G, $TB|_{X_G}$ on X_G, ∇^{TX_G} is the Levi-Civita connection on (TX_G, g^{TX_G}), and A is the associated second fundamental form.

Denote by $\mathrm{vol}(Gx)$ ($x \in U$) the volume of the orbit Gx equipped with the metric induced by g^{TX}. Following [40, (3.10)], let $h(x)$ be the function on U defined by

$$h(x) = (\mathrm{vol}(Gx))^{1/2}. \tag{0.10}$$

Then h reduces to a function on B.

Denote by $I_{\mathbb{C}\otimes E}$ the projection from $\Lambda(T^{*(0,1)}X) \otimes E$ onto $\mathbb{C} \otimes E$ under the decomposition $\Lambda(T^{*(0,1)}X) \otimes E = \mathbb{C} \otimes E \oplus \Lambda^{>0}(T^{*(0,1)}X) \otimes E$, and $I_{\mathbb{C}\otimes E_B}$ the corresponding projection on B.

In the whole paper, for any $x_0 \in X_G$, $Z \in T_{x_0}B$, we write $Z = Z^0 + Z^\perp$, with $Z^0 \in T_{x_0}X_G$, $Z^\perp \in N_{G,x_0}$.

Let $\tau_{Z^0} Z^\perp \in N_{G,\exp^{X_G}_{x_0}(Z^0)}$ be the parallel transport of $Z^\perp$ with respect to the connection ∇^{N_G} along the geodesic in X_G, $[0,1] \ni t \to \exp^{X_G}_{x_0}(tZ^0)$.

For $\varepsilon_0 > 0$ small enough, we identify $Z \in T_{x_0}B$, $|Z| < \varepsilon_0$ with $\exp^B_{\exp^{X_G}_{x_0}(Z^0)}(\tau_{Z^0}Z^\perp) \in B$. Then for $x_0 \in X_G$, $Z, Z' \in T_{x_0}B$, $|Z|, |Z'| < \varepsilon_0$, the map $\Psi : TB|_{X_G} \times TB|_{X_G} \to B \times B$,

$$\Psi(Z, Z') = (\exp^B_{\exp^{X_G}_{x_0}(Z^0)}(\tau_{Z^0}Z^\perp), \exp^B_{\exp^{X_G}_{x_0}(Z'^0)}(\tau_{Z'^0}Z'^\perp))$$

is well defined.

We identify $(E_p)_{B,Z}$ to $(E_p)_{B,x_0}$ by using parallel transport with respect to $\nabla^{(E_p)_B}$ along $[0,1] \ni u \to uZ$.

Let $\pi_B : TB|_{X_G} \times TB|_{X_G} \to X_G$ be the natural projection from the fiberwise product of $TB|_{X_G}$ on X_G onto X_G.

From Theorem 0.1, we only need to understand $P_p^G \circ \Psi$, and under our identification, $P_p^G \circ \Psi(Z, Z')$ is a smooth section of

$$\pi_B^*(\mathrm{End}(E_p)_B) = \pi_B^*(\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B)$$

on $TB|_{X_G} \times TB|_{X_G}$.

Let $|\quad|_{\mathscr{C}^{m'}(X_G)}$ be the $\mathscr{C}^{m'}$-norm on $\mathscr{C}^\infty(X_G, \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B)$ induced by $\nabla^{\mathrm{Cliff}_B}$, ∇^{E_B}, h^E and g^{TX}. The norm $|\quad|_{\mathscr{C}^{m'}(X_G)}$ induces naturally a $\mathscr{C}^{m'}$-norm along X_G on $\mathscr{C}^\infty(TB|_{X_G} \times TB|_{X_G}, \pi_B^*(\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B))$, we still denote it by $|\quad|_{\mathscr{C}^{m'}(X_G)}$.

Let dv_B, dv_{X_G}, dv_{N_G} be the Riemannian volume forms on (B, g^{TB}), (X_G, g^{TX_G}), (N_G, g^{N_G}) respectively. Let $\kappa \in \mathscr{C}^\infty(TB|_{X_G}, \mathbb{R})$, with $\kappa = 1$ on X_G, be defined by that for $Z \in T_{x_0}B$, $x_0 \in X_G$,

$$dv_B(x_0, Z) = \kappa(x_0, Z) dv_{T_{x_0}B}(Z) = \kappa(x_0, Z) dv_{X_G}(x_0) dv_{N_{G,x_0}}. \tag{0.11}$$

The following result is one of the main results of this paper.

Theorem 0.2. *Assume that G acts freely on $\mu^{-1}(0)$ and $\mathbf{J} = J$ on $\mu^{-1}(0)$. Then there exist $\mathcal{Q}_r(Z, Z') \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{B,x_0}$ ($x_0 \in X_G, r \in \mathbb{N}$), polynomials in Z, Z' with the same parity as r, whose coefficients are polynomials in A, R^{TB}, R^{Cliff_B}, R^{E_B}, μ^E, μ^{Cliff} (resp. r^X, R^{det}, R^E; resp. h, R^L, R^{L_B}; resp. μ) and their derivatives at x_0 to order $r-1$ (resp. $r-2$; resp. r, resp. $r+1$), such that if we denote by*

$$P_{x_0}^{(r)}(Z, Z') = \mathcal{Q}_r(Z, Z') P(Z, Z'), \quad \mathcal{Q}_0(Z, Z') = I_{\mathbb{C} \otimes E_B}, \tag{0.12}$$

with

$$\begin{aligned} P(Z, Z') = &\exp\Big(-\frac{\pi}{2}|Z^0 - Z'^0|^2 - \pi\sqrt{-1}\,\big\langle J_{x_0} Z^0, Z'^0 \big\rangle \Big) \\ &\times 2^{\frac{n_0}{2}} \exp\Big(-\pi\big(|Z^\perp|^2 + |Z'^\perp|^2\big)\Big), \end{aligned} \tag{0.13}$$

then there exists $C'' > 0$ such that for any $k, m, m', m'' \in \mathbb{N}$, there exists $C > 0$ such that for $x_0 \in X_G$, $Z, Z' \in T_{x_0}B$, $|Z|, |Z'| \le \varepsilon_0$, [1]

$$\begin{aligned} &(1 + \sqrt{p}|Z^\perp| + \sqrt{p}|Z'^\perp|)^{m''} \sup_{|\alpha|+|\alpha'| \le m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \right. \\ &\left. \left(p^{-n+\frac{n_0}{2}} (h\kappa^{\frac{1}{2}})(Z)(h\kappa^{\frac{1}{2}})(Z') P_p^G \circ \Psi(Z, Z') - \sum_{r=0}^{k} P_{x_0}^{(r)}(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} \right) \right|_{\mathscr{C}^{m'}(X_G)} \\ &\le C p^{-(k+1-m)/2} (1 + \sqrt{p}|Z^0| + \sqrt{p}|Z'^0|)^{2(n+k+m'+2)+m} \exp(-\sqrt{C''}\sqrt{p}|Z - Z'|) + \mathscr{O}(p^{-\infty}). \end{aligned} \tag{0.14}$$

Furthermore, the expansion is uniform in the following sense: for any fixed $k, m, m', m'' \in \mathbb{N}$, assume that the derivatives of g^{TX}, h^L, ∇^L, h^E, ∇^E, and J with order $\leqslant 2n + k +$

[1] In the exponential factor of [29, (7)], we missed m' as in the last line of (0.14) here.

$m+m'+3$ run over a set bounded in the $\mathscr{C}^{m'}$–norm taken with respect to the parameters and, moreover, g^{TX} runs over a set bounded below. Then the constant C is independent of g^{TX}; and the $\mathscr{C}^{m'}$-norm in (0.14) *includes also the derivatives on the parameters.*

In (0.14), the term $\mathscr{O}(p^{-\infty})$ means that for any $l, l_1 \in \mathbb{N}$, there exists $C_{l,l_1} > 0$ such that its $\mathscr{C}^{l_1}$-norm is dominated by $C_{l,l_1}p^{-l}$.

It is interesting to see that the kernel $P(Z, Z')$ is the product of two kernels : along $T_{x_0}X_G$, it is the classical Bergman kernel on $T_{x_0}X_G$ with complex structure J_{x_0}, while along N_G, it is the kernel of a harmonic oscillator on N_{G,x_0}.

Remark 0.3. i) Theorem 0.2 is a special case of Theorem 2.23 where we do not assume $\mathbf{J} = J$ on $P = \mu^{-1}(0)$. In Theorem 3.2, we get explicit informations on $P^{(r)}$ when $\mathbf{J}$ verifies (3.2).

ii) If G does not act freely on P, then X_G is an orbifold. In Section 4.1, we explain how to modify our arguments to get the asymptotic expansion, Theorem 4.1. Analogous to the usual orbifold case [17, (5.27)], $P_p^G(x, x)(x \in P)$ does not have a uniform asymptotic expansion if the singular set of X_G is not empty.

iii) Let $\mathcal{V}$ be an irreducible representation of G, let $P_p^{\mathcal{V}}$ be the orthogonal projection from $\Omega^{0,\bullet}(X, L^p \otimes E)$ on $\mathrm{Hom}_G(\mathcal{V}, \mathrm{Ker}\, D_p) \otimes \mathcal{V} \subset \mathrm{Ker}\, D_p$. In Section 4.2, we get the asymptotic expansion of the kernel $P_p^{\mathcal{V}}(x, x')$ from Theorems 0.1, 0.2.

iv) When $G = \{1\}$, Theorem 0.2 is [17, Theorem 4.18′].

v) If we take $Z = Z' = 0$ in (0.14), then we get for $x_0 \in X_G$,

$$P_{x_0}^{(0)}(0, 0) = 2^{\frac{n_0}{2}} I_{\mathbb{C}\otimes E_B}, \tag{0.15}$$

and

$$\left| p^{-n+\frac{n_0}{2}} h^2(x_0) P_p^G(x_0, x_0) - \sum_{r=0}^{k} P_{x_0}^{(2r)}(0, 0) p^{-r} \right|_{\mathscr{C}^{m'}(X_G)} \leq C p^{-k-1}. \tag{0.16}$$

In Section 4.3, we show that (0.15) and (0.16) are direct consequences of the full off-diagonal asymptotic expansion of the Bergman kernel [17, Theorem 4.18′]. In fact, one possible way to get Theorem 0.2 is to average the full off-diagonal asymptotic expansion of the Bergman kernel on X [17, Theorem 4.18′] with respect to a Haar measure on G. However, we do not know how to get the full off-diagonal expansion, especially the fast decay along N_G in (0.14) in this way.

In this paper we will apply the analytic localization techniques to get Theorem 0.2, and this method also gives us an effective way to compute the coefficients in the asymptotic expansion (cf. §3.2). The key observation is that the G-invariant Bergman kernel is exactly the kernel of the orthogonal projection to the zero space of a deformation of D_p^2 by the Casimir operator (i.e., to consider $D_p^2 - p\mathrm{Cas}$). This plays an essential role in proving Theorems 0.1, 0.2.

Let $\mathscr{I}_p$ be a section of $\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B$ on X_G defined by

$$\mathscr{I}_p(x_0) = \int_{Z \in N_G, |Z| \leq \varepsilon_0} h^2(x_0, Z) P_p^G \circ \Psi((x_0, Z), (x_0, Z)) \kappa(x_0, Z) dv_{N_G}(Z). \tag{0.17}$$

By Theorem 0.1, modulo $\mathscr{O}(p^{-\infty})$, $\mathscr{I}_p(x_0)$ does not depend on ε_0, and

$$\begin{aligned}(0.18)\quad \dim(\operatorname{Ker} D_p)^G &= \int_X \operatorname{Tr}[P_p^G(y,y)]dv_X(y) = \int_U \operatorname{Tr}[P_p^G(y,y)]dv_X(y) + \mathscr{O}(p^{-\infty}) \\ &= \int_B h^2(y)\operatorname{Tr}[P_p^G(y,y)]dv_B(y) + \mathscr{O}(p^{-\infty}) \\ &= \int_{X_G} \operatorname{Tr}[\mathscr{I}_p(x_0)]dv_{X_G}(x_0) + \mathscr{O}(p^{-\infty}).\end{aligned}$$

A direct consequence of Theorem 0.2 is the following corollary.

Corollary 0.4. *Taken $Z = Z' \in N_{G,x_0}$, $m = 0$ in (0.14), we get*

$$\begin{aligned}(0.19)\quad &\Big|p^{-n+\frac{n_0}{2}}(h^2\kappa)(Z)P_p^G(Z,Z) - \sum_{r=0}^{k} P_{x_0}^{(r)}(\sqrt{p}Z,\sqrt{p}Z)p^{-r/2}\Big|_{\mathscr{C}^{m'}(X_G)} \\ &\qquad \le Cp^{-(k+1)/2}(1+\sqrt{p}|Z|)^{-m''} + \mathscr{O}(p^{-\infty}).\end{aligned}$$

In particular, there exist $\Phi_r \in \operatorname{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0}$ $(r \in \mathbb{N})$ which are polynomials in A, R^{TB}, R^{Cliff_B}, R^{E_B}, μ^E, μ^{Cliff}, (resp. r^X, $R^{\det}$, R^E; resp. h, R^{L_B}, R^L; resp. μ), and their derivatives at x_0 up to order $2r-1$ (resp. $2r-2$; resp. $2r$; resp. $2r+1$), and $\Phi_0 = I_{\mathbb{C}\otimes E_B}$, such that for any $k, m' \in \mathbb{N}$, there exists $C_{k,m'} > 0$ such that for any $x_0 \in X_G$, $p \in \mathbb{N}$,

$$(0.20)\qquad \Big|p^{-n+n_0}\mathscr{I}_p(x_0) - \sum_{r=0}^{k}\Phi_r(x_0)p^{-r}\Big|_{\mathscr{C}^{m'}} \le C_{k,m'}p^{-k-1}.$$

In the rest of Introduction, we will specify our results in the Kähler case.

We suppose now that (X,ω,J) is a compact Kähler manifold and $\mathbf{J} = J$ on X. Assume also that (L,h^L,∇^L), (E,h^E,∇^E) are holomorphic Hermitian vector bundles with holomorphic Hermitian connections, and the action of G on X, L, E is holomorphic.

Let $\overline{\partial}^{L^p\otimes E,*}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p\otimes E}$, then

$$(0.21)\qquad D_p = \sqrt{2}(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*}),$$

and

$$(0.22)\qquad D_p^2 = 2\left(\overline{\partial}^{L^p\otimes E}\overline{\partial}^{L^p\otimes E,*} + \overline{\partial}^{L^p\otimes E,*}\overline{\partial}^{L^p\otimes E}\right)$$

preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(X, L^p\otimes E)$.

By the Kodaira vanishing theorem, for p large enough,

$$(0.23)\qquad (\operatorname{Ker} D_p)^G = H^0(X, L^p\otimes E)^G.$$

Thus for p large enough, $P_p^G(x,x') \in (L^p\otimes E)_x \otimes (L^p\otimes E)^*_{x'}$ and so $P_p^G(x,x) \in \operatorname{End}(E_x)$, $\mathscr{I}_p(x_0) \in \operatorname{End}(E_{x_0})$. In particular, in (0.15),

$$(0.24)\qquad P_{x_0}^{(0)}(0,0) = 2^{\frac{n_0}{2}}\operatorname{Id}_{E_G}.$$

Remark 0.5. In the special case of $E = \mathbb{C}$, $P_p^G(x_0, x_0)$ is a non-negative function on X_G, and (0.16) has been proved in [33, Theorem 1] without knowing the informations on $P_{x_0}^{(2r)}(0,0)$, while in [34, Theorem 1], it was claimed that $P_{x_0}^{(0)}(0,0) = 1$. In [33, Prop. 1], Paoletti knew that for any $l \in \mathbb{N}$, there is $C > 0$ such that for any p, $|P_p^G(x,x)| \leq Cp^{-l}$ uniformly on any compact subset of $X \setminus (\mu^{-1}(0) \cup R)$, with R the subset of unstable points of the action of G. In [34], some Toeplitz operator type properties on X_G was also claimed from the analysis of Toeplitz structures of Boutet de Monvel–Guillemin [11], Boutet de Monvel-Sjöstrand [12] and Shiffman-Zelditch [36]. If we suppose moreover that G is a torus, Charles [15] has also a different version on the Toeplitz operator type properties on X_G.

In Section 4.4, we will show that Theorem 0.2 implies properties of Toeplitz operators on X_G (which also hold in the symplectic case). In particular, we recover the results on Toeplitz operators [15], [34].

Let $\widetilde{h}$ denote the restriction to X_G of the function h defined in (0.10).

The second main result of this paper is that we can in fact obtain the scalar curvature r^{X_G} on the symplectic reduction X_G from $\mathscr{I}_p$.

Theorem 0.6. *If (X, ω) is a compact Kähler manifold and L, E are holomorphic vector bundles with holomorphic Hermitian connections ∇^L, ∇^E, $\mathbf{J} = J$, and G acts freely on $\mu^{-1}(0)$, then for p large enough, $\mathscr{I}_p(x_0) \in \mathrm{End}(E_G)_{x_0}$, and in (0.20), $\Phi_r(x_0) \in \mathrm{End}(E_G)_{x_0}$ are polynomials in A, R^{TB}, R^{E_B}, μ^E, R^E (resp. h, R^{L_B}; resp. μ) and their derivatives at x_0 to order $2r-1$ (resp. $2r$, resp. $2r+1$), and $\Phi_0 = \mathrm{Id}_{E_G}$. Moreover*

$$\Phi_1(x_0) = \frac{1}{8\pi} r_{x_0}^{X_G} + \frac{3}{4\pi} \Delta_{X_G} \log \widetilde{h} + \frac{1}{2\pi} R_{x_0}^{E_G}(w_j^0, \overline{w}_j^0). \tag{0.25}$$

Here r^{X_G} is the Riemannian scalar curvature of (TX_G, g^{TX_G}), Δ_{X_G} is the Bochner-Laplacian on X_G (cf. (1.21)), and $\{w_j^0\}$ is an orthonormal basis of $T^{(1,0)}X_G$.

Since the non-equivariant version of this result has already played a crucial role in the work of Donaldson mentioned before, we have reason to believe that Theorem 0.6 might also play a role in the study of stability of projective manifolds. Indeed, as Donaldson usually interprets his results in the framework of geometric quantization, this seems likely to be so.

We recover (0.6) from (0.25) after taking the trace, and then the integration on X_G. Thus (0.25) is a local version of (0.6) in the spirit of the Local Index Theory. The appearance of the term $\frac{3}{4\pi}\Delta_{X_G} \log \widetilde{h}$ is unexpected.

Let T be the torsion of the connection ${}^0\nabla^{TX}$ in (1.2) on U. The curvature Θ of the principal bundle $U \to B$ relates to the torsion T by (1.6).

Following (3.6) and (5.20), we choose $\{e_j^\perp\}$ to be an orthonormal basis of N_{G,x_0} and $\{\frac{\partial}{\partial z_j^0}\} \in T_{x_0}^{(1,0)}X_G$ to be the holomorphic basis of the normal coordinate on X_G, and define $\mathcal{T}_{klm}, \widetilde{\mathcal{T}}_{jkl}$ as in (5.14). In particular, by Remark 5.3, $\widetilde{\mathcal{T}}_{jkl} = 0$ if G is abelian.

The G-invariant section $\widetilde{\mu}^E$ of $TY \otimes \mathrm{End}(E)$ on U is defined by (1.13) and (1.14).

If there is no other specific notification in the next formula (0.26), when we meet the operation $|\quad|^2$, we will first do this operation, then take the sum of the indices.

Theorem 0.7. *Under the assumption of Theorem 0.6, for $p > 0$ large enough, $P_p^G(x,x) \in \mathrm{End}(E_x)$ and $P_{x_0}^{(r)}(0,0) \in \mathrm{End}(E_{x_0})$. Moreover,*

$$
\begin{aligned}
(0.26)\quad P_{x_0}^{(2)}(0,0) = 2^{\frac{n_0}{2}} \Big\{ & \frac{1}{8\pi} r_{x_0}^{X_G} + \frac{1}{\pi} R^{E_G}\big(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0}\big) + \frac{1}{\pi} \Delta_{X_G} \log \widetilde{h} \\
& - \frac{3}{8\pi} \nabla_{e_k^\perp} \nabla_{e_k^\perp} \log h - \frac{2}{\pi} \sqrt{-1} \nabla_{JT(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \overline{z}_l^0})} \log h - \frac{3}{\pi} \Big| \nabla_{\frac{\partial}{\partial \overline{z}_j^0}} \log h \Big|^2 \\
& - \frac{5}{4\pi} \Big| \nabla_{e_j^\perp} \log h \Big|^2 + \frac{1}{2\pi} |T(e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0})|^2 - \frac{1}{2\pi} \Big| \sum_j T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0}) \Big|^2 \\
& + \frac{1}{2\pi} \Big| T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0}) \Big|^2 + \frac{1}{24\pi} \mathcal{T}_{klm}^2 + \frac{1}{2^6 \pi} \widetilde{\mathcal{T}}_{ijk} (3\widetilde{\mathcal{T}}_{kji} - \widetilde{\mathcal{T}}_{ijk}) + \frac{1}{2\pi} \left\langle \widetilde{\mu}_{x_0}^E, \widetilde{\mu}_{x_0}^E \right\rangle_{g^{TY}} \\
& + \frac{1}{\pi} \Big\langle \widetilde{\mu}^E, T(\tfrac{\partial}{\partial z_l^0}, \tfrac{\partial}{\partial \overline{z}_l^0}) \Big\rangle + \frac{3\sqrt{-1}}{2\pi} \left\langle \widetilde{\mu}^E, J e_k^\perp \right\rangle \nabla_{e_k^\perp} \log h + \frac{\sqrt{-1}}{4\pi} \Big\langle J e_k^\perp, \nabla_{e_k^\perp}^{TY} \widetilde{\mu}^E \Big\rangle \Big\}.
\end{aligned}
$$

Remark 0.8. Certainly, if we only assume that $\mathbf{J} = J$ on U, a neighborhood of $P = \mu^{-1}(0)$, then we still have $\Phi_r(x_0) \in \mathrm{End}(E_G)_{x_0}$, as we work on the kernel of the Dirac operator D_p. Set $\mathscr{I}_{p,0} = I_{\mathbb{C}\otimes E_G} \mathscr{I}_p I_{\mathbb{C}\otimes E_G}$, the component of $\mathscr{I}_p$ on $\mathbb{C} \otimes E_G$. As the computation is local, we still have Theorem 0.6 with $\mathscr{I}_p$ replaced by $\mathscr{I}_{p,0}$ and $\mathscr{I}_p - \mathscr{I}_{p,0} = \mathscr{O}(p^{-\infty})$ (cf. (5.18)). If we only work on the $\overline{\partial}$-operator, i.e. the holomorphic sections, in Section 5.6, we explain how to reduce the case of general $\mathbf{J}$ to the case $\mathbf{J} = J$. Same remark holds for $P_p^G(x_0, x_0)$.

Let $i : P \hookrightarrow X$ be the natural injection.

Let $\pi_G : \mathscr{C}^\infty(P, L^p \otimes E)^G \to \mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$ be the natural identification.

By a result of Zhang [43, Theorem 1.1 and Proposition 1.2], one sees that for p large enough, the map

$$\pi_G \circ i^* : \mathscr{C}^\infty(X, L^p \otimes E)^G \to \mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$$

induces a natural isomorphism

$$(0.27) \qquad \sigma_p = \pi_G \circ i^* : H^0(X, L^p \otimes E)^G \to H^0(X_G, L_G^p \otimes E_G).$$

(When $E = \mathbb{C}$, this result was first proved in [20, Theorem 3.8].)

The following result is a symplectic version of the above isomorphism which is proved in Corollary 4.10, as a simple application of the Toeplitz operator type properties proved in that subsection. It might be regarded as an "asymptotic symplectic quantization identification", generalizing the corresponding holomorphic identification (0.27).

Theorem 0.9. *If X is a compact symplectic manifold and $\mathbf{J} = J$, then the natural map $\sigma_p : (\mathrm{Ker}\, D_p)^G \to \mathrm{Ker}\, D_{G,p}$ defined in (4.108) is an isomorphism for p large enough.*

Let $\langle\ ,\ \rangle_{L_G^p \otimes E_G}$ be the metric on $L_G^p \otimes E_G$ induced by h^{L_G} and h^{E_G}.

In view of [40, (3.54)], the natural Hermitian product on $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$ is the following weighted Hermitian product $\langle\ ,\ \rangle_{\widetilde{h}}$:

$$\langle s_1, s_2\rangle_{\widetilde{h}} = \int_{X_G} \langle s_1, s_2\rangle_{L_G^p\otimes E_G}(x_0)\widetilde{h}^2(x_0)\, dv_{X_G}(x_0). \tag{0.28}$$

In fact, $\pi_G : (\mathscr{C}^\infty(P, L^p\otimes E)^G, \langle\ ,\ \rangle) \to (\mathscr{C}^\infty(X_G, L_G^p\otimes E_G), \langle\ ,\ \rangle_{\widetilde{h}})$ is an isometry.

We still denote by $\langle\quad\rangle$ the scalar product on $H^0(X, L^p\otimes E)^G$ induced by (0.23).

Theorem 0.10. *The isomorphism $(2p)^{-\frac{n_0}{4}}\sigma_p$ is an asymptotic isometry from $(H^0(X, L^p\otimes E)^G, \langle\ ,\ \rangle)$ onto $(H^0(X_G, L_G^p\otimes E_G), \langle\ ,\ \rangle_{\widetilde{h}})$, i.e., if $\{s_i^p\}_{i=1}^{d_p}$ is an orthonormal basis of $(H^0(X, L^p\otimes E)^G, \langle\ ,\ \rangle)$, then*

$$(2p)^{-\frac{n_0}{2}}\langle \sigma_p s_i^p, \sigma_p s_j^p\rangle_{\widetilde{h}} = \delta_{ij} + \mathscr{O}\left(\frac{1}{p}\right). \tag{0.29}$$

From the explicit formula (0.26), one can also get the coefficient of p^{-1} in the expansion (0.29). We leave it to the interested readers.

Let $\widetilde{P}_p^{X_G}$ denote the orthogonal projection from $(\mathscr{C}^\infty(X_G, L_G^p\otimes E_G), \langle\ ,\ \rangle_{\widetilde{h}})$ onto $H^0(X, L_G^p\otimes E_G)$. Let $\widetilde{P}_p^{X_G}(x_0, x_0')$ $(x_0,\, x_0' \in X_G)$ be the smooth kernel of the operator $\widetilde{P}_p^{X_G}$ with respect to $\widetilde{h}^2(x_0')dv_{X_G}(x_0')$.

The following result is an easy consequence of [17, Theorem 1.3].

Theorem 0.11. *Under the assumption of Theorem 0.6, there exist smooth coefficients $\widetilde{\Phi}_r(x_0) \in \mathrm{End}(E_G)_{x_0}$ which are polynomials in R^{TX_G}, R^{E_G} (resp. $\widetilde{h}$), and their derivatives at x_0 to order $2r-1$ (resp. $2r$), and $\widetilde{\Phi}_0 = \mathrm{Id}_{E_G}$, such that for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $x_0 \in X_G$, $p \in \mathbb{N}$,*

$$\left| p^{-n+n_0}\widetilde{h}^2(x_0)\widetilde{P}_p^{X_G}(x_0, x_0) - \sum_{r=0}^{k}\widetilde{\Phi}_r(x_0)p^{-r}\right|_{\mathscr{C}^l} \leq C_{k,l}p^{-k-1}. \tag{0.30}$$

Moreover, the following identity holds,

$$\widetilde{\Phi}_1(x_0) = \frac{1}{8\pi}r_{x_0}^{X_G} + \frac{1}{2\pi}\Delta_{X_G}\log\widetilde{h} + \frac{1}{2\pi}R_{x_0}^{E_G}(w_j^0, \overline{w}_j^0). \tag{0.31}$$

Remark 0.12. From (0.25) and (0.31), one sees that in general $\Phi_1 \neq \widetilde{\Phi}_1$, if $\widetilde{h}$ is not constant on X_G. This reflects a subtle difference between the Bergman kernel and the geometric quantization.

From the works [17], [26] and the present paper, we see clearly that the asymptotic expansion of Bergman kernel is parallel to the small time asymptotic expansion of the heat kernel. To localize the problem, the spectral gap property (2.13) and the finite propagation speed of solutions of hyperbolic equations play essential roles.

Let U be a G-neighborhood of $\mu^{-1}(0)$ as in Theorem 0.2, in this paper, we will then work on U/G.

Indeed, after doing suitable rescaling on the coordinate, we get the limit operator $\mathscr{L}_2^0$ (cf. (3.13)) which is the sum of two parts, along $T_{x_0}X_G$, its kernel is infinite dimensional and gives us the classical Bergman kernel as in $\mathbb{C}^{n-n_0}$, while along N_G, it is a harmonic

oscillator and its kernel is one dimensional. This explains well why we can expect to get the fast decay estimate along N_G in (0.14).

This paper is organized as follows. In Section 1, we study connections and Laplacians associated to a principal bundle. In Section 2, we localize the problem by using the spectral gap property and finite propagation speed, then we use the rescaling technique in local index theory to prove Theorem 2.23 which is a version of Theorem 0.2 without assumption on $\mathbf{J}$. We assume G acts freely on $P = \mu^{-1}(0)$ in Sections 2.5-2.8, and in Section 4.1 we explain Theorem 4.1, the version of Theorem 0.2 where we only assume that μ is regular at 0. In Section 3, we get explicit informations on the coefficients $P^{(r)}$ when $\mathbf{J}$ verifies (3.2), thus we get an effective way to compute its first coefficients of the asymptotic expansion (0.14). Especially, we establish (0.12) and (0.13). In Section 4, we explain various applications of our Theorem 0.2, including Toeplitz properties, etc. In Section 5, we compute the coefficients Φ_1 in Theorem 0.6 and $P^{(2)}_{x_0}(0,0)$ in Theorem 0.7 and in the general case: $\mathbf{J} \neq J$. In Section 6, we prove Theorems 0.10, 0.11.

Some results of this paper have been announced in [29, 30].

Notation : In the whole paper, if there is no other specific notification, when in a formula a subscript index appears two times, we sum up with this index.

1. Connections and Laplacians associated to a principal bundle

In this Section, for $\pi : X \to B = X/G$ a G-principal bundle, we will study the associated connections and Bochner-Laplacians. The results in this Section extend the corresponding ones in [2, §1d)] and [1, §5.1, 5.2] where the metric along the fiber is parallel along the horizontal direction. These results will be used in Proposition 2.7 and in Sections 3.3, 5.

If G acts only infinitesimal freely on X, then $B = X/G$ is an orbifold. The results in this Section can be extended easily to this situation, as will be explained in Section 4.1.

This Section is organized as follows. In Section 1.1, we study the Levi-Civita connection for a principal bundle which extends the results of [2, §1d)]. In Section 1.2, we study the relation of the Laplacians on the total and base manifolds.

1.1. Connections associated to a principal bundle.

Let a compact connected Lie group G acts smoothly on the left on a smooth manifold X and $\dim G = n_0$. We suppose temporary that G acts freely on X. Then

$$\pi : X \to B = X/G$$

is a G-principal bundle. We denote by TY the relative tangent bundle for the fibration $\pi : X \to B$.

Let g^{TX} be a G-invariant metric on TX. Let ∇^{TX} be the Levi-Civita connection on TX. By the explicit equation for $\langle \nabla^{TX}_{\cdot} \cdot, \cdot \rangle$ in [1, (1.18)], for W, Z, Z' vector fields on X,

$$\begin{aligned} 2\left\langle \nabla^{TX}_W Z, Z' \right\rangle = {} & W\left\langle Z, Z' \right\rangle + Z\left\langle W, Z' \right\rangle - Z'\left\langle W, Z \right\rangle \\ & - \left\langle W, [Z, Z'] \right\rangle - \left\langle Z, [W, Z'] \right\rangle + \left\langle Z', [W, Z] \right\rangle . \end{aligned} \tag{1.1}$$

Let $T^H X$ be the orthogonal complement of TY in TX.

For $U \in TB$, let $U^H \in T^H X$ be the lift of U.

Let $g^{TY}, g^{T^H X}$ be G-invariant metrics on $TY, T^H X$ induced by g^{TX}. Let P^{TY}, $P^{T^H X}$ be the orthogonal projections from TX onto TY, $T^H X$.

Let g^{TB} be the metric on TB induced by $g^{T^H X}$. Let ∇^{TB} be the Levi-Civita connection on (TB, g^{TB}) with curvature R^{TB}. Set

$$\nabla^{T^H X} = \pi^* \nabla^{TB}, \quad \nabla^{TY} = P^{TY} \nabla^{TX} P^{TY}, \quad {}^0\nabla^{TX} = \nabla^{TY} \oplus \nabla^{T^H X}. \tag{1.2}$$

Then $\nabla^{T^H X}$, ${}^0\nabla^{TX}$ define Euclidean connections on $T^H X$, TX, and ∇^{TY} is the connection on TY induced by ∇^{TX} (cf. [2, Def. 1.6]).

Let T be the torsion of ${}^0\nabla^{TX}$, and let $S \in T^*X \otimes \mathrm{End}(TX)$, $\dot{g}^{TY}_{\cdot} \in T^*B \otimes \mathrm{End}(TY)$ be defined by

$$S = \nabla^{TX} - {}^0\nabla^{TX}, \qquad \dot{g}^{TY}_U = (g^{TY})^{-1}(L_{U^H} g^{TY}) \quad \text{for } U \in TB. \tag{1.3}$$

Then S is a 1-form on X taking values in the skew-adjoint endomorphisms of TX.

By [6, Theorem 1.2] (cf. [5, Theorems 1.1 and 1.2]) the proof of which can also be found in [1, Prop. 10.2] where one applies directly (1.1), we know that ∇^{TY} is the

Levi-Civita connection on TY along the fiber Y, and for $U \in TB$,

$$\nabla^{TY}_{U^H} = L_{U^H} + \frac{1}{2}(g^{TY})^{-1}(L_{U^H} g^{TY}) = L_{U^H} + \frac{1}{2}\dot{g}^{TY}_U. \tag{1.4}$$

Let $\mathfrak{g}$ be the Lie algebra of G. For $K \in \mathfrak{g}$, we denote by $K^X_x = \frac{\partial}{\partial t} e^{-tK} x|_{t=0}$ the corresponding vector field on X, then $gK^X_x = (\mathrm{Ad}_g(K))^X_{gx}$. Thus we can identify the trivial bundle $X \times \mathfrak{g}$ with Ad-action of G on $\mathfrak{g}$ to the G-equivariant bundle TY by the map $K \to K^X$.

Let $\theta : TX \to \mathfrak{g}$ be the connection form of the principal bundle $\pi : X \to B$ such that $T^H X = \mathrm{Ker}\,\theta$, and Θ its curvature.

For $K_1, K_2 \in \mathfrak{g}$, $U, V \in TB$, as U^H is G-invariant, we have

$$L_{U^H} K^X_1 = -[K^X_1, U^H] = 0. \tag{1.5}$$

By (1.4), (1.5), we get $T \in \Lambda^2(T^*X) \otimes TY$ and

$$\begin{aligned} &T(U^H, V^H) = \Theta(U^H, V^H) = -P^{TY}[U^H, V^H], \quad T(K^X_1, K^X_2) = 0, \\ &T(U^H, K^X_1) = \frac{1}{2}(g^{TY})^{-1}(L_{U^H} g^{TY}) K^X_1 = \frac{1}{2}\dot{g}^{TY}_U K^X_1. \end{aligned} \tag{1.6}$$

And by (1.1), (1.4), (1.5) and (1.6), for $W \in TX$, we have (cf. also [2, (1.28)], [1, Prop. 10.6]),

$$\begin{aligned} &S(W)(TY) \subset T^H X, \quad S(U^H) V^H \in TY, \\ &2\left\langle S(U^H) K^X_1, V^H \right\rangle = 2\left\langle S(K^X_1) U^H, V^H \right\rangle = \left\langle T(U^H, V^H), K^X_1 \right\rangle, \\ &\left\langle S(K^X_2) U^H, K^X_1 \right\rangle = -\left\langle S(K^X_2) K^X_1, U^H \right\rangle \\ &\qquad = \frac{1}{2} U^H \left\langle K^X_2, K^X_1 \right\rangle = \left\langle T(U^H, K^X_1), K^X_2 \right\rangle. \end{aligned} \tag{1.7}$$

Let $\{e_i\}$ be an orthonormal basis of TB. By (1.3) and (1.7), for Y a section of TY,

$$\nabla^{TX}_{U^H} Y = \nabla^{TY}_{U^H} Y + \frac{1}{2}\left\langle T(U^H, e^H_i), Y \right\rangle e^H_i. \tag{1.8}$$

Proposition 1.1. *Let $\{f_l\}_{l=1}^{n_0}$ be a G-invariant orthonormal frame of TY, then*

$$\sum_{l=1}^{n_0} \nabla^{TY}_{f_l} f_l = 0. \tag{1.9}$$

Proof. (1.9) is analogue to the fact that any left invariant volume form on G is also right invariant. We only need to work on a fiber Y_b, $b \in B$.

Let dv_Y be the Riemannian volume form on Y_b.

By using $L_{f_k} f_l = \nabla^{TY}_{f_k} f_l - \nabla^{TY}_{f_l} f_k$ and dv_Y is preserved by ∇^{TY} on Y_b, we get

$$L_{f_k} dv_Y = \sum_{l=1}^{n_0} \left\langle \nabla^{TY}_{f_l} f_k, f_l \right\rangle dv_Y. \tag{1.10}$$

Now from $L_{f_k} = i_{f_k} d^Y + d^Y i_{f_k}$ and $\left\langle \nabla^{TY}_{f_l} f_k, f_l \right\rangle$ is G-invariant and (1.10), we get

$$0 = \int_{Y_b} L_{f_k} dv_Y = \sum_{l=1}^{n_0} \left\langle \nabla^{TY}_{f_l} f_k, f_l \right\rangle \int_{Y_b} dv_Y. \tag{1.11}$$

From (1.11), we get (1.9). □

Remark 1.2. If g^{TY} is induced by a family of Ad-invariant metric on $\mathfrak{g}$ under the isomorphism from $X \times \mathfrak{g}$ to TY defined by $K \to K^X$, then (1.9) is trivial. In this case, as in [19, Theorem 11.3], for Y_1, Y_2 two G-invariant sections of TY, by (1.1), we have

$$\nabla^{TY}_{Y_1} Y_2 = \frac{1}{2}[Y_1, Y_2]. \tag{1.12}$$

1.2. **Curvatures and Laplacians associated to a principal bundle.** Let (F, h^F) be a G-equivariant Hermitian vector bundle on X with a G-invariant Hermitian connection ∇^F on X. For any $K \in \mathfrak{g}$, denote by L_K the infinitesimal action induced by K on the corresponding vector bundles.

Let μ^F be the section of $\mathfrak{g}^* \otimes \mathrm{End}(F)$ on X defined by,

$$\mu^F(K) = \nabla^F_{K^X} - L_K \quad \text{for } K \in \mathfrak{g}. \tag{1.13}$$

By using the identification $X \times \mathfrak{g} \to TY$, μ^F defines a G-invariant section $\widetilde{\mu}^F$ of $TY \otimes \mathrm{End}(F)$ on X such that

$$\left\langle \widetilde{\mu}^F, K^X \right\rangle = \mu^F(K). \tag{1.14}$$

The curvature R^F_μ of the Hermitian connection $\nabla^F - \mu^F(\theta)$ on F is G-invariant. Moreover as ∇^F is G-invariant, by (1.13),

$$R^F_\mu(K^X, v) = [L_K, \nabla^F - \mu^F(\theta)](v) = 0 \tag{1.15}$$

for $K \in \mathfrak{g}$, $v \in TX$, and

$$R^F_\mu = R^F - \nabla^F(\mu^F(\theta)) + \mu^F(\theta) \wedge \mu^F(\theta). \tag{1.16}$$

The Hermitian vector bundle (F, h^F) induces a Hermitian vector bundle (F_B, h^{F_B}) on B by identifying G-invariant sections of F on X.

For $s \in \mathscr{C}^\infty(B, F_B) \simeq \mathscr{C}^\infty(X, F)^G$, we define

$$\nabla^{F_B}_U s = \nabla^F_{U^H} s. \tag{1.17}$$

Then ∇^{F_B} is a Hermitian connection on F_B with curvature R^{F_B}.

Observe that ∇^{F_B} is the restriction of the connection $\nabla^F - \mu^F(\theta)$ to $\mathscr{C}^\infty(X, F)^G$, and R^{F_B} is the section induced by R^F_μ. From (1.16), for $U_1, U_2 \in TB$, we get

$$R^{F_B}(U_1, U_2) = R^F(U_1^H, U_2^H) - \mu^F(\Theta)(U_1, U_2). \tag{1.18}$$

Let dv_X be the Riemannian volume form on (X, g^{TX}). We define a scalar product on $\mathscr{C}^\infty(X, F)$ by

$$\langle s_1, s_2 \rangle = \int_X \langle s_1, s_2 \rangle_F(x)\, dv_X(x)\,. \tag{1.19}$$

As in (1.19), h^{F_B}, g^{TB} induce a natural scalar product $\langle \quad \rangle$ on $\mathscr{C}^\infty(B, F_B)$.

Denote by $\mathrm{vol}(Gx)$ $(x \in X)$ the volume of the orbit Gx equipped with the metric induced by g^{TX}. The function

$$h(x) = \sqrt{\mathrm{vol}(Gx)}, \qquad x \in X,$$

as in (0.10) is G-invariant and defines a function on B.

Denote by $\pi_G : \mathscr{C}^\infty(X,F)^G \to \mathscr{C}^\infty(B,F_B)$ the natural identification. Then the map

$$\Phi = h\pi_G : (\mathscr{C}^\infty(X,F)^G, \langle\ ,\ \rangle) \to (\mathscr{C}^\infty(B,F_B), \langle\ ,\ \rangle) \tag{1.20}$$

is an isometry.

Let $\{e_a\}_{a=1}^m$ be an orthonormal frame of TX.

Let (E,h^E) be a Hermitian vector bundle on X and let ∇^E be a Hermitian connection on E. The usual Bochner Laplacians Δ^E, Δ_X are defined by

$$\Delta^E := -\sum_{a=1}^m \left((\nabla^E_{e_a})^2 - \nabla^E_{\nabla^{TX}_{e_a} e_a} \right), \quad \Delta_X = \Delta^{\mathbb{C}}. \tag{1.21}$$

Let $\{f_l\}_{l=1}^{n_0}$ be a G-invariant orthonormal frame of TY, and $\{f^l\}$ its dual basis, and let $\{e_i\}$ be an orthonormal frame of TB, then $\{e_i^H, f_l\}$ is an orthonormal frame of TX.

To simplify the notation, for $\sigma_1, \sigma_2 \in TY \otimes \mathrm{End}(F)$, we denote by $\langle \sigma_1, \sigma_2 \rangle_{g^{TY}} \in \mathrm{End}(F)$ the contraction of $\sigma_1 \otimes \sigma_2$ on the part of TY by g^{TY}. In particular,

$$\langle \widetilde{\mu}^F, \widetilde{\mu}^F \rangle_{g^{TY}} = \sum_{l=1}^{n_0} \langle \widetilde{\mu}^F, f_l \rangle^2 \in \mathrm{End}(F). \tag{1.22}$$

The following result extends [1, Prop. 5.6, 5.10] where $F = X \times_G V$ for a G-representation V, and where g^{TY} is induced by a fixed Ad-invariant metric on $\mathfrak{g}$ under the isomorphism from $X \times \mathfrak{g}$ to TY defined by $K \to K^X$ (Thus h is constant on B).

Theorem 1.3. *As an operator on $\mathscr{C}^\infty(B,F_B)$, we have*

$$\Phi \Delta^F \Phi^{-1} = \Delta^{F_B} - \langle \widetilde{\mu}^F, \widetilde{\mu}^F \rangle_{g^{TY}} - \frac{1}{h} \Delta_B h. \tag{1.23}$$

Proof. At first by (1.6) and (1.7),

$$\begin{aligned} \frac{1}{h}(e_i h) &= \frac{1}{2}(L_{e_i^H} dv_Y)/dv_Y = \frac{1}{2} \left\langle L_{e_i^H} f^l, f^l \right\rangle = -\frac{1}{2} \left\langle L_{e_i^H} f_l, f_l \right\rangle \\ &= \frac{1}{4}(L_{e_i^H} g^{TY})(f_l, f_l) = \frac{1}{2} \left\langle T(e_i^H, f_l), f_l \right\rangle = -\frac{1}{2} \left\langle S(f_l) f_l, e_i^H \right\rangle. \end{aligned} \tag{1.24}$$

As $\widetilde{\mu}^F$ is G-invariant, then $\langle \widetilde{\mu}^F, f_l \rangle$ is also a G-invariant section of $\mathrm{End}(F)$.

By (1.13), $\nabla^F_{f_l} = \langle \widetilde{\mu}^F, f_l \rangle$ on $\mathscr{C}^\infty(X,F)^G$, and by (1.3), $\nabla^{TX}_{f_l} f_l = \nabla^{TY}_{f_l} f_l + S(f_l) f_l$, thus by (1.20), we get for $1 \le l \le n_0$,

$$\Phi[(\nabla^F_{f_l})^2 - \nabla^F_{\nabla^{TX}_{f_l} f_l}]\Phi^{-1} = \langle \widetilde{\mu}^F, f_l \rangle^2 - \langle \widetilde{\mu}^F, \nabla^{TY}_{f_l} f_l \rangle - h \nabla^{F_B}_{S(f_l) f_l} h^{-1}. \tag{1.25}$$

From (1.7), (1.9), (1.21), (1.22), (1.24) and (1.25), we have

$$\begin{aligned} \Phi \Delta^F \Phi^{-1} &= -\sum_{i=1}^{2n-n_0} \Phi \Big[(\nabla^F_{e_i^H})^2 - \nabla^F_{\nabla^{TX}_{e_i^H} e_i^H} \Big] \Phi^{-1} - \sum_{l=1}^{n_0} \Phi \Big[(\nabla^F_{f_l})^2 - \nabla^F_{\nabla^{TX}_{f_l} f_l} \Big] \Phi^{-1} \\ &= h \Delta^{F_B} h^{-1} - \sum_{l=1}^{n_0} \langle \widetilde{\mu}^F, f_l \rangle^2 - 2(e_i h) \nabla^{F_B}_{e_i} h^{-1} = \Delta^{F_B} - \langle \widetilde{\mu}^F, \widetilde{\mu}^F \rangle_{g^{TY}} - \frac{1}{h} \Delta_B h. \end{aligned} \tag{1.26}$$

□

2. G-invariant Bergman kernels

In this Section, we study the uniform estimate with its derivatives on $t = \frac{1}{\sqrt{p}}$ of the G-invariant Bergman kernel $P_p^G(x, x')$ of D_p^2 as $p \to \infty$.

The first main difficulty is to localize the problem to arbitrary small neighborhoods of $P = \mu^{-1}(0)$, so that one can study the G-invariant Bergman kernel in the spirit of [17]. Our observation here is that the G-invariant Bergman kernel is exactly the kernel of the orthogonal projection on the zero space of an operator $\mathcal{L}_p$, which is a deformation of D_p^2 by the Casimir operator. Moreover, $\mathcal{L}_p$ has a spectral gap property (cf. (2.22), (2.23)). In the spirit of [17, §3], this allows us to localize the problem to a problem near a G-neighborhood of Gx. By combining with the Lichnerowicz formula, we get Theorem 0.1 in Section 2.4.

After localizing the problem to a problem near P, we first replace X by $G \times \mathbb{R}^{2n-n_0}$, then we reduce it to a problem on $\mathbb{R}^{2n-n_0}$. On $\mathbb{R}^{2n-n_0}$, the problem in Section 2.7 is similar to a problem on $\mathbb{R}^{2n}$ considered in [17, §3.3].

Comparing with the operator in [17, §3.3], we have an extra quadratic term along the normal direction of X_G. This allows us to improve the estimate in the normal direction. After suitable rescaling, we will introduce a family of Sobolev norms defined by the rescaled connection on L^p and the rescaled moment map in this situation, then we can extend the functional analysis techniques developed in [17, §3.3] and [7, §11].

This section is organized as follows. In Section 2.1, we recall a basic property on the Casimir operator of a compact connected Lie group. In Section 2.2, we recall the definition of spin^c Dirac operators for an almost complex manifold. In Section 2.3, we introduce the operator $\mathcal{L}_p$ to study the G-invariant Bergman kernel P_p^G of D_p^2. In Section 2.4, we explain that the asymptotic expansion of $P_p^G(x, x')$ is localized on a G-neighborhood of Gx, and we establish Theorem 0.1. In Section 2.5, we show that our problem near P is equivalent to a problem on U/G for any open G-neighborhood U of P. In Section 2.6, we derive an asymptotic expansion of $\Phi\mathcal{L}_p\Phi^{-1}$ in coordinates of U/G. In Section 2.7, we study the uniform estimate with its derivatives on t of the Bergman kernel associated to the rescaled operator $\mathscr{L}_2^t$ from $\Phi\mathcal{L}_p\Phi^{-1}$ using heat kernel. In Theorem 2.21, we estimate uniformly the remainder term of the Taylor expansion of $e^{-u\mathscr{L}_2^t}$ for $u \geq u_0 > 0$, $0 < t \leq t_0 \leq 1$. In Section 2.8, we identify $J_{r,u}$, the coefficient of the Taylor expansion of $e^{-u\mathscr{L}_2^t}$, with the Volterra expansion of the heat kernel, thus giving a way to compute the coefficient $P_{x_0}^{(r)}$ in Theorem 0.2. In Section 2.8, we prove Theorem 0.2 except (0.12) and (0.13).

We use the notation in Section 1. In Sections 2.5-2.8, we assume G acts freely on $P = \mu^{-1}(0)$.

2.1. **Casimir operator.**

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and $\dim G = n_0$. We choose an Ad-invariant metric on $\mathfrak{g}$ such that it is the minus Killing form on the semi-simple part of $\mathfrak{g}$.

Let $\{K_j\}_{j=1}^{\dim G}$ be an orthogonal basis of $\mathfrak{g}$ and $\{K^j\}$ be its dual basis of $\mathfrak{g}^*$.

The Casimir operator Cas of $\mathfrak{g}$ is defined as the following element of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$,

$$\mathrm{Cas} := \sum_{j=1}^{\dim G} K_j K_j. \tag{2.1}$$

Then Cas is independent of the choice of $\{K_j\}$ and belongs to the center of $U(\mathfrak{g})$.

Let $\mathfrak{t}$ be the Lie algebra of a maximum torus T of G, and $\mathfrak{t}^*$ its dual. Let $|\quad|$ denote the norm on $\mathfrak{t}^*$ induced by the Ad-invariant metric on $\mathfrak{g}$.

Let $\mathcal{W} \subset \mathfrak{t}^*$ be the fundamental Weyl chamber associated to the set of positive roots Δ^+ of G, and its closure $\overline{\mathcal{W}} \subset \mathfrak{t}^*$.

Let $I = \{K \in \mathfrak{t}; \exp(2\pi K) = 1 \in T\}$ the integer lattice such that $T = \mathfrak{t}/2\pi I$, and $P = \{\alpha \in \mathfrak{t}^*; \alpha(I) \subset \mathbb{Z}\}$ the lattice of integral forms.

Let ϱ_G be the half sum of the positive roots of G.

By the Weyl character formula [19, Theorem 8.21], the irreducible representations correspond one to one to $\vartheta \in \overline{\mathcal{W}} \cap P$, the highest weight of the representation.

Moreover, for any irreducible representation $\rho : G \to \mathrm{End}(E)$ with highest weight $\vartheta \in \overline{\mathcal{W}} \cap P$, classically, the action of Cas on E is given by (cf. [19, Theorem 10.6]),

$$\rho(\mathrm{Cas}) = -(|\vartheta + \varrho_G|^2 - |\varrho_G|^2)\,\mathrm{Id}_E\,. \tag{2.2}$$

Set

$$\nu_1 := \inf_{0 \neq \vartheta \in \overline{\mathcal{W}} \cap P} (|\vartheta + \varrho_G|^2 - |\varrho_G|^2) > 0. \tag{2.3}$$

By (2.2), for any representation $\rho : G \to \mathrm{End}(E)$, if the G-invariant subspace E^G of E is zero, then

$$-\rho(\mathrm{Cas}) \geq \nu_1\,\mathrm{Id}_E\,. \tag{2.4}$$

2.2. **Spin^c Dirac operator.** Let (X, ω) be a compact symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle L over X endowed with a Hermitian connection ∇^L with the property that

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega,$$

where $R^L = (\nabla^L)^2$ is the curvature of (L, ∇^L).

Let (E, h^E) be a Hermitian vector bundle on X with Hermitian connection ∇^E and its curvature R^E.

Let g^{TX} be a Riemannian metric on X.

Let $\mathbf{J} : TX \longrightarrow TX$ be the skew–adjoint linear map which satisfies the relation

$$\omega(u, v) = g^{TX}(\mathbf{J}u, v) \tag{2.5}$$

for $u, v \in TX$.

Let J be an almost complex structure such that

$$g^{TX}(Ju, Jv) = g^{TX}(u, v), \qquad \omega(Ju, Jv) = \omega(u, v), \tag{2.6}$$

and that $\omega(\cdot, J\cdot)$ defines a metric on TX. Then J commutes with $\mathbf{J}$ and

$$-\langle J\mathbf{J}\cdot, \cdot\rangle = \omega(\cdot, J\cdot)$$

is positive by our assumption. Thus $-J\mathbf{J} \in \mathrm{End}(TX)$ is symmetric and positive, and one verifies easily that

$$-J\mathbf{J} = (-\mathbf{J}^2)^{1/2}, \quad J = \mathbf{J}(-\mathbf{J}^2)^{-1/2}. \tag{2.7}$$

The almost complex structure J induces a splitting

$$TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X,$$

where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Let $T^{*(1,0)}X$ and $T^{*(0,1)}X$ be the corresponding dual bundles.

For any $v \in TX \otimes_{\mathbb{R}} \mathbb{C}$ with decomposition $v = v_{1,0} + v_{0,1} \in T^{(1,0)}X \oplus T^{(0,1)}X$, let $\overline{v}^*_{1,0} \in T^{*(0,1)}X$ be the metric dual of $v_{1,0}$. Then

$$c(v) := \sqrt{2}(\overline{v}^*_{1,0} \wedge -i_{v_{0,1}}) \tag{2.8}$$

defines the Clifford action of v on $\Lambda(T^{*(0,1)}X)$, where $\wedge$ and i denote the exterior and interior multiplications respectively.

Set

$$\nu_0 := \inf_{u \in T^{(1,0)}_x X,\, x \in X} R^L_x(u, \overline{u})/|u|^2_{g^{TX}} > 0. \tag{2.9}$$

Let ∇^{TX} be the Levi-Civita connection of the metric g^{TX} with curvature R^{TX}. We denote by $P^{T^{(1,0)}X}$ the projection from $TX \otimes_{\mathbb{R}} \mathbb{C}$ to $T^{(1,0)}X$.

Let $\nabla^{T^{(1,0)}X} = P^{T^{(1,0)}X} \nabla^{TX} P^{T^{(1,0)}X}$ be the Hermitian connection on $T^{(1,0)}X$ induced by ∇^{TX} with curvature $R^{T^{(1,0)}X}$.

By [22, pp.397–398], ∇^{TX} induces canonically a Clifford connection ∇^{Cliff} on $\Lambda(T^{*(0,1)}X)$ and its curvature R^{Cliff} (cf. also [25, §2]).

Let $\{e_a\}_a$ be an orthonormal basis of TX. Then

$$R^{\mathrm{Cliff}} = \frac{1}{4}\sum_{ab} \langle R^{TX} e_a, e_b\rangle c(e_a)c(e_b) + \frac{1}{2}\,\mathrm{Tr}\left[R^{T^{(1,0)}X}\right]. \tag{2.10}$$

Let ∇^{E_p} be the connection on

$$E_p := \Lambda(T^{*(0,1)}X) \otimes L^p \otimes E \tag{2.11}$$

induced by ∇^{Cliff}, ∇^L and ∇^E.

Let $\langle\ \ \rangle_{E_p}$ be the metric on E_p induced by g^{TX}, h^L and h^E.

The L^2–scalar product $\langle\ \ \rangle$ on $\Omega^{0,\bullet}(X, L^p \otimes E)$, the space of smooth sections of E_p, is given by (1.19). We denote the corresponding norm by $\|\cdot\|_{L^2}$.

Definition 2.1. The spin^c Dirac operator D_p is defined by

$$D_p := \sum_{a=1}^{2n} c(e_a)\nabla^{E_p}_{e_a} : \Omega^{0,\bullet}(X, L^p \otimes E) \longrightarrow \Omega^{0,\bullet}(X, L^p \otimes E)\,. \tag{2.12}$$

Clearly, D_p is a formally self–adjoint, first order elliptic differential operator on $\Omega^{0,\bullet}(X, L^p\otimes E)$, which interchanges $\Omega^{0,\mathrm{even}}(X, L^p\otimes E)$ and $\Omega^{0,\mathrm{odd}}(X, L^p\otimes E)$.

If A is any operator, we denote by Spec(A) the spectrum of A.

The following result was proved in [25, Theorems 1.1, 2.5] by applying directly the Lichnerowicz formula (cf. also [8, Theorem 1] in the holomorphic case).

Theorem 2.2. *There exists $C_L > 0$ such that for any $p \in \mathbb{N}$ and any $s \in \Omega^{>0}(X, L^p \otimes E) = \bigoplus_{q\geqslant 1} \Omega^{0,q}(X, L^p \otimes E)$,*

$$\|D_p s\|^2_{L^2} \geqslant (2p\nu_0 - C_L)\|s\|^2_{L^2}\,. \tag{2.13}$$

Moreover $\mathrm{Spec}(D_p^2) \subset \{0\} \cup [2p\nu_0 - C_L, +\infty[$.

2.3. **G-invariant Bergman kernel.** Suppose that the compact connected Lie group G acts on the left of X, and the action of G lifts on L, E and preserves the metrics and connections, ω and the almost complex structure J.

Let $\mu : X \to \mathfrak{g}^*$ be defined by

$$2\pi\sqrt{-1}\mu(K) := \mu^L(K) = \nabla^L_{K^X} - L_K,\ K \in \mathfrak{g}. \tag{2.14}$$

Then μ is the corresponding moment map (cf. [1, Example. 7.9]), i.e. for any $K \in \mathfrak{g}$,

$$d\mu(K) = i_{K^X}\omega. \tag{2.15}$$

For V a subspace of $\Omega^{0,\bullet}(X, L^p \otimes E)$, we denote by $V^\perp$ the orthogonal complement of V in $(\Omega^{0,\bullet}(X, L^p \otimes E), \langle\ \ \rangle)$.

Let $\Omega^{0,\bullet}(X, L^p \otimes E)^G$, $(\mathrm{Ker}\, D_p)^G$ be the G-invariant subspaces of $\Omega^{0,\bullet}(X, L^p \otimes E)$, $\mathrm{Ker}\, D_p$. Let P_p^G be the orthogonal projection from $\Omega^{0,\bullet}(X, L^p \otimes E)$ on $(\mathrm{Ker}\, D_p)^G$.

Definition 2.3. The G-invariant Bergman kernel $P_p^G(x, x')$ $(x, x' \in X)$ of D_p is the smooth kernel of P_p^G with respect to the Riemannian volume form $dv_X(x')$.

Let $\{S_i^p\}_{i=1}^{d_p}$ ($d_p = \dim(\mathrm{Ker}\, D_p)^G$) be any orthonormal basis of $(\mathrm{Ker}\, D_p)^G$ with respect to the norm $\|\ \|_{L^2}$, then

$$P_p^G(x, x') = \sum_{i=1}^{d_p} S_i^p(x) \otimes (S_i^p(x'))^* \in (E_p)_x \otimes (E_p^*)_{x'}. \tag{2.16}$$

Especially, $P_p^G(x, x) \in \mathrm{End}(E_p)_x \simeq \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_x$.

We use the notation μ^F in (1.13) now.

Recall that the Lie derivative L_K on TX is given by

$$L_K V = \nabla^{TX}_{K^X} V - \nabla^{TX}_V K^X. \tag{2.17}$$

Thus

$$\mu^{TX}(K) = \nabla^{TX}_{\cdot} K^X \in \mathrm{End}(TX), \tag{2.18}$$

and the action on $\Lambda(T^{*(0,1)}X)$ induced by $\mu^{TX}(K)$ is given by

$$\mu^{\mathrm{Cliff}}(K) = \frac{1}{4}\sum_{a=1}^{2n} c(e_a)c(\nabla^{TX}_{e_a} K^X) + \frac{1}{2}\,\mathrm{Tr}[P^{T^{(1,0)}X}\nabla^{TX}_{\cdot} K^X]. \tag{2.19}$$

Thus the action L_K of K on smooth sections of $\Lambda(T^{*(0,1)}X)$ is given by (cf. [40, (1.24)])

$$L_K = \nabla^{\mathrm{Cliff}}_{K^X} - \mu^{\mathrm{Cliff}}(K). \tag{2.20}$$

By (2.14) and (2.20), the action L_K of K on $\Omega^{0,\bullet}(X, L^p \otimes E)$ is $\nabla^{E_p}_{K^X} - \mu^{E_p}(K)$ with

$$\mu^{E_p}(K) = 2\pi\sqrt{-1}p\mu(K) + \mu^E(K) + \mu^{\mathrm{Cliff}}(K). \tag{2.21}$$

Definition 2.4. The (formally) self-adjoint operator $\mathcal{L}_p$ acting on $(\Omega^{0,\bullet}(X, L^p \otimes E), \langle\ ,\ \rangle)$ is defined by,

$$\mathcal{L}_p = D_p^2 - p\sum_{i=1}^{\dim G} L_{K_i} L_{K_i}. \tag{2.22}$$

The following result will play a crucial role in the whole paper.

Theorem 2.5. *The projection P_p^G is the orthogonal projection from $\Omega^{0,\bullet}(X, L^p \otimes E)$ onto* $\mathrm{Ker}(\mathcal{L}_p)$. *Moreover, there exist ν, $C_L > 0$ such that for any $p \in \mathbb{N}$,*

$$\begin{aligned} &\mathrm{Ker}(\mathcal{L}_p) = (\mathrm{Ker}\, D_p)^G, \\ &\mathrm{Spec}(\mathcal{L}_{\mathrm{p}}) \subset \{0\} \cup [2p\nu - C_L, +\infty[. \end{aligned} \tag{2.23}$$

Proof. By (2.22), for any $s \in \Omega^{0,\bullet}(X, L^p \otimes E)$,

$$\langle \mathcal{L}_p s, s\rangle = \|D_p s\|^2_{L^2} + p\sum_{i=1}^{\dim G} \|L_{K_i} s\|^2_{L^2}. \tag{2.24}$$

Thus $\mathcal{L}_p s = 0$ is equivalent to

$$D_p s = L_{K_i} s = 0. \tag{2.25}$$

This means s is fixed by the G-action. Thus we get the first equation of (2.23).

For $s \in (\mathrm{Ker}\, \mathcal{L}_p)^\perp$, there exist $s_1 \in \Omega^{0,\bullet}(X, L^p \otimes E)^G \cap (\mathrm{Ker}\, D_p)^\perp$, $s_2 \in (\Omega^{0,\bullet}(X, L^p \otimes E)^G)^\perp$, such that $s = s_1 + s_2$. Clearly,

$$D_p s_1 \in \Omega^{0,\bullet}(X, L^p \otimes E)^G, \quad D_p s_2 \in (\Omega^{0,\bullet}(X, L^p \otimes E)^G)^\perp.$$

By Theorem 2.2 and (2.4),

$$\begin{aligned} \langle \mathcal{L}_p s, s\rangle &= -p\langle \rho(\mathrm{Cas}) s_2, s_2\rangle + \|D_p s_2\|^2_{L^2} + \|D_p s_1\|^2_{L^2} \\ &\geq p\nu_1 \|s_2\|^2_{L^2} + (2p\nu_0 - C_L)\|s_1\|^2_{L^2}, \end{aligned} \tag{2.26}$$

from which we get (2.23). □

We assume that $0 \in \mathfrak{g}^*$ is a regular value of μ. Then $X_G = \mu^{-1}(0)/G$ is an orbifold (X_G is smooth if G acts freely on $P = \mu^{-1}(0)$). Furthermore, ω descends to a symplectic form ω_G on X_G. Thus one gets the Marsden-Weinstein symplectic reduction (X_G, ω_G).

Moreover, (L, ∇^L), (E, ∇^E) descend to (L_G, ∇^{L_G}), (E_G, ∇^{E_G}) over X_G so that the corresponding curvature condition holds [20] :

$$\frac{\sqrt{-1}}{2\pi} R^{L_G} = \omega_G. \tag{2.27}$$

The G-invariant almost complex structure J also descends to an almost complex structure J_G on TX_G, and h^L, h^E, g^{TX} descend to h^{L_G}, h^{E_G}, g^{TX_G}.

We can construct the corresponding spin^c Dirac operator $D_{G,p}$ on X_G.

Let $P_{G,p}$ be the orthogonal projection from $\Omega^{0,\bullet}(X_G, L_G^p \otimes E_G)$ on $\operatorname{Ker} D_{G,p}$, and let $P_{G,p}(x,x')$ be the smooth kernel of $P_{G,p}$ with respect to the Riemannian volume form $dv_{X_G}(x')$.

The purpose of this paper is to study the asymptotic expansion of $P_p^G(x,x')$ when $p \to \infty$, and we will relate it to the asymptotic expansion of the Bergman kernel $P_{G,p}$ on X_G.

2.4. **Localization of the problem.** Let a^X be the injectivity radius of (X, g^{TX}), and $\varepsilon \in (0, a^X/4)$. If $x \in X$, $Z \in T_xX$, let $\mathbb{R} \ni u \to x_u = \exp_x^X(uZ) \in X$ be the geodesic in (X, g^{TX}), such that $x_0 = x$, $\frac{dx_u}{du}|_{u=0} = Z$.

For $x \in X$, we denote by $B^X(x,\varepsilon)$ and $B^{T_xX}(0,\varepsilon)$ the open balls in X and T_xX with center x and radius ε, respectively. The map $T_xX \ni Z \to \exp_x^X(Z) \in X$ is a diffeomorphism from $B^{T_xX}(0,\varepsilon)$ on $B^X(x,\varepsilon)$ for $\varepsilon \leq a^X$.

From now on, we identify $B^{T_xX}(0,\varepsilon)$ with $B^X(x,\varepsilon)$ for $\varepsilon \leq a^X/4$.

Let $f : \mathbb{R} \to [0,1]$ be a smooth even function such that

$$f(v) = \begin{cases} 1 & \text{for} \quad |v| \leq \varepsilon/2, \\ 0 & \text{for} \quad |v| \geq \varepsilon. \end{cases} \tag{2.28}$$

Set

$$F(a) = \Big(\int_{-\infty}^{+\infty} f(v) dv \Big)^{-1} \int_{-\infty}^{+\infty} e^{iva} f(v) dv. \tag{2.29}$$

Then $F(a)$ is an even function and lies in Schwartz space $\mathcal{S}(\mathbb{R})$ and $F(0) = 1$.

Let $\widetilde{F}$ be the holomorphic function on $\mathbb{C}$ such that $\widetilde{F}(a^2) = F(a)$. The restriction of $\widetilde{F}$ to $\mathbb{R}$ lies in the Schwartz space $\mathcal{S}(\mathbb{R})$.

Let $\widetilde{F}(\mathcal{L}_p)(x,x')$ be the smooth kernel of $\widetilde{F}(\mathcal{L}_p)$ with respect to the volume form $dv_X(x')$.

Proposition 2.6. *For any $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ such that for $p \geq C_L/\nu$,*

$$|\widetilde{F}(\mathcal{L}_p)(x,x') - P_p^G(x,x')|_{\mathscr{C}^m(X\times X)} \leq C_{l,m} p^{-l}. \tag{2.30}$$

Here the $\mathscr{C}^m$ norm is induced by ∇^L, ∇^E, ∇^{Cliff}, h^L, h^E and g^{TX}.

Proof. For $a \in \mathbb{R}$, set

$$\phi_p(a) = 1_{[p\nu, +\infty[}(a) \widetilde{F}(a). \tag{2.31}$$

Then by Theorem 2.5, for $p > C_L/\nu$,

$$\widetilde{F}(\mathcal{L}_p) - P_p^G = \phi_p(\mathcal{L}_p). \tag{2.32}$$

By (2.29), for any $m \in \mathbb{N}$ there exists $C_m > 0$ such that

$$\sup_{a \in \mathbb{R}} |a|^m |\widetilde{F}(a)| \leq C_m. \tag{2.33}$$

As X is compact, there exist $\{x_i\}_{i=1}^r \subset X$ such that $\{U_i = B^X(x_i,\varepsilon)\}_{i=1}^r$ is a covering of X. We identify $B^{T_{x_i}X}(0,\varepsilon)$ with $B^X(x_i,\varepsilon)$ by geodesics as above.

We identify $(E_p)_Z$ for $Z \in B^{T_{x_i}X}(0,\varepsilon)$ to $(E_p)_{x_i}$ by parallel transport with respect to the connection ∇^{E_p} along the curve $\gamma_Z : [0,1] \ni u \to \exp^X_{x_i}(uZ)$.

Let $\{e_j\}_{j=1}^{2n}$ be an orthonormal basis of $T_{x_i}X$. Let $\widetilde{e}_j(Z)$ be the parallel transport of e_j with respect to ∇^{TX} along the above curve.

Let $\Gamma^E, \Gamma^L, \Gamma^{\mathrm{Cliff}}$ be the corresponding connection forms of ∇^E, ∇^L and ∇^{Cliff} with respect to any fixed frame for E, L, $\Lambda(T^{*(0,1)}X)$ which is parallel along the curve γ_Z under the trivialization on U_i. Then Γ^L is a usual 1-form.

Denote by ∇_U the ordinary differentiation operator on $T_{x_i}X$ in the direction U. Then

$$\nabla^{E_p} = \nabla + p\,\Gamma^L + \Gamma^{\mathrm{Cliff}} + \Gamma^E, \quad D_p = c(\widetilde{e}_j)\nabla^{E_p}_{\widetilde{e}_j}. \tag{2.34}$$

Let $\{\varphi_i\}$ be a partition of unity subordinate to $\{U_i\}$.

For $l \in \mathbb{N}$, we define a Sobolev norm on the l-th Sobolev space $H^l(X, E_p)$ by

$$\|s\|^2_{H^l_p} = \sum_i \sum_{k=0}^{l} \sum_{i_1,\cdots,i_k=1}^{2n} \|\nabla_{e_{i_1}} \cdots \nabla_{e_{i_k}}(\varphi_i s)\|^2_{L^2} \tag{2.35}$$

Then by (2.34), there exist $C, C', C'' > 0$ such that for $p \geq 1$, $s \in H^2(X, E_p)$,

$$C'\|D_p^2 s\|_{L^2} - C'' p^2 \|s\|_{L^2} \leq \|s\|_{H^2_p} \leq C(\|D_p^2 s\|_{L^2} + p^2\|s\|_{L^2}). \tag{2.36}$$

Observe that D_p commutes with the G-action, thus

$$[D_p, L_{K_j}] = 0. \tag{2.37}$$

By (2.22), (2.37), and the facts that D_p is self-adjoint and L_{K_j} is skew-adjoint, we know

$$\begin{aligned} \|\mathcal{L}_p s\|^2_{L^2} &= \|D_p^2 s\|^2_{L^2} + p^2 \|\sum_j L_{K_j} L_{K_j} s\|^2_{L^2} - 2p\,\mathrm{Re}\sum_j \langle D_p^2 s, L_{K_j} L_{K_j} s\rangle \\ &= \|D_p^2 s\|^2_{L^2} + p^2 \|\sum_j L_{K_j} L_{K_j} s\|^2_{L^2} + 2p \sum_j \|L_{K_j} D_p s\|^2_{L^2}. \end{aligned} \tag{2.38}$$

From (2.36), and (2.38), there exists $C > 0$ such that

$$\|s\|_{H^2_p} \leq C(\|\mathcal{L}_p s\|_{L^2} + p^2\|s\|_{L^2}). \tag{2.39}$$

Let Q be a differential operator of order $m \in \mathbb{N}$ with scalar principal symbol and with compact support in U_i, then

$$[\mathcal{L}_p, Q] = [D_p^2, Q] - p\sum_j [L_{K_j} L_{K_j}, Q] \tag{2.40}$$

is a differential operator of order $m+1$. Moreover, by (2.21), (2.34), the leading term of order $m-1$ differential operator in $[L_{K_j} L_{K_j}, Q]$ is $p^2[((\Gamma^L - 2\pi\sqrt{-1}\mu)(K_j))^2, Q]$. Thus by (2.39) and (2.40),

$$\begin{aligned} \|Qs\|_{H^2_p} &\leq C(\|\mathcal{L}_p Q s\|_{L^2} + p^2\|Qs\|_{L^2}) \\ &\leq C(\|Q\mathcal{L}_p s\|_{L^2} + p\|s\|_{H^{m+1}_p} + p^2\|s\|_{H^m_p} + p^3\|s\|_{H^{m-1}_p}). \end{aligned} \tag{2.41}$$

This means

$$\|s\|_{H_p^{2m+2}} \leq C_m p^{2m+2} \sum_{j=0}^{m+1} \|\mathcal{L}_p^j s\|_{L^2}. \tag{2.42}$$

Moreover, from

$$\langle \mathcal{L}_p^{m'} \phi_p(\mathcal{L}_p) Q s, s' \rangle = \langle s, Q^* \phi_p(\mathcal{L}_p) \mathcal{L}_p^{m'} s' \rangle,$$

(2.31) and (2.33), we know that for any $l, m' \in \mathbb{N}$, there exists $C_{l,m'} > 0$ such that for $p \geq 1$,

$$\|\mathcal{L}_p^{m'} \phi_p(\mathcal{L}_p) Q s\|_{L^2} \leq C_{l,m'} p^{-l+m} \|s\|_{L^2}. \tag{2.43}$$

We deduce from (2.42) and (2.43) that if Q_1, Q_2 are differential operators of order m, m' with compact support in U_i, U_j respectively, then for any $l > 0$, there exists $C_l > 0$ such that for $p \geq 1$,

$$\|Q_1 \phi_p(\mathcal{L}_p) Q_2 s\|_{L^2} \leq C_l p^{-l} \|s\|_{L^2}. \tag{2.44}$$

On $U_i \times U_j$, by using Sobolev inequality and (2.32), we get Proposition 2.6. □

Observe that K_j^X are vector fields along the orbits of the G-action, thus the contribution of $pL_{K_j} L_{K_j}$ in the wave operator $\exp(\sqrt{-1} t \sqrt{\mathcal{L}_p})$ will propagate along the G-orbits, and the principal symbol of $\mathcal{L}_p$ is given by

$$\sigma(\mathcal{L}_p)(\xi) = |\xi|^2 + p \sum_j \langle K_j^X, \xi \rangle^2 \quad \text{for } \xi \in T^*X.$$

By the finite propagation speed for solutions of hyperbolic equations [16, §7.8], [37, §4.4], [38, I. §2.6, §2.8], $\widetilde{F}(\mathcal{L}_p)(x, x')$ only depends on the restriction of $\mathcal{L}_p$ to $G \cdot B^X(x, \varepsilon)$ and

$$\widetilde{F}(\mathcal{L}_p)(x, x') = 0, \quad \text{if } d^X(Gx, x') \geq \varepsilon. \tag{2.45}$$

(When we apply the proof of [38, §2.6, §2.8], we need to suppose that Σ_1, Σ_2 therein are G-space-like surfaces for the operator $\frac{\partial^2}{\partial t^2} - D_p^2$).

Combining with Proposition 2.6, we know that the asymptotic of $P_p^G(x, x')$ as $p \to \infty$ is localized on a neighborhood of Gx.

Proof of Theorem 0.1. From Proposition 2.6 and (2.45), we get (0.7) for any $x, x' \in X$, $d^X(Gx, x') \geq \varepsilon_0$. Now we will establish (0.7) for $x, x' \in X \setminus U$.

Recall that U is a G-open neighborhood of $P = \mu^{-1}(0)$.

As 0 is a regular value of μ, there exists $\epsilon_0 > 0$ such that $\mu : X_{2\epsilon_0} = \mu^{-1}(B^{\mathfrak{g}^*}(0, 2\epsilon_0)) \to B^{\mathfrak{g}^*}(0, 2\epsilon_0)$ is a submersion, $X_{2\epsilon_0}$ is a G-open subset of X.

Fix $\varepsilon, \epsilon_0 > 0$ small enough such that $X_{2\epsilon_0} \subset U$, and $d^X(x, y) > 4\varepsilon$ for any $x \in X_{\epsilon_0}$, $y \in X \setminus U$. Then $V_{\epsilon_0} = X \setminus X_{\epsilon_0}$ is a smooth G-manifold with boundary ∂V_{ϵ_0}.

Consider the operator $\mathcal{L}_p$ on V_{ϵ_0} with the Dirichlet boundary condition. We denote it by $\mathcal{L}_{p,D}$. Note that $\mathcal{L}_{p,D}$ is self-adjoint.

Still from [38, §2.6, §2.8], the wave operator $\exp(\sqrt{-1}t\sqrt{\mathcal{L}_{p,D}})$ is well defined and $\exp(\sqrt{-1}t\sqrt{\mathcal{L}_{p,D}})(x,x')$ only depends on the restriction of $\mathcal{L}_p$ to $G\cdot B^X(x,t)\cap V_{\epsilon_0}$, and is zero if $d^X(Gx,x')\geq t$. Thus, by (2.29),

$$\widetilde{F}(\mathcal{L}_p)(x,x')=\widetilde{F}(\mathcal{L}_{p,D})(x,x'),\quad \text{if } x,x'\in X\setminus U. \tag{2.46}$$

Now for $s\in\mathscr{C}_0^\infty(V_{\epsilon_0},E_p)$, after taking an integration over G, we can get the decomposition $s=s_1+s_2$ with $s_1\in\Omega^{0,\bullet}(X,L^p\otimes E)^G$, $s_2\in(\Omega^{0,\bullet}(X,L^p\otimes E)^G)^\perp$ and $\operatorname{supp} s_i\subset V_{\epsilon_0}\setminus\partial V_{\epsilon_0}$.

Since $\sum_{i=1}^{\dim G}L_{K_i}L_{K_i}$ commutes with the G-action, $\mathcal{L}_p s_1\in\Omega^{0,\bullet}(X,L^p\otimes E)^G$, $\mathcal{L}_p s_2\in(\Omega^{0,\bullet}(X,L^p\otimes E)^G)^\perp$ and, by (2.22), (2.26),

$$\begin{aligned}\langle\mathcal{L}_p s,s\rangle&=\langle\mathcal{L}_p s_1,s_1\rangle+\langle\mathcal{L}_p s_2,s_2\rangle\\&=\|D_p s_2\|^2_{L^2}-p\langle\rho(\mathrm{Cas})s_2,s_2\rangle+\langle D_p^2 s_1,s_1\rangle\\&\geq p\nu_1\|s_2\|^2_{L^2}+\langle D_p^2 s_1,s_1\rangle.\end{aligned} \tag{2.47}$$

To estimate the term $\langle D_p^2 s_1,s_1\rangle$, we will use the Lichnerowicz formula.

Recall that the Bochner-Laplacian Δ^{E_p} on E_p is defined by (1.21).

Let r^X be the Riemannian scalar curvature of (TX,g^{TX}).

Let $\{w_a\}$ be an orthonormal frame of $(T^{(1,0)}X,g^{TX})$. Set

$$\begin{aligned}\omega_d&=-\sum_{a,b}R^L(w_a,\overline{w}_b)\,\overline{w}^b\wedge i_{\overline{w}_a}\,,\\ \tau(x)&=\sum_a R^L(w_a,\overline{w}_a)\,,\qquad R^E_\tau=\sum_a R^E(w_a,\overline{w}_a)\,,\\ \mathbf{c}(R)&=\sum_{a<b}\Big(R^E+\tfrac{1}{2}\operatorname{Tr}[R^{T^{(1,0)}X}]\Big)(e_a,e_b)\,c(e_a)\,c(e_b)\,.\end{aligned} \tag{2.48}$$

The Lichnerowicz formula [1, Theorem 3.52] (cf. [25, Theorem 2.2]) for D_p^2 is

$$D_p^2=\Delta^{E_p}-2p\omega_d-p\tau+\tfrac{1}{4}r^X+\mathbf{c}(R). \tag{2.49}$$

Especially, as $\operatorname{supp} s_i\subset V_{\epsilon_0}\setminus\partial V_{\epsilon_0}$, from (2.49), we get

$$\langle D_p^2 s_1,s_1\rangle=\|\nabla^{E_p}s_1\|^2_{L^2}-p\langle(2\omega_d+\tau)s_1,s_1\rangle+\langle(\tfrac{1}{4}r^X+\mathbf{c}(R))s_1,s_1\rangle. \tag{2.50}$$

Since $s_1\in\Omega^{0,\bullet}(X,L^p\otimes E)^G$, from (1.13), for any $K\in\mathfrak{g}$,

$$\nabla^{E_p}_{K^X}s_1=(L_K+\mu^{E_p}(K))s_1=\mu^{E_p}(K)s_1. \tag{2.51}$$

From (2.21) and (2.51), there exist $C,C'>0$ such that

$$\begin{aligned}\|\nabla^{E_p}s_1\|^2_{L^2}&\geq C\sum_j\|\nabla^{E_p}_{K_j^X}s_1\|^2_{L^2}=C\sum_j\|\mu^{E_p}(K_j)s_1\|^2_{L^2}\\&\geq Cp^2\||\mu|s_1\|^2_{L^2}-C'\|s_1\|^2_{L^2}\geq C\epsilon_0^2p^2\|s_1\|^2_{L^2}-C'\|s_1\|^2_{L^2}.\end{aligned} \tag{2.52}$$

From (2.47)-(2.52), for p large enough,

$$\langle\mathcal{L}_p s,s\rangle\geq p\nu_1\|s_2\|^2_{L^2}+Cp^2\|s_1\|^2_{L^2}. \tag{2.53}$$

Thus there are $C, C' > 0$ such that for $p \geq 1$,

$$\mathrm{Spec}(\mathcal{L}_{p,D}) \subset [Cp - C', \infty[. \tag{2.54}$$

At first as $K_j^X \in T\partial V_{\epsilon_0}$ for any j, thus L_{K_j} preserves the Dirichlet boundary condition. We get for $l \in \mathbb{N}$,

$$L_{K_j}\phi_p(\mathcal{L}_{p,D}) = \phi_p(\mathcal{L}_{p,D})L_{K_j}, \quad (\mathcal{L}_{p,D})^l\phi_p(\mathcal{L}_{p,D}) = \phi_p(\mathcal{L}_{p,D})(\mathcal{L}_{p,D})^l. \tag{2.55}$$

Thus from (2.22) and (2.55),

$$D_{p,D}^2 \leq \mathcal{L}_{p,D}, \tag{2.56}$$

and for $l \in \mathbb{N}$, $(D_{p,D}^2)^l$ commutes with the operator $\phi_p(\mathcal{L}_{p,D})$.

Let $\phi_p(\mathcal{L}_{p,D})(x, x')$ be the smooth kernel of $\phi_p(\mathcal{L}_{p,D})$ with respect to $dv_X(x')$.

Then from the above argument we get that for any $l, k \in \mathbb{N}$, $(D_{p,x}^2)^l(D_{p,x'}^2)^k\phi_p(\mathcal{L}_{p,D})(x, x')$ verifies the Dirichlet boundary condition for x, x' respectively.

By (2.34) and the elliptic estimate for Laplacian with Dirichlet boundary condition [38, Theorem 5.1.3], there exists $C > 0$ such that for $s \in H^{2m+2}(X, E_p) \cap H_0^1(X, E_p)$, $p \in \mathbb{N}$, we have

$$\|s\|_{H^{2m+2}} \leq C(\|D_p^2 s\|_{H^{2m}} + p^2\|s\|_{H^{2m+1}}). \tag{2.57}$$

Thus if Q_1, Q_2 are differential operators of order $2m, 2m'$ with compact support in U_i, U_j respectively, by (2.57) and (2.56), as in (2.42), we get for $s \in \mathscr{C}_0^\infty(V_{\epsilon_0}, E_p)$,

$$\begin{aligned} \|Q_1\phi_p(\mathcal{L}_{p,D})Q_2 s\|_{L^2} &\leq Cp^{2m+2m'}\sum_{j_1=0}^{m}\sum_{j_2=0}^{m'}\|(D_{p,D}^2)^{j_1}\phi_p(\mathcal{L}_{p,D})(D_{p,D}^2)^{j_2}s\|_{L^2} \\ &\leq Cp^{2m+2m'}\sum_{j_1=0}^{m}\sum_{j_2=0}^{m'}\|(\mathcal{L}_{p,D})^{j_1}\phi_p(\mathcal{L}_{p,D})(\mathcal{L}_{p,D})^{j_2}s\|_{L^2}. \end{aligned} \tag{2.58}$$

From (2.54), (2.58), as in (2.44), we get

$$\|Q_1\phi_p(\mathcal{L}_{p,D})Q_2 s\|_{L^2} \leq C_l p^{-l}\|s\|_{L^2}. \tag{2.59}$$

By using Sobolev inequality as in the proof of Proposition 2.6, from (2.30), (2.46) and (2.59), we get Theorem 0.1. □

2.5. **Induced operator on U/G.** Let U be a G-neighborhood of $P = \mu^{-1}(0)$ in X such that G acts freely on $\overline{U}$, the closure of U. We will use the notation as in Introduction and Sections 1.1, 1.2 with X therein replaced by U, especially $B = U/G$.

Let $\pi : U \to B$ be the natural projection with fiber Y. Let TY be the sub-bundle of TU generated by the G-action, let g^{TY}, g^{TP} be the metrics on TY, TP induced by g^{TX}.

Let T^HU, T^HP be the orthogonal complements of TY in TU, (TP, g^{TP}). Let g^{T^HU} be the metric on T^HU induced by g^{TX}, and it induces naturally a Riemannian metric g^{TB} on B.

Let dv_B be the Riemannian volume form on (B, g^{TB}).

Recall that in (1.20), we defined the isometry $\Phi = h\pi_G : (\mathscr{C}^\infty(U, E_p)^G, \langle\ ,\ \rangle) \to (\mathscr{C}^\infty(B, E_{p,B}), \langle\ ,\ \rangle)$.

By (1.14), μ^{E_p} defines a G-invariant section $\widetilde{\mu}^{E_p}$ of $TY \otimes \mathrm{End}(E_p)$ on U.

Remark that ω_d, τ, $\mathbf{c}(R)$ in (2.48) are G-invariant. We still denote by ω_d, τ, $\mathbf{c}(R)$ the induced sections on B.

As a direct corollary of Theorem 1.3 and (2.49), we get the following result,

Proposition 2.7. *As an operator on $\mathscr{C}^\infty(B, E_{p,B})$,*

$$\Phi\mathcal{L}_p\Phi^{-1} = \Phi D_p^2\Phi^{-1} = \Delta^{E_{p,B}} - \langle\widetilde{\mu}^{E_p}, \widetilde{\mu}^{E_p}\rangle_{g^{TY}} - \frac{1}{h}\Delta_B h - 2p\omega_d - p\tau + \tfrac{1}{4}r^X + \mathbf{c}(R). \tag{2.60}$$

From Theorem 0.1, Prop. 2.6 and (2.45), modulo $\mathscr{O}(p^{-\infty})$, $P_p^G(x, x')$ depends only the restriction of $\mathcal{L}_p$ on U.

To get a complete picture on $P_p^G(x, x')$, we explain now that modulo $\mathscr{O}(p^{-\infty})$, $P_p^G(x, x')$ depends only on the restriction of $\Phi\mathcal{L}_p\Phi^{-1}$ on any neighborhood of X_G in B.

As in the proof of Theorem 0.1, we will fix $\epsilon_0 > 0$ small enough such that $X_{2\epsilon_0} = \mu^{-1}(B^{\mathfrak{g}^*}(0, 2\epsilon_0)) \subset U$, and the constant $\varepsilon > 0$ which will be fixed later, verifying that $d^X(x, y) > 4\varepsilon$ for any $x \in X_{\epsilon_0}$, $y \in X \setminus U$. Set $B_{\epsilon_0} = \pi(X_{\epsilon_0})$.

First we will extend all objects from a neighborhood of P to the total space of the normal bundle N of P in X.

Let $\pi_N : N \to P$ be the normal bundle of P in X. We identify N to the orthogonal complement of TP in (TX, g^{TX}). Then G acts on N and the action extends naturally on $\pi_N^*(L|_P)$, $\pi_N^*(E|_P)$.

By (2.5), we have an orthogonal decomposition of TX,

$$TX|_P = T^H P \oplus TY|_P \oplus N, \quad \text{and } TY|_P \simeq P \times \mathfrak{g}, \quad N = \mathbf{J}TY|_P \simeq P \times \mathfrak{g}. \tag{2.61}$$

Denote by P^{TY}, P^{TP}, P^{N_P} the orthogonal projections from TX on TY, TP and $N|_P$ by this identification.

From (2.61), we have

$$TN \simeq \pi_N^* TX|_P \simeq \pi_N^*(TP \oplus \mathfrak{g}). \tag{2.62}$$

For $\varepsilon > 0$, we denote by $B_\varepsilon^N = \{(y, Z) \in N, y \in P, |Z|_{g^{TX}} \leq \varepsilon\}$.

Then for ε_0 small enough, the map $(y, Z) \in N \to \exp_y^X(Z) \in X$ is a diffeomorphism from $B_{2\varepsilon_0}^N$ onto a tubular neighborhood $\mathcal{U}_{2\varepsilon_0}$ of P in X.

From now on, we use the notation (y, Z) instead of $\exp_y^X(Z)$. We identify $y \in P$ with $(y, 0) \in N$. From (2.61), (2.62), we may and we will identify TN to $\pi_N^* TP \oplus \mathfrak{g}$.

For $Z \in N_y$, $|Z| \leq 2\varepsilon_0$, we identify L_Z, E_Z to L_y, E_y by using parallel transport with respect to ∇^L, ∇^E along the curve $[0, 1] \ni u \to uZ$. In this way, we identify the Hermitian bundles $(\pi_N^* L|_P, \pi_N^* h^L)$, $(\pi_N^* E|_P, \pi_N^* h^E)$ to $(L, h^L), (E, h^E)$ on $B_{2\varepsilon_0}^N$.

Let $\varepsilon > 0$ with $\varepsilon < \varepsilon_0/2$. Let $\varphi : \mathbb{R} \to [0, 1]$ be a smooth even function such that

$$\varphi(v) = 1 \ \text{ if } \ |v| < 2; \quad \varphi(v) = 0 \ \text{ if } |v| > 4. \tag{2.63}$$

Let $\psi_\varepsilon : N \to N$ be the map defined by $\psi_\varepsilon(Z) = \varphi(|Z|/\varepsilon)Z \in N_y$ for $Z \in N_y$.

Let $g_Z^{TN} = g_{\psi_\varepsilon(Z)}^{TX}$, $J_Z^N = J_{\psi_\varepsilon(Z)}$ be the induced metric and almost-complex structure on N.

Let $\nabla^{\pi_N^*E}=\psi_\varepsilon^*\nabla^E$, then $\nabla^{\pi_N^*E}$ is the extension of ∇^E on $B^N_{2\varepsilon}$.

Let $\nabla^{\pi_N^*L}$ be the Hermitian connection on $(\pi_N^*L,\pi_N^*h^L)$ defined by that for $Z\in N_y$,

$$\nabla^{\pi_N^*L}=\psi_\varepsilon^*\nabla^L+(1-\varphi(\tfrac{|Z|}{\varepsilon}))R^L_y(Z,P^{TP}\cdot)+\frac{1}{2}(1-\varphi^2(\tfrac{|Z|}{\varepsilon}))R^L_y(Z,P^{N_P}\cdot). \tag{2.64}$$

Then by using the identification (2.61), and (2.62), we calculate directly that its curvature $R^{\pi_N^*L}=(\nabla^{\pi_N^*L})^2$ is

$$\begin{aligned}R^{\pi_N^*L}_Z&=\psi_\varepsilon^*R^L+d\Big((1-\varphi(\tfrac{|Z|}{\varepsilon}))R^L_y(Z,P^{TP}\cdot)+\frac{1}{2}(1-\varphi^2(\tfrac{|Z|}{\varepsilon}))R^L_y(Z,P^{N_P}\cdot)\Big)\\
&=R^L_{\psi_\varepsilon(Z)}(P^{TP}\cdot,P^{TP}\cdot)+R^L_y(P^{N_P}\cdot,\cdot)+\varphi^2(\tfrac{|Z|}{\varepsilon})(R^L_{\psi_\varepsilon(Z)}-R^L_y)(P^{N_P}\cdot,P^{N_P}\cdot)\\
&\qquad+\varphi(\tfrac{|Z|}{\varepsilon})(R^L_{\psi_\varepsilon(Z)}-R^L_y)(P^{N_P}\cdot,P^{TP}\cdot)\\
&\qquad-\varphi'(\tfrac{|Z|}{\varepsilon})\frac{Z^*}{\varepsilon|Z|}\wedge[R^L_y(Z,P^{TP}\cdot)-R^L_{\psi_\varepsilon(Z)}(Z,P^{TP}\cdot)]\\
&\qquad-(\varphi\varphi')(\tfrac{|Z|}{\varepsilon})\frac{Z^*}{\varepsilon|Z|}\wedge[R^L_y(Z,P^{N_P}\cdot)-R^L_{\psi_\varepsilon(Z)}(Z,P^{N_P}\cdot)]\\
&\qquad+d_y\Big((1-\varphi(\tfrac{|Z|}{\varepsilon}))R^L_y(Z,P^{TP}\cdot)+\frac{1}{2}(1-\varphi^2(\tfrac{|Z|}{\varepsilon}))R^L_y(Z,P^{N_P}\cdot)\Big).\end{aligned} \tag{2.65}$$

Here $Z^*\in N^*$ is the dual of $Z\in N$ with respect to the metric g^N.

From (2.65), one deduces that $R^{\pi_N^*L}$ is positive in the sense of (2.9) when ε is small enough, with the corresponding constant ν_0 for $R^{\pi_N^*L}$ being larger than $\frac{4}{5}\nu_0$.

Note that G acts naturally on the normal bundle N, and under our identification, the G-actions on L,E on B^N_ε are exactly the G-actions on $L|_P,E|_P$ on P.

Now we define the G-actions on π_N^*L,π_N^*E by their G-actions on P, then they extend the G-actions on L,E on B^N_ε to N.

By (2.14), the moment map $\mu_N:N\to\mathfrak{g}^*$ of the G-action on N is defined by

$$-2\pi\sqrt{-1}\mu_N(K)=L_K-\nabla^{\pi_N^*L}_{K^N},\ K\in\mathfrak{g}. \tag{2.66}$$

Observe that $\psi_{\varepsilon*}K^N_{(y,Z)}=K^P_y\in TP$, thus from (2.14), (2.61), and (2.64),

$$\begin{aligned}2\pi\sqrt{-1}\mu_N(K)_{(y,Z)}&=(1-\varphi(|Z|/\varepsilon))R^L_y(Z,K^P)+2\pi\sqrt{-1}\mu(K)_{\psi_\varepsilon(Z)}\\
&=R^L_y(Z,K^P)+\mathscr{O}(\varphi^2(|Z|/\varepsilon)|Z|^2).\end{aligned} \tag{2.67}$$

Thus $\mu_N^{-1}(0)=\mu^{-1}(0)=P$ for ε small enough, and for $|Z|\geq 4\varepsilon$,

$$2\pi\sqrt{-1}\mu_N(K)_{(y,Z)}=R^L_y(Z,K^P). \tag{2.68}$$

From now on, we fix ε as above.

Let $\widetilde{F}(\Phi\mathcal{L}_p\Phi^{-1})(x,x')$ $(x,x'\in B_{\epsilon_0})$ be the smooth kernel of $\widetilde{F}(\Phi\mathcal{L}_p\Phi^{-1})$ with respect to $dv_B(x')$. We will also view $\widetilde{F}(\Phi\mathcal{L}_p\Phi^{-1})$ as a $G\times G$-invariant section of $\mathrm{pr}_1^*E_p\otimes\mathrm{pr}_2^*E_p^*$ on $X_{\epsilon_0}\times X_{\epsilon_0}$.

Theorem 2.8. *For any $l,m\in\mathbb{N}$, there exists $C_{l,m}>0$ such that for $p\geq 1$, $x,x'\in X_{\epsilon_0}$,*

$$|h(x)h(x')P^G_p(x,x')-\widetilde{F}(\Phi\mathcal{L}_p\Phi^{-1})(\pi(x),\pi(x'))|_{\mathscr{C}^m(X_{\epsilon_0}\times X_{\epsilon_0})}\leq C_{l,m}p^{-l}. \tag{2.69}$$

Proof. Let D_p^N be the Dirac operator on N associated to the above data by the construction in Section 2.2. By the argument in [25, p. 656-657] and the proof of Theorem 2.5, we know that Theorems 2.2, 2.5 still hold for D_p^N.

Let $\mathcal{L}_p^N$ be the operator on N defined as in (2.22). Then there exists $C > 0$ such that for $p \geq 1$,

$$\mathrm{Spec}\left(\mathcal{L}_p^N\right) \subset \{0\} \cup [p\nu - C, +\infty[. \tag{2.70}$$

Let $P_p^{N,G}$ be the orthogonal projection from $\Omega^{0,\bullet}(N, \pi_N^*(L^p \otimes E))$ on $(\mathrm{Ker}\, D_p^N)^G$, then by (2.70) and the arguments as in the proof of Theorem 2.6, for any $l, m \in \mathbb{N}$, $V \subset N$ a compact subset of N, there exists $C_{l,m} > 0$ such that for $p \geq 1$, $x, x' \in V$,

$$|\widetilde{F}(\mathcal{L}_p^N)(x, x') - P_p^{N,G}(x, x')|_{\mathscr{C}^m(V \times V)} \leq C_{l,m} p^{-l}. \tag{2.71}$$

Let $P_p^{N/G}$ be the projection from $(L^2(N/G, (\Lambda(T^{*(0,1)}N) \otimes \pi_N^*(L^p \otimes E))_{N/G}), \langle\, ,\, \rangle)$ onto $\mathrm{Ker}(\Phi \mathcal{L}_p^N \Phi^{-1})$, and let $P_p^{N/G}(z, z')$ be the smooth kernel of the operator $P_p^{N/G}$ with respect to $dv_{N/G}(z')$.

We still denote by $\mathrm{pr}_1, \mathrm{pr}_2$ the projections from $N \times N$ onto the first and second factor N. We will also view $P_p^{N/G}(z, z')$ as a $G \times G$-invariant section of

$$\mathrm{pr}_1^*(\Lambda(T^{*(0,1)}N) \otimes \pi_N^*(L^p \otimes E)) \otimes \mathrm{pr}_2^*(\Lambda(T^{*(0,1)}N) \otimes \pi_N^*(L^p \otimes E))^*$$

on $N \times N$.

As Φ in (1.20) defines an isometry from $(\mathrm{Ker}\, D_p^N)^G = \mathrm{Ker}\, \mathcal{L}_p^N$ onto $\mathrm{Ker}(\Phi \mathcal{L}_p^N \Phi^{-1})$, one has

$$h(x)h(x')P_p^{N,G}(x, x') = P_p^{N/G}(\pi(x), \pi(x')). \tag{2.72}$$

On N/G, by the arguments as in the proof of Theorem 2.6, we get

$$|\widetilde{F}(\Phi \mathcal{L}_p \Phi^{-1})(z, z') - P_p^{N/G}(z, z')|_{\mathscr{C}^m(V/G \times V/G)} \leq C_{l,m} p^{-l}. \tag{2.73}$$

By the finite propagation speed (2.45), we know that for $x, x' \in X_{\epsilon_0}$,

$$\widetilde{F}(\mathcal{L}_p^N)(x, x') = \widetilde{F}(\mathcal{L}_p)(x, x'). \tag{2.74}$$

Now we get (2.69) from (2.30), (2.71)-(2.74). □

Let $d^B(\cdot, \cdot)$ be the Riemannian distance on B.

By (2.60) and the finite propagation speed for solutions of hyperbolic equations [16, §7.8], [37, §4.4], $\widetilde{F}(\Phi \mathcal{L}_p \Phi^{-1})(x, x')$ only depends on the restriction of $\Phi \mathcal{L}_p \Phi^{-1}$ to $B^B(x, \varepsilon)$ and

$$\widetilde{F}(\Phi \mathcal{L}_p \Phi^{-1})(x, x') = 0, \quad \text{if } d^B(x, x') \geq \varepsilon. \tag{2.75}$$

Thus we have localized our problem near X_G.

Theorem 2.8 helps us to understand that the asymptotic of $P_p^G(x, x')$ is local near X_G. In the rest, we will not use directly Theorem 2.8, but the argument of its proof will be used in Section 2.6.

2.6. **Rescaling and a Taylor expansion of the operator $\Phi\mathcal{L}_p\Phi^{-1}$.** Recall that N_G is the normal bundle of X_G in B, and we identify N_G as the orthogonal complement of TX_G in (TB, g^{TB}).

Let P^{TX_G}, P^{N_G} be the orthogonal projection from TB on TX_G, N_G on X_G.

Recall that $\nabla^{N_G}, {}^0\nabla^{TB}$ are connections on N_G, TB on X_G, and A is the associated second fundamental form defined in (0.9).

We fix $x_0 \in X_G$.

If $W \in T_{x_0}X_G$, let $\mathbb{R} \ni t \to x_t = \exp^{X_G}_{x_0}(tW) \in X_G$ be the geodesic in X_G such that $x_t|_{t=0} = x_0$, $\frac{dx}{dt}|_{t=0} = W$.

If $W \in T_{x_0}X_G$, $|W| \leq \varepsilon$, $V \in N_{x_0}$, let $\tau_W V \in N_{G,\exp^{X_G}_{x_0}(W)}$ be the natural parallel transport of V with respect to the connection ∇^{N_G} along the curve $[0,1] \ni t \to \exp^{X_G}_{x_0}(tW)$.

If $Z \in T_{x_0}B$, $Z = Z^0 + Z^\perp$, $Z^0 \in T_{x_0}X_G$, $Z^\perp \in N_{x_0}$, $|Z^0|, |Z^\perp| \leq \varepsilon$, we identify Z with $\exp^B_{\exp^{X_G}_{x_0}(Z^0)}(\tau_{Z^0}Z^\perp)$. This identification is a diffeomorphism from $B^{TX_G}_{x_0}(0,\varepsilon) \times B^{N_G}_{x_0}(0,\varepsilon)$ into an open neighborhood $\mathscr{U}(x_0)$ of x_0 in B. We denote it by Ψ, and $\mathscr{U}(x_0) \cap X_G = B^{TX_G}_{x_0}(0,\varepsilon) \times \{0\}$.

From now on, we use indifferently the notation $B^{TX_G}_{x_0}(0,\varepsilon) \times B^{N_G}_{x_0}(0,\varepsilon)$ or $\mathscr{U}(x_0)$, x_0 or $0, \cdots$.

We identify $(L_B)_Z, (E_B)_Z$ and $(E_{p,B})_Z$ to $(L_B)_{x_0}, (E_B)_{x_0}$ and $(E_{p,B})_{x_0}$ by using parallel transport with respect to $\nabla^{L_B}, \nabla^{E_B}$ and $\nabla^{E_{p,B}}$ along the curve $\gamma_u : [0,1] \ni u \to uZ$.

Recall that $T^HU \subset TX$ is the horizontal bundle for $\pi : U \to B$ defined in Section 2.5.

Let P^{T^HU} be the orthogonal projection from TX onto T^HU.

For $W \in TB$, let $W^H \in T^HU$ be the lift of W.

For $y_0 \in \pi^{-1}(x_0)$, we define the curve $\widetilde{\gamma}_u : [0,1] \to X$ to be the lift of the curve γ_u with $\widetilde{\gamma}_0 = y_0$ and $\frac{\partial\widetilde{\gamma}_u}{\partial u} \in T^HU$. Then on $\pi^{-1}(B^{TB}(0,\varepsilon))$, we use the parallel transport with respect to ∇^L, ∇^E and ∇^{E_p} along the curve $\widetilde{\gamma}_u$ to trivialized the corresponding bundles. By (1.17), the previous trivialization is naturally induced by this one.

Let $\{e^0_i\}$, $\{e^\perp_j\}$ be orthonormal basis of $T_{x_0}X_G$, N_{G,x_0}, then $\{e_i\} = \{e^0_i, e^\perp_j\}$ is an orthonormal basis of $T_{x_0}B$. Let $\{e^i\}$ be its dual basis. We will also denote $\Psi_*(e^0_i), \Psi_*(e^\perp_j)$ by $e^0_i, e^\perp_j$. Thus in our coordinate,

$$\frac{\partial}{\partial Z^0_i} = e^0_i, \qquad \frac{\partial}{\partial Z^\perp_j} = e^\perp_j. \tag{2.76}$$

For $\varepsilon > 0$ small enough, we will extend the geometric objects on $B^{TB}(x_0,\varepsilon)$ to $\mathbb{R}^{2n-n_0} \simeq T_{x_0}B$ (here we identify $(Z_1, \cdots, Z_{2n-n_0}) \in \mathbb{R}^{2n-n_0}$ to $\sum_i Z_i e_i \in T_{x_0}B$) such that D_p will become the restriction of a spin^c Dirac operator on $G \times \mathbb{R}^{2n-n_0}$ associated to a Hermitian line bundle with positive curvature. In this way, we can replace X by $G \times \mathbb{R}^{2n-n_0}$.

First of all, we denote by L_0, E_0 the trivial bundles $L|_{Gy_0}, E|_{Gy_0}$ on $X_0 = G \times \mathbb{R}^{2n-n_0}$, and we still denote by ∇^L, ∇^E, h^L etc. the connections and metrics on L_0, E_0 on $\pi^{-1}(B^{T_{x_0}B}(0,4\varepsilon))$ induced by the above identification. Then h^L, h^E is identified with the constant metrics $h^{L_0} = h^{L_{y_0}}$, $h^{E_0} = h^{E_{y_0}}$.

Set

$$\mathcal{R}^{\perp}=\sum_j Z_j^{\perp}e_j^{\perp}=Z^{\perp},\quad \mathcal{R}^0=\sum_i Z_i^0 e_i^0=Z^0,\quad \mathcal{R}=\mathcal{R}^{\perp}+\mathcal{R}^0=Z. \tag{2.77}$$

Then $\mathcal{R}$ is the radial vector field on $\mathbb{R}^{2n-n_0}$.

Let $\varphi_\varepsilon : X_0 \to X_0$ be the map defined by $\varphi_\varepsilon(g,Z)=(g,\varphi(|Z|/\varepsilon)Z)$ for $(g,Z)\in G\times\mathbb{R}^{2n-n_0}$.

Let $g^{TX_0}(g,Z)=g^{TX}(\varphi_\varepsilon(g,Z))$, $J_0(g,Z)=J(\varphi_\varepsilon(g,Z))$ be the metric and almost-complex structure on X_0.

Let $\nabla^{E_0}=\varphi_\varepsilon^*\nabla^E$, then ∇^{E_0} is the extension of ∇^E on $\pi^{-1}(B^{T_{x_0}B}(0,\varepsilon))$.

Let ∇^{L_0} be the Hermitian connection on (L_0,h^{L_0}) on $G\times\mathbb{R}^{2n-n_0}$ defined by for $Z\in\mathbb{R}^{2n-n_0}$,

$$\nabla^{L_0}=\varphi_\varepsilon^*\nabla^L+\Big(1-\varphi(\tfrac{|Z|}{\varepsilon})\Big)R^L_{y_0}(\mathcal{R}^H,P^{TY}_{y_0}\cdot)+\frac{1}{2}\Big(1-\varphi^2(\tfrac{|Z|}{\varepsilon})\Big)R^L_{y_0}(\mathcal{R}^H,P^{T^HU}_{y_0}\cdot). \tag{2.78}$$

As in (2.65), its curvature R^{L_0} is positive in the sense of (2.9) for ε small enough, and the corresponding constant ν_0 for R^{L_0} is bigger than $\frac{4}{5}\nu_0$ uniformly for $y_0\in P$.

From now on, we fix ε as above.

Now G acts naturally on X_0, and under our identification, the G-action on L,E on $G\times B^{T_{x_0}B}(0,\varepsilon)$ is exactly the G-action on $L|_{Gy_0},E|_{Gy_0}$.

We define a G-action on L_0,E_0 by its G-action on Gy_0, then it extends the G-action on L,E on $G\times B^{T_{x_0}B}(0,\varepsilon)$ to X_0.

By (2.15), for any $K\in\mathfrak{g}$, $W\in TP$ on $P=\mu^{-1}(0)$, we have

$$\begin{aligned}&R^L(W,K^X)=-2\pi\sqrt{-1}\omega(W,K^X)=2\pi\sqrt{-1}W(\mu(K))=0,\\ &R^L_{(1,Z^0)}(\mathcal{R}^H,K^X)=R^L_{(1,Z^0)}((\mathcal{R}^{\perp})^H,K^X).\end{aligned} \tag{2.79}$$

Observe that for $(1,Z)\in G\times\mathbb{R}^{2n-n_0}$, $\varphi_{\varepsilon *}K^{X_0}_{(1,Z)}=K^X_{y_0}$ for $K\in\mathfrak{g}$, by (2.14), the moment map $\mu_{X_0}:X_0\to\mathfrak{g}^*$ of the G-action on X_0 is given by

$$2\pi\sqrt{-1}\mu_{X_0}(K)_{(1,Z)}=(1-\varphi(\tfrac{|Z|}{\varepsilon}))R^L_{y_0}(\mathcal{R}^H,K^X_{y_0})+2\pi\sqrt{-1}\mu(K)_{\varphi_\varepsilon(1,Z)}. \tag{2.80}$$

Now from the choice of our coordinate, we know that $\mu_{X_0}=0$ on $G\times\mathbb{R}^{2n-2n_0}\times\{0\}$. Moreover,

$$2\pi\sqrt{-1}\mu(K)_{\varphi_\varepsilon(1,Z)}=R^L_{(1,Z)}(\varphi(\tfrac{|Z|}{\varepsilon})(\mathcal{R}^{\perp})^H,K^X)+\mathscr{O}(\varphi(\tfrac{|Z|}{\varepsilon})|Z||Z^{\perp}|). \tag{2.81}$$

From our construction, (2.80) and (2.81), we know that

$$\mu_{X_0}^{-1}(0)=G\times\mathbb{R}^{2n-2n_0}\times\{0\}. \tag{2.82}$$

By (2.79) and (2.80), for $Z\in T_{x_0}B$, $|Z|\geq 4\varepsilon$,

$$2\pi\sqrt{-1}\mu_{X_0}(K)_{(1,Z)}=R^L_{y_0}((\mathcal{R}^{\perp})^H,K^X_{y_0}). \tag{2.83}$$

Let $D_p^{X_0}$ be the Dirac operator on X_0 associated to the above data by the construction in Section 2.2. As in (2.70), the analogue of Theorems 2.2, 2.5 still holds for $D_p^{X_0}$.

Let g^{TB_0} be the metric on $B_0=\mathbb{R}^{2n-n_0}$ induced by g^{TX_0}, and let dv_{B_0} be the Riemannian volume form on (B_0,g^{TB_0}).

The operator $\Phi\mathcal{L}_p^{X_0}\Phi^{-1}$ is also well-defined on $T_{x_0}B\simeq\mathbb{R}^{2n-n_0}$.

Let $P_{x_0,p}$ be the orthogonal projection from $L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X_0)\otimes L_0^p\otimes E_0)_{B_0})$ onto $\mathrm{Ker}(\Phi\mathcal{L}_p^{X_0}\Phi^{-1})$ on $\mathbb{R}^{2n-n_0}$. Let $P_{x_0,p}(Z,Z')$ $(Z,Z'\in\mathbb{R}^{2n-n_0})$ be the smooth kernel of $P_{x_0,p}$ with respect to $dv_{B_0}(Z')$. As before, we view $P_{x_0,p}$ as a $G\times G$-invariant section of

$$\mathrm{pr}_1^*(\Lambda(T^{*(0,1)}X_0)\otimes L_0^p\otimes E_0)\otimes\mathrm{pr}_2^*(\Lambda(T^{*(0,1)}X_0)\otimes L_0^p\otimes E_0)^*$$

on $X_0\times X_0$.

Let $P_{0,p}^G$ be the orthogonal projection from $\Omega^{0,\bullet}(X_0,L_0^p\otimes E_0)$ onto $(\mathrm{Ker}\,D_p^{X_0})^G$, and let $P_{0,p}^G(x,x')$ be the smooth kernel of $P_{0,p}^G$ with respect to the volume form $dv_{X_0}(x')$.

Note that Φ in (1.20) defines an isometry from $(\mathrm{Ker}\,D_p^{X_0})^G=\mathrm{Ker}\,\mathcal{L}_p^{X_0}$ onto $\mathrm{Ker}(\Phi\mathcal{L}_p^{X_0}\Phi^{-1})$, as in (2.72), we get

$$h(x)h(x')P_{0,p}^G(x,x')=P_{x_0,p}(\pi(x),\pi(x')). \tag{2.84}$$

Proposition 2.9. *For any $l,m\in\mathbb{N}$, there exists $C_{l,m}>0$ such that for $x,x'\in G\times B^{T_{x_0}B}(0,\varepsilon)$,*

$$\left|(P_{0,p}^G-P_p^G)(x,x')\right|_{\mathscr{C}^m}\leq C_{l,m}p^{-l}. \tag{2.85}$$

Proof. By the analogue of Theorems 2.2, 2.5, we know that for $x,x'\in G\times B^{T_{x_0}B}(0,\varepsilon)$, $P_{0,p}^G-\widetilde{F}(\mathcal{L}_p^{X_0})$ verifies also (2.30), and for $x,x'\in G\times B^{T_{x_0}B}(0,\varepsilon)$,

$$\widetilde{F}(\mathcal{L}_p^{X_0})(x,x')=\widetilde{F}(\mathcal{L}_p)(x,x')$$

by finite propagation speed. Thus we get (2.85). □

Let $T^{*(0,1)}X_0$ be the anti-holomorphic cotangent bundle of (X_0,J_0). Since $J_0(g,Z)=J(\varphi_\varepsilon(g,Z))$, $T^{*(0,1)}_{Z,J_0}X_0$ is naturally identified with $T^{*(0,1)}_{\varphi_\varepsilon(g,Z),J}X_0$.

Let $\nabla^{\mathrm{Cliff}_0}$ be the Clifford connection on $\Lambda(T^{*(0,1)}X_0)$ induced by the Levi-Civita connection ∇^{TX_0} on (X_0,g^{TX_0}). Let $R^{E_0},R^{TX_0},R^{\mathrm{Cliff}_0}$ be the corresponding curvatures on E_0,TX_0 and $\Lambda(T^{*(0,1)}X_0)$ (cf. (2.10)).

We identify $\Lambda(T^{*(0,1)}X_0)_{(g,Z)}$ with $\Lambda(T^{*(0,1)}_{(g,0)}X)$ by identifying first $\Lambda(T^{*(0,1)}X_0)_{(g,Z)}$ with $\Lambda(T^{*(0,1)}_{\varphi_\varepsilon(g,Z),J}X_0)$, which in turn is identified with $\Lambda(T^{*(0,1)}_{Gy_0}X)$ by using parallel transport along $u\to u\varphi_\varepsilon(g,Z)$ with respect to $\nabla^{\mathrm{Cliff}_0}$. We also trivialize $\Lambda(T^{*(0,1)}X_0)$ in this way.

Let S_L be a G-invariant unit section of $L|_{Gy_0}$. Using S_L and the above discussion, we get an isometry

$$\Lambda(T^{*(0,1)}X_0)\otimes E_0\otimes L_0^p\simeq(\Lambda(T^{*(0,1)}X)\otimes E)|_{\pi^{-1}(x_0)}=\mathbf{E}|_{\pi^{-1}(x_0)}.$$

For any $1\leq i\leq 2n-n_0$, let $\widetilde{e}_i(Z)$ be the parallel transport of e_i with respect to the connection ${}^0\nabla^{TB}$ along $[0,1]\ni u\to uZ^0$, and with respect to the connection ∇^{TB} along $[1,2]\ni u\to Z^0+(u-1)Z^\perp$.

If $\alpha=(\alpha_1,\cdots,\alpha_{2n-n_0})$ is a multi-index, set $Z^\alpha=Z_1^{\alpha_1}\cdots Z_{2n-n_0}^{\alpha_{2n-n_0}}$.

Recall that A, $\mathcal{R}^\perp$ have been defined in (0.9), (2.77).

The following Lemma extends [1, Prop. 1.28] (cf. also [17, Lemma 4.5]).

Lemma 2.10. *The Taylor expansion of $\widetilde{e}_i(Z)$ with respect to the basis $\{e_i\}$ to order r is a polynomial of the Taylor expansion of the curvature coefficients of R^{TB} to order $r-2$ and A to order $r-1$.*

Proof. Let $\partial_i = \nabla_{e_i}$ be the partial derivatives along e_i.

Let Γ^{TB} be the connection form of ∇^{TB} with respect to the frame $\{\widetilde{e}_i\}$ of TB. By the definition of our fixed frame, we have $i_{\mathcal{R}^\perp}\Gamma^{TB} = 0$. As in [1, (1.12)],

$$L_{\mathcal{R}^\perp}\Gamma^{TB} = [i_{\mathcal{R}^\perp}, d]\Gamma^{TB} = i_{\mathcal{R}^\perp}(d\Gamma^{TB} + \Gamma^{TB}\wedge\Gamma^{TB}) = i_{\mathcal{R}^\perp}R^{TB}. \tag{2.86}$$

Let $\Theta(Z) = (\theta_j^i(Z))_{i,j=1}^{2n-n_0}$ be the $(2n-n_0)\times(2n-n_0)$-matrix such that

$$e_i = \sum_j \theta_i^j(Z)\widetilde{e}_j(Z), \quad \widetilde{e}_j(Z) = (\Theta(Z)^{-1})_j^k e_k. \tag{2.87}$$

Set $\theta^j(Z) = \sum_i \theta_i^j(Z)e^i$ and

$$\theta = \sum_j e^j \otimes e_j = \sum_j \theta^j \widetilde{e}_j \in T^*B \otimes TB. \tag{2.88}$$

As ∇^{TB} is torsion free, $\nabla^{TB}\theta = 0$. Thus the $\mathbb{R}^{2n-n_0}$-valued one-form $\theta = (\theta^j(Z))$ satisfies the structure equation,

$$d\theta + \Gamma^{TB}\wedge\theta = 0. \tag{2.89}$$

By the same proof of [1, Prop. 1.27], we have

$$\mathcal{R}^\perp = \sum_j Z_j^\perp \widetilde{e}_j^\perp(Z), \quad i_{\mathcal{R}^\perp}\theta = \sum_j Z_j^\perp e_j^\perp = Z^\perp. \tag{2.90}$$

Here under our trivialization by $\{\widetilde{e}_i\}$, we consider $Z^\perp = (0, Z_1^\perp, \cdots, Z_{n_0}^\perp)$ as a $\mathbb{R}^{2n-n_0}$-valued function.

Substituting (2.90) and $(L_{\mathcal{R}^\perp} - 1)Z^\perp = 0$ into the identity $i_{\mathcal{R}^\perp}(d\theta + \Gamma^{TB}\wedge\theta) = 0$, we obtain

$$(L_{\mathcal{R}^\perp} - 1)L_{\mathcal{R}^\perp}\theta = (L_{\mathcal{R}^\perp} - 1)(dZ^\perp + \Gamma^{TB}Z^\perp) = (L_{\mathcal{R}^\perp}\Gamma^{TB})Z^\perp = (i_{\mathcal{R}^\perp}R^{TB})Z^\perp. \tag{2.91}$$

Here we consider R^{TB} as a matrix of 2-forms, so that $R^{TB}Z^\perp$ is a vector of 2-forms, and θ is a $\mathbb{R}^{2n-n_0}$-valued 1-form.

By (2.90) and (2.91), we get

$$i_{e_j}(L_{\mathcal{R}^\perp} - 1)L_{\mathcal{R}^\perp}\theta^i(Z) = \left\langle R^{TB}(\mathcal{R}^\perp, e_j)\mathcal{R}^\perp, \widetilde{e}_i\right\rangle(Z). \tag{2.92}$$

We will denote by $\partial^\perp$, ∂^0 the partial derivatives along N_G, TX_G respectively. Then we have the following Taylor expansions of (2.92): for $j \in \{2(n-n_0)+1, \cdots, 2n-n_0\}$, i.e. $e_j \in N_G$, by $L_{\mathcal{R}^\perp}e^j = e^j$, we have

$$\sum_{|\alpha^\perp|\geq 1}(|\alpha^\perp|^2 + |\alpha^\perp|)((\partial^\perp)^{\alpha^\perp}\theta_j^i)(Z^0)\frac{(Z^\perp)^{\alpha^\perp}}{\alpha^\perp!} = \left\langle R^{TB}(\mathcal{R}^\perp, e_j)\mathcal{R}^\perp, \widetilde{e}_i\right\rangle(Z). \tag{2.93}$$

and for $j \in \{1, \cdots, 2(n-n_0)\}$, i.e. $e_j \in TX_G$, by $L_{\mathcal{R}^\perp}e^j = 0$, we have

$$\sum_{|\alpha^\perp|\geq 1}(|\alpha^\perp|^2 - |\alpha^\perp|)((\partial^\perp)^{\alpha^\perp}\theta_j^i)(Z^0)\frac{(Z^\perp)^{\alpha^\perp}}{\alpha^\perp!} = \left\langle R^{TB}(\mathcal{R}^\perp, e_j)\mathcal{R}^\perp, \widetilde{e}_i\right\rangle(Z). \tag{2.94}$$

From (2.93), (2.94), we still need to obtain the Taylor expansions for $\theta_j^i(Z^0)$, $(1 \leq i, j \leq 2n-n_0)$ and $(\partial_k^\perp\theta_j^i)(Z^0)$, $(1 \leq j \leq 2(n-n_0))$.

By our construction, we know that for i or $j \in \{2(n-n_0)+1, \cdots, 2n-n_0\}$,

$$\widetilde{e}_k^{\perp}(Z^0) = e_k^{\perp}(Z^0), \quad \theta_j^i(Z^0) = \delta_{i,j}. \tag{2.95}$$

By [1, (1.21)] (cf. [17, (4.35)]), we know that on $\mathbb{R}^{2n-2n_0} \times \{0\}$, for $i, j \in \{1, \cdots, 2(n-n_0)\}$,

$$\begin{aligned} &\theta_j^i(0) = \delta_{i,j}, \\ &\sum_{|\alpha^0| \geq 1} (|\alpha^0|^2 + |\alpha^0|)((\partial^0)^{\alpha^0}\theta_j^i)(0)\frac{(Z^0)^{\alpha^0}}{\alpha^0!} = \left\langle R^{TX_G}(\mathcal{R}^0, e_j)\mathcal{R}^0, \widetilde{e}_i \right\rangle (Z^0). \end{aligned} \tag{2.96}$$

while by (0.9), (2.87), and $[e_i^{\perp}, e_j^{\perp}] = 0$, we get

$$\begin{aligned} (\partial_k^{\perp}\theta_j^i)(Z^0) &= e_k^{\perp}\langle e_j^0, \widetilde{e}_i^0\rangle(Z^0) = \langle \nabla^{TB}_{e_k^{\perp}} e_j^0, \widetilde{e}_i^0\rangle(Z^0) \\ &= \langle \nabla^{TB}_{e_j^0} e_k^{\perp}, \widetilde{e}_i^0\rangle(Z^0) = -\langle \nabla^{TB}_{e_j^0}\widetilde{e}_i^0, e_k^{\perp}\rangle(Z^0) = -\langle A(e_j^0)\widetilde{e}_i^0, e_k^{\perp}\rangle(Z^0). \end{aligned} \tag{2.97}$$

Let R^{TX_G}, R^{N_G} be the curvatures of $\nabla^{TX_G}, \nabla^{N_G}$. By (0.9),

$$R^{TX_G} + R^{N_G} + A^2 + {}^0\nabla^{TB}_{\cdot} A = R^{TB}|_{X_G} \in \Lambda^2(TX_G) \otimes \mathrm{End}(TB). \tag{2.98}$$

For $1 \leq j \leq 2(n-n_0)$, $2(n-n_0)+1 \leq i \leq 2n-n_0$, $i' = i - 2(n-n_0)$, by $[e_k^{\perp}, e_j^0] = 0$, as in (2.97), we get

$$(\partial_k^{\perp}\theta_j^i)(Z^0) = e_k^{\perp}\langle e_j^0, \widetilde{e}_{i'}^{\perp}\rangle(Z^0) = \langle \nabla^{TB}_{e_j^0} e_k^{\perp}, \widetilde{e}_{i'}^{\perp}\rangle(Z^0) = \langle \nabla^{N_G}_{e_j^0} e_k^{\perp}, e_{i'}^{\perp}\rangle(Z^0). \tag{2.99}$$

By [1, Prop. 1.18] (cf. (2.104)) and (2.99), the Taylor expansion of $(\partial_k^{\perp}\theta_j^i)(Z^0)$ at 0 to order r only determines by those of R^{N_G} to order $r-1$.

Now by (2.87), (2.93)-(2.99) determine the Taylor expansion of $\theta_j^i(Z)$ to order m in terms of the Taylor expansion of the curvature coefficients of R^{TB} to order $m-2$ and A to order $m-1$.

By (2.87), we get Lemma 2.10. □

Let dv_{TB} be the Riemannian volume form on $(T_{x_0}B, g^{TB})$.

Let $\kappa(Z)$ ($Z \in \mathbb{R}^{2n-n_0}$) be the smooth positive function defined by the equation

$$dv_{B_0}(Z) = \kappa(Z) dv_{TB}(Z), \tag{2.100}$$

with $\kappa(0) = 1$.

For $s \in \mathscr{C}^{\infty}(\mathbb{R}^{2n-n_0}, \mathbf{E}_{x_0})$ and $Z \in \mathbb{R}^{2n-n_0}$, for $t = \frac{1}{\sqrt{p}}$, set

$$\begin{aligned} &(S_t s)(Z) = s(Z/t), \quad \nabla_t = S_t^{-1} t \kappa^{\frac{1}{2}} \nabla^{E_{p,B_0}} \kappa^{-\frac{1}{2}} S_t, \\ &\mathscr{L}_2^t = S_t^{-1} t^2 \kappa^{\frac{1}{2}} \Phi D_p^{X_0,2} \Phi^{-1} \kappa^{-\frac{1}{2}} S_t. \end{aligned} \tag{2.101}$$

As in (1.18), we denote by $R^{L_B}, R^{E_B}, R^{\mathrm{Cliff}_B}$ the curvatures on L_B, E_B, $\Lambda(T^{*(0,1)}X)_B$ induced by $\nabla^L, \nabla^E, \nabla^{\mathrm{Cliff}}$ on X.

As in (1.14), $\widetilde{\mu} \in TY$, $\widetilde{\mu}^E \in TY \otimes \mathrm{End}(E)$, $\widetilde{\mu}^{\mathrm{Cliff}} \in TY \otimes \mathrm{End}(\Lambda(T^{*(0,1)}X))$ are sections induced by $\mu, \mu^E, \mu^{\mathrm{Cliff}}$ in (2.15), (2.21).

Denote by ∇_V the ordinary differentiation operator on $T_{x_0}B$ in the direction V.

Denote by $(\partial^{\alpha} R^{L_B})_{x_0}$ the tensor $(\partial^{\alpha} R^{L_B})_{x_0}(e_i, e_j) := \partial^{\alpha}(R^{L_B}(e_i, e_j))_{x_0}$.

Theorem 2.11. *There exist $\mathcal{A}_{i,j,r}$ (resp. $\mathcal{B}_{i,r}$, $\mathcal{C}_r$) ($r \in \mathbb{N}, i,j \in \{1,\cdots,2n-n_0\}$) polynomials in Z, and $\mathcal{A}_{i,j,r}$ is a monomial in Z with degree r, the degree on Z of $\mathcal{B}_{i,r}$ (resp. $\mathcal{C}_r$) has the same parity with $r-1$ (resp. r), with the following properties:*

– the coefficients of $\mathcal{A}_{i,j,r}$ are polynomials in R^{TB} (resp. A) and their derivatives at x_0 to order $r-2$ (resp. $r-1$);

– the coefficients of $\mathcal{B}_{i,r}$ are polynomials in R^{TB}, A, R^{Cliff_B}, R^{E_B}, (resp. R^{L_B}) and their derivatives at x_0 to order $r-1$ (resp. r);

– the coefficients of $\mathcal{C}_r$ are polynomials in R^{TB}, A, R^{Cliff_B}, R^{E_B}, $\widetilde{\mu}^E$, $\widetilde{\mu}^{\mathrm{Cliff}}$ (resp. r^X, $\mathrm{Tr}[R^{T^{(1,0)}X}]$, R^E; resp. h, R^L, R^{L_B}; resp. μ) and their derivatives at x_0 to order $r-1$ (resp. $r-2$; resp. r; resp. $r+1$).

– if we denote by

$$\begin{aligned}
&\mathcal{O}_r = \mathcal{A}_{i,j,r}\nabla_{e_i}\nabla_{e_j} + \mathcal{B}_{i,r}\nabla_{e_i} + \mathcal{C}_r,\\
&\mathscr{L}_2^0 = -\sum_{j=1}^{2n-n_0}\Big(\nabla_{e_j} + \frac{1}{2}R^{L_B}_{x_0}(\mathcal{R},e_j)\Big)^2 - 2\omega_{d,x_0} - \tau_{x_0} + 4\pi^2|P^{TY}\mathbf{J}_{x_0}\mathcal{R}|^2,
\end{aligned} \tag{2.102}$$

then

$$\mathscr{L}_2^t = \mathscr{L}_2^0 + \sum_{r=1}^{m} t^r\mathcal{O}_r + \mathscr{O}(t^{m+1}). \tag{2.103}$$

Moreover, there exists $m' \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, $t \le 1$, $|tZ| \le \varepsilon$, the derivatives of order $\le k$ of the coefficients of the operator $\mathscr{O}(t^{m+1})$ are dominated by $Ct^{m+1}(1+|Z|)^{m'}$.

Proof. Let Γ^{E_B}, Γ^{L_B} and $\Gamma^{\mathrm{Cliff}_B}$ be the connection forms of ∇^{E_B}, ∇^{L_B} and $\nabla^{\mathrm{Cliff}_B}$ with respect to any fixed frames for E_B, L_B and $\Lambda(T^{*(1,0)}X)_B$ which are parallel along the curve $\gamma_u : [0,1] \ni u \to uZ$ under our trivialization on $B^{T_{x_0}B}(0,\varepsilon)$. Then Γ^{E_B} is $\mathrm{End}(\mathbb{C}^{\dim E})$-valued 1-form on $\mathbb{R}^{2n-n_0}$ and Γ^{L_B} is 1-form on $\mathbb{R}^{2n-n_0}$.

Now for $\Gamma^\bullet = \Gamma^{E_B}, \Gamma^{L_B}$ or $\Gamma^{\mathrm{Cliff}_B}$ and $R^\bullet = R^{E_B}, R^{L_B}$ or R^{Cliff_B} respectively, by the definition of our fixed frame and [1, Proposition 1.18] (cf. also [17, (4.45)]), the Taylor coefficients of $\Gamma^\bullet(e_j)(Z)$ at x_0 to order r only determines by those of $R^\bullet$ to order $r-1$, and

$$\sum_{|\alpha|=r}(\partial^\alpha\Gamma^\bullet)_{x_0}(e_j)\frac{Z^\alpha}{\alpha!} = \frac{1}{r+1}\sum_{|\alpha|=r-1}(\partial^\alpha R^\bullet)_{x_0}(\mathcal{R},e_j)\frac{Z^\alpha}{\alpha!}. \tag{2.104}$$

Especially,

$$\Gamma^\bullet_Z(e_j) = \frac{1}{2}R^\bullet_{x_0}(\mathcal{R},e_j) + \mathscr{O}(|Z|^2). \tag{2.105}$$

By (2.101), for $t = 1/\sqrt{p}$, if $|Z| \le \sqrt{p}\varepsilon$, then

$$\nabla_t = \kappa^{\frac{1}{2}}(tZ)\Big(\nabla + (t\Gamma^{\mathrm{Cliff}_B} + t\Gamma^{E_B} + \frac{1}{t}\Gamma^{L_B})(tZ)\Big)\kappa^{-\frac{1}{2}}(tZ). \tag{2.106}$$

Moreover, set

$$(\nabla^{TB}_{e_i}e_j)(Z) = \Gamma^k_{ij}(Z)e_k, \quad g_{ij}(Z) = g^{TB}(e_i,e_j)(Z) = \theta^k_i\theta^k_j(Z), \tag{2.107}$$

then Γ^k_{ij} is the connection form of ∇^{TB} with respect to the frame $\{e_i\}$.

Let (g^{ij}) be the inverse matrix of (g_{ij}), then

$$\Delta^{E_{p,B}} = -\sum_{ij} g^{ij}\Big(\nabla^{E_{p,B}}_{e_i}\nabla^{E_{p,B}}_{e_j} - \Gamma^k_{ij}\nabla^{E_{p,B}}_{e_k}\Big), \tag{2.108}$$

and by (1.1), (2.100),

$$\begin{aligned}\kappa(Z) &= (\det g_{ij})^{1/2}(Z),\\ \Gamma^k_{ij} &= \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).\end{aligned} \tag{2.109}$$

By (2.60), (2.101) and (2.108),

$$\begin{aligned}\mathscr{L}^t_2(Z) = -g^{ij}(tZ)(\nabla_{t,e_i}\nabla_{t,e_j} - t\Gamma^k_{ij}(tZ)\nabla_{t,e_k}) - \langle t\widetilde{\mu}^{E_p}, t\widetilde{\mu}^{E_p}\rangle_{g^{TY}}(tZ)\\ - 2\omega_d(tZ) - \tau(tZ) + t^2\Big(\frac{1}{4}r^X + \mathbf{c}(R) - \frac{1}{h}\Delta_{B_0}h\Big)(tZ).\end{aligned} \tag{2.110}$$

By (2.21),

$$\langle t\widetilde{\mu}^{E_p}, t\widetilde{\mu}^{E_p}\rangle_{g^{TY}} = -4\pi^2|\frac{1}{t}\widetilde{\mu}|^2_{g^{TY}} + \langle 4\pi\sqrt{-1}\widetilde{\mu} + t^2(\widetilde{\mu}^{\mathrm{Cliff}} + \widetilde{\mu}^E), \widetilde{\mu}^{\mathrm{Cliff}} + \widetilde{\mu}^E\rangle_{g^{TY}}. \tag{2.111}$$

By (2.5), (2.15), and $\widetilde{\mu}_{y_0} = 0$, for $y_0 \in P, \pi(y_0) = x_0$, we get for $K \in \mathfrak{g}$,

$$-\langle \mathbf{J}e^H_i, K^X\rangle_{y_0} = \omega(K^X, e^H_i) = \nabla_{e^H_i}(\mu(K)) = \langle \nabla^{TY}_{e^H_i}\widetilde{\mu}, K^X\rangle_{y_0}, \tag{2.112}$$

thus

$$|\widetilde{\mu}|^2_{g^{TY}}(Z) = |\nabla^{TY}_{\mathcal{R}}\widetilde{\mu}|^2_{g^{TY}} + \mathscr{O}(|Z|^3) = |P^{TY}\mathbf{J}_{x_0}\mathcal{R}|^2 + \mathscr{O}(|Z|^3). \tag{2.113}$$

By Lemma 2.10, (2.104), (2.106), (2.110) and (2.113), we know that $\mathscr{L}^t_2$ has the expansion (2.103), in particular, we get the formula $\mathscr{L}^0_2$ in (2.102).

By (2.98), (2.104) and (2.110), we get the properties on $\mathcal{A}_{i,j,r}$, $\mathcal{B}_{i,r}$.

By (2.98), (2.110) and (2.111), we get the properties on $\mathcal{C}_r$.

The proof of Theorem 2.11 is complete. □

2.7. **Uniform estimate on the G-invariant Bergman kernel.** Recall that the operators $\mathscr{L}^t_2$, ∇_t were defined in (2.101), and $\mathbf{E}_0 = \Lambda(T^{*(0,1)}X_0)\otimes E_0$. We have trivialized the bundle $\mathbf{E}_{0,B_0}$ to $\mathbf{E}_{B,x_0}$ in Section 2.6. We still denote by $h^{\mathbf{E}_{0,B_0}}$ the metric on the trivial bundle $\mathbf{E}_{B,x_0}$ on $\mathbb{R}^{2n-n_0}$ induced by the corresponding metric on $\mathbf{E}_{0,B_0}$. Note that $h^{\mathbf{E}_{0,B_0}}$ is not a constant metric on $\mathbb{R}^{2n-n_0}$.

We also denote by $\langle\ ,\ \rangle_{0,L^2}$ and $\|\ \|_{0,L^2}$ the scalar product and the L^2 norm on $\mathscr{C}^\infty(T_{x_0}B, \mathbf{E}_{B,x_0})$ induced by $g^{T_{x_0}B}, h^{\mathbf{E}_{0,B_0}}$ as in (1.19).

Let $\widetilde{\mu}_{X_0}, \widetilde{\mu}^{E_{0,p}}$ be the G-invariant sections of TY, $TY\otimes\mathrm{End}(E_{0,p})$ on X_0 induced by μ_{X_0}, $\mu^{E_{0,p}}$ as in (1.14).

Let $\{f_l\}$ be a G-invariant orthonormal frame of TY on $\pi^{-1}(B^B(x_0,\varepsilon))$, then $(f_{0,l})_Z = (f_l)_{\varphi_\varepsilon(Z)}$ is a G-invariant orthonormal frame of TY_0 on X_0.

Definition 2.12. Set

$$\mathcal{D}_t = \{\nabla_{t,e_i}, 1\le i\le 2n-n_0; \frac{1}{t}\langle\widetilde{\mu}_{X_0}, f_{0,l}\rangle(tZ), 1\le j\le n_0\}. \tag{2.114}$$

For $k \in \mathbb{N}^*$, let $\mathcal{D}_t^k$ be the family of operators acting on $\mathscr{C}^\infty(T_{x_0}B, \mathbf{E}_{B,x_0})$ which can be written in the form $Q = Q_1 \cdots Q_k$, $Q_i \in \mathcal{D}_t$.

For $s \in \mathscr{C}^\infty(T_{x_0}B, \mathbf{E}_{B,x_0})$, $k \geq 1$, set

$$\begin{aligned} \|s\|_{t,0}^2 &= \int_{\mathbb{R}^{2n-n_0}} |s(Z)|^2_{h^{\mathbf{E}_{0,B_0}(tZ)}} dv_{T_{x_0}B}(Z) = t^{-2n+n_0} \|S_t s\|^2_{0,L^2}, \\ \|s\|_{t,k}^2 &= \|s\|_{t,0}^2 + \sum_{l=1}^{k} \sum_{Q \in \mathcal{D}_t^l} \|Qs\|_{t,0}^2. \end{aligned} \tag{2.115}$$

We denote by $\langle s', s \rangle_{t,0}$ the inner product on $\mathscr{C}^\infty(T_{x_0}B, \mathbf{E}_{B,x_0})$ corresponding to $\| \quad \|^2_{t,0}$.

Let H_t^m be the Sobolev space of order m with norm $\| \quad \|_{t,m}$. Let H_t^{-1} be the Sobolev space of order -1 and let $\| \quad \|_{t,-1}$ be the norm on H_t^{-1} defined by $\|s\|_{t,-1} = \sup_{0 \neq s' \in H_t^1} |\langle s, s' \rangle_{t,0}| / \|s'\|_{t,1}$.

If $A \in \mathscr{L}(H_t^m, H_t^{m'})$ $(m, m' \in \mathbb{Z})$, we denote by $\|A\|_t^{m,m'}$ the norm of A with respect to the norms $\| \quad \|_{t,m}$ and $\| \quad \|_{t,m'}$.

Then $\mathscr{L}_2^t$ is a formally self-adjoint elliptic operator with respect to $\| \quad \|^2_{t,0}$, and is a smooth family of operators with respect to the parameter $x_0 \in X_G$.

Theorem 2.13. *There exist constants $C_1, C_2, C_3 > 0$ such that for $t \in]0,1]$ and any $s, s' \in C_0^\infty(\mathbb{R}^{2n-n_0}, \mathbf{E}_{B,x_0})$,*

$$\begin{aligned} &\left\langle \mathscr{L}_2^t s, s \right\rangle_{t,0} \geq C_1 \|s\|_{t,1}^2 - C_2 \|s\|_{t,0}^2, \\ &|\left\langle \mathscr{L}_2^t s, s' \right\rangle_{t,0}| \leq C_3 \|s\|_{t,1} \|s'\|_{t,1}. \end{aligned} \tag{2.116}$$

Proof. By (2.83) and our construction for L_0, E_0 on X_0, we know for $Z \in T_{x_0}B$, $|Z| > 4\varepsilon$,

$$\mu^{E_{0,p}}(K)_{(1,Z)} = p\, R^L_{y_0}((\mathcal{R}^\perp)^H, K^X_{y_0}). \tag{2.117}$$

Thus from (2.110) and (2.115),

$$\begin{aligned} \left\langle \mathscr{L}_2^t s, s \right\rangle_{t,0} &= \|\nabla_t s\|^2_{t,0} - t^2 \left\langle \langle \widetilde{\mu}^{E_{0,p}}, \widetilde{\mu}^{E_{0,p}} \rangle_{g^{TY}}(tZ) s, s \right\rangle_{t,0} \\ &\quad + \left\langle \left(-2S_t^{-1}\omega_d - S_t^{-1}\tau + t^2 S_t^{-1}(\tfrac{1}{4} r^X + \mathbf{c}(R) - \frac{1}{h}\Delta_{B_0} h) \right) s, s \right\rangle_{t,0}. \end{aligned} \tag{2.118}$$

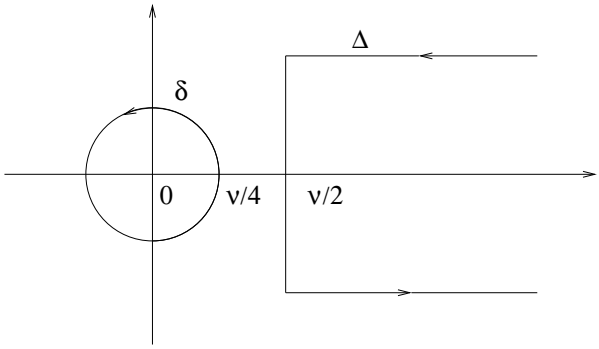
From (2.80), (2.111), (2.117), and our construction on ∇^{E_0},

$$-t^2 \left\langle \langle \widetilde{\mu}^{E_{0,p}}, \widetilde{\mu}^{E_{0,p}} \rangle_{g^{TY}}(tZ) s, s \right\rangle_{t,0} \geq 2\pi^2 \sum_{l=1}^{n_0} \left\| \frac{1}{t} \langle \widetilde{\mu}_{X_0}, f_{0,l} \rangle (tZ) s \right\|^2_{t,0} - Ct \|s\|^2_{t,0}. \tag{2.119}$$

From (2.118) and (2.119), we get (2.116). □

Recall that ν is the constant in (2.23).

Let δ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\nu/4$, and let Δ be the oriented path in $\mathbb{C}$ which goes parallel to the real axis from $+\infty + i$ to $\frac{\nu}{2} + i$ then parallel to the imaginary axis to $\frac{\nu}{2} - i$ and the parallel to the real axis to $+\infty - i$.



Theorem 2.14. *There exist $t_0 > 0$, $C > 0$ such that for $t \in]0, t_0]$, $\lambda \in \delta \cup \Delta$ and $x_0 \in X_G$, $(\lambda - \mathscr{L}_2^t)^{-1}$ exists and*

$$\begin{aligned} &\|(\lambda - \mathscr{L}_2^t)^{-1}\|_t^{0,0} \leq C, \\ &\|(\lambda - \mathscr{L}_2^t)^{-1}\|_t^{-1,1} \leq C(1 + |\lambda|^2). \end{aligned} \tag{2.120}$$

Proof. By (2.23), (2.60) for $D_p^{X_0}$, and (2.101), there exists $t_0 > 0$ such that for $t \in]0, t_0]$,

$$\mathrm{Spec}\left(\mathscr{L}_2^t\right) \subset \{0\} \cup [\nu, +\infty[\,. \tag{2.121}$$

Thus $(\lambda - \mathscr{L}_2^t)^{-1}$ exists for $\lambda \in \delta \cup \Delta$.

The first inequality of (2.120) is from (2.121).

By (2.116), for $\lambda_0 \in \mathbb{R}$, $\lambda_0 \leq -2C_2$, $(\lambda_0 - \mathscr{L}_2^t)^{-1}$ exists, and we have $\|(\lambda_0 - \mathscr{L}_2^t)^{-1}\|_t^{-1,1} \leq \frac{1}{C_1}$. Now,

$$(\lambda - \mathscr{L}_2^t)^{-1} = (\lambda_0 - \mathscr{L}_2^t)^{-1} - (\lambda - \lambda_0)(\lambda - \mathscr{L}_2^t)^{-1}(\lambda_0 - \mathscr{L}_2^t)^{-1}. \tag{2.122}$$

Thus for $\lambda \in \delta \cup \Delta$, from (2.122), we get

$$\|(\lambda - \mathscr{L}_2^t)^{-1}\|_t^{-1,0} \leq \frac{1}{C_1}\Big(1 + \frac{4}{\nu}|\lambda - \lambda_0|\Big). \tag{2.123}$$

Now we change the last two factors in (2.122), and apply (2.123), we get

$$\begin{aligned} \|(\lambda - \mathscr{L}_2^t)^{-1}\|_t^{-1,1} &\leq \frac{1}{C_1} + \frac{|\lambda - \lambda_0|}{{C_1}^2}\Big(1 + \frac{4}{\nu}|\lambda - \lambda_0|\Big) \\ &\leq C(1 + |\lambda|^2). \end{aligned} \tag{2.124}$$

The proof of our Theorem is complete. □

Proposition 2.15. *Take $m \in \mathbb{N}^*$. There exists $C_m > 0$ such that for $t \in]0, 1]$, $Q_1, \cdots, Q_m \in \mathcal{D}_t \cup \{Z_i\}_{i=1}^{2n-n_0}$ and $s, s' \in \mathscr{C}_0^\infty(\mathbb{R}^{2n-n_0}, \mathbf{E}_{B,x_0})$,*

$$\Big|\big\langle [Q_1, [Q_2, \ldots, [Q_m, \mathscr{L}_2^t]] \ldots] s, s' \big\rangle_{t,0}\Big| \leq C_m \|s\|_{t,1} \|s'\|_{t,1}. \tag{2.125}$$

Proof. Note that $[\nabla_{t,e_i}, Z_j] = \delta_{ij}$. By (2.110), we know that $[Z_j, \mathscr{L}_2^t]$ verifies (2.125).

Recall that by (2.80) and (2.83), $(\nabla_{e_i}\langle \widetilde{\mu}_{X_0}, f_{0,l}\rangle)(tZ)$ is uniformly bounded with its derivatives for $t \in [0, 1]$ and

$$\nabla_{e_i}\langle \widetilde{\mu}_{X_0}, f_{0,l}\rangle = (e_i\langle \widetilde{\mu}_{X_0}, f_{0,l}\rangle)_{x_0} = \omega(f_{0,l}, e_i)_{x_0} \tag{2.126}$$

for $|Z| \geq 4\varepsilon$. Thus $[\frac{1}{t}\langle \widetilde{\mu}_{X_0}, f_{0,l}\rangle(tZ), \mathscr{L}_2^t]$ also verifies (2.125).

Note that by (2.101),

$$[\nabla_{t,e_i},\nabla_{t,e_j}]=\left(R^{L_{0,B_0}}(tZ)+t^2R^{\mathbf{E}_{0,B_0}}(tZ)\right)(e_i,e_j). \tag{2.127}$$

Thus from (2.110), (2.126) and (2.127), we know that $[\nabla_{t,e_k},\mathscr{L}_2^t]$ has the same structure as $\mathscr{L}_2^t$ for $t\in]0,1]$, i.e. $[\nabla_{t,e_k},\mathscr{L}_2^t]$ has the type as

$$\begin{aligned}\sum_{ij}a_{ij}(t,tZ)\nabla_{t,e_i}\nabla_{t,e_j}+\sum_i c_i(t,tZ)\nabla_{t,e_i}\\ +\sum_l\left[c_l'(t,tZ)\frac{1}{t}\langle\widetilde{\mu}_{X_0},f_{0,l}\rangle(tZ)+a|\frac{1}{t}\widetilde{\mu}_{X_0}|^2_{g^{TY}}(tZ)\right]+c(t,tZ),\end{aligned} \tag{2.128}$$

where $a\in\mathbb{C}$; $a_{ij}(t,Z),c_i(t,Z),c_j'(t,Z),c(t,Z)$ and their derivatives on Z are uniformly bounded for $Z\in\mathbb{R}^{2n-n_0},t\in[0,1]$; moreover, they are polynomials in t. In fact, for $[\nabla_{t,e_k},\mathscr{L}_2^t]$, $a=0$ in (2.128).

Let $(\nabla_{t,e_i})^*$ be the adjoint of ∇_{t,e_i} with respect to $\langle\ ,\ \rangle_{t,0}$, then by (2.115),

$$(\nabla_{t,e_i})^*=-\nabla_{t,e_i}-t(k^{-1}\nabla_{e_i}k)(tZ), \tag{2.129}$$

the last term of (2.129) and its derivatives in Z are uniformly bounded in $Z\in\mathbb{R}^{2n-n_0},t\in[0,1]$.

By (2.128) and (2.129), (2.125) is verified for $m=1$.

By iteration, we know that $[Q_1,[Q_2,\ldots,[Q_m,\mathscr{L}_2^t]]\ldots]$ has the same structure (2.128) as $\mathscr{L}_2^t$. By (2.129), we get Proposition 2.15. □

Theorem 2.16. *For any $t\in]0,t_0]$, $\lambda\in\delta\cup\Delta$, $m\in\mathbb{N}$, the resolvent $(\lambda-\mathscr{L}_2^t)^{-1}$ maps H_t^m into H_t^{m+1}. Moreover for any $\alpha\in\mathbb{Z}^{2n-n_0}$, there exist $N\in\mathbb{N}$, $C_{\alpha,m}>0$ such that for $t\in]0,t_0]$, $\lambda\in\delta\cup\Delta$, $s\in\mathscr{C}_0^\infty(\mathbb{R}^{2n-n_0},\mathbf{E}_{B,x_0})$,*

$$\|Z^\alpha(\lambda-\mathscr{L}_2^t)^{-1}s\|_{t,m+1}\leq C_{\alpha,m}(1+|\lambda|^2)^N\sum_{\alpha'\leq\alpha}\|Z^{\alpha'}s\|_{t,m}. \tag{2.130}$$

Proof. For $Q_1,\cdots,Q_m\in\mathcal{D}_t$, $Q_{m+1},\cdots,Q_{m+|\alpha|}\in\{Z_i\}_{i=1}^{2n-n_0}$, we can express $Q_1\cdots Q_{m+|\alpha|}(\lambda-\mathscr{L}_2^t)^{-1}$ as a linear combination of operators of the type

$$[Q_1,[Q_2,\ldots[Q_{m'},(\lambda-\mathscr{L}_2^t)^{-1}]]\ldots]Q_{m'+1}\cdots Q_{m+|\alpha|},\quad m'\leq m+|\alpha|. \tag{2.131}$$

Let $\mathscr{R}_t$ be the family of operators

$$\mathscr{R}_t=\{[Q_{j_1},[Q_{j_2},\ldots[Q_{j_l},\mathscr{L}_2^t]]\ldots]\}.$$

Clearly, any commutator $[Q_1,[Q_2,\ldots[Q_{m'},(\lambda-\mathscr{L}_2^t)^{-1}]]\ldots]$ is a linear combination of operators of the form

$$(\lambda-\mathscr{L}_2^t)^{-1}R_1(\lambda-\mathscr{L}_2^t)^{-1}R_2\cdots R_{m'}(\lambda-\mathscr{L}_2^t)^{-1} \tag{2.132}$$

with $R_1,\cdots,R_{m'}\in\mathscr{R}_t$.

By Proposition 2.15, the norm $\|\quad\|_t^{1,-1}$ of the operators $R_j\in\mathscr{R}_t$ is uniformly bound by C.

By Theorem 2.14, we find that there exist $C>0$, $N\in\mathbb{N}$ such that the norm $\|\quad\|_t^{0,1}$ of operators (2.132) is dominated by $C(1+|\lambda|^2)^N$. □

Let $\pi_B : TB \times_B TB \to B$ be the natural projection from the fiberwise product of TB on B.

Let $e^{-u\mathscr{L}_2^t}(Z,Z')$, $(\mathscr{L}_2^t e^{-u\mathscr{L}_2^t})(Z,Z')$ be the smooth kernels of the operators $e^{-u\mathscr{L}_2^t}$, $\mathscr{L}_2^t e^{-u\mathscr{L}_2^t}$ with respect to $dv_{T_{x_0}B}(Z')$.

Note that $\mathscr{L}_2^t$ are families of differential operators with coefficients in $\mathrm{End}(\mathbf{E}_{B,x_0}) = \mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0}$. Thus we can view $e^{-u\mathscr{L}_2^t}(Z,Z')$, $(\mathscr{L}_2^t e^{-u\mathscr{L}_2^t})(Z,Z')$ as smooth sections of $\pi_B^*(\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_B)$ on $TB\times_B TB$.

Let $\nabla^{\mathrm{End}(\mathbf{E}_B)}$ be the connection on $\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_B$ induced by $\nabla^{\mathrm{Cliff}_B}$ and ∇^{E_B}. And $\nabla^{\mathrm{End}(\mathbf{E}_B)}$, h^E and g^{TX} induce naturally a $\mathscr{C}^m$-norm for the parameter $x_0 \in X_G$.

As in Introduction, for $Z \in T_{x_0}B$, we will write $Z = Z^0 + Z^\perp$, with $Z^0 \in T_{x_0}X_G$, $Z^\perp \in N_{G,x_0}$.

Theorem 2.17. *There exists $C'' > 0$ such that for any $m, m', m'', r \in \mathbb{N}$, $u_0 > 0$, there exists $C > 0$ such that for $t \in]0, t_0]$, $u \geq u_0$, $Z, Z' \in T_{x_0}B$,*

$$\begin{aligned}
&\sup_{|\alpha|+|\alpha'|\leq m}(1+|Z^\perp|+|Z'^\perp|)^{m''}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}}\frac{\partial^r}{\partial t^r}e^{-u\mathscr{L}_2^t}(Z,Z')\Big|_{\mathscr{C}^{m'}(X_G)}\\
&\qquad\leq C(1+|Z^0|+|Z'^0|)^{2(n+r+m'+1)+m}\exp\Big(\frac{1}{2}\nu u-\frac{2C''}{u}|Z-Z'|^2\Big),\\
&\sup_{|\alpha|+|\alpha'|\leq m}(1+|Z^\perp|+|Z'^\perp|)^{m''}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}}\frac{\partial^r}{\partial t^r}(\mathscr{L}_2^t e^{-u\mathscr{L}_2^t})(Z,Z')\Big|_{\mathscr{C}^{m'}(X_G)}\\
&\qquad\leq C(1+|Z^0|+|Z'^0|)^{2(n+r+m'+1)+m}\exp\Big(-\frac{1}{4}\nu u-\frac{2C''}{u}|Z-Z'|^2\Big),
\end{aligned}\tag{2.133}$$

where $\mathscr{C}^{m'}(X_G)$ is the $\mathscr{C}^{m'}$ norm for the parameter $x_0 \in X_G$.

Proof. By (2.121), for any $k \in \mathbb{N}^*$,

$$\begin{aligned}
e^{-u\mathscr{L}_2^t} &= \frac{(-1)^{k-1}(k-1)!}{2\pi i u^{k-1}}\int_{\delta\cup\Delta} e^{-u\lambda}(\lambda-\mathscr{L}_2^t)^{-k}d\lambda,\\
\mathscr{L}_2^t e^{-u\mathscr{L}_2^t} &= \frac{(-1)^{k-1}(k-1)!}{2\pi i u^{k-1}}\int_{\Delta} e^{-u\lambda}\Big[\lambda(\lambda-\mathscr{L}_2^t)^{-k}-(\lambda-\mathscr{L}_2^t)^{-k+1}\Big]d\lambda.
\end{aligned}\tag{2.134}$$

From Theorem 2.16, we deduce that if $Q \in \cup_{l=1}^m \mathcal{D}_t^l$, there are $N \in \mathbb{N}$, $C_m > 0$ such that for any $\lambda \in \delta \cup \Delta$,

$$\|Q(\lambda-\mathscr{L}_2^t)^{-m}\|_t^{0,0} \leq C_m(1+|\lambda|^2)^N. \tag{2.135}$$

Recall that $\mathscr{L}_t^2$ is self-adjoint with respect to $\|\ \|_{t,0}$. After taking the adjoint of (2.135), we get

$$\|(\lambda-\mathscr{L}_2^t)^{-m}Q\|_t^{0,0} \leq C_m(1+|\lambda|^2)^N. \tag{2.136}$$

From (2.134), (2.135) and (2.136), we get if $Q, Q' \in \cup_{l=1}^m \mathcal{D}_t^l$,

$$\begin{aligned}
&\|Qe^{-u\mathscr{L}_2^t}Q'\|_t^{0,0} \leq C_m e^{\frac{1}{4}\nu u},\\
&\|Q(\mathscr{L}_2^t e^{-u\mathscr{L}_2^t})Q'\|_t^{0,0} \leq C_m e^{-\frac{1}{2}\nu u}.
\end{aligned}\tag{2.137}$$

Let $|\ \ |_m$ be the usual Sobolev norm on $\mathscr{C}^\infty(\mathbb{R}^{2n-n_0}, \mathbf{E}_{B,x_0})$ induced by $h^{\mathbf{E}_{B,x_0}} = h^{(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0}}$ and the volume form $dv_{T_{x_0}B}(Z)$ as in (2.115).

Observe that by (2.106), (2.115), there exists $C > 0$ such that for $s \in \mathscr{C}^\infty(T_{x_0}B, \mathbf{E}_{B,x_0})$, $\mathrm{supp}(s) \subset B^{T_{x_0}B}(0,q)$, $m \geq 0$,

$$\frac{1}{C}(1+q)^{-m}\|s\|_{t,m} \leq |s|_m \leq C(1+q)^m\|s\|_{t,m}. \tag{2.138}$$

Now (2.137), (2.138) together with Sobolev's inequalities imply that if $Q, Q' \in \cup_{l=1}^m \mathcal{D}_t^l$, for $\mathcal{K}_u(\mathscr{L}_2^t) = e^{-\frac{1}{4}\nu u}e^{-u\mathscr{L}_2^t}$ or $e^{\frac{1}{2}\nu u}\mathscr{L}_2^t e^{-u\mathscr{L}_2^t}$, we have

$$\sup_{|Z|,|Z'|\leq q} |Q_Z Q'_{Z'}\mathcal{K}_u(\mathscr{L}_2^t)(Z,Z')| \leq C(1+q)^{2n+2}. \tag{2.139}$$

By (2.80), (2.81) and (2.83),

$$\sum_{l=1}^{n_0}\Big|\frac{1}{t}\langle\widetilde{\mu}_{X_0}, f_{0,l}\rangle(tZ)\Big|^2 = |\frac{1}{t}\widetilde{\mu}_{X_0}|^2_{g^{TY}}(tZ) \geq C|Z^\perp|^2. \tag{2.140}$$

Thus by (2.106), (2.139), (2.140), we derive (2.133) with the exponentials $e^{\frac{1}{4}\nu u}$, $e^{-\frac{1}{2}\nu u}$ for the case when $r = m' = 0$ and $C'' = 0$, i.e.

$$\begin{aligned}\sup_{|\alpha|+|\alpha'|\leq m} (1+|Z^\perp|+|Z'^\perp|)^{m''}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}}\mathcal{K}_u(\mathscr{L}_2^t)\,(Z,Z')\,\Big| \\ \leq C(1+|Z^0|+|Z'^0|)^{2n+m+2}.\end{aligned} \tag{2.141}$$

To obtain (2.133) in general, we proceed as in the proof of [4, Theorem 11.14].

Note that the function f is defined in (2.28). For $\varrho > 1$, put

$$K_{u,\varrho}(a) = \int_{-\infty}^{+\infty}\exp(iv\sqrt{2u}a)\exp(-\frac{v^2}{2})\Big(1 - f(\frac{1}{\varrho}\sqrt{2u}v)\Big)\frac{dv}{\sqrt{2\pi}}. \tag{2.142}$$

Then there exist $C', C_1 > 0$ such that for any $c > 0$, $m, m' \in \mathbb{N}$, there is $C > 0$ such that for $u \geq u_0$, $a \in \mathbb{C}, |\mathrm{Im}(a)| \leq c$, we have

$$|a|^m|K_{u,\varrho}^{(m')}(a)| \leq C\exp\Big(C'c^2u - \frac{C_1}{u}\varrho^2\Big). \tag{2.143}$$

For any $c > 0$, let V_c be the images of $\{\lambda \in \mathbb{C}, |\mathrm{Im}(\lambda)| \leq c\}$ by the map $\lambda \to \lambda^2$. Then

$$V_c = \{\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) \geq \frac{1}{4c^2}\mathrm{Im}(\lambda)^2 - c^2\},$$

and $\delta \cup \Delta \subset V_c$ for c large enough.

Let $\widetilde{K}_{u,\varrho}$ be the holomorphic function such that $\widetilde{K}_{u,\varrho}(a^2) = K_{u,\varrho}(a)$.

By (2.143), for $\lambda \in V_c$,

$$|\lambda|^m|\widetilde{K}_{u,\varrho}^{(m')}(\lambda)| \leq C\exp\Big(C'c^2u - \frac{C_1}{u}\varrho^2\Big). \tag{2.144}$$

Using finite propagation speed of solutions of hyperbolic equations and (2.142), we find that there exists a fixed constant (which depends on ε) $c' > 0$ such that

$$\widetilde{K}_{u,\varrho}(\mathscr{L}_2^t)(Z,Z') = e^{-u\mathscr{L}_2^t}(Z,Z') \quad \text{if } |Z - Z'| \geq c'\varrho. \tag{2.145}$$

By (2.144), we see that given $k \in \mathbb{N}$, there is a unique holomorphic function $\widetilde{K}_{u,\varrho,k}(\lambda)$ defined on a neighborhood of V_c such that it verifies the same estimates as $\widetilde{K}_{u,\varrho}$ in (2.144) and $\widetilde{K}_{u,\varrho,k}(\lambda) \to 0$ as $\lambda \to +\infty$; moreover

$$\widetilde{K}^{(k-1)}_{u,\varrho,k}(\lambda)/(k-1)! = \widetilde{K}_{u,\varrho}(\lambda). \tag{2.146}$$

Thus as in (2.134),

$$\begin{aligned} &\widetilde{K}_{u,\varrho}(\mathscr{L}_2^t) = \frac{1}{2\pi i}\int_{\delta\cup\Delta} \widetilde{K}_{u,\varrho,k}(\lambda)(\lambda - \mathscr{L}_2^t)^{-k} d\lambda, \\ &\mathscr{L}_2^t \widetilde{K}_{u,\varrho}(\mathscr{L}_2^t) = \frac{1}{2\pi i}\int_{\Delta} \widetilde{K}_{u,\varrho,k}(\lambda)\Big[\lambda(\lambda - \mathscr{L}_2^t)^{-k} - (\lambda - \mathscr{L}_2^t)^{-k+1}\Big] d\lambda. \end{aligned} \tag{2.147}$$

By (2.135), (2.136) and by proceeding as in (2.137)-(2.139), we find that for $\mathbf{K}_u(a) = \widetilde{K}_{u,\varrho}(a)$ or $a\widetilde{K}_{u,\varrho}(a)$, for $|Z|, |Z'| \le q$,

$$\begin{aligned} \sup_{|\alpha|+|\alpha'|\le m} &(1+|Z^{\perp}|+|Z'^{\perp}|)^{2n+m+m''+2}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\mathbf{K}_u(\mathscr{L}_2^t)(Z,Z')\Big| \\ &\le C(1+q)^{2n+2+m}\exp(C'c^2u - \frac{C_1}{u}\varrho^2). \end{aligned} \tag{2.148}$$

Setting $\varrho \in \mathbb{N}^*$, $|\varrho - \frac{1}{c'}|Z - Z'|| < 1$ in (2.148), we get for α, α' verifying $|\alpha| + |\alpha'| \le m$,

$$\begin{aligned} (1+|Z^{\perp}|+|Z'^{\perp}|)^{m''}&\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\mathbf{K}_u(\mathscr{L}_2^t)(Z,Z')\Big| \\ &\le C(1+|Z^0|+|Z'^0|)^{2n+m+2}\exp(C'c^2u - \frac{C_1}{2c'^2u}|Z-Z'|^2). \end{aligned} \tag{2.149}$$

Take $\delta_1 = \frac{C'c^2+\frac{1}{4}\nu}{C'c^2+\frac{1}{2}\nu}$, from $(2.141)^{\delta_1}\times$ $(2.149)^{1-\delta_1}$ and (2.145), we get (2.133) for $r = m' = 0$.

To get (2.133) for $r \ge 1$, note that from (2.134), for $k \ge 1$

$$\frac{\partial^r}{\partial t^r}e^{-u\mathscr{L}_2^t} = \frac{(-1)^{k-1}(k-1)!}{2\pi i u^{k-1}}\int_{\delta\cup\Delta} e^{-u\lambda}\frac{\partial^r}{\partial t^r}(\lambda - \mathscr{L}_2^t)^{-k} d\lambda. \tag{2.150}$$

We have the similar equation for $\frac{\partial^r}{\partial t^r}(\mathscr{L}_2^t e^{-u\mathscr{L}_2^t})$.

Set

$$I_{k,r} = \Big\{(\mathbf{k},\mathbf{r}) = (k_i, r_i) | \sum_{i=0}^{j} k_i = k + j, \sum_{i=1}^{j} r_i = r,\ k_i, r_i \in \mathbb{N}^*\Big\}. \tag{2.151}$$

Then there exist $a^{\mathbf{k}}_{\mathbf{r}} \in \mathbb{R}$ such that

$$\begin{aligned} &A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t) = (\lambda - \mathscr{L}_2^t)^{-k_0}\frac{\partial^{r_1}\mathscr{L}_2^t}{\partial t^{r_1}}(\lambda - \mathscr{L}_2^t)^{-k_1}\cdots\frac{\partial^{r_j}\mathscr{L}_2^t}{\partial t^{r_j}}(\lambda - \mathscr{L}_2^t)^{-k_j}, \\ &\frac{\partial^r}{\partial t^r}(\lambda - \mathscr{L}_2^t)^{-k} = \sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a^{\mathbf{k}}_{\mathbf{r}} A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t). \end{aligned} \tag{2.152}$$

We claim that $A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)$ is well defined and for any $m\in\mathbb{N}$, $k>2(m+r+1)$, $Q,Q'\in\cup_{l=1}^{m}\mathcal{D}^l_t$, there exist $C>0$, $N\in\mathbb{N}$ such that for $\lambda\in\delta\cup\Delta$,

$$\|QA^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q's\|_{t,0}\leq C(1+|\lambda|)^N\sum_{|\beta|\leq 2r}\|Z^\beta s\|_{t,0}. \tag{2.153}$$

In fact, by (2.110), $\frac{\partial^r}{\partial t^r}\mathscr{L}^t_2$ is a combination of $\frac{\partial^{r_1}}{\partial t^{r_1}}(g^{ij}(tZ))$, $(\frac{\partial^{r_2}}{\partial t^{r_2}}\nabla_{t,e_i})$, $\frac{\partial^{r_3}}{\partial t^{r_3}}(q(tZ))$, $\frac{\partial^{r_4}}{\partial t^{r_4}}(t\langle\widetilde{\mu}^{E_{0,p}},f_{0,l}(tZ)\rangle)$, where q runs over the functions r^X, etc., appearing in (2.110).

Now $\frac{\partial^{r_1}}{\partial t^{r_1}}(q(tZ))$ (resp. $\frac{\partial^{r_1}}{\partial t^{r_1}}\ (t\langle\widetilde{\mu}^{E_{0,p}},f_{0,l}\rangle(tZ))$, $\frac{\partial^{r_1}}{\partial t^{r_1}}\nabla_{t,e_i}$) ($r_1\geq 1$) are functions of the type as $q'(tZ)Z^\beta$, $|\beta|\leq r_1$ (resp. r_1+1) (where q', as q, runs over the functions r^X, etc., appearing in (2.110)), with $q'(Z)$ and its derivatives on Z being bounded smooth functions on Z.

Let $\mathscr{R}'_t$ be the family of operators of the type

$$\mathscr{R}'_t=\{[f_{j_1}Q_{j_1},[f_{j_2}Q_{j_2},\dots[f_{j_l}Q_{j_l},\mathscr{L}^t_2]]\dots]\}$$

with f_{j_i} smooth bounded (with its derivatives) functions and $Q_{j_i}\in\mathcal{D}_t\cup\{Z_j\}_{j=1}^{2n-n_0}$.

Now for the operator $A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q'$, we will move first all the term Z^β in $d'(tZ)Z^\beta$ as above to the right hand side of this operator, to do so, we always use the commutator trick, i.e., each time, we consider only the commutation for Z_i, not for Z^β with $|\beta|>1$.

Then $A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q'$ is as the form $\sum_{|\beta|\leq 2r}L^t_\beta Q''_\beta Z^\beta$, and Q''_β is obtained from Q' and its commutation with Z^β.

Now we move all the terms ∇_{t,e_i}, $\langle\frac{1}{t}\widetilde{\mu},f_{0,l}\rangle(tZ)$ in $\frac{\partial^{r_j}\mathscr{L}^t_2}{\partial t^{r_j}}$ to the right hand side of the operator L^t_β.

Then as in the proof of Theorem 2.16, we get finally that $QA^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q'$ is as the form $\sum_\beta\mathscr{L}^t_\beta Z^\beta$ where $\mathscr{L}^t_\beta$ is a linear combination of operators of the form

$$Q(\lambda-\mathscr{L}^t_2)^{-k'_0}R_1(\lambda-\mathscr{L}^t_2)^{-k'_1}R_2\cdots R_{l'}(\lambda-\mathscr{L}^t_2)^{-k'_{l'}}Q'''Q'', \tag{2.154}$$

with $R_1,\cdots,R_{l'}\in\mathscr{R}'_t$, $Q'''\in\cup_{l=1}^{2r}\mathcal{D}^l_t$, $Q''\in\cup_{l=1}^{m}\mathcal{D}^l_t$, $|\beta|\leq 2r$, and Q'' is obtained from Q' and its commutation with Z^β.

By the argument as in (2.135) and (2.136), as $k>2(m+r+1)$, we can split the above operator to two parts

$$\begin{aligned}&Q(\lambda-\mathscr{L}^t_2)^{-k'_0}R_1(\lambda-\mathscr{L}^t_2)^{-k'_1}R_2\cdots R_i(\lambda-\mathscr{L}^t_2)^{-k''_i};\\&(\lambda-\mathscr{L}^t_2)^{-(k'_i-k''_i)}\cdots R_{l'}(\lambda-\mathscr{L}^t_2)^{-k'_{l'}}Q'''Q'',\end{aligned}$$

and the $\|\quad\|^{0,0}_t$-norm of each part is bounded by $C(1+|\lambda|^2)^N$.

Thus the proof of (2.153) is complete.

By (2.150), (2.152) and (2.153), we get the similar estimate (2.141), (2.149) for $\frac{\partial^r}{\partial t^r}e^{-u\mathscr{L}^t_2}$, $\frac{\partial^r}{\partial t^r}(\mathscr{L}^t_2e^{-u\mathscr{L}^t_2})$ with the exponential $2n+m+2r+2$ instead of $2n+m+2$ therein.

Thus we get (2.133) for $m'=0$.

Finally, for $U\in TX_G$ a vector on X_G,

$$\nabla_U^{\pi^*\operatorname{End}(\mathbf{E}_B)}e^{-u\mathscr{L}^t_2}=\frac{(-1)^{k-1}(k-1)!}{2\pi i u^{k-1}}\int_{\delta\cup\Delta}e^{-u\lambda}\nabla_U^{\pi^*\operatorname{End}(\mathbf{E}_B)}(\lambda-\mathscr{L}^t_2)^{-k}d\lambda. \tag{2.155}$$

Now, by using the similar formula (2.152) for $\nabla_U^{\pi^*\mathrm{End}(\mathbf{E}_B)}(\lambda-\mathscr{L}_2^t)^{-k}$ by replacing $\frac{\partial^{r_1}\mathscr{L}_2^t}{\partial t^{r_1}}$ by $\nabla_U^{\pi^*\mathrm{End}(\mathbf{E}_B)}\mathscr{L}_2^t$, and remark that $\nabla_U^{\pi^*\mathrm{End}(\mathbf{E}_B)}\mathscr{L}_2^t$ is a differential operator on $T_{x_0}B$ with the same structure as $\mathscr{L}_2^t$.

Then by the above argument, we get (2.133) for $m'\geq 1$. □

Let $P_{0,t}$ be the orthogonal projection from $\mathscr{C}^\infty(T_{x_0}B,\mathbf{E}_{B,x_0})$ to the kernel of $\mathscr{L}_2^t$ with respect to $\langle\ ,\ \rangle_{t,0}$. Set

$$F_u(\mathscr{L}_2^t)=\frac{1}{2\pi i}\int_\Delta e^{-u\lambda}(\lambda-\mathscr{L}_2^t)^{-1}d\lambda. \tag{2.156}$$

By (2.121),

$$F_u(\mathscr{L}_2^t)=e^{-u\mathscr{L}_2^t}-P_{0,t}=\int_u^{+\infty}\mathscr{L}_2^t e^{-u_1\mathscr{L}_2^t}du_1. \tag{2.157}$$

Let $P_{0,t}(Z,Z')$, $F_u(\mathscr{L}_2^t)(Z,Z')$ be the smooth kernels of $P_{0,t}, F_u(\mathscr{L}_2^t)$ with respect to $dv_{T_{x_0}B}(Z')$.

Corollary 2.18. *With the notation in Theorem 2.17,*

$$\begin{aligned}\sup_{|\alpha|+|\alpha'|\leq m}&(1+|Z^\perp|+|Z'^\perp|)^{m''}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\frac{\partial^r}{\partial t^r}F_u(\mathscr{L}_2^t)\,(Z,Z')\Big|_{\mathscr{C}^{m'}(P)}\\ &\leq C(1+|Z^0|+|Z'^0|)^{2n+m+2m'+2r+2}\exp(-\frac{1}{8}\nu u-\sqrt{C''\nu}|Z-Z'|).\end{aligned} \tag{2.158}$$

Proof. Note that $\frac{1}{8}\nu u+\frac{2C''}{u}|Z-Z'|^2\geq\sqrt{C''\nu}|Z-Z'|$, thus

$$\begin{aligned}\int_u^{+\infty}e^{-\frac{1}{4}\nu u_1-\frac{2C''}{u_1}|Z-Z'|^2}du_1&\leq e^{-\sqrt{C''\nu}|Z-Z'|}\int_u^{+\infty}e^{-\frac{1}{8}\nu u_1}du_1\\ &=\frac{8}{\nu}e^{-\frac{1}{8}\nu u-\sqrt{C''\nu}|Z-Z'|}.\end{aligned} \tag{2.159}$$

By (2.133), (2.157) and (2.159), we get (2.158). □

For k large enough, set

$$\begin{aligned}F_{r,u}&=\frac{(-1)^{k-1}(k-1)!}{2\pi i\,r!u^{k-1}}\int_\Delta e^{-u\lambda}\sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}}a_{\mathbf{r}}^{\mathbf{k}}A_{\mathbf{r}}^{\mathbf{k}}(\lambda,0)d\lambda,\\ J_{r,u}&=\frac{(-1)^{k-1}(k-1)!}{2\pi i\,r!u^{k-1}}\int_{\delta\cup\Delta}e^{-u\lambda}\sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}}a_{\mathbf{r}}^{\mathbf{k}}A_{\mathbf{r}}^{\mathbf{k}}(\lambda,0)d\lambda,\\ F_{r,u,t}&=\frac{1}{r!}\frac{\partial^r}{\partial t^r}F_u(\mathscr{L}_2^t)-F_{r,u},\quad J_{r,u,t}=\frac{1}{r!}\frac{\partial^r}{\partial t^r}e^{-u\mathscr{L}_2^t}-J_{r,u}.\end{aligned} \tag{2.160}$$

Certainly, as $t\to 0$, the limit of $\|\quad\|_{t,m}$ exists, and we denote it by $\|\quad\|_{0,m}$.

Theorem 2.19. *For any $r \geq 0$, $k > 0$, there exist $C > 0$, $N \in \mathbb{N}$ such that for $t \in [0, t_0], \lambda \in \delta \cup \Delta$,*

(2.161)
$$\Big\| \Big(\frac{\partial^r \mathscr{L}_2^t}{\partial t^r} - \frac{\partial^r \mathscr{L}_2^t}{\partial t^r}|_{t=0} \Big) s \Big\|_{t,-1} \leq Ct \sum_{|\alpha| \leq r+3} \|Z^\alpha s\|_{0,1},$$
$$\Big\| \Big(\frac{\partial^r}{\partial t^r} (\lambda - \mathscr{L}_2^t)^{-k} - \sum_{(\mathbf{k},\mathbf{r}) \in I_{k,r}} a_{\mathbf{r}}^{\mathbf{k}} A_{\mathbf{r}}^{\mathbf{k}}(\lambda, 0) \Big) s \Big\|_{0,0} \leq Ct(1 + |\lambda|^2)^N \sum_{|\alpha| \leq 4r+3} \|Z^\alpha s\|_{0,0}.$$

Proof. Note that by (2.106), (2.115), for $t \in [0, 1]$, $k \geq 1$

$$\|s\|_{t,0} \leq C\|s\|_{0,0}, \quad \|s\|_{t,k} \leq C \sum_{|\alpha| \leq k} \|Z^\alpha s\|_{0,k}. \tag{2.162}$$

An application of Taylor expansion for (2.110) leads to the following equation, if s, s' have compact support,

$$\Big| \Big\langle \Big(\frac{\partial^r \mathscr{L}_2^t}{\partial t^r} - \frac{\partial^r \mathscr{L}_2^t}{\partial t^r}|_{t=0} \Big) s, s' \Big\rangle_{0,0} \Big| \leq Ct \|s'\|_{t,1} \sum_{|\alpha| \leq r+3} \|Z^\alpha s\|_{0,1}. \tag{2.163}$$

Thus we get the first inequality of (2.161).

Note that

$$(\lambda - \mathscr{L}_2^t)^{-1} - (\lambda - \mathscr{L}_2^0)^{-1} = (\lambda - \mathscr{L}_2^t)^{-1} (\mathscr{L}_2^t - \mathscr{L}_2^0)(\lambda - \mathscr{L}_2^0)^{-1}. \tag{2.164}$$

Now from (2.120), (2.163) and (2.164),

$$\big\| \big((\lambda - \mathscr{L}_2^t)^{-1} - (\lambda - \mathscr{L}_2^0)^{-1} \big) s \big\|_{0,0} \leq Ct(1 + |\lambda|^4) \sum_{|\alpha| \leq 3} \|Z^\alpha s\|_{0,0}. \tag{2.165}$$

After taking the limit, we know that Theorems 2.14-2.16 still hold for $t = 0$.

Note that $\nabla_{0,e_j} = \nabla_{e_j} + \frac{1}{2} R^{L_B}_{x_0}(\mathcal{R}, e_j)$ by (2.106).

If we denote by $\mathscr{L}_{\lambda,t} = \lambda - \mathscr{L}_2^t$, then

$$\begin{aligned} A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t) - A_{\mathbf{r}}^{\mathbf{k}}(\lambda, 0) = &\sum_{i=1}^{j} \mathscr{L}_{\lambda,t}^{-k_0} \cdots \left(\frac{\partial^{r_i} \mathscr{L}_2^t}{\partial t^{r_i}} - \frac{\partial^{r_i} \mathscr{L}_2^t}{\partial t^{r_i}}|_{t=0} \right) \mathscr{L}_{\lambda,0}^{-k_i} \cdots \mathscr{L}_{\lambda,0}^{-k_j} \\ &+ \sum_{i=0}^{j} \mathscr{L}_{\lambda,t}^{-k_0} \cdots \big(\mathscr{L}_{\lambda,t}^{-k_i} - \mathscr{L}_{\lambda,0}^{-k_i} \big) \left(\frac{\partial^{r_{i+1}} \mathscr{L}_2^t}{\partial t^{r_{i+1}}}|_{t=0} \right) \cdots \mathscr{L}_{\lambda,0}^{-k_j}. \end{aligned} \tag{2.166}$$

Now from the first inequality of (2.161), (2.120), (2.152), (2.165) and (2.166), we get (2.161). □

Theorem 2.20. *There exist $C > 0$, $N \in \mathbb{N}$ such that for $t \in]0, t_0]$, $u \geq u_0$, $q \in \mathbb{N}$, $Z, Z' \in T_{x_0}B$, $|Z|, |Z'| \leq q$,*

$$\begin{aligned} \Big| F_{r,u,t}(Z, Z') \Big| &\leq Ct^{\frac{1}{2(2n - n_0 + 1)}} (1 + q)^N e^{-\frac{1}{8}\nu u}, \\ \Big| J_{r,u,t}(Z, Z') \Big| &\leq Ct^{\frac{1}{2(2n - n_0 + 1)}} (1 + q)^N e^{\frac{1}{2}\nu u}. \end{aligned} \tag{2.167}$$

Proof. Let $J^0_{x_0,q}$ be the vector space of square integrable sections of $\mathbf{E}_{B,x_0}$ over $\{Z \in T_{x_0}B, |Z| \leq q+1\}$.

If $s \in J^0_{x_0,q}$, put $\|s\|^2_{(q)} = \int_{|Z|\leq q+1} |s|^2_{\mathbf{E}_{B,x_0}} dv_{TB}(Z)$. Let $\|A\|_{(q)}$ be the operator norm of $A \in \mathscr{L}(J^0_{x_0,q})$ with respect to $\| \quad \|_{(q)}$.

By (2.150), (2.160) and (2.161), we get: there exist $C > 0, N \in \mathbb{N}$ such that for $t \in]0, t_0], u \geq u_0$,

$$\begin{aligned} &\|F_{r,u,t}\|_{(q)} \leq Ct(1+q)^N e^{-\frac{1}{2}\nu u}, \\ &\|J_{r,u,t}\|_{(q)} \leq Ct(1+q)^N \ e^{\frac{1}{4}\nu u}. \end{aligned} \tag{2.168}$$

Let $\phi : \mathbb{R} \to [0,1]$ be a smooth function with compact support, equal 1 near 0, such that $\int_{T_{x_0}B} \phi(Z) dv_{T_{x_0}B}(Z) = 1$.

Take $\varsigma \in]0,1]$.

By the proof of Theorem 2.17, $F_{r,u}$ verifies the similar inequality as in (2.158). Thus by (2.158), there exists $C > 0$ such that if $|Z|, |Z'| \leq q$, $U, U' \in \mathbf{E}_{B,x_0}$,

$$\begin{aligned} &\Big| \langle F_{r,u,t}(Z,Z')U,U'\rangle - \int_{T_{x_0}B\times T_{x_0}B} \langle F_{r,u,t}(Z-W,Z'-W')U,U'\rangle \\ &\qquad \frac{1}{\varsigma^{4n-2n_0}} \phi(W/\varsigma)\phi(W'/\varsigma) dv_{T_{x_0}B}(W) dv_{T_{x_0}B}(W') \Big| \leq C\varsigma(1+q)^N e^{-\frac{1}{8}\nu u}|U||U'|. \end{aligned} \tag{2.169}$$

On the other hand, by (2.168),

$$\begin{aligned} &\Big| \int_{T_{x_0}B\times T_{x_0}B} \langle F_{r,u,t}(Z-W,Z'-W')U,U'\rangle \ \frac{1}{\varsigma^{4n-2n_0}} \phi(W/\varsigma)\phi(W'/\varsigma) \\ &\qquad dv_{T_{x_0}B}(W) dv_{T_{x_0}B}(W') \Big| \leq Ct \frac{1}{\varsigma^{2n-n_0}} (1+q)^N e^{-\frac{1}{2}\nu u}|U||U'|. \end{aligned} \tag{2.170}$$

By taking $\varsigma = t^{1/2(2n-n_0+1)}$, we get (2.167).

In the same way, we get (2.167) for $J_{r,u,t}$. □

Theorem 2.21. *There exists $C'' > 0$ such that for any $k, m, m', m'' \in \mathbb{N}$, there exist $N \in \mathbb{N}$, $C > 0$ such that if $t \in]0, t_0], u \geq u_0$, $Z, Z' \in T^H_{x_0}U$, $\alpha, \alpha \in \mathbb{Z}^{2n-n_0}$, $|\alpha| + |\alpha'| \leq m$,*

$$\begin{aligned} &(1+|Z^\perp| + |Z'^\perp|)^{m''} \Big| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \Big(F_u(\mathscr{L}^t_2) - \sum_{r=0}^{k} F_{r,u} t^r \Big)(Z,Z') \Big|_{\mathscr{C}^{m'}(X_G)} \\ &\qquad \leq Ct^{k+1}(1+|Z^0| + |Z'^0|)^{2(n+k+m'+2)+m} \exp(-\frac{1}{8}\nu u - \sqrt{C''\nu}|Z-Z'|), \\ &(1+|Z^\perp| + |Z'^\perp|)^{m''} \Big| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \Big(e^{-u\mathscr{L}^t_2} - \sum_{r=0}^{k} J_{r,u} t^r \Big)(Z,Z') \Big|_{\mathscr{C}^{m'}(X_G)} \\ &\qquad \leq Ct^{k+1}(1+|Z^0| + |Z'^0|)^{2(n+k+m'+2)+m} \exp(\frac{1}{2}\nu u - \frac{2C''}{u}|Z-Z'|^2). \end{aligned} \tag{2.171}$$

Proof. By (2.160) and (2.167),

$$\frac{1}{r!}\frac{\partial^r}{\partial t^r} F_u(\mathscr{L}^t_2)|_{t=0} = F_{r,u}, \quad \frac{1}{r!}\frac{\partial^r}{\partial t^r} e^{-u\mathscr{L}^t_2}|_{t=0} = J_{r,u}. \tag{2.172}$$

Now by Theorem 2.17 and (2.160), $J_{r,u}, F_{r,u}$ have the same estimates as $\frac{\partial^r}{\partial t^r}e^{-u\mathscr{L}_2^t}$, $\frac{\partial^r}{\partial t^r}F_u(\mathscr{L}_2^t)$ in (2.133), (2.158).

Again from (2.133), (2.158), (2.160), (2.167), and the Taylor expansion

$$G(t)-\sum_{r=0}^{k}\frac{1}{r!}\frac{\partial^r G}{\partial t^r}(0)t^r=\frac{1}{k!}\int_0^t(t-t_0)^k\frac{\partial^{k+1}G}{\partial t^{k+1}}(t_0)dt_0, \tag{2.173}$$

we get (2.171). □

2.8. **Evaluation of $J_{r,u}$.** For $u>0$, we will write $u\Delta_j$ for the rescaled simplex $\{(u_1,\cdots,u_j)|\ 0\le u_1\le u_2\le\cdots\le u_j\le u\}$.

Let $e^{-u\mathscr{L}_2^0}(Z,Z')$ be the smooth kernel of $e^{-u\mathscr{L}_2^0}$ with respect to $dv_{T_{x_0}B}(Z')$.

Recall that the $\mathcal{O}_r$'s have been defined in (2.102).

Theorem 2.22. *For $r\ge 0$, we have*

$$J_{r,u}=\sum_{\sum_{i=1}^j r_i=r,\, r_i\ge 1}(-1)^j\int_{u\Delta_j}e^{-(u-u_j)\mathscr{L}_2^0}\mathcal{O}_{r_j}e^{-(u_j-u_{j-1})\mathscr{L}_2^0}\cdots\mathcal{O}_{r_1}e^{-u_1\mathscr{L}_2^0}du_1\cdots du_j, \tag{2.174}$$

where the product in the integrand is the convolution product. Moreover,

$$J_{r,u}(Z,Z')=(-1)^rJ_{r,u}(-Z,-Z'). \tag{2.175}$$

Proof. We introduce an even extra-variable σ such that $\sigma^{r+1}=0$.

Set $[\ \ ]^{[r]}$ the coefficient of σ^r, $\mathscr{L}_\sigma=\mathscr{L}_2^0+\sum_{j=1}^r\mathcal{O}_j\sigma^j$.

From (2.160), (2.172), we know

$$J_{r,u}(Z,Z')=\frac{1}{r!}\frac{\partial^r}{\partial t^r}e^{-u\mathscr{L}_2^t}(Z,Z')|_{t=0}=[e^{-u\mathscr{L}_\sigma}]^{[r]}(Z,Z'). \tag{2.176}$$

Now from (2.176) and the Volterra expansion of $e^{-u\mathscr{L}_\sigma}$ (cf. [1, §2.4]), we get (2.174).

We prove (2.175) by iteration.

From (2.102)

$$\mathscr{L}_2^0=-\sum_{j=1}^{2n-n_0}(\nabla_{e_j})^2-\pi^2\langle((P^{TH_U}\mathbf{J}P^{TH_U})^2+4P^{TH_U}\mathbf{J}P^{TY}\mathbf{J}P^{TH_U})_{x_0}\mathcal{R},\mathcal{R}\rangle+2\pi\sqrt{-1}\nabla_{P^{TH_U}\mathbf{J}P^{TH_U}\mathcal{R}}-2\omega_{d,x_0}-\tau_{x_0}. \tag{2.177}$$

Here the matrix $((P^{TH_U}\mathbf{J}P^{TH_U})^2+4P^{TH_U}\mathbf{J}P^{TY}\mathbf{J}P^{TH_U})_{x_0}$ need not commute with $P^{TH_U}\mathbf{J}P^{TH_U}$.

Thus [3, (6.37), (6.38)] does not apply directly here, and we could not get a precise formula for $e^{-u\mathscr{L}_2^0}$ as in [17, (4.106)].

By the uniqueness of the solution of heat equations and (2.177), we know

$$e^{-u\mathscr{L}_2^0}(Z,Z')=e^{-u\mathscr{L}_2^0}(-Z,-Z'). \tag{2.178}$$

By (2.174),

$$J_{0,u}(Z,Z')=e^{-u\mathscr{L}_2^0}(Z,Z'). \tag{2.179}$$

Thus we get (2.175) for $r = 0$.

If (2.175) holds for $r \leq k$, then by (2.174), (2.179),

$$J_{k+1,u} = -\sum_{j=1}^{k+1} \int_0^u e^{-(u-u_1)\mathscr{L}_2^0} \mathcal{O}_j J_{k+1-j,u_1} du_1. \tag{2.180}$$

By the iteration, Theorem 2.11 and (2.179), and note that ∇_{e_i} in $\mathcal{O}_j$ will change the parity of the polynomials we obtained, we get (2.175) for $r = k+1$. □

2.9. **Proof of Theorem 0.2.** By (2.157) and (2.171), for any $u > 0$ fixed, there exists $C_u > 0$ such that for $t = \frac{1}{\sqrt{p}}$, $Z, Z' \in T_{x_0}B$, $x_0 \in P$, $\alpha, \alpha' \in \mathbb{Z}^{2n-n_0}$, $|\alpha| + |\alpha'| \leq m$, we have

$$\begin{aligned}(1 + |Z^{\perp}| + |Z'^{\perp}|)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha} \partial Z'^{\alpha'}} \Big(P_{0,t} - \sum_{r=0}^{k} t^r (J_{r,u} - F_{r,u}) \Big)(Z, Z') \right|_{\mathscr{C}^{m'}(X_G)} \\ \leq C_u t^{k+1} (1 + |Z^0| + |Z'^0|)^{2(n+k+m'+2)+m} \exp(-\sqrt{C''\nu}|Z - Z'|).\end{aligned} \tag{2.181}$$

Set

$$P^{(r)} = J_{r,u} - F_{r,u}. \tag{2.182}$$

Then $P^{(r)}$ does not depend on $u > 0$ by (2.181), as $P_{0,t}$ does not depend on u.

Moreover, by taking the limit of (2.158) as $t \to 0$,

$$\begin{aligned}(1 + |Z^{\perp}| + |Z'^{\perp}|)^{m''} \Big| F_{r,u}(Z, Z') \Big|_{\mathscr{C}^{m'}(X_G)} \\ \leq C(1 + |Z^0| + |Z'^0|)^{2n+2r+2m'+2} \exp(-\frac{1}{8}\nu u - \sqrt{C''\nu}|Z - Z'|).\end{aligned} \tag{2.183}$$

Thus

$$J_{r,u}(Z, Z') = P^{(r)}(Z, Z') + F_{r,u}(Z, Z') = P^{(r)}(Z, Z') + \mathcal{O}(e^{-\frac{1}{8}\nu u}), \tag{2.184}$$

uniformly on any compact set of $T_{x_0}B \times T_{x_0}B$.

Especially, from (2.175), (2.184), we get

$$P^{(r)}(Z, Z') = (-1)^r P^{(r)}(-Z, -Z'). \tag{2.185}$$

By (2.101), for $Z, Z' \in T_{x_0}B$,

$$P_{x_0,p}(Z, Z') = p^{n-\frac{n_0}{2}} \kappa^{-\frac{1}{2}}(Z) P_{0,t}(Z/t, Z'/t) \kappa^{-\frac{1}{2}}(Z'). \tag{2.186}$$

We note in passing that, as a consequence of (2.181) and (2.186), we obtain the following estimate.

Theorem 2.23. *For any $k, m, m', m'' \in \mathbb{N}$, there exists $C > 0$ such that for $Z, Z' \in T_{x_0}B$, $|Z|, |Z'| \leq \varepsilon$, $x_0 \in X_G$,*

$$\begin{aligned}(2.187)\quad &\sup_{|\alpha|+|\alpha'|\leq m} (1+\sqrt{p}|Z^{\perp}|+\sqrt{p}|Z'^{\perp}|)^{m''}\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\right.\\ &\left.\left(p^{-n+\frac{n_0}{2}}P_{x_0,p}(Z,Z')-\sum_{r=0}^{k}P^{(r)}(\sqrt{p}Z,\sqrt{p}Z')\kappa^{-\frac{1}{2}}(Z)\kappa^{-\frac{1}{2}}(Z')p^{-r/2}\right)\right|_{\mathscr{C}^{m'}(X_G)}\\ &\leq Cp^{-(k+1-m)/2}(1+\sqrt{p}|Z^0|+\sqrt{p}|Z'^0|)^{2(n+k+m'+2)+m}\exp(-\sqrt{C''\nu}\sqrt{p}|Z-Z'|).\end{aligned}$$

From (2.84), (2.85), (2.109) and (2.187), we get Theorem 0.2 without knowing the properties (0.12), (0.13) for $P^{(r)}$.

To prove the uniformity part of Theorem 0.2, we notice that in the proof of Theorem 2.17, we only use the derivatives of the coefficients of $\mathscr{L}_2^t$ with order $\leqslant 2n+m+m'+r+2$. Thus the constants in Theorems 2.17 and 2.20, (resp. Theorem 2.21) are uniformly bounded, if with respect to a fixed metric g_0^{TX}, the $\mathscr{C}^{2n+m+m'+r+2}$ (resp. $\mathscr{C}^{2n+m+m'+k+3}$)–norms on X of the data (g^{TX}, h^L, ∇^L, h^E, ∇^E, J) are bounded, and g^{TX} is bounded below.

Moreover, taking derivatives with respect to the parameters we obtain a similar equation as (2.155), where $x_0 \in X_G$ plays now a role of a parameter. Thus the $\mathscr{C}^{m'}$–norm in (2.187) can also include the parameters if the $\mathscr{C}^{m'}$–norms (with respect to the parameter $x_0 \in X_G$) of the derivatives of above data with order $\leqslant 2n+k+m+3$ are bounded.

Thus we can take $C_{k,l}$ in (0.10) independent of g^{TX} under our condition.

This achieves the proof of Theorem 0.2 except (0.12) and (0.13) which will be proved in Theorem 3.2 under the condition in Theorem 0.2.

3. Evaluation of $P^{(r)}$

In this Section, inspired by the method in [26, §1.4, 1.5], we develop a direct and effective method to compute $P^{(r)}$. In particular, we get (0.12) and (0.13) under the condition in Theorem 0.2.

This section is organized as follows. In Section 3.1, we study the spectrum of the limited operator $\mathscr{L}_2^0$. In Section 3.2, we get a direct method to evaluate $P^{(r)}$ in (0.12), especially, we prove (0.12) and (0.13). In Section 3.3, we compute explicitly $\mathcal{O}_1$ in (2.103), and get a general formula for $P^{(2)}$ by using the operators $\mathcal{O}_1$, $\mathcal{O}_2$. In Section 3.4, we compute explicitly an interesting example: the line bundle $\mathcal{O}(2)$ on $(\mathbb{C}P^1, 2\,\omega_{FS})$. We verify that Theorem 0.2 coincides with our computation here if 0 is a regular value of the moment map μ, but it does not hold if 0 is a singular value.

We use the notations in Section 2.6, and we suppose that (3.2) is verified.

3.1. **Spectrum of $\mathscr{L}_2^0$.** Recall that T^HP is the orthogonal complement of TY in (TP, g^{TP}). Note that by (2.5) and (2.15), we have the following orthogonal splitting of vector bundles on $P = \mu^{-1}(0)$,

$$TX = T^HP \oplus TY \oplus \mathbf{J}TY, \qquad TP = T^HP \oplus TY. \tag{3.1}$$

In the rest of this Section, we suppose that on P

$$\mathbf{J}T^HP = T^HP, \qquad \mathbf{J}^2TY = TY. \tag{3.2}$$

(2.7) and (3.2) imply that $-J\mathbf{J}$ preserves TY and $\mathbf{J}TY$.

As g^{TX} is J-invariant, we get

$$\mathbf{J}\,TY = J\,TY, \quad J\,T^HP = T^HP. \tag{3.3}$$

Thus $(\mathbf{J}TY)_B|_{X_G}$ is the orthogonal complement of TX_G in TB, and $\mathbf{J}$ induces naturally $\mathbf{J}_G \in \mathrm{End}(TX_G)$. We will identify $(\mathbf{J}TY)_B|_{X_G}$ to the normal bundle of X_G in B.

For $U, V \in T_{x_0}B$, $x_0 \in X_G$, we have

$$\omega(U^H, V^H) = \omega_G(P^{TX_G}U, P^{TX_G}V). \tag{3.4}$$

From the above discussion, for $x_0 \in X_G$, we can choose $\{w_j^0\}_{j=1}^{n-n_0}$, $\{e_j^\perp\}_{j=1}^{n_0}$ orthonormal basis of $T^{(1,0)}_{x_0}X_G$, $(\mathbf{J}TY)_{B,x_0} \subset TB$ such that

$$\begin{aligned} \mathbf{J}|_{T^{(1,0)}_{x_0}X_G} &= \frac{\sqrt{-1}}{2\pi}\mathrm{diag}(a_1, \cdots, a_{n-n_0}) \in \mathrm{End}(T^{(1,0)}_{x_0}X_G), \\ \mathbf{J}^2|_{(\mathbf{J}TY)_B} &= \frac{-1}{4\pi^2}\mathrm{diag}(a_1^{\perp,2}, \cdots, a_{n_0}^{\perp,2}) \in \mathrm{End}((\mathbf{J}TY)_{B,x_0}), \end{aligned} \tag{3.5}$$

with $a_j, a_j^\perp > 0$, and let $\{w^{0,j}\}_{j=1}^{n-n_0}$, $\{e^{\perp j}\}_{j=1}^{n_0}$ be their dual basis, then

$$e^0_{2j-1} = \tfrac{1}{\sqrt{2}}(w_j^0 + \overline{w}_j^0) \quad \text{and} \quad e^0_{2j} = \tfrac{\sqrt{-1}}{\sqrt{2}}(w_j^0 - \overline{w}_j^0)\,,$$

$j = 1, \ldots, n - n_0$, forms an orthonormal basis of $T_{x_0}X_G$.

From now on, we use the coordinate in Section 2.6 induced by the above basis.

Denote by $Z^0 = (Z_1^0, \cdots, Z_{2n-2n_0}^0)$, $Z^\perp = (Z_1^\perp, \cdots, Z_{n_0}^\perp)$, then $Z = (Z^0, Z^\perp)$.

In what follows we will use the complex coordinates $z^0 = (z_1^0, \cdots, z_{n-n_0}^0)$, thus $Z^0 = z^0 + \overline{z}^0$, and $w_i^0 = \sqrt{2}\frac{\partial}{\partial z_i^0}$, $\overline{w}_i^0 = \sqrt{2}\frac{\partial}{\partial \overline{z}_i^0}$, and

$$e_{2i-1}^0 = \tfrac{\partial}{\partial z_i^0} + \tfrac{\partial}{\partial \overline{z}_i^0}, \quad e_{2i}^0 = \sqrt{-1}(\tfrac{\partial}{\partial z_i^0} - \tfrac{\partial}{\partial \overline{z}_i^0}). \tag{3.6}$$

We will also identify z^0 to $\sum_i z_i^0 \frac{\partial}{\partial z_i^0}$ and $\overline{z}^0$ to $\sum_i \overline{z}_i^0 \frac{\partial}{\partial \overline{z}_i^0}$ when we consider z^0 and $\overline{z}^0$ as vector fields. Remark that

$$\left|\tfrac{\partial}{\partial z_i^0}\right|^2 = \left|\tfrac{\partial}{\partial \overline{z}_i^0}\right|^2 = \frac{1}{2}, \quad \text{so that } |z^0|^2 = |\overline{z}^0|^2 = \frac{1}{2}|Z^0|^2. \tag{3.7}$$

It is very useful to rewrite $\mathscr{L}_2^0$ by using the creation and annihilation operators. Set

$$\begin{aligned} &b_i = -2\tfrac{\partial}{\partial z_i^0} + \frac{1}{2}a_i \overline{z}_i^0, \quad b_i^+ = 2\tfrac{\partial}{\partial \overline{z}_i^0} + \frac{1}{2}a_i z_i^0, \quad b = (b_1, \cdots, b_{n-n_0}); \\ &b_j^\perp = -\tfrac{\partial}{\partial Z_j^\perp} + a_j^\perp Z_j^\perp, \quad b_j^{\perp +} = \tfrac{\partial}{\partial Z_j^\perp} + a_j^\perp Z_j^\perp, \quad b^\perp = (b_1^\perp, \cdots, b_{n_0}^\perp). \end{aligned} \tag{3.8}$$

Then for any polynomial $g(Z^0, Z^\perp)$ on Z^0 and $Z^\perp$,

$$\begin{aligned} &[b_i, b_j^+] = b_i b_j^+ - b_j^+ b_i = -2a_i\delta_{i\,j}, && [b_i, b_j] = [b_i^+, b_j^+] = 0\,, \\ &[g, b_j] = 2\tfrac{\partial}{\partial z_j^0} g, && [g, b_j^+] = -2\tfrac{\partial}{\partial \overline{z}_j^0} g\,, \\ &[b_i^\perp, b_j^{\perp +}] = -2a_i^\perp \delta_{i\,j}, && [b_j^\perp, b_k^\perp] = [b_j^{\perp +}, b_k^{\perp +}] = 0, \\ &[g, b_j^\perp] = -[g, b_j^{\perp +}] = \tfrac{\partial}{\partial Z_j^\perp} g\,. \end{aligned} \tag{3.9}$$

Set

$$\mathscr{L} = \sum_{j=1}^{n-n_0} b_j b_j^+, \quad \mathscr{L}^\perp = \sum_{j=1}^{n_0} b_j^\perp b_j^{\perp +}, \quad \nabla_{0,\cdot} = \nabla_\cdot + \frac{1}{2}R_{x_0}^{L_B}(\mathcal{R}, \cdot). \tag{3.10}$$

From (1.18) and (3.4), for $U, V \in T_{x_0}B$, we get

$$R_{x_0}^{L_B}(U, V) = -2\pi\sqrt{-1}\left\langle \mathbf{J} P^{TX_G} U, P^{TX_G} V \right\rangle. \tag{3.11}$$

By (2.48), (3.5), (3.8), (3.10) and (3.11), we have

$$\begin{aligned} &b_i = -2\nabla_{0, \frac{\partial}{\partial z_i^0}}, \quad b_i^+ = 2\nabla_{0, \frac{\partial}{\partial \overline{z}_i^0}}, \quad \nabla_{0, e_j^\perp} = \nabla_{e_j^\perp}, \\ &\tau_{x_0} = \sum_j a_j + \sum_j a_j^\perp. \end{aligned} \tag{3.12}$$

From (2.102), (3.10) and (3.12), we get

$$\begin{aligned} \mathscr{L}_2^0 &= -\sum_{j=1}^{2n-2n_0} (\nabla_{0,e_j^0})^2 - \sum_{j=1}^{n_0} \left((\nabla_{e_j^\perp})^2 - |a_j^\perp Z_j^\perp|^2 \right) - 2\omega_{d,x_0} - \tau_{x_0} \\ &= \mathscr{L} + \mathscr{L}^\perp - 2\omega_{d,x_0}. \end{aligned} \tag{3.13}$$

By [38, §8.6], [26, Theorem 1.15], we know

Theorem 3.1. *The spectrum of the restriction of $\mathscr{L}$ on $L^2(\mathbb{R}^{2n-2n_0})$ is given by*

$$\mathrm{Spec}\left(\mathscr{L}|_{L^2(\mathbb{R}^{2n-2n_0})}\right) = \left\{ 2\sum_{i=1}^{n-n_0} \alpha_i^0 a_i \,:\, \alpha^0 = (\alpha_1^0, \cdots, \alpha_{n-n_0}^0) \in \mathbb{N}^{n-n_0} \right\}, \tag{3.14}$$

and an orthogonal basis of the eigenspace of $2\sum_{i=1}^{n-n_0}\alpha_i^0 a_i$ *is given by*

$$b^{\alpha^0}\Big((z^0)^\beta \exp\Big(-\frac{1}{4}\sum_i a_i|z_i^0|^2\Big)\Big), \quad \text{with } \beta\in\mathbb{N}^{n-n_0}. \tag{3.15}$$

The spectrum of the restriction of $\mathscr{L}^\perp$ *on* $L^2(\mathbb{R}^{n_0})$ *is given by*

$$\mathrm{Spec}\big(\mathscr{L}^\perp|_{L^2(\mathbb{R}^{n_0})}\big)=\Big\{2\sum_{i=1}^{n_0}\alpha_i^\perp a_i^\perp\,:\,\alpha^\perp=(\alpha_1^\perp,\cdots,\alpha_{n_0}^\perp)\in\mathbb{N}^{n_0}\Big\}, \tag{3.16}$$

and the eigenspace of $2\sum_{i=1}^{n_0}\alpha_i^\perp a_i^\perp$ *is one dimensional and an orthonormal basis is given by*

$$\Big(\prod_{i=1}^{n_0}\sqrt{\frac{\pi}{a_i^\perp}}(2a_i^\perp)^{\alpha_i^\perp}(\alpha_i^\perp!)\Big)^{-1/2}(b^\perp)^{\alpha^\perp}\exp\Big(-\frac{1}{2}\sum_i a_i^\perp|Z_i^\perp|^2\Big). \tag{3.17}$$

Especially, the orthonormal basis of $\mathrm{Ker}(\mathscr{L}|_{L^2(\mathbb{R}^{2n-2n_0})})$; $\mathrm{Ker}(\mathscr{L}^\perp|_{L^2(\mathbb{R}^{n_0})})$ are

$$\begin{aligned}&\Big(\frac{a^\beta}{2^{|\beta|}\beta!}\prod_{i=1}^{n-n_0}\frac{a_i}{2\pi}\Big)^{\frac{1}{2}}\Big((z^0)^\beta\exp\Big(-\frac{1}{4}\sum_{j=1}^{n-n_0}a_j|z_j^0|^2\Big),\ \beta\in\mathbb{N}^{n-n_0};\\ &G^\perp(Z^\perp)=\Big(\prod_{i=1}^{n_0}\frac{a_i^\perp}{\pi}\Big)^{\frac{1}{4}}\exp\Big(-\frac{1}{2}\sum_{i=1}^{n_0}a_i^\perp|Z_i^\perp|^2\Big).\end{aligned} \tag{3.18}$$

Let $P_{\mathscr{L}}(Z^0,Z'^0)$, $P_{\mathscr{L}^\perp}(Z^\perp,Z'^\perp)$ (resp. $P(Z,Z')$) be the kernels of the orthogonal projections $P_{\mathscr{L}}$, $P_{\mathscr{L}^\perp}$ (resp. P) from $L^2(\mathbb{R}^{2n-2n_0})$ onto $\mathrm{Ker}(\mathscr{L})$, $L^2(\mathbb{R}^{n_0})$ onto $\mathrm{Ker}(\mathscr{L}^\perp)$ (resp. $L^2(\mathbb{R}^{2n-n_0})$ onto $\mathrm{Ker}(\mathscr{L}+\mathscr{L}^\perp)$).

From (3.18), we get

$$\begin{aligned}P_{\mathscr{L}}(Z^0,Z'^0)=&\Big(\prod_{i=1}^{n-n_0}\frac{a_i}{2\pi}\Big)\ \exp\Big(-\frac{1}{4}\sum_{i=1}^{n-n_0}a_i\big(|z_i^0|^2+|z_i'^0|^2-2z_i^0\overline{z}_i'^0\big)\Big),\\ P_{\mathscr{L}^\perp}(Z^\perp,Z'^\perp)=&\Big(\prod_{i=1}^{n_0}\sqrt{\frac{a_i^\perp}{\pi}}\Big)\exp\Big(-\frac{1}{2}\sum_{i=1}^{n_0}a_i^\perp(|Z_i^\perp|^2+|Z_i'^\perp|^2)\Big),\\ P(Z,Z')=&P_{\mathscr{L}}(Z^0,Z'^0)P_{\mathscr{L}^\perp}(Z^\perp,Z'^\perp).\end{aligned} \tag{3.19}$$

Let P^N be the orthogonal projection from $L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X)\otimes E)_{x_0})$ onto $N=\mathrm{Ker}(\mathscr{L}_2^0)$. Let $P^N(Z,Z')$ be the associated kernel.

Recall that the projection $I_{\mathbb{C}\otimes E_B}$ from $(\Lambda(T^{*(0,1)}X)\otimes E)_B$ onto $\mathbb{C}\otimes E_B$ is defined in Introduction.

By (2.7), (2.9), (2.48) and (3.5),

$$-\omega_{d,x_0}\geq\nu_0 \quad \text{on } \Lambda^{>0}(T^{*(0,1)}X), \tag{3.20}$$

thus

$$P^N(Z,Z')=P(Z,Z')I_{\mathbb{C}\otimes E_B}. \tag{3.21}$$

If $\mathbf{J} = J$ on P, then by (3.19) and (3.21),

$$
\begin{aligned}
P^N(Z,Z') &= \exp\Big(-\frac{\pi}{2}\sum_{i=1}^{n-n_0}\big(|z_i^0|^2+|z_i'^0|^2-2z_i^0\overline{z}_i'^0\big)\Big)\\
&\quad\times 2^{\frac{n_0}{2}}\exp\Big(-\pi\big(|Z^\perp|^2+|Z'^\perp|^2\big)\Big)I_{\mathbb{C}\otimes E_B},\\
P^N((0,Z^\perp),(0,Z^\perp)) &= 2^{\frac{n_0}{2}}\exp\Big(-2\pi|Z^\perp|^2\Big)I_{\mathbb{C}\otimes E_B}.
\end{aligned}
\tag{3.22}
$$

3.2. **Evaluation of $P^{(r)}$: a proof of (0.12) and (0.13).** Recall that δ is the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\nu/4$.

By (2.121),

$$
P_{0,t} = \frac{1}{2\pi i}\int_\delta(\lambda-\mathscr{L}_2^t)^{-1}d\lambda. \tag{3.23}
$$

Let $f(\lambda,t)$ be a formal power series with values in $\mathrm{End}(L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0}))$

$$
f(\lambda,t)=\sum_{r=0}^{\infty}t^r f_r(\lambda),\quad f_r(\lambda)\in\mathrm{End}(L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0})). \tag{3.24}
$$

By (2.103), consider the equation of formal power series for $\lambda\in\delta$,

$$
(\lambda-\mathscr{L}_2^0-\sum_{r=1}^{\infty}t^r\mathcal{O}_r)f(\lambda,t)=\mathrm{Id}_{L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0})}. \tag{3.25}
$$

Let $N^\perp$ be the orthogonal space of N in $L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0})$, and $P^{N^\perp}$ be the orthogonal projection from $L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0})$ onto $N^\perp$.

We decompose $f(\lambda,t)$ according the splitting $L^2(\mathbb{R}^{2n-n_0},(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0}) = N\oplus N^\perp$,

$$
g_r(\lambda)=P^N f_r(\lambda),\quad f_r^\perp(\lambda)=P^{N^\perp}f_r(\lambda). \tag{3.26}
$$

Using Theorem 3.1, (3.13), (3.20), (3.26) and identifying the powers of t in (3.25), we find that

$$
\begin{aligned}
g_0(\lambda)&=\frac{1}{\lambda}P^N,\quad f_0^\perp(\lambda)=(\lambda-\mathscr{L}_2^0)^{-1}P^{N^\perp},\\
f_r^\perp(\lambda)&=(\lambda-\mathscr{L}_2^0)^{-1}\sum_{j=1}^{r}P^{N^\perp}\mathcal{O}_j f_{r-j}(\lambda),\\
g_r(\lambda)&=\frac{1}{\lambda}\sum_{j=1}^{r}P^N\mathcal{O}_j f_{r-j}(\lambda).
\end{aligned}
\tag{3.27}
$$

Recall that $P^{(r)}$ ($r\in\mathbb{N}$) is defined in (2.182) and (2.187).

Theorem 3.2. *There exist $J_r(Z,Z')$ polynomials in Z,Z' with the same parity as r, whose coefficients are polynomials in A, R^{TB}, R^{Cliff_B}, R^{E_B}, μ^E, μ^{Cliff} (resp. r^X, $\mathrm{Tr}[R^{T^{(1,0)}X}]$, R^E; resp. h, R^L, R^{L_B}; resp. μ) and their derivatives at x_0 up to order $r-1$ (resp. $r-2$;*

resp. r; resp. $r+1$), and in the inverses of the linear combination of the eigenvalues of $\mathbf{J}$ *at* x_0 *, such that*

$$P^{(r)}(Z,Z') = J_r(Z,Z')P(Z,Z'). \tag{3.28}$$

Moreover,

$$P^{(0)}(Z,Z') = P^N(Z,Z') = P(Z,Z')I_{\mathbb{C}\otimes E_B}. \tag{3.29}$$

Proof. By (3.23), for $M>0$, by combining Theorems 2.13-2.16 and the argument as in [26, §1.3], we get another proof of the existence of the asymptotic expansion of $P_{0,t}(Z,Z')$ for $|Z|,|Z'|\leq M$ when $t\to 0$.

By (2.84), (2.85) and (2.186), this gives another proof of Theorems 0.2, 2.23 for $|Z|,|Z'|\leq M/\sqrt{p}$. Moreover, by (2.150), (2.160) and (3.26),

$$P^{(r)} = \frac{1}{2\pi i}\int_\delta g_r(\lambda)d\lambda + \frac{1}{2\pi i}\int_\delta f_r^\perp(\lambda)d\lambda. \tag{3.30}$$

From (3.27), (3.30), we get (3.29).

Generally, from Theorems 2.11, 3.1, (3.9), (3.27), (3.30) and the residue formula, we conclude Theorem 3.2. □

Proof of (0.12) *and* (0.13). As $\mathbf{J}=J$ on $\mu^{-1}(0)$, the condition (3.2) is verified.

From Theorem 3.2, (3.22), we get (0.12) and (0.13). □

From Theorem 3.1, (3.27), (3.30), and the residue formula, we can get $P^{(r)}$ by using the operators $(\mathscr{L}_2^0)^{-1}$, $P^N, P^{N^\perp}, \mathcal{O}_k$ $(k\leq r)$.

This gives us a direct method to compute $P^{(r)}$ in view of Theorem 3.1. In particular,

$$P^{(1)} = -P^N\mathcal{O}_1P^{N^\perp}(\mathscr{L}_2^0)^{-1}P^{N^\perp} - P^{N^\perp}(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1P^N, \tag{3.31}$$

and

$$\begin{aligned}
P^{(2)} =& \frac{1}{2\pi i}\int_\delta \Big[(\lambda-\mathscr{L}_2^0)^{-1}P^{N^\perp}(\mathcal{O}_1f_1+\mathcal{O}_2f_0)(\lambda) + \frac{1}{\lambda}P^N(\mathcal{O}_1f_1+\mathcal{O}_2f_0)(\lambda)\Big]d\lambda\\
=& \frac{1}{2\pi i}\int_\delta \Big\{(\lambda-\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big[\mathcal{O}_1\Big((\lambda-\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1+\frac{1}{\lambda}P^N\mathcal{O}_1\Big)+\mathcal{O}_2\Big]\\
&+\frac{1}{\lambda}P^N\Big[\mathcal{O}_1\Big((\lambda-\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1+\frac{1}{\lambda}P^N\mathcal{O}_1\Big)+\mathcal{O}_2\Big]\Big\}(\lambda-\mathscr{L}_2^0)^{-1}d\lambda\\
=&(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1P^N - P^{N^\perp}(\mathscr{L}_2^0)^{-2}\mathcal{O}_1P^N\mathcal{O}_1P^N\\
&+(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp} - (\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_2P^N\\
&+P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp} - P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-2}P^{N^\perp}\mathcal{O}_1P^N\\
&-P^N\mathcal{O}_1P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-2}P^{N^\perp} - P^N\mathcal{O}_2(\mathscr{L}_2^0)^{-1}P^{N^\perp}.
\end{aligned} \tag{3.32}$$

In the next Subsection we will prove $P^N\mathcal{O}_1P^N=0$, thus the second and seventh terms in (3.32) are zero.

3.3. **A formula for $\mathcal{O}_1$.** We will use the notation in Section 1. All tensors in this Subsection will be evaluated at the base point $x_0 \in X_G$.

For ψ a tensor on X, we denote by $\nabla^X\psi$ its covariant derivative induced by ∇^{TX}.

If ψ_1 is a G-equivariant tensor, then we can consider it as a tensor on $B = U/G$ with the covariant derivative $\nabla^B\psi_1$, we will denote by

$$(\nabla^B\nabla^B\psi_1)_{(c_je_j,c'_ke_k)} := c_jc'_k(\nabla^B_{e_j}\nabla^B_{e_k}\psi_1)_{x_0},$$

etc.

We denote by $\{e_a\}$ an orthonormal basis of (TX, g^{TX}).

To simplify the notation, we often denote by U the lift U^H of $U \in TB$.

Recall that $\widetilde{\mu} \in TY$ is defined by (1.14) and the moment map μ (2.14), and that A is the second fundamental form of X_G defined by (0.10).

Lemma 3.3. *The following identities hold,*

$$\begin{aligned}
&(\nabla^{TY}_{\mathcal{R}}\widetilde{\mu})_{x_0} = -\mathbf{J}\mathcal{R}^{\perp},\\
&(\nabla^{TY}_{\cdot}\nabla^{TY}_{\cdot}\widetilde{\mu})_{(\mathcal{R},\mathcal{R})} := (\nabla^{TY}_{e^H_j}\nabla^{TY}_{e^H_i}\widetilde{\mu})_{x_0}Z_jZ_i\\
&\quad = -P^{TY}\left((\nabla^X_{\mathcal{R}^0}\mathbf{J})(\mathcal{R}^0 + 2\mathcal{R}^{\perp}) + (\nabla^X_{\mathcal{R}^{\perp}}\mathbf{J})\mathcal{R}^{\perp}\right)\\
&\qquad - \mathbf{J}A(\mathcal{R}^0)\mathcal{R}^0 - \frac{1}{2}T(\mathcal{R}^0, \mathbf{J}\mathcal{R}^0) + T(\mathcal{R}^{\perp}, \mathbf{J}\mathcal{R}^{\perp}).
\end{aligned} \tag{3.33}$$

Proof. Recall that P^{TY}, P^{T^HX} are the orthogonal projections from TX onto TY, T^HX defined in Section 1.1. Note that on P, by (3.3),

$$\mathbf{J}e_i^{\perp,H} \in TY, \quad \mathbf{J}e_i^{0,H} = (\mathbf{J}_Ge_i^0)^H \in T^HP. \tag{3.34}$$

By (1.14) and (2.15), for $K \in \mathfrak{g}$,

$$-\left\langle \mathbf{J}e_i^H, K^X\right\rangle = \nabla_{e_i^H}\mu(K) = \left\langle \nabla^{TY}_{e_i^H}\widetilde{\mu}, K^X\right\rangle + \left\langle \widetilde{\mu}, \nabla^{TY}_{e_i^H}K^X\right\rangle. \tag{3.35}$$

From (1.4), (1.5), (1.6) and (3.35),

$$\nabla^{TY}_{e_i^H}\widetilde{\mu} = -P^{TY}\mathbf{J}e_i^H - \frac{1}{2}\dot{g}^{TY}_{e_i^H}\widetilde{\mu} = -P^{TY}\mathbf{J}e_i^H - T(e_i^H, \widetilde{\mu}). \tag{3.36}$$

From (3.36) and the fact that $\widetilde{\mu} = 0$ on P, one gets the first equation in (3.33).

Now for W (resp. Y) a smooth section of TX (resp. TY), by (1.8),

$$\begin{aligned}
\left\langle \nabla^{TY}_{e_j^H}P^{TY}W, Y\right\rangle &= e_j^H\langle W, Y\rangle - \left\langle P^{TY}W, \nabla^{TY}_{e_j^H}Y\right\rangle\\
&= \left\langle \nabla^{TX}_{e_j^H}W, Y\right\rangle + \frac{1}{2}\left\langle T(e_j^H, P^{T^HX}W), Y\right\rangle.
\end{aligned} \tag{3.37}$$

By (3.37),

$$\nabla^{TY}_{e_j^H}P^{TY}W = P^{TY}\nabla^{TX}_{e_j^H}W + \frac{1}{2}T(e_j^H, P^{T^HX}W). \tag{3.38}$$

By (3.36) and (3.38),

$$\nabla^{TY}_{e^H_j}\nabla^{TY}_{e^H_i}\widetilde{\mu} = -P^{TY}(\nabla^X_{e^H_j}\mathbf{J})e^H_i - P^{TY}\mathbf{J}\nabla^{TX}_{e^H_j}e^H_i - \frac{1}{2}T(e^H_j, P^{T^HX}\mathbf{J}e^H_i) - \frac{1}{2}(\nabla^{TY}_{e^H_j}\dot{g}^{TY}_{e^H_i})\widetilde{\mu} - \frac{1}{2}\dot{g}^{TY}_{e^H_i}(\nabla^{TY}_{e^H_j}\widetilde{\mu}). \tag{3.39}$$

By (1.3) and (1.7), for U_1, U_2 sections of TB on B,

$$\nabla^{TX}_{U^H_2}U^H_1 = (\nabla^{TB}_{U_2}U_1)^H - \frac{1}{2}T(U^H_2, U^H_1). \tag{3.40}$$

By the definition of our basis $\{e^0_i, e^\perp_j\}$ in Section 2.6,

$$(\nabla^{TB}_{e^0_i}e^0_j)_{x_0} = A(e^0_i)e^0_j, \quad (\nabla^{TB}_{e^0_i}e^\perp_j)_{x_0} = (\nabla^{TB}_{e^\perp_j}e^0_i)_{x_0} = A(e^0_i)e^\perp_j, \quad (\nabla^{TB}_{e^\perp_j}e^\perp_i)_{x_0} = 0. \tag{3.41}$$

Thus by (1.6), (3.2), (3.36), (3.39), (3.40), (3.41) and the facts that A exchanges N_G and TX_G on X_G, and that $\widetilde{\mu} = 0$ on P, we get

$$(\nabla^{TY}_{\cdot}\nabla^{TY}_{\cdot}\widetilde{\mu})_{(\mathcal{R},\mathcal{R})} = -P^{TY}(\nabla^X_{\mathcal{R}}\mathbf{J})\mathcal{R} - \mathbf{J}A(\mathcal{R}^0)\mathcal{R}^0 - \frac{1}{2}T(\mathcal{R}, \mathbf{J}\mathcal{R}^0) + T(\mathcal{R}, \mathbf{J}\mathcal{R}^\perp). \tag{3.42}$$

We use the closeness of ω to complete the proof of (3.33).

From (0.2), for $U, V, W \in TX$,

$$\langle(\nabla^X_U\mathbf{J})V, W\rangle = (\nabla^X_U\omega)(V, W), \tag{3.43}$$

thus

$$\langle(\nabla^X_U\mathbf{J})V, W\rangle + \langle(\nabla^X_V\mathbf{J})W, U\rangle + \langle(\nabla^X_W\mathbf{J})U, V\rangle = d\omega(U, V, W) = 0. \tag{3.44}$$

By (1.3), (1.7), (3.44) and (3.34), for Y a smooth section of TY,

$$\langle\mathbf{J}\nabla^{TX}_Y e^0_j, e^\perp_i\rangle = -\langle\nabla^{TX}_Y e^0_j, \mathbf{J}e^\perp_i\rangle = -\langle T(e^0_j, \mathbf{J}e^\perp_i), Y\rangle$$

and

$$\begin{aligned}\langle T(e^\perp_i, \mathbf{J}e^0_j), Y\rangle &= -2\langle\nabla^{TX}_Y(\mathbf{J}e^0_j), e^\perp_i\rangle = -2\langle(\nabla^X_Y\mathbf{J})e^0_j, e^\perp_i\rangle + 2\langle T(e^0_j, \mathbf{J}e^\perp_i), Y\rangle \\ &= 2\langle(\nabla^X_{e^0_j}\mathbf{J})e^\perp_i, Y\rangle - 2\langle(\nabla^X_{e^\perp_i}\mathbf{J})e^0_j, Y\rangle + 2\langle T(e^0_j, \mathbf{J}e^\perp_i), Y\rangle.\end{aligned} \tag{3.45}$$

From (3.42), (3.45), we get the second equation of (3.33). □

Theorem 3.4. *The following identity holds,*

$$\begin{aligned}\mathcal{O}_1 = &-\frac{2}{3}(\partial_j R^{L_B})_{x_0}(\mathcal{R}, e_i)Z_j\nabla_{0,e_i} - \frac{1}{3}(\partial_i R^{L_B})_{x_0}(\mathcal{R}, e_i) \\ &- 2\left\langle A(e^0_i)e^0_j, \mathcal{R}^\perp\right\rangle\nabla_{0,e^0_i}\nabla_{0,e^0_j} - \pi\sqrt{-1}\left\langle(\nabla^X_{\mathcal{R}}\mathbf{J})e_a, e_b\right\rangle c(e_a)\,c(e_b) \\ &+ 4\pi^2\left\langle(\nabla^X_{\mathcal{R}^0}\mathbf{J})(\mathcal{R}^0 + 2\mathcal{R}^\perp) + (\nabla^X_{\mathcal{R}^\perp}\mathbf{J})\mathcal{R}^\perp - T(\mathcal{R}^\perp, \mathbf{J}\mathcal{R}^\perp), \mathbf{J}\mathcal{R}^\perp\right\rangle \\ &+ 4\pi^2\left\langle\mathbf{J}A(\mathcal{R}^0)\mathcal{R}^0 + \frac{1}{2}T(\mathcal{R}^0, \mathbf{J}\mathcal{R}^0), \mathbf{J}\mathcal{R}^\perp\right\rangle + 4\pi\sqrt{-1}\left\langle\widetilde{\mu}^{\mathrm{Cliff}} + \widetilde{\mu}^E, \mathbf{J}\mathcal{R}^\perp\right\rangle.\end{aligned} \tag{3.46}$$

Proof. For $\psi \in (T^*X \otimes \mathrm{End}(\Lambda(T^{*(0,1)}X)))_B \simeq (T^*X \otimes (C(TX) \otimes_{\mathbb{R}} \mathbb{C}))_B$, where $C(TX)$ is the Clifford bundle of TX, we denote by $\nabla^X\psi$ the covariant derivative of ψ induced by ∇^{TX}.

From $[\nabla^{\mathrm{Cliff}}_W, c(e_a)] = c(\nabla^{TX}_W e_a)$, we observe that for $W \in TB$,

$$\begin{aligned}(3.47)\qquad \nabla^X_W(\psi(e_a)c(e_a)) &= (\nabla^X_W\psi)(e_a)c(e_a) + \psi(\nabla^{TX}_{W^H}e_a)c(e_a) + \psi(e_a)c(\nabla^{TX}_{W^H}e_a)\\ &= (\nabla^X_W\psi)(e_a)c(e_a).\end{aligned}$$

Thus by (2.48) and (3.47), for $k \geq 2$,

$$\begin{aligned}(3.48)\quad -(2\omega_d+\tau)(tZ) &= \frac{1}{2}\left(R^L(e_a,e_b)\,c(e_a)\,c(e_b)\right)(tZ)\\ &= \frac{1}{2}\sum_{r=0}^{k}\frac{\partial^r}{\partial t^r}\left[(R^L(e_a,e_b)\,c(e_a)\,c(e_b))(tZ)\right]|_{t=0}\frac{t^r}{r!} + \mathscr{O}(t^{k+1})\\ &= \frac{1}{2}\Big(R^L_{x_0} + t(\nabla^X_{\mathcal{R}}R^L)_{x_0}\Big)(e_a,e_b)\,c(e_a)\,c(e_b) + \mathscr{O}(t^2).\end{aligned}$$

By Lemma 3.3 and (2.111), we have

$$\begin{aligned}(3.49)\qquad -t^2\langle\widetilde{\mu}^{E_p},\widetilde{\mu}^{E_p}\rangle(tZ) =& 4\pi^2\sum_{k=2}^{3}\frac{1}{k!}\frac{\partial^k}{\partial t^k}\Big(|\widetilde{\mu}|^2_{g^{TY}}(tZ)\Big)\Big|_{t=0}t^{k-2}\\ &+ 4\pi\sqrt{-1}t\left\langle\widetilde{\mu}^{\mathrm{Cliff}} + \widetilde{\mu}^E, \mathbf{J}\mathcal{R}^{\perp}\right\rangle_{x_0} + \mathscr{O}(t^2).\end{aligned}$$

The following two formulas are clear,

$$\begin{aligned}&\frac{1}{2}\,\frac{\partial^2}{\partial t^2}|\widetilde{\mu}|^2_{g^{TY}}(tZ)\Big|_{t=0} = \frac{1}{2}\left(\nabla\nabla|\widetilde{\mu}|^2_{g^{TY}}(Z)\right)_{(\mathcal{R},\mathcal{R})}\Big|_{Z=0} = |\nabla^{TY}_{\mathcal{R}}\widetilde{\mu}|^2,\\ (3.50)\qquad &\frac{1}{3!}\,\frac{\partial^3}{\partial t^3}|\widetilde{\mu}|^2_{g^{TY}}(tZ)\Big|_{t=0} = \frac{1}{6}\left(\nabla\nabla\nabla|\widetilde{\mu}|^2_{g^{TY}}(Z)\right)_{(\mathcal{R},\mathcal{R},\mathcal{R})}\Big|_{Z=0}\\ &\qquad\qquad = \langle(\nabla^{TY}\nabla^{TY}\widetilde{\mu})_{(\mathcal{R},\mathcal{R})}, \nabla^{TY}_{\mathcal{R}}\widetilde{\mu}\rangle.\end{aligned}$$

From Lemma 3.3 and (3.49)-(3.50), we see that the contribution from $-t^2\langle\widetilde{\mu}^{E_p},\widetilde{\mu}^{E_p}\rangle(tZ)$ is the last three terms of (3.46).

By (2.104), (2.106) and (3.10), we have

$$(3.51)\qquad \nabla_{t,e_i} = \nabla_{0,e_i} + \frac{t}{3}(\partial_j R^{L_B})_{x_0}Z_j(\mathcal{R},e_i) - \frac{t}{2}(\frac{1}{\kappa}\nabla_{e_i}\kappa)(tZ) + \mathscr{O}(t^2).$$

By $g_{ij}(Z) = \theta^k_i(Z)\theta^k_j(Z)$ and (2.95)-(2.97), we know

$$\begin{aligned}g_{ij}(Z) =& \delta_{ij} - 2\left\langle A(e^0_i)e^0_j, \mathcal{R}^{\perp}\right\rangle + \mathscr{O}(|Z|^2) \quad \text{for } 1 \leq i,j \leq 2(n-n_0),\\ &\delta_{ij} + \mathscr{O}(|Z|^2) \quad \text{otherwise;}\\ \kappa(Z) =& \det(g_{ij}(Z))^{1/2} = 1 - \langle A(e^0_i)e^0_i, \mathcal{R}^{\perp}\rangle + \mathscr{O}(|Z|^2).\end{aligned} \tag{3.52}$$

From (3.41), (3.51) and (3.52), the first three terms of the right hand side of (3.46) is the coefficient t^1 of the Taylor expansion of $-g^{ij}(tZ)(\nabla_{t,e_i}\nabla_{t,e_j} - t\nabla_{t,\nabla^{TB}_{e_i}e_j(tZ)})$.

By (2.110), (3.43) and the above argument, the proof of Theorem 3.4 is complete. □

Theorem 3.5. *We have the relation*

$$P^N\mathcal{O}_1P^N=0. \tag{3.53}$$

Proof. By (3.8) and (3.19),

$$\begin{aligned}&b_i^+P^N=b_i^{\perp+}P^N=0\,,\quad (b_i^\perp P^N)(Z,Z')=2a_i^\perp Z_i^\perp P^N(Z,Z'),\\&(b_iP^N)(Z,Z')=a_i(\overline{z}_i^0-\overline{z}_i'^0)P^N(Z,Z').\end{aligned} \tag{3.54}$$

We learn from (3.54) that for any polynomial $g(Z^\perp)$ in $Z^\perp$, we can write $g(Z^\perp)P^N(Z,Z')$ as sums of $g_{\beta^\perp}(b^\perp)^{\beta^\perp}P^N(Z,Z')$ with constants $g_{\beta^\perp}$. By Theorem 3.1,

$$P_{\mathscr{L}^\perp}(b^\perp)^{\alpha^\perp}g(Z^\perp)P^N=0\,,\quad \text{for } |\alpha^\perp|>0. \tag{3.55}$$

Let $\{w_a\}$ be an orthonormal basis of $(T^{(1,0)}X,g^{TX})$.

Note that if f,g are two $\mathbb{C}$-linear forms, then

$$f(e_a)g(e_a)=f(w_a)g(\overline{w}_a)+f(\overline{w}_a)g(w_a).$$

Thus by Theorem 3.1, (2.8), (3.21) and (3.54),

$$\begin{aligned}&P^N\left\langle(\nabla^X_{\mathcal{R}}\mathbf{J})e_a,e_b\right\rangle c(e_a)\,c(e_b)P^N=-2P^N\left\langle(\nabla^X_{\mathcal{R}}\mathbf{J})w_a,\overline{w}_a\right\rangle P^N\\&=-2\left\langle(\nabla^X_{z^0+\overline{z}'^0}\mathbf{J})w_a,\overline{w}_a\right\rangle P^N=\sqrt{-1}\,\mathrm{Tr}\,|_{TX}[J(\nabla^X_{z^0+\overline{z}'^0}\mathbf{J})]P^N.\end{aligned} \tag{3.56}$$

By (3.8), (3.12), (3.21), (3.46), (3.54)-(3.56), we get

$$\begin{aligned}P^N\mathcal{O}_1P^N=P^N\Big\{&\frac{2}{3}(\partial_{\mathcal{R}}R^{L_B})_{x_0}(\mathcal{R},\tfrac{\partial}{\partial\overline{z}_i^0})b_i-\frac{1}{3}(\partial_{e_i^0}R^{L_B})_{x_0}(\mathcal{R},e_i^0)\\&+\frac{1}{3}(\partial_{\mathcal{R}}R^{L_B})_{x_0}(\mathcal{R},e_j^\perp)b_j^\perp-\frac{1}{3}(\partial_{e_j^\perp}R^{L_B})_{x_0}(\mathcal{R}^0,e_j^\perp)\\&+\pi\,\mathrm{Tr}\,|_{TX}[J(\nabla^X_{\mathcal{R}^0}\mathbf{J})]+8\pi^2\left\langle(\nabla^X_{\mathcal{R}^0}\mathbf{J})\mathcal{R}^\perp,\mathbf{J}\mathcal{R}^\perp\right\rangle\Big\}P^N.\end{aligned} \tag{3.57}$$

By (3.9), (3.54) and (3.55),

$$P^NZ_j^\perp Z_k^\perp P^N=\frac{1}{2a_k^\perp}P^NZ_j^\perp b_k^\perp P^N=\frac{1}{2a_k^\perp}\delta_{jk}P^N. \tag{3.58}$$

For ψ a tensor on X_G, let $\nabla^{X_G}\psi$ be the covariant derivative of ψ induced by the Levi-Civita connection ∇^{TX_G}.

For $U,V,W\in T_{x_0}X_G$, by (3.2), (3.3) and (3.11), we have

$$(\partial_UR^{L_B})_{x_0}(V,W)=-2\pi\sqrt{-1}\left\langle(\nabla^{X_G}_U\mathbf{J}_G)V,W\right\rangle=-2\pi\sqrt{-1}\left\langle(\nabla^X_U\mathbf{J})V,W\right\rangle. \tag{3.59}$$

From (3.2), (3.5), we know that

$$\mathbf{J}e_j^\perp=\frac{a_j}{2\pi}Je_j^\perp. \tag{3.60}$$

By Theorem 3.1, (1.18), (2.7), (3.9), (3.44) and (3.54)-(3.60), we get

$$
\begin{aligned}
(3.61)\quad P^N\mathcal{O}_1P^N &= P^N\Big\{-\frac{4\pi\sqrt{-1}}{3}\Big[2\Big\langle(\nabla^X_{\mathcal{R}^0}\mathbf{J})_{\frac{\partial}{\partial z_i^0}},\frac{\partial}{\partial\overline{z}_i^0}\Big\rangle+\Big\langle(\nabla^X_{\frac{\partial}{\partial z_i^0}}\mathbf{J})\mathcal{R}^0,\frac{\partial}{\partial\overline{z}_i^0}\Big\rangle\\
&\quad-\Big\langle(\nabla^X_{\frac{\partial}{\partial\overline{z}_i^0}}\mathbf{J})\mathcal{R}^0,\frac{\partial}{\partial z_i^0}\Big\rangle\Big]+\pi\,\mathrm{Tr}\,|_{TX}[J(\nabla^X_{\mathcal{R}^0}\mathbf{J})]+2\pi\,\big\langle(\nabla^X_{\mathcal{R}^0}\mathbf{J})e_j^\perp,Je_j^\perp\big\rangle\Big\}P^N\\
&=\pi\Big[-4\sqrt{-1}\Big\langle(\nabla^X_{z^0+\overline{z}'^0}\mathbf{J})_{\frac{\partial}{\partial z_i^0}},\frac{\partial}{\partial\overline{z}_i^0}\Big\rangle+\mathrm{Tr}\,|_{TX}[J(\nabla^X_{z^0+\overline{z}'^0}\mathbf{J})]-2\,\big\langle J(\nabla^X_{z^0+\overline{z}'^0}\mathbf{J})e_j^\perp,e_j^\perp\big\rangle\Big]P^N=0.
\end{aligned}
$$

The proof of Theorem 3.5 is complete. □

From (3.32) and Theorem 3.5, we get the following general formula which will be used in Section 5,

$$
\begin{aligned}
P^{(2)}=&(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1P^N-(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_2P^N\\
&+P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}-P^N\mathcal{O}_2(\mathscr{L}_2^0)^{-1}P^{N^\perp}\\
&+(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_1P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}-P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-2}P^{N^\perp}\mathcal{O}_1P^N.
\end{aligned}\tag{3.62}
$$

3.4. **Example** $(\mathbb{C}P^1,2\,\omega_{FS})$**.** Let ω_{FS} be the Kähler form associated to the Fubini-Study metric $g_{FS}^{T\mathbb{C}P^1}$ on $\mathbb{C}P^1$. We will use the metric $g^{T\mathbb{C}P^1}=2\,g_{FS}^{T\mathbb{C}P^1}$ on $\mathbb{C}P^1$ in this Subsection.

Let L be the holomorphic line bundle $\mathcal{O}(2)$ on $\mathbb{C}P^1$. Recall that $\mathcal{O}(-1)$ is the tautological line bundle of $\mathbb{C}P^1$.

We will use the homogeneous coordinate $(z_0,z_1)\in\mathbb{C}^2$ for $\mathbb{C}P^1\simeq(\mathbb{C}^2\setminus\{0\})/\mathbb{C}^*$.

Denote by $U_i=\{[z_0,z_1]\in\mathbb{C}P^1;z_i\neq0\}$, $(i=0,1)$, the open subsets of $\mathbb{C}P^1$, and the two coordinate charts are defined by $\phi_i:U_i\simeq\mathbb{C}$, $\phi_i([z_0,z_1])=\frac{z_j}{z_i}$, $j\neq i$.

For any $i_0,i_1\in\mathbb{N}$, $z_0^{i_0}z_1^{i_1}$ is naturally identified to a holomorphic section of $\mathcal{O}(-i_0-i_1)^*$ on $\mathbb{C}P^1$. For any $k\in\mathbb{N}$, we have

$$
H^0(\mathbb{C}P^1,\mathcal{O}(k))=\mathbb{C}\{s_{k,i_0}:=z_0^{i_0}z_1^{i_1},\ i_0+i_1=k,\text{ and }i_0,i_1\in\mathbb{N}\}.\tag{3.63}
$$

On U_i, the trivialization of the line bundle L is defined by $L\ni s\to s/z_i^2$, here z_i^2 is considered as a holomorphic section of $\mathcal{O}(2)$.

In the following, we will work on $\mathbb{C}$ by using $\phi_1:U_1\to\mathbb{C}$. Then for $z\in\mathbb{C}$,

$$
\omega_{FS}(z)=\frac{\sqrt{-1}}{2\pi}\overline{\partial}\partial\log((1+|z|^2)^{-1})=\frac{\sqrt{-1}}{2\pi}\frac{dz\wedge d\overline{z}}{(1+|z|^2)^2}.\tag{3.64}
$$

Let h^L be the smooth Hermitian metric on L on $\mathbb{C}P^1$ defined by for $z\in\mathbb{C}$,

$$
|s_{2,0}|^2_{h^L}(z)=(1+|z|^2)^{-2}.\tag{3.65}
$$

Let ∇^L be the holomorphic Hermitian connection of (L,h^L) with its curvature R^L.

By (3.64) and (3.65), under our trivialization on $\mathbb{C}$

$$
\begin{aligned}
&\nabla^L=\overline{\partial}+\partial+\partial\log(|s_{2,0}|^2_{h^L}),\\
&\frac{\sqrt{-1}}{2\pi}R^L=\frac{\sqrt{-1}}{2\pi}\overline{\partial}\partial\log|s_{2,0}|^2_{h^L}=2\,\omega_{FS}=:\omega.
\end{aligned}\tag{3.66}
$$

Let K be the canonical basis of $\mathrm{Lie}\,S^1=\mathbb{R}$, i.e. for $t\in\mathbb{R},\exp(tK)=e^{2\pi\sqrt{-1}t}\in S^1$.

We define an S^1-action on $\mathbb{C}P^1$ by $g\cdot[z_0,z_1]=[gz_0,z_1]$ for $g\in S^1$.

On our local coordinate, $g\cdot z=gz$, and the vector field $K^{\mathbb{C}P^1}$ on $\mathbb{C}P^1$ induced by K is

$$K^{\mathbb{C}P^1}(z):=\tfrac{\partial}{\partial t}\exp(-tK)\cdot z|_{t=0}=-2\pi\sqrt{-1}\Big(z\tfrac{\partial}{\partial z}-\overline{z}\tfrac{\partial}{\partial\overline{z}}\Big). \tag{3.67}$$

Set

$$\mu(K)([z_0,z_1])=\frac{2|z_0|^2}{|z_0|^2+|z_1|^2}-1.$$

Then, on $\mathbb{C}$,

$$\mu(K)=2|z|^2\,(1+|z|^2)^{-1}-1. \tag{3.68}$$

By (3.64), (3.67) and (3.68), we verify easily that μ is a moment map associated to the S^1-action on $(\mathbb{C}P^1,\omega)$ in the sense of (2.15).

The Lie S^1-action on the sections of L defined by (2.14) induces a holomorphical S^1-action on L. In particular, from (3.66)-(3.68),

$$\tfrac{\partial}{\partial t}\exp(-tK)\cdot s_{2,j}|_{t=0}=:L_Ks_{2,j}=2\pi\sqrt{-1}(1-j)\,s_{2,j}. \tag{3.69}$$

By (3.69), the S^1-invariant sub-space of $H^0(\mathbb{C}P^1,L^p)$ and $\mu^{-1}(0)$ are

$$H^0(\mathbb{C}P^1,L^p)^{S^1}=\mathbb{C}\,s_{2p,p},\quad \mu^{-1}(0)=\{z\in\mathbb{C},|z|=1\}, \tag{3.70}$$

and S^1 acts freely on $\mu^{-1}(0)$, thus $(\mathbb{C}P^1)_{S^1}=\{\mathrm{pt}\}$.

Under our trivialization of L, $s_{2p,j}\in H^0(\mathbb{C}P^1,L^p)$ is the function z^j, and from (3.65),

$$\|s_{2p,j}\|^2_{L^2}=\int_{\mathbb{C}}\frac{|z|^{2j}}{(1+|z|^2)^{2p}}2\,\omega_{FS}=\int_0^\infty\frac{2t^j\,dt}{(1+t)^{2p+2}}=\frac{2\,j!\,(2p-j)!}{(2p+1)!}. \tag{3.71}$$

Thus $(\frac{(2p+1)!}{2\,(p!)^2})^{1/2}s_{2p,p}$ is an orthonormal basis of $H^0(\mathbb{C}P^1,L^p)^{S^1}$.

Let $\overline{\partial}^{L^p*}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p}$. For $p\geq 1$, the spin^c Dirac operator D_p in (2.12) and its kernel are given by

$$D_p=\sqrt{2}\left(\overline{\partial}^{L^p}+\overline{\partial}^{L^p*}\right),\quad \mathrm{Ker}\,D_p=H^0(\mathbb{C}P^1,L^p). \tag{3.72}$$

Finally, by Def. 2.3, for $p\geq 1$, we get

$$\begin{aligned}P_p^G(z,z')&=\frac{(2p+1)!}{2\,(p!)^2}s_{2p,p}(z)\otimes s_{2p,p}(z')^*,\\ P_p^G(z,z)&=\frac{(2p+1)!}{2\,(p!)^2}|s_{2p,p}|^2_{h^{L^p}}(z)=\frac{(2p+1)!}{2\,(p!)^2}\frac{|z|^{2p}}{(1+|z|^2)^{2p}}.\end{aligned} \tag{3.73}$$

Note that our trivialization by $s_{2,0}$ is not unitary, thus we do not see directly the off-diagonal decay (0.14) from (3.73).

Here we will only verify that (3.73) is compatible with (0.13), (0.15) and (0.16).

Recall that Stirling's formula [38, (3.A.40)] tells us that as $p\to+\infty$,

$$p!=(2\pi p)^{1/2}p^p\,e^{-p}\left(1+\mathscr{O}\left(\frac{1}{p}\right)\right). \tag{3.74}$$

By (3.74),

$$\frac{(2p+1)!}{2\,(p!)^2}=\frac{\sqrt{p}}{\sqrt{\pi}\,e}\,2^{2p}\Big(1+\frac{1}{2p}\Big)^{2p}\Big(1+\mathscr{O}(\frac{1}{p})\Big)=\sqrt{\frac{p}{\pi}}\,2^{2p}\Big(1+\mathscr{O}(\frac{1}{p})\Big). \tag{3.75}$$

Now, $\mathbb{C}^*$ is an open neighborhood of $\mu^{-1}(0)$ and $B = \mathbb{C}^*/S^1 \simeq \mathbb{R}^+$ by mapping $z \in \mathbb{C}^*$ to $r = |z| \in \mathbb{R}^+$.

By (3.64), the metrics on $\{|z| = r\} = \{re^{2\pi\sqrt{-1}\theta}; \theta \in \mathbb{R}/\mathbb{Z}\}$, $B \simeq \mathbb{R}^+$ induced by $\omega = 2\,\omega_{FS}$ is

$$8\pi\, r^2\,(1+r^2)^{-2}\, d\theta \otimes d\theta, \qquad g^{TB} = \frac{2}{\pi}(1+r^2)^{-2}\, dr \otimes dr. \tag{3.76}$$

From (3.76), the fiberwise volume function $h^2(r)$ in (0.10) on $\mathbb{R}^+$ is

$$h^2(r) = \sqrt{8\pi}\, r\,(1+r^2)^{-1}. \tag{3.77}$$

From (3.73), (3.75) and (3.77), we get for $|z| = r$,

$$h^2(r) P_p^G(z,z) = \sqrt{8\pi}\frac{(2p+1)!}{2\,(p!)^2}\Big(\frac{r}{1+r^2}\Big)^{2p+1} = \sqrt{2\,p}\Big(\frac{2r}{1+r^2}\Big)^{2p+1}\Big(1+\mathscr{O}(\frac{1}{p})\Big). \tag{3.78}$$

When $|z| = 1$, from (3.78), we re-find (0.15) and (0.16).

From (3.76), $\sqrt{2\pi}\frac{\partial}{\partial r}$ is an orthonormal basis of (B, g^{TB}) at $r = 1$, thus the normal coordinate $Z^\perp$ has the form $r - 1 = \sqrt{2\pi}(Z^\perp + \mathscr{O}(|Z^\perp|^2)$. Thus

$$(2r\,(1+r^2)^{-1})^{2p+1} = e^{(2p+1)\log(1-\pi(Z^\perp)^2+\mathscr{O}(|Z^\perp|^3))} = e^{-2\pi p(Z^\perp)^2} + \cdots. \tag{3.79}$$

This means that (3.78), (3.79) are compatible with (0.13) and (3.22).

If we consider the sub-space $H^0(\mathbb{C}P^1, L^p)_{-p}$ of $H^0(\mathbb{C}P^1, L^p)$ with the weight $-p$ of S^1-action, then by (2.14) as in (3.69), and (3.71), $\sqrt{p+\frac{1}{2}}\, s_{2p,0}$ is an orthonormal basis of $H^0(\mathbb{C}P^1, L^p)_{-p}$.

Thus the smooth kernel $P_p^{-p}(z,z')$ of the orthogonal projection from $\mathscr{C}^\infty(\mathbb{C}P^1, L^p)$ onto $H^0(\mathbb{C}P^1, L^p)_{-p}$ is

$$P_p^{-p}(z,z') = (p+\frac{1}{2})\, s_{2p,0}(z) \otimes s_{2p,0}(z')^*, \quad P_p^{-p}(z,z) = (p+\frac{1}{2})(1+|z|^2)^{-2p}. \tag{3.80}$$

Note that $\mu^{-1}(-1) = \{0\}$, i.e. -1 is a singular value of μ.

Let μ_1 be the moment map defined by $\mu_1(K) = \mu(K) + 1$, then $H^0(\mathbb{C}P^1, L^p)_{-p}$ is the corresponding S^1-invariant holomorphic sections of L^p with respect to the corresponding S^1-action.

Thus 0 is a singular value of μ_1 and this explains why we have a factor p in (3.80) instead of $p^{1/2}$ in (3.78).

4. Applications

This Section is organized as follows. In Section 4.1, we explain Theorem 4.1, the version of Theorem 0.2 when we only assume that μ is regular at 0. In Section 4.2, We explain how to handle the ϑ-weight Bergman kernel. In Section 4.3, we deduce (0.15), and (0.16) from [17, Theorem 4.18′]. In Section 4.4, we explain Theorem 0.2 implies Toeplitz operator type properties on X_G. In Section 4.5, we extend our results for non-compact manifolds and for covering spaces. In Section 4.6, we explain the relation on the G-invariant Bergman kernel on X and the Bergman kernel on X_G.

We use the notation in Introduction.

4.1. **Orbifold case.** In this Subsection, we only suppose that $0 \in \mathfrak{g}^*$ is a regular value of μ, then G acts only infinitesimal freely on $P = \mu^{-1}(0)$, thus $X_G = P/G$ is a compact symplectic orbifold.

Let $G^0 = \{g \in G, g \cdot x = x \text{ for any } x \in P\}$, then G^0 is a finite normal sub-group of G and G/G_0 acts effectively on P.

We will use the notation for the orbifold as in [24, §1], [17, §4.2].

Let U be a G-neighborhood of $P = \mu^{-1}(0)$ in X such that G acts infinitesimal freely on $\overline{U}$, the closure of U. From the construction in Section 1.2, any G-equivariant vector bundle F on U induces an orbifold vector bundle F_B on the orbifold $B = U/G$.

The function h in (0.10) is only $\mathscr{C}^\infty$ on the regular part of the orbifold B, and we extend continuously h to U/G from its regular part, which is $\mathscr{C}^\infty$ and we denote it by $\widehat{h}$, then $\widehat{h}$ is also $\mathscr{C}^\infty$ on U.

As we work on P in Section 2.5, we need not to modify this part.

We need to modify Section 2.6 as follows.

Observe first that the construction in Section 1.1 works well if we only assume that G acts locally freely on X therein.

Denote by ∇^{T^HU} the connection on T^HU as in Section 1.1, and on P, let ∇^N, ∇^{T^HP}, ${}^0\nabla^{T^HU}$ be the connections on N, T^HU in Section 2.5 as in (0.9).

For $y_0 \in P$, $W \in T^HU$ (resp.T^HP), we define $\mathbb{R} \ni t \to x_t = \exp^{T^HU}_{y_0}(tW) \in U$ (resp. $\exp^{T^HP}_{y_0}(tW) \in P$) the curve such that $x_t|_{t=0} = y_0$, $\frac{dx}{dt}|_{t=0} = W$, $\frac{dx}{dt} \in T^HU$, $\nabla^{T^HU}_{\frac{dx}{dt}}\frac{dx}{dt} = 0$ (resp. $\frac{dx}{dt} \in T^HP$, $\nabla^{T^HP}_{\frac{dx}{dt}}\frac{dx}{dt} = 0$).

By proceeding as in Section 2.6, we identify $B^{T^HU}(y_0, \varepsilon)$ to a subset of U as following, for $Z \in B^{T^HU}(y_0, \varepsilon)$, $Z = Z^0 + Z^\perp$, $Z^0 \in T^H_{y_0}P$, $Z^\perp \in N_{x_0}$, we identify Z with $\exp^{T^HU}_{\exp^{T^HP}_{x_0}(Z^0)}(\tau_{Z^0}Z^\perp)$.

Set $G_{y_0} = \{g \in G, gy_0 = y_0\}$, then $G \cdot B^{T^HU}(y_0, \varepsilon) = G \times_{G_{y_0}} B^{T^HU}(y_0, \varepsilon)$ is a G-neighborhood of Gy_0, and $(G_{y_0}, B^{T^HU}(y_0, \varepsilon))$ is a local coordinate of B.

As the construction in Section 2.6 is G_{y_0}-equivariant, we extend the geometric objects on $G \times_{G_{y_0}} B^{T^HU}(y_0, \varepsilon)$ to $G \times_{G_{y_0}} \mathbb{R}^{2n-n_0} = X_0$.

Thus we get the corresponding geometric objects on $G \times \mathbb{R}^{2n-n_0}$ by using the covering $G \times \mathbb{R}^{2n-n_0} \to G \times_{G_{y_0}} \mathbb{R}^{2n-n_0}$, especially, $\widehat{\mathcal{L}}^{X_0}_p$ (where we use the $\widehat{\cdot}$ notation to indicate the modification) is defined similarly on $G \times \mathbb{R}^{2n-n_0}$, and Theorem 2.5 holds for $\widehat{\mathcal{L}}^{X_0}_p$.

Let $\widehat{\pi}_G : G \times \mathbb{R}^{2n-n_0} \to \mathbb{R}^{2n-n_0}$ be the natural projection and as in (1.20), we define $\widehat{\Phi} = \widehat{h}\widehat{\pi}_G$, then the operator $\widehat{\Phi}\widehat{\mathcal{L}}^{X_0}_p\widehat{\Phi}^{-1}$ is well-defined on $T^H_{y_0}U \simeq \mathbb{R}^{2n-n_0}$.

Let $g^{T^HX_0}$ be the metric on $\mathbb{R}^{2n-n_0}$ induced by g^{TX_0}, and let $dv_{T^HX_0}$ be the Riemannian volume form on $(\mathbb{R}^{2n-n_0}, g^{T^HX_0})$.

Let $P_{y_0,p}$ be the orthogonal projection from $L^2(\mathbb{R}^{2n-n_0}, (\Lambda(T^{*(0,1)}X) \otimes L^p \otimes E)_{y_0})$ onto $\mathrm{Ker}(\widehat{\Phi}\widehat{\mathcal{L}}^{X_0}_p\widehat{\Phi}^{-1})$ on $\mathbb{R}^{2n-n_0}$. Let $P_{y_0,p}(Z, Z')$ $(Z, Z' \in \mathbb{R}^{2n-n_0})$ be the smooth kernel of $P_{y_0,p}$ with respect to $dv_{T^HX_0}(Z')$.

Let $P^G_{0,p}$ be the orthogonal projection from $\Omega^{0,\bullet}(X_0, L^p_0 \otimes E_0)$ on $(\mathrm{Ker}\, D^{X_0}_p)^G$, and let $P^G_{0,p}(x, x')$ be the smooth kernel of $P^G_{0,p}$ with respect to the volume form $dv_{X_0}(x')$.

Let $P_p^{X_0/G}(y,y')$ $(y,y'\in X_0/G)$ be the smooth kernel associated to the operator on X_0/G induced by $\widehat{\Phi}\widehat{\mathcal{L}}_p^{X_0}\widehat{\Phi}^{-1}$ as in (2.72).

Note that our trivialization of the restriction of L on $B^{TH U}(y_0,\varepsilon)$ as in Section 2.6 is not G_{y_0}-invariant, except that G_{y_0} acts trivially on L_{y_0}.

For $x,x'\in X_0$, with their representatives $\widetilde{x},\widetilde{x}'\in\mathbb{R}^{2n-n_0}$, we have

$$\widehat{h}(x)\widehat{h}(x')P_{0,p}^G(x,x')=P_p^{X_0/G}(\pi(x),\pi(x'))=\frac{1}{|G^0|}\sum_{g\in G_{y_0}}(g,1)\cdot P_{y_0,p}(g^{-1}\widetilde{x},\widetilde{x}'). \tag{4.1}$$

The second equation of (4.1) is from direct computation (cf. [17, (5.19)]).

As we work on $G\times\mathbb{R}^{2n-n_0}$, for the operator $\widehat{\Phi}\widehat{\mathcal{L}}_p^{X_0}\widehat{\Phi}^{-1}$, Prop. 2.9 and Sections 2.7-2.9 still holds.

From Theorem 2.23 for $P_{y_0,p}$ and (4.1), we get

Theorem 4.1. *Under the same notation in Theorems 0.2, 2.23, for $\alpha,\alpha'\in\mathbb{N}^{2n-n_0}$, $|\alpha|+|\alpha'|\leq m$, we have*

$$\begin{aligned}(1+\sqrt{p}|Z^{\perp}|+\sqrt{p}|Z'^{\perp}|)^{m''}&\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\Big(p^{-n+\frac{n_0}{2}}(\widehat{h}\kappa^{\frac{1}{2}})(Z)(\widehat{h}\kappa^{\frac{1}{2}})(Z')P_p^G\circ\Psi(Z,Z')\right.\\&\left.-\frac{1}{|G^0|}\sum_{r=0}^{k}\sum_{g\in G_{y_0}}(g,1)\cdot P_{y_0}^{(r)}(g^{-1}\sqrt{p}Z,\sqrt{p}Z')p^{-\frac{r}{2}}\Big)\right|_{\mathscr{C}^{m'}(P)}\\&\leq Cp^{-(k+1-m)/2}(1+\sqrt{p}|Z^0|+\sqrt{p}|Z'^0|)^{2(n+k+m'+2)+m}\exp(-\sqrt{C''\nu p}\inf_{g\in G_{y_0}}|g^{-1}Z-Z'|)\\&\qquad+\mathscr{O}(p^{-\infty}).\end{aligned} \tag{4.2}$$

If $Z=Z'=Z^0$, then for $g\in G_{y_0}$, $gZ^0=Z^0$, we use Theorem 2.23 for $Z=Z'=0$ with the base point Z^0, and for the rest element in G_{y_0}, we use Theorem 2.23 for $Z=Z'=Z^0$ with the base point y_0, then we get

$$\begin{aligned}\Big|p^{-n+\frac{n_0}{2}}(\widehat{h}^2\kappa)(Z^0)P_p^G\circ\Psi(Z^0,Z^0)-\frac{1}{|G^0|}\sum_{r=0}^{k}\sum_{g\in G_{y_0},gZ^0=Z^0}(g,1)\cdot P_{Z^0}^{(2r)}(0,0)p^{-r}\\-\frac{1}{|G^0|}\sum_{r=0}^{2k}\sum_{g\in G_{y_0},gZ^0\neq Z^0}(g,1)\cdot P_{y_0}^{(r)}(g^{-1}\sqrt{p}Z^0,\sqrt{p}Z^0)p^{-\frac{r}{2}}\Big|\\\leq Cp^{-(2k+1)/2}\Big(1+(1+\sqrt{p}|Z^0|)^{2(n+2k+2)}\exp(-\sqrt{C'''\nu p}|Z^0|)\Big).\end{aligned} \tag{4.3}$$

Note that if $g\in G_{y_0}$ acts as the multiplication by $e^{i\theta}$ on L_{y_0}, then $(g,1)\cdot P_{y_0}^{(r)}$, $(g,1)\cdot P_{Z^0}^{(r)}$ in (4.3) have a factor $e^{i\theta p}$ which depends on p.

Of course, after replacing L by some power of L, we can assume that G_{y_0} acts as identity on L for any $y_0\in P$, in this case, $(g,1)\cdot P_{y_0}^{(r)}$, $(g,1)\cdot P_{Z^0}^{(r)}$ do not depend on p.

From Theorem 3.2 and (4.3), if the singular set of X_G is not empty, analogous to the usual orbifold case [17, (5.27)], $p^{-n+\frac{n_0}{2}}\ P_p^G(y_0,y_0)$, $(y_0\in P)$ does not have a uniform asymptotic expansion in the form $\sum_{r=0}^{\infty}c_r(y_0)p^{-r}$.

4.2. **ϑ-weight Bergman kernel on X.** In this section, we assume that G acts on $P = \mu^{-1}(0)$ freely.

Let $\mathcal{V}$ be a finite dimensional irreducible representation of G, we denote it by $\rho^{\mathcal{V}} : G \to \mathrm{End}(\mathcal{V})$. Let ϑ be the highest weight of the representation $\mathcal{V}$. Let $\mathcal{V}^*$ be the trivial bundle on X with G-action $\rho^{\mathcal{V}^*}$ induced by $\rho^{\mathcal{V}}$.

Let $P_p^{\mathcal{V}}$ be the orthogonal projection from $\Omega^{0,\bullet}(X, L^p \otimes E)$ on $\mathrm{Hom}_G(\mathcal{V}, \mathrm{Ker}\, D_p) \otimes \mathcal{V} \subset \mathrm{Ker}\, D_p$. Let $P_p^{\mathcal{V}}(x, x')$, $(x, x' \in X)$, be the smooth kernel of $P_p^{\mathcal{V}}$ with respect to $dv_X(x')$.

We call $P_p^{\mathcal{V}}(x, x')$ the ϑ-weight Bergman kernel of D_p.

We explain now the asymptotic expansion of $P_p^{\mathcal{V}}(x, x')$ as $p \to \infty$.

We will consider the corresponding objects in Sections 1-3 by replacing E by $E \otimes \mathcal{V}^*$. Especially, we denote by $D_p^{\mathcal{V}^*}$ the corresponding spin^c Dirac operator associated to the bundle $L^p \otimes E \otimes \mathcal{V}^*$.

Certainly, all results in Sections 1-3 still hold for the bundle $E \otimes \mathcal{V}^*$.

Let P_p^{ϑ} be the orthogonal projection from $\mathscr{C}^\infty(X, E_p \otimes \mathcal{V}^*)$ onto $(\mathrm{Ker}\, D_p^{\mathcal{V}^*})^G$, and $P_p^{\vartheta}(x, x')$, $(x, x' \in X)$ the smooth kernel of P_p^{ϑ} with respect to $dv_X(x')$.

As $\mathcal{V}$ is an irreducible representation of G, we get

$$\mathrm{Ker}\, D_p^{\mathcal{V}^*} = (\mathrm{Ker}\, D_p) \otimes \mathcal{V}^*, \quad (\mathrm{Ker}\, D_p^{\mathcal{V}^*})^G = \mathrm{Hom}_G(\mathcal{V}, \mathrm{Ker}\, D_p). \tag{4.4}$$

Let $\{v_i\}$ be an orthonormal basis of $\mathcal{V}$ with respect to a G-invariant metric on $\mathcal{V}$ and $\{v_i^*\}$ the corresponding dual basis.

Let dg be a Haar measure on G. By Schur Lemma,

$$\int_G g \cdot (v_j \otimes v_i^*) dg = \frac{1}{\dim_{\mathbb{C}} \mathcal{V}} \delta_{ij}\, \mathrm{Id}_{\mathcal{V}}. \tag{4.5}$$

Thus if W is a finite dimensional representation of G with the highest weight ϑ, then for any $s \in W$, we have

$$s = (\dim_{\mathbb{C}} \mathcal{V}) \Big(\int_G g \cdot (s \otimes v_i^*) dg \Big) \otimes v_i \in \mathrm{Hom}_G(\mathcal{V}, W) \otimes \mathcal{V} = W. \tag{4.6}$$

From (4.6) and the $G \times G$-invariance of the kernel $P_p^{\vartheta}(x, x')$, we get

$$\begin{aligned} P_p^{\mathcal{V}}(x, x') &= (\dim_{\mathbb{C}} \mathcal{V}) \sum_i (P_p^{\vartheta}(x, x') v_i^*, v_i), \\ P_p^{\mathcal{V}}(x, x) &= (\dim_{\mathbb{C}} \mathcal{V})\ \mathrm{Tr}_{\mathcal{V}^*}\, P_p^{\vartheta}(x, x) \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_x. \end{aligned} \tag{4.7}$$

In fact, let $\{\psi_j\}$ be an orthonormal basis of $\mathrm{Ker}(D_p^{\mathcal{V}^*})^G$, then for any j fixed, in view of the second equality in (4.4), one sees that

$$\psi_j^* \psi_j \in \mathrm{End}_G(\mathcal{V}) \quad \text{and} \quad \mathrm{Tr}_{\mathcal{V}}[\psi_j^* \psi_j] = \|\psi_j\|_{L^2}^2 = 1. \tag{4.8}$$

Thus by Schur Lemma,

$$\psi_j^* \psi_j = \frac{1}{\dim_{\mathbb{C}} \mathcal{V}}\, \mathrm{Id}_{\mathcal{V}} \tag{4.9}$$

and $\{(\dim_{\mathbb{C}} \mathcal{V})^{\frac{1}{2}} \psi_j v_i\}$ is an orthonormal basis of $\mathrm{Hom}_G(\mathcal{V}, \mathrm{Ker}\, D_p) \otimes \mathcal{V} \subset \mathrm{Ker}\, D_p$.

Let U be a G-neighborhood of $P = \mu^{-1}(0)$ as in Theorem 0.2, P_p^ϑ is viewed as a smooth section of $\mathrm{pr}_1^*(E_p \otimes \mathcal{V}^*)_B \otimes \mathrm{pr}_2^*(E_p \otimes \mathcal{V}^*)_B^*$ on $B \times B$, or as a $G \times G$-invariant smooth section of $\mathrm{pr}_1^*(E_p \otimes \mathcal{V}^*) \otimes \mathrm{pr}_2^*(E_p \otimes \mathcal{V}^*)^*$ on $U \times U$.

Moreover, v_i, v_i^* are smooth (not G-invariant) sections of $U \times \mathcal{V}$, $U \times \mathcal{V}^*$ on U. Thus from (4.7), $P_p^{\mathcal{V}}$ is not a $G \times G$-invariant section of $\mathrm{pr}_1^*(E_p) \otimes \mathrm{pr}_2^*(E_p^*)$ on $U \times U$.

Now Theorem 0.2 applies well to the bundle $E \otimes \mathcal{V}^*$, thus we get the asymptotic expansion of $P_p^\vartheta(x, x')$ as $p \to +\infty$, and the leading term in the expansion of $p^{-n+\frac{n_0}{2}}(h\kappa^{\frac{1}{2}})(x)(h\kappa^{\frac{1}{2}})(x')P_p^\vartheta(x, x')$ is $P(\sqrt{p}Z, \sqrt{p}Z')I_{\mathbb{C}\otimes(E\otimes\mathcal{V}^*)_B}$.

By (4.7), the leading term of the asymptotic expansion of $p^{-n+\frac{n_0}{2}}(h\kappa^{\frac{1}{2}})(x)(h\kappa^{\frac{1}{2}})(x')P_p^{\mathcal{V}}(x, x')$ is

$$(\dim_{\mathbb{C}} \mathcal{V})^2 \, P(\sqrt{p}Z, \sqrt{p}Z')I_{\mathbb{C}\otimes E_B}, \quad P(0,0) = 2^{n_0/2}. \tag{4.10}$$

Let Θ be the curvature of $P \to X_G$ as in Section 1.1. Let $\rho_*^{\mathcal{V}^*}$ denote the differential of $\rho^{\mathcal{V}^*}$. By (1.18),

$$R^{(E\otimes\mathcal{V}^*)_G} = R^{E_G} + \rho_*^{\mathcal{V}^*}(\Theta). \tag{4.11}$$

In the same way, we can define $\mathscr{I}_p^{\mathcal{V}}$ a section of $\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B$ on X_G by (0.17) for $P_p^{\mathcal{V}}$. From (0.25) (which will be proved in Section 5), (4.7), (4.10) and (4.11), we get

Theorem 4.2. *Under the condition of Theorem 0.6, the first coefficients of the asymptotic expansion of $\mathscr{I}_p^{\mathcal{V}} \in \mathrm{End}(E_G)$ in (0.20) is*

$$\begin{aligned} \Phi_0 &= (\dim_{\mathbb{C}} \mathcal{V})^2, \\ \Phi_1 &= \frac{1}{8\pi}(\dim_{\mathbb{C}} \mathcal{V})^2 \left(r_{x_0}^{X_G} + 6\Delta_{X_G} \log h + 4R_{x_0}^{E_G}(w_j^0, \overline{w}_j^0) \right) \\ &\quad + \frac{1}{2\pi}(\dim_{\mathbb{C}} \mathcal{V}) \, \mathrm{Tr}_{\mathcal{V}^*} \left[\rho_*^{\mathcal{V}^*}(\Theta)(w_j^0, \overline{w}_j^0) \right]. \end{aligned} \tag{4.12}$$

4.3. **Averaging the Bergman kernel: a direct proof of** (0.15) **and** (0.16)**.** We use the same assumption and notation in Theorem 0.2.

Let $P_p(x, x')$ be the smooth kernel of the orthogonal projection P_p from $\Omega^{0,\bullet}(X, L^p \otimes E)$ onto $\mathrm{Ker}\, D_p$ with respect to $dv_X(x')$. Then $P_p(x, x')$ is the usual Bergman kernel associated to D_p.

Let dg be a Haar measure on G. By Schur Lemma,

$$P_p^G(x, x') = \int_G ((g, 1) \cdot P_p)(x, x')dg = \int_G (g, 1) \cdot P_p(g^{-1}x, x')dg. \tag{4.13}$$

One possible way to get Theorem 0.2 is to apply the full off-diagonal expansion [17, Theorem 4.18′] to (4.13).

Unfortunately, we do not know how to get the full off-diagonal expansion, especially the fast decay along N_G in (0.14) in this way.

However, it is easy to get (0.15) and (0.16) as direct consequences of [17, Theorem 4.18′] and (4.13).

As in Section 2.5, we denote by TY the sub-bundle of TX on a neighborhood of $P = \mu^{-1}(0)$ generated by the G-action and by T^HP the orthogonal complement of TY in (TP, g^{TP}).

Take $y_0 \in P$. Let $\{e_i\}_{i=1}^{2(n-n_0)}$, $\{f_l\}_{l=1}^{n_0}$ be orthonormal basis of $T^H_{y_0}P$, $T_{y_0}Y$. Then $\{e_i\}_{i=1}^{2(n-n_0)} \cup \{f_l, J_{y_0}f_l\}_{l=1}^{n_0}$ is an orthonormal basis of $T_{y_0}X$. We use this orthonormal basis to get a local coordinate of X by using the exponential map $\exp^X_{y_0}$.

We identify $B^{T_{y_0}X}(0,\varepsilon)$ to $B^X(y_0,\varepsilon)$ by the exponential map $Z \to \exp^X_{y_0}(uZ)$.

Let $\nabla^{\mathrm{Cliff}\otimes E}$ be the connection on $\Lambda(T^{*(0,1)}X)\otimes E$ induced by ∇^{Cliff} and ∇^E.

For $Z \in B^{T_{y_0}X}(0,\varepsilon)$, we identify L_Z, $(\Lambda(T^{*(0,1)}X)\otimes E)_Z$, $(E_p)_Z$ to L_{y_0}, $(\Lambda(T^{*(0,1)}X)\otimes E)_{y_0}$, $(E_p)_{y_0}$ by parallel transport with respect to the connections ∇^L, $\nabla^{\mathrm{Cliff}\otimes E}$, ∇^{E_p} along the curve $\gamma_Z : [0,1] \ni u \to uZ$.

Under this identification, for $Z, Z' \in B^{T_{y_0}X}(0,\varepsilon)$, one has

$$P_p(Z,Z') \in \mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{y_0}.$$

Let $\kappa_1(Z)$ be the function on $B^{T_{y_0}X}(0,\varepsilon)$ defined by

$$dv_X(Z) = \kappa_1(Z)dv_{T_{x_0}X}. \tag{4.14}$$

By [17, Theorem 4.18′] (i.e. Theorem 0.2 for $G = \{1\}$), there exist $J_r(Z') \in \mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{y_0}$, polynomials in Z' with the same parity as r, such that for any $k, m' \in \mathbb{N}$, there exist $C, M > 0$ such that for $Z' \in T_{y_0}X$, $|Z'| \le \varepsilon$,

$$\begin{aligned}\left|\frac{1}{p^n}P_p(Z',0) - \sum_{r=0}^{k} J_r(\sqrt{p}Z')\kappa_1^{-1}(Z')e^{-\frac{\pi}{2}p|Z'|^2}p^{-\frac{r}{2}}\right|_{\mathscr{C}^{m'}(P)} \\ \le Cp^{-(k+1)/2}(1+\sqrt{p}|Z'|)^M \exp(-\sqrt{C''\nu}\sqrt{p}|Z'|) + \mathscr{O}(p^{-\infty}),\end{aligned} \tag{4.15}$$

and

$$J_0(Z) = I_{\mathbb{C}\otimes E}. \tag{4.16}$$

For $K \in \mathfrak{g}$, $|K|$ small, e^K maps $(\Lambda(T^{*(0,1)}X)\otimes E)_{e^{-K}y_0}$, $L_{e^{-K}y_0}$ to $(\Lambda(T^{*(0,1)}X)\otimes E)_{y_0}$, L_{y_0}, and under our identification, we denote these maps by

$$f^E(K) \in \mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{y_0}, \quad f^L(K) \in \mathrm{End}(L_{y_0}) \simeq \mathbb{C}. \tag{4.17}$$

By [1, Prop. 5.1], if we denote by

$$j_{\mathfrak{g}}(K) = \det_{\mathfrak{g}}\Big(\frac{1-e^{-ad\,K}}{ad\,K}\Big) \tag{4.18}$$

for $K \in \mathfrak{g}$, then in exponential coordinates of G,

$$d(e^K) = j_{\mathfrak{g}}(K)dK. \tag{4.19}$$

By [17, Prop. 4.1] (i.e. Theorem 0.1 for $G = \{1\}$), (4.13), as G acts freely on P, we know

$$P^G_p(y_0,y_0) = \int_{K\in\mathfrak{g},|K|\le\varepsilon} f^E(K)(f^L(K))^p P_p(e^{-K}y_0,y_0)j_{\mathfrak{g}}(K)dK + \mathscr{O}(p^{-\infty}). \tag{4.20}$$

Let S^L be the section of L on $B^{T_{y_0}X}(0,\varepsilon)$ obtained by parallel transport of a unit vector of L_{y_0} with respect to the connection ∇^L along the curve γ_Z. Let Γ^L be the connection form of L with respect to this trivialization.

Recall that for $K \in \mathfrak{g}$, the corresponding vector field K^X on X is defined in Section 1.1. Recall that $\{K_i\}$ is a basis of $\mathfrak{g}$.

By (2.105), for $K \in \mathfrak{g}$,

$$\begin{aligned} &(e^K \cdot S^L)(0) = e^K \cdot S^L(e^{-K}y_0) = f^L(K)S^L(0), \text{ with } f^L(0) = 1, \\ &\Gamma^L_Z(K^X) = \frac{1}{2}R^L_{y_0}(Z, K^X) + \mathscr{O}(|Z|^2). \end{aligned} \tag{4.21}$$

By (2.14), (2.15), (4.21) and $\mu = 0$ on P, we get

$$\begin{aligned} (L_{K_j}(L_{K_i}S^L))(0) &= (\nabla^L_{K_j^X}(\nabla^L_{K_i^X}S^L - 2\pi\sqrt{-1}\mu(K_i)S^L))(0) \\ &= \frac{1}{2}R^L_{y_0}(K_j^X, K_i^X)S^L(0) = \pi\sqrt{-1}\langle d\mu(K_i), K_j^X\rangle S^L(0) = 0. \end{aligned} \tag{4.22}$$

By (2.14), (4.21), (4.22) and $\mu = 0$ on P, we get

$$\begin{aligned} \frac{\partial f^L}{\partial K_i}(0)S^L(0) &= (L_{K_i}S^L)(0) = (\nabla^L_{K_i^X}S^L)(0) = 0, \\ \frac{\partial^2 f^L}{\partial K_i \partial K_j}(0)S^L(0) &= \frac{\partial^2}{\partial t_1 \partial t_2}(e^{t_1K_i + t_2K_j} \cdot S^L)(0)|_{t_1=t_2=0} \\ &= (L_{K_j}(L_{K_i}S^L) + L_{K_i}(L_{K_j}S^L))(0) = 0. \end{aligned} \tag{4.23}$$

Thus from (4.23),

$$(f^L(K))^p = (1 + \mathscr{O}(|K|^3))^p. \tag{4.24}$$

Moreover,

$$f^E(K) = \mathrm{Id}_{(\Lambda(T^{*(0,1)}X)\otimes E)_{x_0}} + \mathscr{O}(|K|), \quad \kappa_1(Z) = 1 + \mathscr{O}(|Z|^2). \tag{4.25}$$

Let dv_Y be the Riemannian volume form on (TY, g^{TY}). Observe also that if we denote by $i_{y_0} : G \to Gy_0$ the map defined by $i_{y_0}(g) = gy_0$, then

$$\frac{1}{h^2(y)}dv_Y(y) = i^{-1}_{y_0 *}dg, \tag{4.26}$$

this gives us a factor $\frac{1}{h^2(y_0)}$ when we take the integral on $\mathfrak{g}$ instead on the normal coordinate on X.

By (4.13), (4.15), (4.20), (4.24)-(4.26) and the Taylor expansion for κ_1, f^E, f^L, as in [1, Theorems 5.8, 5.9], we know that there exist $J'_r(Z)$ polynomials in Z with same parity on r, and $J'_0 = I_{\mathbb{C}\otimes E}$, such that

$$P^G_p(y_0, y_0) \sim p^n \frac{1}{h^2(y_0)} \int_{K\in\mathfrak{g}, |K|\leq\varepsilon} e^{-\frac{\pi}{2}p|K|^2} \sum_{r=0}^{\infty} J'_r(\sqrt{p}K)p^{-r/2}dK. \tag{4.27}$$

Moreover,

$$\int_{K\in\mathfrak{g}} e^{-\frac{\pi}{2}p|K|^2}dK = 2^{\frac{n_0}{2}}p^{-\frac{n_0}{2}}. \tag{4.28}$$

After taking the integral on $\mathfrak{g}$, from (4.27) and (4.28), we get (0.15) and (0.16).

By (4.7), (4.27) and (4.28), we get also the asymptotic expansion for $P_p^{\mathcal{V}}(y_0, y_0)$, $y_0 \in P$.

4.4. **Toeplitz operators on X_G.** In this Subsection, we suppose that (X, ω) is a Kähler manifold, $\mathbf{J} = J$, and L, E are holomorphic vector bundles with holomorphic Hermitian connections ∇^L, ∇^E. Let G be a compact connected Lie group acting holomorphically on X, L, E which preserves h^L and h^E.

We suppose that G acts freely on $P = \mu^{-1}(0)$. Then (X_G, ω_G) is Kähler and L_G, E_G are holomorphic on X_G.

In this case, there exists a natural isomorphism from $(\operatorname{Ker} D_p)^G$ onto $\operatorname{Ker} D_{G,p}$.

At the end of this Subsection, we will explain the corresponding result in the symplectic case, especially, for $p \gg 1$, we construct a natural isomorphism from $(\operatorname{Ker} D_p)^G$ onto $\operatorname{Ker} D_{G,p}$.

In the current situation, the spin^c Dirac operator D_p was given by (0.21) and D_p^2 preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(X, L^p \otimes E)$. Similar properties hold for $D_{G,p}$.

As in Section 2.3, let $P_{G,p}$ be the orthogonal projection from $\Omega^{0,\bullet}(X_G, L_G^p, \otimes E_G)$ onto $\operatorname{Ker} D_{G,p}$, and let $P_{G,p}(x, x')$ be the corresponding smooth kernel.

By the Kodaira vanishing theorem, for p large enough,

$$(\operatorname{Ker} D_p)^G = H^0(X, L^p \otimes E)^G, \quad \operatorname{Ker} D_{G,p} = H^0(X_G, L_G^p \otimes E_G). \tag{4.29}$$

As $D_p^2, D_{G,p}^2$ preserve the $\mathbb{Z}$-gradings of $\Omega^{0,\bullet}(X, L^p \otimes E)$, $\Omega^{0,\bullet}(X_G, L_G^p \otimes E_G)$ respectively, we only need to take care of their restrictions on $\mathscr{C}^\infty(X, L^p \otimes E)$ and $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$. In this way,

$$\begin{aligned} &P_p^G(x, x') \in \mathscr{C}^\infty(X \times X, \operatorname{pr}_1^*(L^p \otimes E) \otimes \operatorname{pr}_2^*(L^p \otimes E)^*), \\ &P_{G,p}(x_0, x_0') \in \mathscr{C}^\infty(X_G \times X_G, \operatorname{pr}_1^*(L_G^p \otimes E_G) \otimes \operatorname{pr}_2^*(L_G^p \otimes E_G)^*). \end{aligned} \tag{4.30}$$

Recall that the morphism $\sigma_p : H^0(X, L^p \otimes E)^G \to H^0(X_G, L_G^p \otimes E_G)$ was defined in (0.27). Set

$$\sigma_p^G = \sigma_p \circ P_p^G : \mathscr{C}^\infty(X, L^p \otimes E) \to H^0(X_G, L_G^p \otimes E_G). \tag{4.31}$$

Let σ_p^{G*} be the adjoint of σ_p^G with respect to the natural inner products (cf. (1.19)) on $\mathscr{C}^\infty(X, L^p \otimes E)$, $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$. Set

$$\mathcal{P}_p^{X_G} := p^{-\frac{n_0}{2}} \sigma_p^G \circ \sigma_p^{G*}. \tag{4.32}$$

Let $\{s_{p,i}\}_{i=1}^{d_p}$ be an orthonormal basis of $H^0(X, L^p \otimes E)^G$. For $y_0 \in X_G$, x, $x' \in X$, one verifies

$$\begin{aligned} &P_p^G(x, x') = \sum_{i=1}^{d_p} s_{p,i}(x) \otimes s_{p,i}(x')^*, \\ &\sigma_p^G(y_0, x) = P_p^G(y_0, x), \quad \sigma_p^{G*}(x, y_0) = P_p^G(x, y_0), \end{aligned} \tag{4.33}$$

where by $P_p^G(y_0, x)$ (resp. $P_p^G(x, y_0)$) we mean $P_p^G(y, x)$ (resp. $P_p^G(x, y)$) for any $y \in \pi_G^{-1}(y_0)$, which is well-defined by the G-invariance of P_p^G.

From (0.27), we know that $\mathcal{P}_p^{X_G}$ commutes with the operator $P_{G,p}$ and

$$\mathcal{P}_p^{X_G} = P_{G,p}\mathcal{P}_p^{X_G}P_{G,p}. \tag{4.34}$$

Let $P_p^G|_P$ be the restriction of the smooth kernel $P_p^G(x,x')$ on $P\times P$. Then

$$P_p^G|_P(x,x') \in \mathscr{C}^\infty(P\times P, \mathrm{pr}_1^*(L^p\otimes E)\otimes \mathrm{pr}_2^*(L^p\otimes E)^*)$$

is $G\times G$-invariant. By composing with π_G,

$$(\pi_G\circ P_p^G|_P)(x_0,x_0') \in \mathscr{C}^\infty(X_G\times X_G, \mathrm{pr}_1^*(L_G^p\otimes E_G)\otimes \mathrm{pr}_2^*(L_G^p\otimes E_G)^*).$$

We denote by $\pi_G\circ P_p^G|_P$ the operator defined by the smooth kernel $(\pi_G\circ P_p^G|_P)(x_0,x_0')$ and the volume form $dv_{X_G}(x_0')$. Then from (4.33), we verify that

$$\mathcal{P}_p^{X_G}(x_0,x_0') = p^{-\frac{n_0}{2}}P_p^G(x_0,x_0') = p^{-\frac{n_0}{2}}\pi_G\circ P_p^G|_P(x_0,x_0'). \tag{4.35}$$

Definition 4.3. A family of operators $T_p : H^0(X_G, L_G^p\otimes E_G)\to H^0(X_G, L_G^p\otimes E_G)$ is a Toeplitz operator if there exists a sequence of smooth sections $g_l\in\mathscr{C}^\infty(X_G,\mathrm{End}(E_G))$ with an asymptotic expansion $g(\cdot,p)$ of the form $\sum_{l=0}^\infty p^{-l}g_l(x)$ such that for any $k\in\mathbb{N}^*$, there exists $C>0$ such that for any $p\in\mathbb{N}$,

$$\|T_p - P_{G,p}\sum_{l=0}^k p^{-l}g_l(x)P_{G,p}\|^{0,0}\le Cp^{-(k+1)}. \tag{4.36}$$

Here $\|\quad\|^{0,0}$ is the operator norm with respect to the norm $\|\quad\|_{L^2}$. We call $g_0(x)$ the principal symbol of T_p. If T_p is self-adjoint, then we call T_p is a self-adjoint Toeplitz operator.

Recall that h is the fiberwise volume function defined by (0.10).

Let dg be a Haar measure on G.

The main result of this Subsection is the following result.

Theorem 4.4. *Let f be a smooth section of $\mathrm{End}(E)$ on X. Let $f^G\in\mathscr{C}^\infty(X_G,\mathrm{End}(E_G))$ be the G-invariant part of f on P defined by $f^G(x)=\int_G gf(g^{-1}x)dg$. Then $\mathcal{T}_{f,p} = p^{-\frac{n_0}{2}}\sigma_p^G f\sigma_p^{G*}$ is a Toeplitz operator with principal symbol $2^{\frac{n_0}{2}}\frac{f^G}{h^2}(x)$. In particular $\mathcal{P}_p^{X_G}$ is a Toeplitz operator with principal symbol $2^{\frac{n_0}{2}}/h^2(x)$.*

Proof. Let f^* be the adjoint of f. By writing

$$f = \frac{f+f^*}{2} + \sqrt{-1}\frac{f-f^*}{2\sqrt{-1}},$$

we may and we will assume from now on that f is self-adjoint.

We need to find a family of sections $g_l\in\mathscr{C}^\infty(X_G,\mathrm{End}(E_G))$ such that for any $m\ge 1$,

$$\mathcal{T}_{f,p} = \sum_{l=0}^m P_{G,p}\, g_l\, p^{-l}\, P_{G,p} + \mathscr{O}(p^{-m-1}). \tag{4.37}$$

Moreover, we can make these g_l's to be self-adjoint.

Let U be a G-neighborhood of $P=\mu^{-1}(0)$ as in Theorem 0.2.

Let ψ be a G-invariant function on X such that $\psi=1$ on a neighborhood of P and $\mathrm{supp}(\psi)\subset\{y\in X, d(y,P)<\varepsilon_0/2\}\cap U$.

Write

$$\sigma_p^G f \sigma_p^{G*} = \sigma_p^G \psi f \sigma_p^{G*} + \sigma_p^G (1-\psi) f \sigma_p^{G*}. \tag{4.38}$$

For $x_0, x_0' \in X_G$, let $x, x' \in P$ such that $\pi(x) = x_0$, $\pi(x') = x_0'$. By (4.33),

$$(\sigma_p^G((1-\psi)f)\sigma_p^{G*})(x_0, x_0') = \int_X P_p^G(x,y)((1-\psi)f)(y) P_p^G(y,x') dv_X(y). \tag{4.39}$$

From Theorem 0.1, (4.39) and $\mathrm{supp}((1-\psi)f) \cap P = \emptyset$, we know that for any $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ such that for any $p \in \mathbb{N}, x_0, x_0' \in X_G$,

$$|(\sigma_p^G((1-\psi)f)\sigma_p^{G*})(x_0, x_0')|_{\mathscr{C}^m(X_G \times X_G)} \leq C_{l,m} p^{-l}. \tag{4.40}$$

We define $f_B \in \mathscr{C}^\infty(B, \mathrm{End}(E_B))$ by

$$f_B(x_0) = \int_G g(\psi f)(g^{-1}x) dg \tag{4.41}$$

for $x_0 \in B, x \in U$ such that $\pi(x) = x_0$. Clearly, if $x_0 \in P$, as $\psi|_P = 1$, one gets

$$f_B(x_0) = f^G(x_0). \tag{4.42}$$

From (4.41), for $x_0,\ x_0' \in B,\ x,\ x' \in U$ such that $\pi(x) = x_0,\ \pi(x') = x_0'$, one gets

$$\begin{aligned} \sigma_p^G \psi f \sigma_p^{G*}(x_0, x_0') &= \int_U P_p^G(x,y)(\psi f)(y) P_p^G(y,x') dv_X(y) \\ &= \int_B P_p^G(x_0, y_0) f_B(y_0) P_p^G(y_0, x_0') h^2(y_0) dv_B(y_0). \end{aligned} \tag{4.43}$$

For $x_0 \in X_G$, we will work on the normal coordinate of X_G with center x_0 as in Theorem 0.2.

Recall that $P_{\mathscr{L}}(Z^0, Z'^0)$ was defined by (3.19) with $a_i = a_i^\perp = 2\pi$ therein.

By (4.39), (4.40) and (4.43), for $|Z^0|, |Z'^0| \leq \varepsilon_0/2$,

$$\mathcal{T}_{f,p}(Z^0, Z'^0) - p^{-n_0/2} \int_{\substack{|W| \leq \varepsilon_0, \\ W \in T_{x_0} B}} P_p^G(Z^0, W)(f_B h^2)(W) P_p^G(W, Z'^0) dv_B(W) = \mathscr{O}(p^{-\infty}). \tag{4.44}$$

By Theorem 0.2, (4.44) and the Taylor expansion of f_B, there exist $Q_{0,r} \in \mathrm{End}(E_{G,x_0})$ polynomials on Z^0, Z'^0 with same parity on r such that the following formula, obtained through compositions, holds,

$$\begin{aligned} &\left| p^{-n+n_0} \mathcal{T}_{f,p}(Z^0, Z'^0) - \sum_{r=0}^{k} (Q_{0,r} P_{\mathscr{L}})(\sqrt{p} Z^0, \sqrt{p} Z'^0) p^{-\frac{r}{2}} \right|_{\mathscr{C}^{m'}(X_G)} \\ &\quad \leq C p^{-(k+1)/2} (1 + \sqrt{p}|Z^0| + \sqrt{p}|Z'^0|)^M \exp(-\sqrt{C''\nu}\sqrt{p}|Z^0 - Z'^0|) + \mathscr{O}(p^{-\infty}). \end{aligned} \tag{4.45}$$

On the normal coordinate in X_G, under the trivialization induced by the parallel transport of $\nabla^{(L^p \otimes E)_G}$ along the geodesic, by [17, Theorem 4.18′] (i.e. Theorem 0.2 for $G = 1$), we get : there exist $J_r(Z^0, Z'^0) \in \mathrm{End}(E_G)_{x_0}$, polynomials in Z^0, Z'^0 with the

same parity as r, such that for any $k, m' \in \mathbb{N}$, there exist $M \in \mathbb{N}, C > 0$ such that for $x_0 \in X_G$, $Z^0, Z'^0 \in T_{x_0}X_G$, $|Z^0|, |Z'^0| \leq \varepsilon$,

$$\begin{aligned}(4.46)\quad &\left|p^{-n+n_0}P_{G,p}(Z^0,Z'^0) - \sum_{r=0}^{k}(J_rP_{\mathscr{L}})(\sqrt{p}Z^0,\sqrt{p}Z'^0)\kappa^{-\frac{1}{2}}(Z^0)\kappa^{-\frac{1}{2}}(Z'^0)p^{-\frac{r}{2}}\right|_{\mathscr{C}^{m'}(X_G)}\\ &\leq Cp^{-(k+1)/2}(1+\sqrt{p}|Z^0|+\sqrt{p}|Z'^0|)^M\exp(-\sqrt{C''\nu}\sqrt{p}|Z^0-Z'^0|)+\mathscr{O}(p^{-\infty}).\end{aligned}$$

By using the Taylor expansion of $\kappa^{-1/2}$, from (4.46), there exist $J_{0,r} \in \mathrm{End}(E_{G,x_0})$, polynomials on Z^0, Z'^0 with same parity as r, such that

$$\begin{aligned}(4.47)\quad &\left|p^{-n+n_0}P_{G,p}(Z^0,Z'^0) - \sum_{r=0}^{k}(J_{0,r}P_{\mathscr{L}})(\sqrt{p}Z^0,\sqrt{p}Z'^0)p^{-\frac{r}{2}}\right|_{\mathscr{C}^{m'}(X_G)}\\ &\leq Cp^{-(k+1)/2}(1+\sqrt{p}|Z^0|+\sqrt{p}|Z'^0|)^M\exp(-\sqrt{C''\nu}\sqrt{p}|Z^0-Z'^0|)+\mathscr{O}(p^{-\infty}).\end{aligned}$$

Moreover, by (0.13) and (4.42), for $Q_{0,0}, J_{0,0}$ in (4.45) and (4.47), we have

$$(4.48)\qquad Q_{0,0} = 2^{\frac{n_0}{2}}\frac{f^G}{h^2}(x_0), \quad J_{0,0} = \mathrm{Id}_{E_G}.$$

In what follows, all operators will be defined by their kernels with respect to $dv_{T_{x_0}X_G}$. We will add a subscript z^0 or z'^0 when we need to specify the operator acting on the variables Z^0 or Z'^0.

By Theorem 3.1, we know that

$$(4.49)\qquad b^+_{j,z^0}P_{\mathscr{L}} = 0, \quad (b_jP_{\mathscr{L}})(Z^0,Z'^0) = b_{j,z^0}P_{\mathscr{L}}(Z^0,Z'^0) = 2\pi(\overline{z}^0_j - \overline{z}'^0_j)P_{\mathscr{L}}(Z^0,Z'^0).$$

Thus for $F(Z^0,Z'^0)$ a polynomial on Z^0, Z'^0, by (3.9), Theorem 3.1, (4.49), we can replace $\overline{z}^0_j$ in F of $(FP_{\mathscr{L}})(Z^0,Z'^0)$ by the combination of b_{j,z^0} and $\overline{z}'^0_j$, thus there exist polynomials F_{α^0} ($\alpha^0 \in \mathbb{N}^{n-n_0}$) on z^0, Z'^0 (resp. $F_{\alpha^0,0}$ on z^0, $\overline{z}'^0$) such that

$$\begin{aligned}(4.50)\qquad &(FP_{\mathscr{L}})(Z^0,Z'^0) = \sum_{\alpha^0} b^{\alpha^0}_{z^0}(F_{\alpha^0}P_{\mathscr{L}})(Z^0,Z'^0),\\ &((FP_{\mathscr{L}})\circ P_{\mathscr{L}})(Z^0,Z'^0) = \sum_{\alpha^0} b^{\alpha^0}_{z^0}F_{\alpha^0,0}(z^0,\overline{z}'^0)P_{\mathscr{L}}(Z^0,Z'^0).\end{aligned}$$

In fact, by Theorem 3.1, the coefficient of $P_{\mathscr{L}}(Z^0,Z'^0)$ in the right hand side of the second equation of (4.50) is anti-holomorphic on z'^0. Moreover, by Theorem 3.1, $|\alpha^0| + \deg F_{\alpha^0}$, $|\alpha^0| + \deg F_{\alpha^0,0}$ have the same parity with the degree of F on Z^0, Z'^0. In particular, $F_{0,0}(z^0,\overline{z}'^0)$ is a polynomial on z^0, $\overline{z}'^0$ and its degree has the same parity with $\deg F$.

We will denote by

$$(4.51)\qquad F_{\mathscr{L}} := F_0, \quad F_{\mathscr{L},0} = F_{0,0}.$$

Let $(FP_{\mathscr{L}})_p$ be the operator defined by the kernel $p^{n-n_0}(FP_{\mathscr{L}})(\sqrt{p}Z^0,\sqrt{p}Z'^0)$.

By Theorem 3.1, (4.47), (4.49), there exist polynomials $H_r(F)$ on $Z^0, Z'^0 \in T_{x_0}X_G$, with the same parity with $\deg F + r$, such that we have the following asymptotic at

center x_0,

$$P_{G,p}(FP_{\mathscr{L}})_pP_{G,p}\sim\sum_{r=0}^{\infty}(H_r(F)P_{\mathscr{L}})_pp^{-r/2};\quad H_0(F)=F_{0,0}=F_{\mathscr{L},0}, \tag{4.52}$$

with the reminder term estimated in the sense of (4.45) and (4.47).

By Theorem 3.1, (4.47), (4.50) and (4.52), the coefficient of $p^{-k/2}$ in the expansion (4.52) of $\sum_{r=0}^{k}P_{G,p}(Q_{0,r}P_{\mathscr{L}})_pp^{-\frac{r}{2}}P_{G,p}$ is

$$((Q_{0,k})_{\mathscr{L},0}P_{\mathscr{L}})_p+\sum_{r=1}^{k}(H_r(Q_{0,k-r})P_{\mathscr{L}})_p. \tag{4.53}$$

Now, by (4.31),

$$\mathcal{T}_{f,p}=p^{-\frac{n_0}{2}}\sigma_p^Gf\sigma_p^{G*}=P_{G,p}\mathcal{T}_{f,p}P_{G,p}. \tag{4.54}$$

Thus by (4.45), (4.53) and (4.54), we get

$$Q_{0,k}=(Q_{0,k})_{\mathscr{L},0}+\sum_{r=0}^{k-1}H_{k-r}(Q_{0,r}). \tag{4.55}$$

By (4.47) and (4.48), for $\mathbf{f}\in\mathscr{C}^\infty(X_G,\mathrm{End}(E_G))$, there exist polynomials $G_r(\mathbf{f})(Z^0,Z'^0)$ with the same parity as r such that in the normal coordinates as above,

$$P_{G,p}\mathbf{f}P_{G,p}\sim\sum_{r=0}^{\infty}(G_r(\mathbf{f})P_{\mathscr{L}})_p\,p^{-\frac{r}{2}};\quad\text{with } G_0(\mathbf{f})(Z^0,Z'^0)=\mathbf{f}(x_0). \tag{4.56}$$

By (4.47), (4.56), $(P_{G,p})^2=P_{G,p}$ and by proceeding as in (4.55), we get

$$G_r(\mathbf{f})=\sum_{i=0}^{r}H_i(G_{r-i}(\mathbf{f}))=(G_r(\mathbf{f}))_{\mathscr{L},0}+\sum_{i=1}^{r}H_i(G_{r-i}(\mathbf{f})). \tag{4.57}$$

From (4.48), we define

$$g_0(x)=2^{\frac{n_0}{2}}\frac{f^G}{h^2}(x). \tag{4.58}$$

Assume that we have found $g_l\in\mathscr{C}^\infty(X_G,\mathrm{End}(E_G))$, $(l\leq k_0)$, self-adjoint sections such that (4.37) holds for $m=k_0$.

We claim that $Q_{0,2k_0+1}$ is determined by g_l, $(l\leq k_0)$, and there exists $g_{k_0+1}\in\mathscr{C}^\infty(X_G,\mathrm{End}(E_G))$ self-adjoint such that $Q_{0,2k_0+2}$ is determined by g_l $(l\leq k_0+1)$.

By (4.37), (4.45) and (4.56), for $0<k\leq 2k_0$,

$$Q_{0,k}=\sum_{2l+j=k}G_j(g_l). \tag{4.59}$$

Then by (4.55), (4.57) and (4.59), for $m = 2k_0 + 1$,

$$
\begin{aligned}
(4.60)\quad Q_{0,m} &= (Q_{0,m})_{\mathscr{L},0} + \sum_{r=0}^{m-1} H_{m-r}\Big(\sum_{l=0}^{[r/2]} G_{r-2l}(g_l)\Big) \\
&= (Q_{0,m})_{\mathscr{L},0} + \sum_{l=0}^{[\frac{m-1}{2}]}\sum_{r=2l}^{m-1} H_{m-r}(G_{r-2l}(g_l)) \\
&= (Q_{0,m})_{\mathscr{L},0} + \sum_{l=0}^{[\frac{m-1}{2}]}\Big(G_{m-2l}(g_l) - (G_{m-2l}(g_l))_{\mathscr{L},0}\Big).
\end{aligned}
$$

Set

$$
(4.61)\qquad \mathcal{F}_m = (Q_{0,m})_{\mathscr{L},0} - \sum_{l=0}^{[\frac{m-1}{2}]} (G_{m-2l}(g_l))_{\mathscr{L},0}.
$$

Then by (4.45), (4.52), $\mathcal{F}_m$ is a polynomial on $z^0, \overline{z}'^0$ with the same parity as m. Moreover, as $T_{f,p}$ and g_l are self-adjoint, we know that

$$
(4.62)\qquad \mathcal{F}^{(i)}_{m,x_0}(z^0, \overline{z}'^0) = (\mathcal{F}^{(i)}_{m,x_0}(z'^0, \overline{z}^0))^*.
$$

Let $\mathcal{F}^{(i)}_m$ be the degree i part of the polynomial $\mathcal{F}_m$ on $z^0, \overline{z}'^0$.

We need to prove that for $m = 2k_0 + 1$,

$$
(4.63)\qquad \mathcal{F}^{(i)}_m = 0 \quad \text{for } i > 0.
$$

Set

$$
(4.64)\qquad
\begin{aligned}
F^{(i)}_m(x_0, y_0) &= \mathcal{F}^{(i)}_{m,x_0}(0, \overline{z}'^0) \in \mathrm{End}(E_{G,x_0}), \\
\widetilde{F}^{(i)}_m(x_0, y_0) &= (F^{(i)}_m(y_0, x_0))^* \in \mathrm{End}(E_{G,y_0}),
\end{aligned}
$$

with $y_0 = \exp^{X_G}_{x_0}(Z'^0)$, they define smooth sections on a neighborhood of the diagonal of $X_G \times X_G$. Clearly, $\widetilde{F}^{(i)}_m(x_0, y_0)$'s need not be polynomials of z^0 and $\overline{z}'^0$.

Let $\psi : \mathbb{R} \to [0, 1]$ be an even function such that $\psi(u) = 1$ for $|u| \leq \varepsilon_0/4$ and 0 for $|u| > \varepsilon_0/2$.

Let $d^{X_0}(x_0, y_0)$ be the Riemannian distance on X_G.

We denote by $(\psi F^{(i)}_m P_{G,p})$, $(P_{G,p}\psi \widetilde{F}^{(i)}_m)$ the operators defined by the kernel $(\psi(d^{X_0})F^{(i)}_m P_{G,p})(x_0, y_0)$, $(P_{G,p}\psi(d^{X_0})\widetilde{F}^{(i)}_m)(x_0, y_0)$ with respect to $dv_{X_G}(y_0)$. Set

$$
(4.65)\qquad P_{p,k_0} = \mathcal{T}_{f,p} - P_{G,p}\sum_{l=0}^{k_0} g_l p^{-l} P_{G,p} - \sum_i (\psi F^{(i)}_{2k_0+1} P_{G,p})\, p^{(-2k_0+i-1)/2}.
$$

By (4.45), (4.56), (4.59) and (4.60),

$$
\begin{aligned}
(4.66)\quad &\Big| p^{-n+n_0}\Big(\mathcal{T}_{f,p} - P_{G,p}\sum_{l=0}^{k_0} g_l p^{-l} P_{G,p}\Big)(0, Z'^0) - p^{-(2k_0+1)/2}(\mathcal{F}_{2k_0+1}P_{\mathscr{L}})(0, \sqrt{p}Z'^0)\Big| \\
&\qquad \leq C p^{-k_0-1}(1 + \sqrt{p}|Z'^0|)^M \exp(-\sqrt{C''\nu}\sqrt{p}|Z'^0|) + \mathscr{O}(p^{-\infty}).
\end{aligned}
$$

Then by (4.53) and (4.66), there exist polynomials Q_{0,r,k_0} on Z^0, Z'^0 with the same parity as r such that for $k > 2k_0 + 2$, the kernel of the operator P_{p,k_0} has the expansion at the normal coordinate of x_0, as

$$\Big|p^{-n+n_0}P_{p,k_0}(0,Z'^0) - \sum_{r=2k_0+2}^{k}(Q_{0,r,k_0}P_{\mathscr{L}})(0,\sqrt{p}Z'^0)p^{-\frac{r}{2}}\Big| \leq Cp^{-(k+1)/2}(1+\sqrt{p}|Z'^0|)^M \exp(-\sqrt{C''\nu}\sqrt{p}|Z'^0|) + \mathscr{O}(p^{-\infty}). \tag{4.67}$$

We denote by Q^p_{0,r,k_0} the operator defined as in (4.65) by the kernel $Q^p_{0,r,k_0}(x_0,y_0) = p^{n-n_0}\psi(d^{X_0}(x_0,y_0))(Q_{0,r,k_0}P_{\mathscr{L}})(0,\sqrt{p}Z'^0)$.

Set

$$K_{p,k}(x_0,y) = \psi(d^{X_0}(x_0,y))P_{p,k_0}(x_0,y) - \sum_{r=2k_0+2}^{k} Q^p_{0,r,k_0}p^{-\frac{r}{2}}(x_0,y). \tag{4.68}$$

Then by (4.67),

$$|K_{p,k}(x_0,y)| \leq Cp^{n-n_0-(k+1)/2}(1+\sqrt{p}d^{X_0}(x_0,y))^M \exp(-\sqrt{C''\nu}\sqrt{p}d^{X_0}(x_0,y)) + \mathscr{O}(p^{-\infty}). \tag{4.69}$$

Thus for any $s \in \mathscr{C}^\infty(X_G, L^p_G \otimes E_G)$,

$$\begin{aligned}\|K_{p,k}s\|^2_{L^2} &\leq \int_{x_0\in X_G}\Big(\int_{y_0\in X_G}|K_{p,k}(x_0,y_0)|dv_{X_G}(y_0)\Big)\\ &\quad\times\Big(\int_{y_0\in X_G}|K_{p,k}(x_0,y_0)||s|^2(y_0)dv_{X_G}(y_0)\Big)dv_{X_G}(x_0)\\ &\leq Cp^{-(k+1)}\|s\|^2_{L^2}.\end{aligned} \tag{4.70}$$

In the same way as in (4.70),

$$\|Q^p_{0,r,k_0}s\|^2_{L^2} \leq C\|s\|^2_{L^2}. \tag{4.71}$$

Moreover, by Theorem 0.1, (4.65), we get

$$|((1-\psi(d^{X_0}))P_{p,k_0})(x_0,y_0)| = \mathscr{O}(p^{-\infty}). \tag{4.72}$$

From (4.69), (4.70), (4.71) and (4.72), we know that there exists $C > 0$ such that for any $s \in \mathscr{C}^\infty(X_G, L^p_G \otimes E_G)$,

$$\|P_{p,k_0}s\|_{L^2} \leq Cp^{-(k_0+1)}\|s\|_{L^2}. \tag{4.73}$$

Let P^*_{p,k_0} be the adjoint of P_{p,k_0}. By (4.73),

$$\|P^*_{p,k_0}s\|_{L^2} \leq Cp^{-(k_0+1)}\|s\|_{L^2}. \tag{4.74}$$

But

$$P^*_{p,k_0} = \mathcal{T}_{f,p} - P_{G,p}\sum_{l=0}^{k_0} g_l p^{-l}P_{G,p} - \sum_i (P_{G,p}\psi\widetilde{F}^{(i)}_{2k_0+1})p^{(-2k_0+i-1)/2}. \tag{4.75}$$

By (4.47) and the Taylor expansion of $\widetilde{F}^{(i)}_{2k_0+1}$ under our trivialization of E_G by using parallel transport along the path $[0,1] \ni u \to uZ'^0$, we have that in the sense of (4.66),

$$p^{-n+n_0}\sum_i (P_{G,p}\widetilde{F}^{(i)}_{2k_0+1})(0,Z'^0)p^{(-2k_0+i-1)/2} \sim \sum_{\alpha,i,r}\left(J_{0,r}P_{\mathscr{L}}\right)(0,\sqrt{p}Z'^0)\frac{\partial^\alpha \widetilde{F}^{(i)}_{2k_0+1}}{\partial (Z'^0)^\alpha}(x_0,0)\frac{(\sqrt{p}Z'^0)^\alpha}{\alpha!}p^{(-2k_0+i-|\alpha|-r-1)/2}. \tag{4.76}$$

By (4.66), (4.74), (4.75) and (4.76), we know that all coefficients of $p^{(-2k_0-1+j)/2}$ for $j>0$ of the right hand side of (4.76) should be zero. Thus we get for any $j>0$,

$$\sum_{i=j}^{\deg F_m}\sum_{|\alpha|+r=0}^{i-j} J_{0,r}(0,\sqrt{p}Z'^0)\frac{\partial^\alpha \widetilde{F}^{(i)}_{2k_0+1}}{\partial (Z'^0)^\alpha}(x_0,0)\frac{(\sqrt{p}Z'^0)^\alpha}{\alpha!}=0. \tag{4.77}$$

From (4.77), we will prove by recurrence that for any $j>0$

$$\frac{\partial^\alpha \widetilde{F}^{(i)}_{2k_0+1}}{\partial (Z'^0)^\alpha}(x_0,0)=0 \quad \text{for } i-|\alpha|\geq j>0. \tag{4.78}$$

In fact, for $j=\deg F_{2k_0+1}$ in (4.77), by (4.48), we get $\widetilde{F}^{(\deg F_{2k_0+1})}_{2k_0+1}(x_0,0)=0$, thus (4.78) holds for $j=\deg F_{2k_0+1}$.

Assume that for $j>j_0>0$, (4.78) holds. Then for $j=j_0$, the coefficient with $r>0$ in (4.77) is zero, thus by (4.48), (4.77) reads as

$$\sum_\alpha \frac{\partial^\alpha \widetilde{F}^{(j_0+|\alpha|)}_{2k_0+1}}{\partial (Z'^0)^\alpha}(x_0,0)\frac{(\sqrt{p}Z'^0)^\alpha}{\alpha!}=0. \tag{4.79}$$

From (4.79), we get (4.78) for $j=j_0$. The proof of (4.78) is complete.

By (4.74), (4.76) and (4.78), by comparing the coefficient of $p^{-(2k_0+1)/2}$ in (4.66) and (4.75), we get

$$\widetilde{F}^{(i)}_{2k_0+1}(x_0,Z'^0)=\mathcal{F}^{(i)}_{2k_0+1,x_0}(0,\overline{z}'^0)+\mathscr{O}(|Z'^0|^{i+1}). \tag{4.80}$$

Thus from (4.64) and (4.80)

$$F^{(i)}_{2k_0+1}(Z'^0,x_0)=(\mathcal{F}^{(i)}_{2k_0+1,x_0}(0,\overline{z}'^0))^*+\mathscr{O}(|Z'^0|^{i+1}). \tag{4.81}$$

Let $j_{X_G}: X_G \to X_G\times X_G$ be the diagonal injection. By (4.64),

$$\frac{\partial}{\partial z'^0_j}F^{(i)}_{2k_0+1}=0 \quad \text{near } j_{X_G}(X_G). \tag{4.82}$$

By (4.64) again and recurrence, for $\alpha\in\mathbb{N}^{n-n_0}$, if $\alpha_j>0$, by taking $\alpha'=(\alpha_0,\cdots,\alpha_j-1,\cdots,\alpha_{n-n_0})$, one has

$$\frac{\partial^\alpha}{\partial z^{0,\alpha}}F^{(i)}_{2k_0+1}(\cdot,x_0)|_{x_0}=\frac{\partial}{\partial z^0_j}j^*_{X_G}\Big(\frac{\partial^{\alpha'}}{\partial z^{0,\alpha'}}F^{(i)}_{2k_0+1}\Big)\Big|_{x_0}-\frac{\partial^{\alpha'}}{\partial z^{0,\alpha'}}\frac{\partial}{\partial z'^0_j}F^{(i)}_{2k_0+1}(\cdot,\cdot)\Big|_{0,0}=0. \tag{4.83}$$

But by (4.81), for $|\alpha|\leq i$,

$$\frac{\partial^\alpha}{\partial z^{0,\alpha}}F^{(i)}_{2k_0+1}(\cdot,x_0)|_{x_0}=\Big(\Big(\frac{\partial^\alpha}{\partial \overline{z}'^{0,\alpha}}\mathcal{F}^{(i)}_{2k_0+1,x_0}\Big)(0,\overline{z}'^0)\Big)^*. \tag{4.84}$$

From (4.83) and (4.84), the α-derivative for $|\alpha| \leq i$ of $F^{(i)}_{m,x_0}(x_0, \cdot)$ is zero at x_0. Thus

$$\mathcal{F}^{(i)}_{m,x_0}(0, \overline{z}'^0) = \mathcal{F}^{(i)}_{m,x_0}(z^0, 0) = 0. \tag{4.85}$$

Now, we consider the operator

$$\frac{1}{\sqrt{p}} P_{G,p} \left(\nabla^{L^p \otimes E}_{\varphi(d^{X_0}(x_0,y_0))(\frac{\partial}{\partial z^0_j} + \frac{\partial}{\partial \overline{z}^0_j})_{x_0}} \left(\mathcal{T}_{f,p} - \sum_{l=0}^{k_0} P_{G,p} g_l p^{-l} P_{G,p} \right) \right) P_{G,p}, \tag{4.86}$$

then the leading term of its asymptotic expansion as in (4.45) is

$$(\tfrac{\partial}{\partial z^0_j} \mathcal{F}_{2k_0+1,x_0})(\sqrt{p} z^0, \sqrt{p} \overline{z}'^0) P_{\mathscr{L}}(\sqrt{p} z^0, \sqrt{p} \overline{z}'^0) p^{-(2k_0+1)/2}. \tag{4.87}$$

Here by (4.85)), $(\frac{\partial}{\partial z^0_j} \mathcal{F}_{2k_0+1,x_0})(z^0, \overline{z}'^0)$ is an even degree polynomial on $z^0, \overline{z}'^0$, and its constant term is zero. Now, by proceeding as (4.65)–(4.84) for the operator (4.86), by (4.62), we get

$$\frac{\partial}{\partial z^0_j} \mathcal{F}^{(i)}_{2k_0+1,x_0}(z^0, \overline{z}'^0) = \frac{\partial}{\partial z'^0_j} \mathcal{F}^{(i)}_{2k_0+1,x_0}(z^0, \overline{z}'^0) = 0. \tag{4.88}$$

By continous this processus, we get (4.63).

This means that $Q_{0,2k_0+1}$ verifies also (4.59).

By the same argument, (4.60) still holds for $m = 2k_0 + 2$. Thus we can define

$$g_{k_0+1}(x_0) = \mathcal{F}^{(0)}_{2k_0+2,x_0} = \mathcal{F}^{(0)}_{2k_0+2,x_0}(0, 0). \tag{4.89}$$

By proceeding exactly the same proof as before, we get (4.63) for $m = 2k_0 + 2$. Thus for $k = 2k_0 + 2$, (4.59) still holds.

As $\pi_{G,p}, \mathcal{T}_{f,p}, g_l$ $(1 \leq l \leq k_0)$ are self-adjoint, g_{k_0+1} is also self-adjoint.

By recurrence, we know that there exist g_l's such that (4.37) holds for any m.

The proof of Theorem 4.4 is complete. □

Corollary 4.5. *For $f_1, f_2 \in \mathscr{C}^\infty(X)$, we have*

$$[\mathcal{T}_{f_1,p}, \mathcal{T}_{f_2,p}] = \frac{2^{n_0}}{\sqrt{-1}\, p} P_{G,p} \left\{ \frac{f_1^G}{h^2}, \frac{f_2^G}{h^2} \right\} P_{G,p} + \mathscr{O}(p^{-2}). \tag{4.90}$$

Here $\{\,,\,\}$ is the Poisson bracket on $(X_G, 2\pi\omega_G)$: for $g_1, g_2 \in \mathscr{C}^\infty(X_G)$, if ξ_{g_2} is the Hamiltonian vector field generated by g_2 which is defined by $2\pi i_{\xi_{g_2}} \omega_G = dg_2$, then

$$\{g_1, g_2\} = \xi_{g_2}(dg_1). \tag{4.91}$$

Proof. By applying [27] or [28, §5.5], (cf. [9] for another approach where they worked for $E = \mathbb{C}$), from Theorem 4.4, we get immediately (4.90). □

Lemma 4.6. *Let*

$$T_p = \sum_{l=0}^{\infty} P_{G,p} g_l p^{-l} P_{G,p} + \mathscr{O}(p^{-\infty}) : H^0(X_G, L^p_G \otimes E_G) \to H^0(X_G, L^p_G \otimes E_G)$$

be a Toeplitz operator with principal symbol $g_0 \in \mathscr{C}^\infty(X_G, \mathrm{End}(E_G))$. Then

i) If g_0 is invertible, then T_p^{-1} is a Toeplitz operator with principal symbol $g^{-1} \mathrm{Id}_{E_G}$.

ii) If $g_0 = g\,\mathrm{Id}_{E_G}$ with $g \in \mathscr{C}^\infty(X_G)$, $g > 0$, and T_p is self-adjoint, then $T_p^{1/2}$ is a self-adjoint Toeplitz operator with principal symbol $g^{1/2}\,\mathrm{Id}_{E_G}$.

Proof. We only prove ii), the proof of i) is similar and simpler.

As $g > 0$, there exist $C_0, C_1 > 0$ such that $C_0 < g < C_1$. Thus for any $s \in H^0(X_G, L_G^p \otimes E_G)$,

$$\langle T_p s, s\rangle = \langle g_0 s, s\rangle + \mathscr{O}\left(\frac{1}{p}\right)\|s\|^2_{L^2} \geq \left(C_0 + \mathscr{O}\left(\frac{1}{p}\right)\right)\|s\|^2_{L^2}. \tag{4.92}$$

Thus for p large enough, $T_p^{1/2} : H^0(X_G, L_G^p \otimes E_G) \to H^0(X_G, L_G^p \otimes E_G)$ is well defined.

Let δ_1 be the smooth bounded closed contour on $\{\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) > 0\}$ such that $[\frac{1}{2}C_0, 2C_1]$ is in the interior domain got by δ_1.

As in the proof of Theorem 4.4, by recurrence, we will find $f_l \in \mathscr{C}^\infty(X_G, \mathrm{End}(E_G))$ such that

$$T_p = (T_{m,p})^2 + \mathscr{O}(p^{-m-1}) \quad \text{with} \quad T_{m,p} = \sum_{l=0}^{m} P_{G,p} f_l p^{-l} P_{G,p}. \tag{4.93}$$

Then for p large enough,

$$\begin{aligned} T_p^{1/2} - T_{m,p} &= \frac{1}{2\pi i}\int_{\lambda\in\delta_1} \lambda^{1/2}\Big[(\lambda - T_p)^{-1} - (\lambda - (T_{m,p})^2)^{-1}\Big]d\lambda \\ &= \frac{1}{2\pi i}\int_{\lambda\in\delta_1} \lambda^{1/2}(\lambda - T_p)^{-1}(T_p - (T_{m,p})^2)(\lambda - (T_{m,p})^2)^{-1}d\lambda. \end{aligned} \tag{4.94}$$

If (4.93) holds, then by (4.94) we know that in the sense of the operator norm,

$$T_p^{1/2} - T_{m,p} = \mathscr{O}(p^{-m-1}). \tag{4.95}$$

To complete the proof of Lemma 4.6, it remains to establish (4.93).

By (4.47), there exist $Q_{0,r} \in \mathrm{End}(E_G)_{x_0}$ such that in the sense of (4.45), (4.47) and (4.52),

$$T_p \sim \sum_{r=0}^{\infty} (Q_{0,r} P_{\mathscr{L}})_p p^{-r/2}. \tag{4.96}$$

We will prove by recurrence that there exist $f_l \in \mathscr{C}^\infty(X_G, \mathrm{End}(E_G))$ self-adjoint such that for any $k \in \mathbb{N}$,

$$\begin{aligned} &\Big|p^{-n+n_0}(T_p - (T_{k,p})^2)(\sqrt{p}Z^0, \sqrt{p}Z'^0)\Big| \\ &\qquad \leq p^{-(2k+1)/2}(1 + \sqrt{p}|Z^0| + \sqrt{p}|Z'^0|)^M \exp(-\sqrt{C''\nu}\sqrt{p}|Z^0 - Z'^0|) + \mathscr{O}(p^{-\infty}). \end{aligned} \tag{4.97}$$

Set $f_0 = g^{1/2}\,\mathrm{Id}_{E_G}$. Then (4.93) is verified for $m = 0$.

Assume that for $k \leq m$, we have found f_l such that (4.93) holds. If we denote the expansion of $(T_{m,p})^2$ in the sense of (4.47),

$$(T_{m,p})^2 \sim \sum_{r=0}^{\infty} (\widetilde{Q}^m_{0,r} P_{\mathscr{L}})_p p^{-r/2}. \tag{4.98}$$

Then by the proof of (4.59) for $2k_0+1$,

$$\widetilde{Q}^m_{0,2m+1} = \sum_{2l+r=2m+1} G_r(g_l) = Q_{0,2m+1}. \tag{4.99}$$

Thus by (4.99), (4.97) stills holds when we replace the factor $p^{-(2m+1)/2}$ by p^{-m-1} at the right hand side of (4.97). Thus

$$T_p - (T_{k,p})^2 \sim \sum_{r=2m+2}^{\infty} ((Q_{0,r} - \widetilde{Q}^m_{0,r})P_{\mathscr{L}})_p p^{-r/2}. \tag{4.100}$$

By (4.52), (4.100), we know that

$$Q_{0,2m+2} - \widetilde{Q}^m_{0,2m+2} = (Q_{0,2m+2} - \widetilde{Q}^m_{0,2m+2})_{\mathscr{L},0}. \tag{4.101}$$

This means that $Q_{0,2m+2} - \widetilde{Q}^m_{0,2m+2}$ is a polynomial on $z^0, \overline{z}'^0$ with even degree.

Set

$$f_{m+1}(x_0) = -\frac{1}{2} g^{-1/2} (Q_{0,2m+2} - \widetilde{Q}^m_{0,2m+2})(0,0). \tag{4.102}$$

Then by the proof of (4.59), for $2k_0+2$, we know that the polynomial $Q_{0,2m+2} - \widetilde{Q}^m_{0,2m+2}$ equals to the constant $-2g^{1/2} f_{m+1}$. Thus we prove (4.97) for $k = m+1$.

By the above argument, we have established (4.93), thus Lemma 4.6. □

Since the isomorphism $\sigma_p : H^0(X, L^p \otimes E)^G \to H^0(X_G, L^p_G \otimes E_G)$ is not an isometry, we define the associated unitary operator,

$$\Sigma_p = \sigma_p^{G*} (\sigma_p^G \circ \sigma_p^{G*})^{-1/2} : H^0(X_G, L^p_G \otimes E_G) \to H^0(X, L^p \otimes E)^G. \tag{4.103}$$

Theorem 4.7. *Let f be a $\mathscr{C}^\infty$ section of* $\mathrm{End}(E)$ *on X. Then*

$$T^G_{f,p} = \Sigma_p^* f \Sigma_p : H^0(X_G, L^p_G \otimes E_G) \to H^0(X_G, L^p_G \otimes E_G) \tag{4.104}$$

is a Toeplitz operator on X_G. Its principal symbol is $f^G \in \mathscr{C}^\infty(X_G, \mathrm{End}(E_G))$.

Proof. By (4.34) and (4.103),

$$T^G_{f,p} = (\mathcal{P}^{X_G}_p)^{-\frac{1}{2}} \mathcal{T}_{f,p} (\mathcal{P}^{X_G}_p)^{-\frac{1}{2}}. \tag{4.105}$$

By Theorem 4.4, (4.32), $\mathcal{P}^{X_G}_p = p^{-\frac{n_0}{2}} \sigma^G_p \circ \sigma^{G*}_p$, $\mathcal{T}_{f,p}$ are Toeplitz operators on X_G with principal symbols $2^{n_0/2}/h^2(x)$, $2^{n_0/2} \frac{f^G}{h^2}(x)$ respectively.

By Lemma 4.6, we know that $(\mathcal{P}^{X_G}_p)^{-\frac{1}{2}}$ is a Toeplitz operator on X_G.

By (4.105), $T^G_{f,p}$ has the expansion as (4.45). By the proof of Theorem 4.4, we then know that $T^G_{f,p}$ is a Toeplitz operator. □

Remark 4.8. i) Certainly, by combining the argument here and Section 4.1, we can get the corresponding version when X_G is an orbifold.

ii) When $E = \mathbb{C}$, and $f = 1$, from Theorem 4.4, $\mathcal{P}^{X_G}_p$ is an elliptic (i.e. its principal symbol is invertible) Toeplitz operator. This is the analytic core result claimed in [34, §8].

iii) When $E = \mathbb{C}$ and G is the torus $\mathbb{T}^{n_0}$, Theorem 4.7 is one of the main results of Charles [15, Theorem 1.2], and in [15, §5.6], he knew also that $\mathcal{P}_p^{X_G}$ is an elliptic Toeplitz operator. Moreover, he established the corresponding version when X_G is an orbifold.

If X is only symplectic and $\mathbf{J} = J$, then as the argument in [40, §3e)], J induces an almost complex structure J_G on $(TX)_B$, and J_G preserves $N_{G,J} = N_G \oplus J_G N_G$ and TX_G. Thus one can construct canonically the Hermitian vector bundles $N_{G,J}^{(1,0)}$ etc, which further gives the canonical identification of Hermitian vector bundles

$$\Lambda(T^{*(0,1)}X)_B|_{X_G} = \Lambda(N_{G,J}^{*(0,1)})\widehat{\otimes}\Lambda(T^{*(0,1)}X_G). \tag{4.106}$$

Let q be the canonical orthogonal projection

$$q : \Lambda(N_{G,J}^{*(0,1)})\widehat{\otimes}\Lambda(T^{*(0,1)}X_G) \otimes L_G^p \otimes E_G \to \Lambda(T^{*(0,1)}X_G) \otimes L_G^p \otimes E_G \tag{4.107}$$

which acts as identity on $\Lambda(T^{*(0,1)}X_G)\otimes L_G^p\otimes E_G$ and maps each $\Lambda^i(N_{G,J}^{*(0,1)})\widehat{\otimes}\Lambda(T^{*(0,1)}X_G)$ $\otimes L_G^p \otimes E_G$, $i \geq 1$, to zero.

We define

$$\sigma_p := P_{G,p} q \pi_G i^* P_p^G : (\operatorname{Ker} D_p)^G \to \operatorname{Ker} D_{G,p}. \tag{4.108}$$

Certainly in the Kähler case, σ_p coincides with (0.27).

By using Theorems 0.1, 0.2 as in the proof of Theorem 4.4 (cf. [27], [28, §5.5] for more details on the Toeplitz operators in the symplectic setting), we get

Theorem 4.9. *Let f be a smooth section of $\operatorname{End}(E)$ on X, then $\mathcal{T}_{f,p} = \sigma_p f \sigma_p^* : \operatorname{Ker} D_{G,p} \to \operatorname{Ker} D_{G,p}$ is a Toeplitz operator with principal symbol $2^{n_0/2}\frac{f^G}{h^2}(x) I_{\mathbb{C}\otimes E_G} \in \operatorname{End}(\Lambda(T^{*(0,1)}X_G) \otimes E_G)$.*

Corollary 4.10. *For p large enough, σ_p in (4.108) is an isomorphism. Thus σ_p defines a natural identification for 'quantization commutes with reduction' in the (asymptotic) symplectic case.*

Proof. From Theorem 4.9 for $f = 1$, we get

$$\sigma_p \sigma_p^* = 2^{n_0/2} P_{G,p} h^{-2} I_{\mathbb{C}\otimes E_G} P_{G,p} + \mathscr{O}(\frac{1}{p}). \tag{4.109}$$

But from the argument as (4.71) and Theorem 0.2 for $G = 1$, we get for any $s \in \Omega^{0,\bullet}(X_G, L_G^p \otimes E_G)$, we have

$$\|(I_{\mathbb{C}\otimes E_G} P_{G,p} - P_{G,p})s\|_{L^2} \leq \frac{C}{\sqrt{p}} \|s\|_{L^2}. \tag{4.110}$$

Thus for p large enough, $\sigma_p \sigma_p^*$ is an isomorphism. Thus σ_p is surjective.

In view of (0.6), σ_p in (4.108) is an isomorphism. □

Remark 4.11. If we replace the condition $\mathbf{J} = J$ by (3.2), then the canonical map σ_p in (4.108) is still well defined. From the argument here, we still know that σ_p is an isomorphism for p large enough,.

4.5. **Generalization to non-compact manifolds.** In this Subsection, let (X,ω) be a symplectic manifold, and (L,∇^L) (resp. (E,∇^E)) be Hermitian line (vector) bundle on X, and the compact connected Lie group G acts on X as in Introduction, especially, $\omega=\frac{\sqrt{-1}}{2\pi}R^L$. But we only suppose that (X,g^{TX}) is a complete manifold.

If $G=1$, these kind results were studied in [26, §3].

By the argument in Section 2.3, if the square of the spin^c Dirac operator D_p^2 has a spectral gap as in (2.13), then we can localize our problem and get a version of Theorems 0.1, 0.2 from Section 2.6. In particular, if the geometric data on X verify the bounded geometry, then D_p^2 verify the spectral gap (2.13).

We explain in more detail now.

We suppose

i) The tensors $R^E, r^X, \mathrm{Tr}[R^{T^{(1,0)}X}]$ are uniformly bounded with respect on (X,g^{TX}).

ii) There exists $c>0$ such that

$$\sqrt{-1}R^L(\cdot,J\cdot)\geq cg^{TX}(\cdot,\cdot). \tag{4.111}$$

Remark 4.12. For the operator $D_p=\sqrt{2}(\overline{\partial}^{L^p\otimes E}+\overline{\partial}^{L^p\otimes E,*})$ in the holomorphic case, the above condition i) can be replaced by

i') The tensors $R^E, R^{T^{(1,0)}X}, \partial T$ is uniformly bounded with respect to (X,g^{TX}), here T is the torsion of (X,ω) as in [26, §3.5].

Then by the argument in [25, p. 656] (cf. [26, §3]), we know that Theorem 2.2 still holds. Thus Theorem 2.5 still holds.

Let P_p^G be the orthogonal projection from $L^2(X,E_p)$ onto $(\mathrm{Ker}\,D_p)^G$, and $P_p^G(x,x')$ $(x,x\in X)$ be its kernel as in Def. 2.3.

Note that $\mathrm{Ker}\,D_p$ and $(\mathrm{Ker}\,D_p)^G$ need not be finite dimensional.

By the proof of Prop. 2.6, we know that for any compact set $K\subset X$, $l,m\in\mathbb{N}$, there exist $C_{l,m}(K)>0$ such that for $p\geq C_L/\nu$,

$$|\widetilde{F}(\mathcal{L}_p)(x,x')-P_p^G(x,x')|_{\mathscr{C}^m(K\times K)}\leq C_{l,m}(K)p^{-l}. \tag{4.112}$$

By the proof of Theorem 0.1, we get

Theorem 4.13. *For any compact set $K\subset X$, $0<\varepsilon_0\leq\delta_0$, $l,m\in\mathbb{N}$, there exists $C_{l,m}>0$ (depend on K, ε) such that for $p\geq 1$, $x,x'\in K, d^X(Gx,x')\geq\varepsilon_0$ or $x,x'\in(X\setminus X_{2\varepsilon_0})\cap K$,*

$$|P_p^G(x,x')|_{\mathscr{C}^m}\leq C_{l,m}p^{-l}. \tag{4.113}$$

From Section 2.6, we get Theorem 0.2, but now the norm $\mathscr{C}^{m'}(X_G)$ in (0.14) should be replaced by $\mathscr{C}^{m'}(K)$ for any compact set $K\subset X_G$.

One interesting case of the above discussion is when $P=\mu^{-1}(0)$ is compact, by the same argument as in Theorems 4.4, 4.9, we can prove a version of Section 4.4.

In fact, when $X=\mathbb{C}^n, G=\mathbb{T}^{n_0}$, L is the trivial line bundle with the metric $|1|_{h^L}(Z)=e^{-|z|^2}$, the Toeplitz operator type properties was studied by Charles [15].

Another interesting case is a version of Theorem 0.2 for covering manifolds.

Let $\widetilde{X}$ be a para-compact smooth manifold, such that there is a discrete group Γ acting freely on $\widetilde{X}$ with a compact quotient $X = \widetilde{X}/\Gamma$.

Let $\pi_\Gamma : \widetilde{X} \longrightarrow X$ be the projection. Assume that all above geometric data on X can be lift on $\widetilde{X}$. We denote by $\widetilde{\mathbf{J}}$, $g^{T\widetilde{X}}$, $\widetilde{\omega}$, $\widetilde{J}$, $\widetilde{L}$, $\widetilde{E}$ the pull-back of the corresponding objects in Section 0 by the projection $\pi_\Gamma : \widetilde{X} \to X$, moreover, we assume that the G-action and the Γ-action on them commute.

By the above arguments (cf. [25, Theorems 4.4 and 4.6]), there exists a spectral gap for the square of the spin^c Dirac operator $\widetilde{D}_p$ on $\widetilde{X}$.

By the finite propagation speed of solutions of hyperbolic equations (2.75), we get an extension of [26, Theorem 3.13] where $G = 1$.

Theorem 4.14. *We fix $0 < \varepsilon_0 < \inf_{x\in X}\{$injectivity radius of $x\}$. For any compact set $K \subset \widetilde{X}$ and $k, l \in \mathbb{N}$, there exists $C_{k,l,K} > 0$ such that for $x, x' \in K$, $p \in \mathbb{N}$,*

$$\begin{aligned} &\left|\widetilde{P}_p^G(x,x') - P_p^G(\pi_\Gamma(x),\pi_\Gamma(x'))\right|_{\mathscr{C}^l(K\times K)} \leqslant C_{k,l,K}\, p^{-k-1}, \quad \text{if } d^X(x,x') < \varepsilon_0, \\ &\left|\widetilde{P}_p^G(x,x')\right|_{\mathscr{C}^l(K\times K)} \leqslant C_{k,l,K}\, p^{-k-1}, \quad \text{if } d^X(x,x') \geqslant \varepsilon_0. \end{aligned} \tag{4.114}$$

Especially, $\widetilde{P}_p^G(x,x)$ has the same asymptotic expansion as $P_p^G(\pi_\Gamma(x),\pi_\Gamma(x))$ in Corollary 0.4 on any compact set $K \subset \widetilde{X}$.

4.6. **Relation on the Bergman kernel on X_G.** From (2.60), if the operator $\Phi\mathcal{L}_p\Phi^{-1}$ has the form $D_{G,p}^2 + \Delta_N + 4\pi|\mu|^2 - 2\pi n_0$ under the splitting (4.106), then we will find the full asymptotic expansion of the Bergman kernel on X_G from $P_p^G(x,x')$.

In this Subsection, we suppose that X is compact and G is a torus $\mathbb{T}^{n_0} = \mathbb{R}^{n_0}/\mathbb{Z}^{n_0}$.

Let $\theta : TP \to \mathfrak{g}$ be a connection form for the G-principal bundle $\pi : P = \mu^{-1}(0) \to X_G$ with curvature Θ. Let $T^HP = \operatorname{Ker}\theta \subset TP$.

Set $M = P \times \mathfrak{g}^*$, $\mathbf{q} : M \to \mathfrak{g}^*$ be the natural projection and

$$\omega^M = \pi^*\omega_G + d\langle \mathbf{q}, \theta\rangle = \pi^*\omega_G + \langle \mathbf{q}, \Theta\rangle + \langle d\mathbf{q}, \theta\rangle. \tag{4.115}$$

By the normal crossing formula [21, Prop. 40.1], we know there exists a symplectic diffeomorphism such that on a neighborhood U of P,

$$\Psi_{sym} : (X,\omega) \simeq (M,\omega^M), \tag{4.116}$$

and under this identification, the moment map μ (cf. (2.14)) is defined by $-\mathbf{q}$.

From now on, we use this neighborhood of P and we will choose metrics and connections.

Let $g^{\mathfrak{g}}$ be the metric on $\mathfrak{g}$ induced by the canonical flat metric on $\mathbb{R}^{n_0}$, and $\{K_i\}$ be the canonical unitary basis of $\mathbb{R}^{n_0}$.

Now we choose J an almost -complex structure on TX compatible with ω such that on T^HP on U, J is induced by an almost-complex structure on TX_G which is compatible with ω_G, and on $\mathfrak{g} \oplus \mathfrak{g}^*$, for $K \in \mathfrak{g}$, $JK \in \mathfrak{g}^*$ is defined by $(JK, K') = \langle K, K'\rangle_{\mathfrak{g}}$ for $K' \in \mathfrak{g}$.

We also suppose Θ is J-invariant.

Let g^{TX} be a J-invariant metric on TX such that

$$g^{TX} = \pi^* g^{TX_G} \oplus g^{\mathfrak{g}} \oplus g^{\mathfrak{g}^*} \quad \text{on } U. \tag{4.117}$$

As $g^{\mathfrak{g}}$ is a constant metric on $TY = \mathfrak{g}$, ∇^{TY} is the trivial connection on TY. By (1.7), on U,

$$\nabla^{TP}_{U_1^H} = \nabla^{TX_G}_{U_1^H} + \nabla^{TY}_{U_1^H} + S(U_1^H). \tag{4.118}$$

Let $\nabla^{\Lambda(N^{*(0,1)}_{G,J})}$ be the trivial connection on the trivial bundle $\Lambda(N^{*(0,1)}_{G,J})$ (cf. (4.106)) on U, and $\nabla^{\mathrm{Cliff}_{X_G}}$ be the Clifford connection on $\Lambda(T^{*(0,1)}X_G)$.

By (4.118), under the identification (4.106), on U, we have

$$\begin{aligned}\nabla^{\mathrm{Cliff}}_{e_i^H} &= \nabla^{\mathrm{Cliff}_{X_G}}_{e_i^H} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\Lambda(N^{*(0,1)}_{G,J})}_{e_i^H} + \frac{1}{2}\langle S(e_i^H)e_j^H, K_l\rangle c(e_j^H)c(K_l) \\ &= \nabla^{\mathrm{Cliff}_{X_G}}_{e_i^H} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\Lambda(N^{*(0,1)}_{G,J})}_{e_i^H} + \frac{1}{4}\langle \Theta(e_i, e_j), K_l\rangle c(e_j^H)c(K_l).\end{aligned} \tag{4.119}$$

However, the last term does not preserve $\Lambda(T^{*(0,1)}X_G)$ and $\Lambda(N^{*(0,1)}_{G,J})$.

From (2.60) and (4.119), in general, $\Phi\mathcal{L}_p\Phi^{-1}$ will not preserve $\Lambda(T^{*(0,1)}X_G)$ and $\Lambda(N^{*(0,1)}_{G,J})$ if Θ is not null.

Now, we suppose that $\Theta = 0$ on X_G.

In this situation, on $B = U/G \subset X_G \times \mathfrak{g}^*$, by (2.60), we have

$$\Phi\mathcal{L}_p\Phi^{-1} = D^2_{G,p} - \sum_l (\nabla^{\Lambda(N^{*(0,1)}_{G,J})}_{K_l})^2 + 4\pi^2|\mathfrak{q}|^2 - 2n_0\pi. \tag{4.120}$$

By Theorem 0.2, Section 3.2 and (3.19), we know that the asymptotic expansion of the Bergman kernel has the following relation for $(x, Z^\perp) \in N_{G,x}$, $(x', Z'^\perp) \in N_{G,x'}$,

$$P^G_p((x, Z^\perp), (x', Z'^\perp)) = P_{G,p}(x, x')P_{\mathscr{L}}(Z^\perp, Z'^\perp) + \mathcal{O}(p^{-\infty}). \tag{4.121}$$

5. Computing the coefficient Φ_1 and $P^{(2)}(0,0)$

In this Section, (X,ω,J) is a compact Kähler manifold, g^{TX} is a G-invariant Riemannian metric on TX which is compatible with J. (E,h^E), (L,h^L) are holomorphic Hermitian vector bundles on X, and ∇^E,∇^L are the holomorphic Hermitian connections on (E,h^E), (L,h^L). Moreover,

$$\frac{\sqrt{-1}}{2\pi}R^L=\omega.$$

The action of G is holomorphic and G acts freely on $P=\mu^{-1}(0)$. Thus (X_G,ω_G,J_G) is a compact Kähler manifold.

In Sections 5.1-5.5, we suppose that in (0.2), $\mathbf{J}=J$ on a G-neighborhood U of $P=\mu^{-1}(0)$.

The main purpose here is to compute the coefficient Φ_1 in (0.20) and $P^{(2)}(0,0)$ in (0.16).

By (0.19) (cf. also Theorem 2.23),

$$\Phi_1(x_0)=\int_{Z\in N_{G,x_0}}P^{(2)}_{x_0}(Z,Z)dv_{N_G}(Z). \tag{5.1}$$

We will first compute explicitly the terms $\mathcal{O}_1$ and $\mathcal{O}_2$ involved in $P^{(2)}$ in (3.32), (3.62), and then compute the integration of $P^{(2)}$ along the normal spaces to X_G.

Sometimes the computations seem to be long and tedious, involving many subtle relations between metrics, connections and curvatures near X_G, but fortunately the final result on Φ_1 is still of a simple form, as expected.

Throughout the computations below, a key idea is to rewrite all operators by using the creation and annihilation operators b_i,b_i^+, $b_j^\perp,b_j^{\perp+}$, then under the help of (3.9) and Theorem 3.1, we can do the operations and to obtain the crucial Lemmas 5.9, 5.11.

To get the final simple formula (0.25), we still need to prove a highly non-trivial identity (5.130).

The formula for $P^{(2)}(0,0)$ in Theorem 0.7 is quite complicate, it involves h, the volume function of the orbit and the curvature for the principal bundle $P\to X_G$.

This Section is organized as follows. In Section 5.1, we explain various relations of the curvature of the fibration $P\to X_G$ and the second fundamental form of P. In Section 5.2, we obtain the explicit formulas for the operators $\mathcal{O}_1$, $\mathcal{O}_2$. In Section 5.3, we apply the formulas in Section 5.2 and (5.1) to (3.31), and we get a formula for the coefficient Φ_1. In Section 5.4, we compute finally Φ_1, thus prove Theorem 0.6. In Section 5.5, we compute $P^{(2)}(0,0)$ in Theorem 0.7. In Section 5.6, we explain how to reduce the general case to the case $\mathbf{J}=J$ which has been worked out in Sections 5.1-5.4.

In the whole Section, if there is no other specific notification, when we meet the operation $|\quad|^2$, we will first do this operation, then take the sum of the indices.

5.1. The second fundamental form of P.

We use the notations in Sections 2.2, 2.3. Then the normal bundle N_G of X_G in U/G is $(JTY)_G$.

Let $\iota:X_G\to U/G$ be the natural embedding.

We will apply the notation in Section 1.1 to $B=U/G$.

Let ∇^{TX_G}, ∇^{N_G} be connections on TX_G, N_G on X_G induced by projections of $\nabla^{TB}|_{X_G}$. Then ∇^{TX_G} is the Levi-Civita connection on (TX_G, g^{TX_G}).

Let

$$ {}^0\nabla^{TB} = \nabla^{TX_G} \oplus \nabla^{N_G} \tag{5.2}$$

be the connection on TB on X_G induced by ∇^{TX_G}, ∇^{N_G} with curvature ${}^0R^{TB}$.

Set

$$ A = \nabla^{TB}|_{X_G} - {}^0\nabla^{TB}. \tag{5.3}$$

Then A is a 1-form on X_G taking values in the skew-adjoint endomorphisms of $(TB)|_{X_G}$ which exchange TX_G and N_G.

We recall the following properties of R^{TB}: for $U, V, W, W_2 \in TB$,

$$ \begin{aligned} &\left\langle R^{TB}(U,V)W, W_2\right\rangle = \left\langle R^{TB}(W,W_2)U, V\right\rangle, \\ &R^{TB}(U,V)W + R^{TB}(V,W)U + R^{TB}(W,U)V = 0. \end{aligned} \tag{5.4}$$

On X_G, let $\{e_i^0\}$ be an orthonormal frame of TX_G, let $\{e_j^\perp\}$ be an orthonormal frame of N_G, then $\{e_i\} = \{e_i^0, e_j^\perp\}$ is an orthonormal frame of TB.

The following result gives detail informations on the torsion T of the fibration, as well as the second fundamental form A.

Theorem 5.1. *On P, the restriction of the tensor $\langle JT(\cdot, J\cdot)\cdot\rangle$ on $(N_G)^{\otimes 3}$ is symmetric, and*

$$ (A(e_i^0)e_j^0)^H = \frac{1}{2} JT(e_i^{0,H}, Je_j^{0,H}), \tag{5.5a}$$

$$ T(e_i^{0,H}, e_j^{0,H}) = T((J_G e_i^0)^H, (J_G e_j^0)^H), \tag{5.5b}$$

$$ T(e_i^{0,H}, e_j^{\perp,H}) = 2T((J_G e_i^0)^H, Je_j^{\perp,H}), \tag{5.5c}$$

$$ \left\langle T(e_i^{0,H}, e_j^{\perp,H}), Je_k^{\perp,H}\right\rangle = \left\langle T(e_i^{0,H}, e_k^{\perp,H}), Je_j^{\perp,H}\right\rangle, \tag{5.5d}$$

$$ \sum_k \left\langle T(e_k^{\perp,H}, e_j^{\perp,H}), Je_k^{\perp,H}\right\rangle = 0. \tag{5.5e}$$

Proof. Observe first that we have

$$ \nabla^{TX} J = 0; \tag{5.6a}$$

$$ (J_G e_i^0)^H = Je_i^{0,H} \quad \text{on} \quad P. \tag{5.6b}$$

Let Z be a smooth section of TY, then $JZ \in JTY \subset T^H X$ on P, by (1.7), (3.1) and (5.6a), on P, we have

$$ \begin{aligned} \left\langle J(A(e_i^0)e_j^0)^H, Z\right\rangle &= -\left\langle \nabla^{T^H X}_{e_i^{0,H}} e_j^{0,H}, JZ\right\rangle = -\left\langle \nabla^{TX}_{e_i^{0,H}} e_j^{0,H}, JZ\right\rangle \\ &= \left\langle \nabla^{TX}_{e_i^{0,H}} (Je_j^{0,H}), Z\right\rangle = \left\langle S(e_i^{0,H}) Je_j^{0,H}, Z\right\rangle = -\frac{1}{2}\left\langle T(e_i^{0,H}, Je_j^{0,H}), Z\right\rangle. \end{aligned} \tag{5.7}$$

Thus we get (5.5a), as $A(e_i^0)e_j^0 \in N_G$.

Note that $[Z, e_i^H] \in TY$, by (1.5), (1.7) and (5.6a),

$$\langle T(e_i^H, e_j^H), Z\rangle = 2\left\langle \nabla^{TX}_{e_i^H} Z, e_j^H\right\rangle = 2\left\langle \nabla^{TX}_Z e_i^H, e_j^H\right\rangle = 2\left\langle \nabla^{TX}_Z (Je_i^H), Je_j^H\right\rangle. \tag{5.8}$$

From (5.6b) and (5.8), we get (5.5b).

From (1.7), (5.8) and $Je_j^{\perp,H}, Je_k^{\perp,H} \in TY$ on P, we get

$$\left\langle T(e_i^{0,H}, e_j^{\perp,H}), Z\right\rangle = 2\left\langle S(Z)(Je_i^{0,H}), Je_j^{\perp,H}\right\rangle = 2\left\langle T(Je_i^{0,H}, Je_j^{\perp,H}), Z\right\rangle. \tag{5.9}$$

Thus we get (5.5c). By (1.6), (5.9), we get

$$\left\langle T(e_i^{0,H}, e_j^{\perp,H}), Je_k^{\perp,H}\right\rangle = 2\left\langle T(Je_i^{0,H}, Je_j^{\perp,H}), Je_k^{\perp,H}\right\rangle = \left\langle T(e_i^{0,H}, e_k^{\perp,H}), Je_j^{\perp,H}\right\rangle. \tag{5.10}$$

Thus we get (5.5d). By (1.7), (5.6a) and $Je_j^{\perp,H} \in TY$ on P,

$$\begin{aligned}\left\langle T(e_i^{\perp,H}, Je_j^{\perp,H}), Je_k^{\perp,H}\right\rangle &= \left\langle \nabla^{TX}_{Je_k^{\perp,H}} e_i^{\perp,H}, Je_j^{\perp,H}\right\rangle \\ &= -\left\langle \nabla^{TX}_{Je_k^{\perp,H}} (Je_i^{\perp,H}), e_j^{\perp,H}\right\rangle = \left\langle \nabla^{TX}_{Je_k^{\perp,H}} e_j^{\perp,H}, Je_i^{\perp,H}\right\rangle = \left\langle T(e_j^{\perp,H}, Je_i^{\perp,H}), Je_k^{\perp,H}\right\rangle.\end{aligned} \tag{5.11}$$

By (1.6) and (5.11), $\langle JT(\cdot, J\cdot), \cdot\rangle$ is symmetric on the horizontal lift of $N_G^{\otimes 3}$.

Note that $\{Je_k^{\perp,H}\}$ is a G-invariant orthonormal frame of TY on P, by (5.8),

$$\left\langle T(e_i^{\perp,H}, e_j^{\perp,H}), Je_k^{\perp,H}\right\rangle = 2\left\langle \nabla^{TY}_{Je_k^{\perp,H}} (Je_i^{\perp,H}), Je_j^{\perp,H}\right\rangle. \tag{5.12}$$

By (1.9) and (5.12), we get (5.5e). The proof of Theorem 5.1 is complete. □

Remark 5.2. From (1.6) and (5.5b), $\Theta|_{X_G}$ is a $(1,1)$-form on X_G. Especially, for any complex representation V of G, $P\times_G V$ is a holomorphic vector bundle on X_G. Moreover, by (5.5a), for $U \in TX_G, V \in N_G$, we have at x_0,

$$A(U)V = \left\langle A(U)V, e_j^0\right\rangle e_j^0 = -\left\langle V, A(U)e_j^0\right\rangle e_j^0 = \frac{1}{2}\left\langle T(U, Je_j^0), JV\right\rangle e_j^0. \tag{5.13}$$

For $x_0 \in X_G$, if $\{e_j^\perp\}$ is a fixed orthonormal basis of N_{G,x_0} as above, then for $U \in T_{x_0}X_G$, we will denote by

$$\mathcal{T}_{ijk} = \left\langle JT(e_i^\perp, Je_j^\perp), e_k^\perp\right\rangle,\ \widetilde{\mathcal{T}}_{ijk} = \left\langle JT(e_i^\perp, e_j^\perp), e_k^\perp\right\rangle,\ \mathcal{T}_{jk}(U) = \left\langle JT(U, e_j^\perp), e_k^\perp\right\rangle. \tag{5.14}$$

By Theorem 5.1, $\mathcal{T}_{ijk}$ is symmetric on i, j, k and $\mathcal{T}_{jk} \in T^*_{x_0}X_G$ is symmetric on j, k, $\widetilde{\mathcal{T}}_{ijk}$ is anti-symmetric on i, j.

Remark 5.3. From Remark 1.2 and (5.12), we know that $\langle JT(\cdot,\cdot),\cdot\rangle$ is anti-symmetric on $(N_G)^{\otimes 3}$ if g^{TY} is induced by a family of Ad-invariant metric on $\mathfrak{g}$. If G is abelian, then by (1.12), (5.12), $T(\cdot,\cdot) = 0$ on $(N_G)^{\otimes 2}$, thus $\widetilde{\mathcal{T}}_{ijk} = 0$.

5.2. **Operators $\mathcal{O}_1$, $\mathcal{O}_2$ in (2.103).** We use the notation in Sections 2.6, 3.1, and all tensors will be evaluated at $x_0 \in X_G$.

Recall that (X, ω) is Kähler and $\mathbf{J} = J$ on a G-neighborhood U of $P = \mu^{-1}(0)$, then in (3.5)

$$a_i = a_j^{\perp} = 2\pi. \tag{5.15}$$

Clearly, on U, the Levi-Civita connection ∇^{TX} preserves $T^{(1,0)}X$ and $T^{(0,1)}X$, and $\nabla^{T^{(1,0)}X} = P^{T^{(1,0)}X}\nabla^{TX}P^{T^{(1,0)}X}$ is the holomorphic Hermitian connection on $T^{(1,0)}X$, while the Clifford connection ∇^{Cliff} on $\Lambda(T^{*(0,1)}X)$ is $\nabla^{\Lambda(T^{*(0,1)}X)}$, the natural connection induced by $\nabla^{T^{(1,0)}X}$.

Let $\overline{\partial}^{L^p\otimes E,*}$ be the canonical formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p\otimes E}$ on $\Omega^{0,\bullet}(U, L^p \otimes E)$. Then the operator D_p in (2.12) is

$$D_p = \sqrt{2}\left(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*}\right). \tag{5.16}$$

Note that D_p^2 preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(U, L^p \otimes E)$.

Set

$$D_{p,i}^2 = D_p^2|_{\Omega^{0,i}(U,L^p\otimes E)}. \tag{5.17}$$

Since ∇^{Cliff} preserves the $\mathbb{Z}$-grading of $\Lambda(T^{*(0,1)}X)$, the operator $\mathscr{L}_2^t$ in (2.101) also preserves the $\mathbb{Z}$-grading on $\Lambda(T^{*(0,1)}X_0)$. Moreover, $\mathscr{L}_2^t$ is invertible on $\oplus_{q=1}^n \Omega^{0,q}(X_0, L_0^p \otimes E_0)$ for t small enough.

From Section 3.2, for $P^{(r)}$ in (0.12),

$$P^{(r)} = I_{\mathbb{C}\otimes E_G} P^{(r)} I_{\mathbb{C}\otimes E_G}. \tag{5.18}$$

Thus we only need to do the computation for $D_{p,0}^2$.

In what follows, we compute everything on $\mathscr{C}^\infty(U, L^p \otimes E)$.

Take $x_0 \in X_G$.

If $Z \in T_{x_0}B$, $Z = Z^0 + Z^\perp$, $Z^0 \in T_{x_0}X_G$, $Z^\perp \in N_{x_0}$, $|Z^0|, |Z^\perp| \leq \varepsilon$, in Section 2.6, we identify Z with $\exp^B_{\exp^{X_G}_{x_0}(Z^0)} \tau_{Z^0}(Z^\perp)$. This identification is a diffeomorphism from $B_{x_0}^{TX_G}(0,\varepsilon) \times B_{x_0}^N(0,\varepsilon)$ into an open neighborhood $\mathscr{U}(x_0)$ of x_0 in B, we denote it by Ψ. Then $\mathscr{U}(x_0) \cap X_G = B_{x_0}^{TX_G}(0,\varepsilon) \times \{0\}$.

In what follows, we use indifferently the notation $B_{x_0}^{TX_G}(0,\varepsilon) \times B_{x_0}^N(0,\varepsilon)$ or $\mathscr{U}(x_0)$, x_0 or $0, \cdots$.

From now on, we replace U/G by $\mathbb{R}^{2n-n_0} \simeq T_{x_0}B$ as in Section 2.6, and we use the notation therein. Especially,

$$\nabla_t = tS_t^{-1}\kappa^{1/2}\nabla^{(L^p\otimes E)_B}\kappa^{-1/2}S_t, \tag{5.19}$$

and $\mathcal{O}_r$ in (2.103) takes value in $\mathrm{End}(E_B)$.

Let $\{e_i^0\}$, $\{e_j^\perp\}$ be orthonormal basis of $T_{x_0}X_G$, N_{G,x_0} respectively. We will also denote $\Psi_*(e_i^0)$, $\Psi_*(e_j^\perp)$ by e_i^0, $e_j^\perp$.

Let $\{e_i\}$ denote the basis $\{e_i^0, e_j^\perp\}$. Thus in our coordinates,

$$\frac{\partial}{\partial Z_i^0} = e_i^0, \qquad \frac{\partial}{\partial Z_j^\perp} = e_j^\perp. \tag{5.20}$$

We denote by $(g^{ij}(Z))$ the inverse of the matrix $(g_{ij}(Z)) = (g^{TB}_{ij}(Z))$.

Recall Γ^l_{ij} is the connection form of ∇^{TB}, with respect to the frame $\{e_i\}$, defined in (2.107). Also recall that $\mathcal{R}$, $\mathcal{R}^0$ and $\mathcal{R}^\perp$ are defined in (2.77).

As in (1.14), the moment map μ induces a G-invariant $\mathscr{C}^\infty$ section $\widetilde{\mu}$ of TY on U.

Note also that by (2.48), $R^E_\tau \in \mathrm{End}(E)$ defines a section of $\mathrm{End}(E_B)$ on $B = U/G$.

Set

$$\begin{aligned}\mathscr{L}^t_3(Z) = &-g^{ij}(tZ)\Big(\nabla_{t,e_i}\nabla_{t,e_j} - t\Gamma^k_{ij}(tZ)\nabla_{t,e_k}\Big)\\ &+ t^2\Big(\frac{1}{h}g^{ij}(\nabla_{e_i}\nabla_{e_j}h - \Gamma^k_{ij}\nabla_{e_k}h)\Big)(tZ) - t^2R^E_\tau(tZ) - 2\pi n.\end{aligned} \tag{5.21}$$

By (2.60), (2.101) and (5.21), we can reformulate (2.102), (2.110), in using the notations in (3.10), as follows,

$$\begin{aligned}&\nabla_{0,\cdot} = \nabla_\cdot + \frac{1}{2}R^{L_B}_{x_0}(\mathcal{R},\cdot) = \nabla_\cdot - \pi\sqrt{-1}\left\langle J_{x_0}Z^0,\cdot\right\rangle_{x_0},\\ &\mathscr{L}^0_2 = \sum_{j=1}^{n-n_0} b_jb_j^+ + \sum_{j=1}^{n_0} b^\perp_j b^{\perp +}_j = -\sum_j(\nabla_{0,e_j})^2 + 4\pi^2|Z^\perp|^2 - 2\pi n,\\ &\mathscr{L}^t_2(Z) = \mathscr{L}^t_3(Z) + 4\pi^2\Big|\frac{1}{t}\widetilde{\mu}\Big|^2_{g^{TY}}(tZ) - \left\langle 4\pi\sqrt{-1}\widetilde{\mu} + t^2\widetilde{\mu}^E, \widetilde{\mu}^E\right\rangle_{g^{TY}}(tZ).\end{aligned} \tag{5.22}$$

If there is no another specification, we will evaluate our tensors at x_0, and most of time, we will omit the subscript x_0.

Set $h_0 = h_{x_0}$, and for $U \in T_{x_0}B$, set

$$\begin{aligned}&B(Z,U) = \frac{1}{2}\sum_{|\alpha|=2}(\partial^\alpha R^{L_B})_{x_0}\frac{Z^\alpha}{\alpha!}(\mathcal{R},U),\\ &I_1 = -B(Z,e_i)\nabla_{0,e_i} - \frac{1}{2}\nabla_{e_i}(B(Z,e_i)),\\ &I_2 = \Big(\Big\langle\frac{1}{3}R^{TX_G}(\mathcal{R}^0,e^0_i)\mathcal{R}^0 + \nabla^{TX_G}_{\mathcal{R}^0}(A(e^0_i)\mathcal{R}^\perp), e^0_j\Big\rangle + \left\langle e^0_i, \nabla^{TX_G}_{\mathcal{R}^0}(A(e^0_j)\mathcal{R}^\perp)\right\rangle\\ &\quad - 3\left\langle A(e^0_i)\mathcal{R}^\perp, A(e^0_j)\mathcal{R}^\perp\right\rangle + \left\langle R^{TB}(\mathcal{R}^\perp,e^0_i)\mathcal{R}^\perp, e^0_j\right\rangle\Big)\nabla_{0,e^0_i}\nabla_{0,e^0_j}\\ &\quad + \Big(\left\langle R^{N_G}(\mathcal{R}^0,e^0_j)\mathcal{R}^\perp, e^\perp_i\right\rangle + \frac{4}{3}\left\langle R^{TB}(\mathcal{R}^\perp,e^0_j)\mathcal{R}^\perp, e^\perp_i\right\rangle\Big)\nabla_{0,e^\perp_i}\nabla_{0,e^0_j}\\ &\quad + \frac{1}{3}\left\langle R^{TB}(\mathcal{R}^\perp,e^\perp_i)\mathcal{R}^\perp, e^\perp_j\right\rangle\nabla_{0,e^\perp_i}\nabla_{0,e^\perp_j}.\end{aligned} \tag{5.23}$$

Recall that the operator $\mathscr{L}$ has been defined in (3.10).

Set also

$$\begin{aligned}\Gamma_{ii}(\mathcal{R}) =&\frac{2}{3}R^{TX_G}_{x_0}(\mathcal{R}^0,e_i^0)e_i^0+\nabla^{TB}_{\mathcal{R}^0}(A(e_i^0)e_i^0)+R^{TB}(\mathcal{R}^\perp,e_i^0)e_i^0\\&+A(e_i^0)A(e_i^0)\mathcal{R}^\perp+\nabla^{TX_G}_{e_i^0}(A(e_i^0)\mathcal{R}^\perp)-A(\mathcal{R}^0)A(e_i^0)e_i^0,\\ K_2(\mathcal{R}) =&\frac{1}{3}\left\langle R^{TX_G}(\mathcal{R}^0,e_i^0)\mathcal{R}^0,e_i^0\right\rangle+\left\langle R^{TB}(\mathcal{R}^\perp,e_i^0)\mathcal{R}^\perp,e_i^0\right\rangle\\&+\frac{1}{3}\left\langle R^{TB}(\mathcal{R}^\perp,e_i^\perp)\mathcal{R}^\perp,e_i^\perp\right\rangle+2\Big(\sum_i\left\langle A(e_i^0)e_i^0,\mathcal{R}^\perp\right\rangle\Big)^2\\&-|A(e_i^0)\mathcal{R}^\perp|^2+2\left\langle\nabla^{TX_G}_{\mathcal{R}^0}(A(e_i^0)\mathcal{R}^\perp),e_i^0\right\rangle.\end{aligned}\tag{5.24}$$

Lemma 5.4. *There exist second order differential operators $\mathcal{O}'_r$ as in Theorem 2.11 such that for $|t|\leq 1$,*

$$\mathscr{L}_3^t=\mathscr{L}_3^0+\sum_{r=1}^m t^r\mathcal{O}'_r+\mathscr{O}(t^{m+1}),\tag{5.25}$$

with

(5.26)

$$\begin{aligned}\mathscr{L}_3^0 =&\mathscr{L}-\sum_{j=1}^{n_0}(\nabla_{e_j^\perp})^2-2\pi n_0=\mathscr{L}_2^0-4\pi^2|Z^\perp|^2,\\ \mathcal{O}'_1 =&-\frac{2}{3}(\partial_jR^{L_B})_{x_0}(\mathcal{R},e_i)Z_j\nabla_{0,e_i}-\frac{1}{3}(\partial_iR^{L_B})_{x_0}(\mathcal{R},e_i)-2\left\langle A(e_i^0)e_j^0,\mathcal{R}^\perp\right\rangle\nabla_{0,e_i^0}\nabla_{0,e_j^0},\\ \mathcal{O}'_2 =&I_1+I_2+\Big[\frac{1}{4}K_2(\mathcal{R})-\frac{3}{8}\Big(\sum_l\left\langle A(e_l^0)e_l^0,\mathcal{R}^\perp\right\rangle\Big)^2,\mathscr{L}_2^0\Big]\\&-2\left\langle A(e_i^0)e_j^0,\mathcal{R}^\perp\right\rangle\Big(\frac{2}{3}(\partial_kR^{L_B})_{x_0}(\mathcal{R},e_j^0)Z_k\nabla_{0,e_i^0}+\frac{1}{3}(\partial^0_jR^{L_B})_{x_0}(\mathcal{R},e_i^0)\Big)\\&+\left\langle\Gamma_{ii}(\mathcal{R}),e_j\right\rangle\nabla_{0,e_j}-\frac{1}{2}\left\langle A(e_l^0)e_l^0,\mathcal{R}^\perp\right\rangle\nabla_{A(e_k^0)e_k^0}+2\left\langle A(e_i^0)e_j^0,\mathcal{R}^\perp\right\rangle\nabla_{A(e_i^0)e_j^0}\\&+\frac{2}{3}\left\langle R^{TB}(\mathcal{R}^\perp,e_i^\perp)e_i^\perp,e_j\right\rangle\nabla_{0,e_j}-R^{E_B}_{x_0}(\mathcal{R},e_i)\nabla_{0,e_i}-R^{E_B}_{\tau,x_0}\\&-\frac{1}{9}\sum_i\Big[\sum_j(\partial_jR^{L_B})_{x_0}(\mathcal{R},e_i)Z_j\Big]^2+\frac{1}{h_0}(\nabla_{e_j}\nabla_{e_j}h-\nabla_{A(e_i^0)e_i^0}h)_{x_0}.\end{aligned}$$

Proof. By (2.104) and (5.19),

$$\begin{aligned}\nabla_{t,e_i}=\kappa^{1/2}(tZ)\Big(\nabla_{e_i}+\Big(&\frac{1}{2}R^{L_B}_{x_0}+\frac{t}{3}(\partial_kR^{L_B})_{x_0}Z_k\\&+\frac{t^2}{4}\sum_{|\alpha|=2}(\partial^\alpha R^{L_B})_{x_0}\frac{Z^\alpha}{\alpha!}+\frac{t^2}{2}R^{E_B}_{x_0}\Big)(\mathcal{R},e_i)+\mathscr{O}(t^3)\Big)\kappa^{-1/2}(tZ).\end{aligned}\tag{5.27}$$

To get (5.26), we could use (2.93)-(2.97), while here we will get it directly from the local computation.

By [1, Prop. 1.28] (cf. [26, (1.31)]) and (2.104),

$$\begin{aligned}
&\big\langle e_i^0, e_j^0\big\rangle_{Z^0} = \delta_{ij} + \frac{1}{3}\big\langle R^{TX_G}_{x_0}(\mathcal{R}^0, e_i^0)\mathcal{R}^0, e_j^0\big\rangle_{x_0} + \mathscr{O}(|Z^0|^3),\\
&(\nabla^{N_G}_{e_k^0}\nabla^{N_G}_{e_i^0} e_j^\perp)_{x_0} = \frac{1}{2} R^{N_G}_{x_0}(e_k^0, e_i^0)e_j^\perp.
\end{aligned} \tag{5.28}$$

Moreover, for $W, V \in N_{x_0}$, $\gamma_s(t) = (Z^0, t(W + sV))$ is a family of geodesics from $(Z^0, 0)$.

Set $Y = \frac{\partial}{\partial t}\gamma_s(t)$, $X(\gamma_s(t)) = \frac{\partial}{\partial s}\gamma_s(t) = tV$.

Since $\nabla^{TB}_Y Y = 0$, $\nabla^{TB}_Y X - \nabla^{TB}_X Y = [Y, X] = \gamma_*[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = 0$, we get

$$0 = \nabla^{TB}_X \nabla^{TB}_Y Y = \nabla^{TB}_Y \nabla^{TB}_Y X - R^{TB}(Y, X)Y. \tag{5.29}$$

Take $V = e_i^\perp$, we get at $s = t = 0$,

$$(\nabla^{TB}_W \nabla^{TB}_W e_i^\perp)_{Z^0} = \frac{1}{3}\nabla^{TB}_Y \nabla^{TB}_Y \nabla^{TB}_Y X = \frac{1}{3} R^{TB}(W, e_i^\perp)W. \tag{5.30}$$

Under our coordinate, we have

$$\begin{aligned}
&(\nabla^{TB}_{e_j^\perp} e_i^\perp)_{x_0} = (\nabla^{TX_G}_{e_i^0} e_j^0)_{x_0} = (\nabla^{N_G}_{e_j^0} e_i^\perp)_{x_0} = 0, \quad (\nabla^{TB}_{e_i^0} e_j^0)_{x_0} = A_{x_0}(e_i^0)e_j^0,\\
&(\nabla^{TB}_{e_j^\perp} e_i^0)_{x_0} = (\nabla^{TB}_{e_i^0} e_j^\perp)_{x_0} = A_{x_0}(e_i^0)e_j^\perp,\\
&(\nabla^{TB}_{\mathcal{R}^\perp} e_i^\perp)_Z = 0,\\
&(\nabla^{TB}_{e_j^\perp} e_i^\perp)_{Z^0} = (\nabla^{N_G}_{\mathcal{R}^0} e_i^\perp)_{Z^0} = 0.
\end{aligned} \tag{5.31}$$

Moreover, by (5.4), (5.28), (5.30) and (5.31) (comparing with [26, (1.31)]), we have at x_0 that

$$\begin{aligned}
&\nabla^{TB}_{e_k^\perp}\nabla^{TB}_{e_j^\perp} e_i^\perp = \frac{1}{3}R^{TB}(e_k^\perp, e_j^\perp)e_i^\perp + \frac{1}{3}R^{TB}(e_k^\perp, e_i^\perp)e_j^\perp,\\
&\nabla^{TB}_{e_k^0}\nabla^{TB}_{e_j^\perp} e_i^\perp = 0,\\
&\nabla^{TB}_{e_k^\perp}\nabla^{TB}_{e_j^\perp} e_i^0 = \nabla^{TB}_{e_k^\perp}\nabla^{TB}_{e_i^0} e_j^\perp = R^{TB}(e_k^\perp, e_i^0)e_j^\perp,\\
&\nabla^{TB}_{e_k^0}\nabla^{TB}_{e_j^\perp} e_i^0 = \nabla^{TB}_{e_k^0}\nabla^{TB}_{e_i^0} e_j^\perp = \nabla^{N_G}_{e_k^0}\nabla^{N_G}_{e_i^0} e_j^\perp + A(e_k^0)A(e_i^0)e_j^\perp + \nabla^{TX_G}_{e_k^0}(A(e_i^0)e_j^\perp)\\
&\quad = \frac{1}{2}R^{N_G}(e_k^0, e_i^0)e_j^\perp + A(e_k^0)A(e_i^0)e_j^\perp + \nabla^{TX_G}_{e_k^0}(A(e_i^0)e_j^\perp),\\
&\nabla^{TB}_{e_j^\perp}\nabla^{TB}_{e_k^0} e_i^0 = R^{TB}(e_j^\perp, e_k^0)e_i^0 + \nabla^{TB}_{e_k^0}\nabla^{TB}_{e_j^\perp} e_i^0,\\
&\nabla^{TB}_{e_k^0}\nabla^{TB}_{e_j^0} e_i^0 = \nabla^{TX_G}_{e_k^0}\nabla^{TX_G}_{e_j^0} e_i^0 + \nabla^{TB}_{e_k^0}(A(e_j^0)e_i^0)\\
&\quad = \frac{1}{3}R^{TX_G}(e_k^0, e_j^0)e_i^0 + \frac{1}{3}R^{TX_G}(e_k^0, e_i^0)e_j^0 + \nabla^{TB}_{e_k^0}(A(e_j^0)e_i^0).
\end{aligned} \tag{5.32}$$

In the following, for a tensor ψ and the covariant derivative ∇^B acting on ψ induced by ∇^{TB}, we denote by

$$(\nabla^B\nabla^B\psi)_{(c_j e_j, c_k' e_k)} = c_j c_k'(\nabla^B_{e_j}\nabla^B_{e_k}\psi)_{x_0}.$$

From (5.32), we get at x_0 the following formula which will be used in (5.37), (5.38), (5.55), (5.56) and (5.157),

$$
\begin{aligned}
&(\nabla^{TB}\nabla^{TB}e_i^0)_{(\mathcal{R}^0,\mathcal{R}^0)} = \frac{1}{3}R^{TX_G}(\mathcal{R}^0,e_i^0)\mathcal{R}^0 + \nabla^{TB}_{\mathcal{R}^0}(A(e_j^0)e_i^0)Z_j^0,\\
&(\nabla^{TB}\nabla^{TB}e_i^0)_{(\mathcal{R}^0,\mathcal{R}^\perp)} = \frac{1}{2}R^{N_G}(\mathcal{R}^0,e_i^0)\mathcal{R}^\perp + A(\mathcal{R}^0)A(e_i^0)\mathcal{R}^\perp + \nabla^{TX_G}_{\mathcal{R}^0}(A(e_i^0)\mathcal{R}^\perp),\\
&(\nabla^{TB}\nabla^{TB}e_i^0)_{(\mathcal{R}^\perp,\mathcal{R}^\perp)} = R^{TB}(\mathcal{R}^\perp,e_i^0)\mathcal{R}^\perp,\\
&(\nabla^{TB}\nabla^{TB}e_j^\perp)_{(\mathcal{R}^0,\mathcal{R}^0)} = A(\mathcal{R}^0)A(\mathcal{R}^0)e_j^\perp + \nabla^{TX_G}_{\mathcal{R}^0}(A(e_k^0)e_j^\perp)Z_k^0,\\
&(\nabla^{TB}\nabla^{TB}e_j^\perp)_{(\mathcal{R}^0,\mathcal{R}^\perp)} = 0,\\
&(\nabla^{TB}\nabla^{TB}e_j^\perp)_{(\mathcal{R}^\perp,\mathcal{R}^\perp)} = \frac{1}{3}R^{TB}(\mathcal{R}^\perp,e_j^\perp)\mathcal{R}^\perp,\\
&(\nabla^{TB}\nabla^{TB}e_j)_{(\mathcal{R}^\perp,\mathcal{R}^0)} = (\nabla^{TB}\nabla^{TB}e_j)_{(\mathcal{R}^0,\mathcal{R}^\perp)} + R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)e_j.
\end{aligned}
\tag{5.33}
$$

Note that by (5.31), $\nabla^{TB}_{\mathcal{R}}(A_{x_0}(e_i^0)e_i^0) = A(\mathcal{R}^0)A_{x_0}(e_i^0)e_i^0$.
From (5.31), (5.32), we get

$$
\begin{aligned}
&(\nabla^{TB}_{e_i^\perp}e_i^\perp)_Z = \frac{2}{3}R^{TB}(\mathcal{R}^\perp,e_i^\perp)e_i^\perp + \mathscr{O}(|Z|^2),\\
&(\nabla^{TB}_{e_i^0}e_i^0)_Z = A_{x_0}(e_i^0)e_i^0 - \nabla^{TB}_{\mathcal{R}}(A_{x_0}(e_i^0)e_i^0) + \frac{2}{3}R^{TX_G}(\mathcal{R}^0,e_i^0)e_i^0 + \nabla^{TB}_{\mathcal{R}^0}(A(e_i^0)e_i^0)\\
&\qquad + A(e_i^0)A(e_i^0)\mathcal{R}^\perp + \nabla^{TX_G}_{e_i^0}(A(e_i^0)\mathcal{R}^\perp) + R^{TB}(\mathcal{R}^\perp,e_i^0)e_i^0 + \mathscr{O}(|Z|^2)\\
&\qquad = A_{x_0}(e_i^0)e_i^0 + \Gamma_{ii}(\mathcal{R}) + \mathscr{O}(|Z|^2),
\end{aligned}
\tag{5.34}
$$

Thus by (5.31), (5.32) and (5.33), at x_0,

$$
\begin{aligned}
\nabla_{\mathcal{R}^0}\nabla_{\mathcal{R}^\perp}\left\langle e_j^\perp,e_i^0\right\rangle &= \left\langle \nabla^{TB}_{\mathcal{R}^0}e_j^\perp,\nabla^{TB}_{\mathcal{R}^\perp}e_i^0\right\rangle + \left\langle e_j^\perp,\nabla^{TB}_{\mathcal{R}^0}\nabla^{TB}_{\mathcal{R}^\perp}e_i^0\right\rangle\\
&= \frac{1}{2}\left\langle R^{N_G}(\mathcal{R}^0,e_i^0)\mathcal{R}^\perp,e_j^\perp\right\rangle.
\end{aligned}
\tag{5.35}
$$

On the other hand, we have the following expansion for $\langle e_j,e_i\rangle_Z$,

$$
\begin{aligned}
&\langle e_i,e_j\rangle_Z = \langle e_i,e_j\rangle_{Z^0} + (\nabla_{\mathcal{R}^\perp}\langle e_i,e_j\rangle)_{Z^0} + \frac{1}{2}(\nabla\nabla\langle e_i,e_j\rangle)_{(\mathcal{R}^\perp,\mathcal{R}^\perp),x_0} + \mathscr{O}(|Z|^3)\\
&= \langle e_i,e_j\rangle_{Z^0} + (\nabla_{\mathcal{R}^\perp}\langle e_i,e_j\rangle)_{x_0} + (\nabla_{\mathcal{R}^0}\nabla_{\mathcal{R}^\perp}\langle e_i,e_j\rangle)_{x_0} + \left\langle \nabla^{TB}_{\mathcal{R}^\perp}e_i,\nabla^{TB}_{\mathcal{R}^\perp}e_j\right\rangle_{x_0}\\
&\qquad + \frac{1}{2}\left\langle (\nabla^{TB}\nabla^{TB}e_i)_{(\mathcal{R}^\perp,\mathcal{R}^\perp)},e_j\right\rangle + \frac{1}{2}\left\langle e_i,(\nabla^{TB}\nabla^{TB}e_j)_{(\mathcal{R}^\perp,\mathcal{R}^\perp)}\right\rangle + \mathscr{O}(|Z|^3).
\end{aligned}
\tag{5.36}
$$

Thus by (5.28), (5.31), (5.33) and (5.35)-(5.36),

$$
\begin{aligned}
\left\langle e_i^0,e_j^0\right\rangle_Z &= \delta_{ij} - 2\left\langle A_{x_0}(e_i^0)e_j^0,\mathcal{R}^\perp\right\rangle + \frac{1}{3}\left\langle R^{TX_G}(\mathcal{R}^0,e_i^0)\mathcal{R}^0,e_j^0\right\rangle\\
&\quad + \left\langle \nabla^{TX_G}_{\mathcal{R}^0}(A(e_i^0)\mathcal{R}^\perp),e_j^0\right\rangle + \left\langle e_i^0,\nabla^{TX_G}_{\mathcal{R}^0}(A(e_j^0)\mathcal{R}^\perp)\right\rangle\\
&\quad + \left\langle A(e_i^0)\mathcal{R}^\perp,A(e_j^0)\mathcal{R}^\perp\right\rangle + \left\langle R^{TB}(\mathcal{R}^\perp,e_i^0)\mathcal{R}^\perp,e_j^0\right\rangle + \mathscr{O}(|Z|^3),
\end{aligned}
\tag{5.37}
$$

and

$$\begin{aligned}
&\big\langle e_i^0, e_j^\perp\big\rangle_Z = \frac{1}{2}\big\langle R^{N_G}(\mathcal{R}^0, e_i^0)\mathcal{R}^\perp, e_j^\perp\big\rangle + \frac{2}{3}\big\langle R^{TB}(\mathcal{R}^\perp, e_i^0)\mathcal{R}^\perp, e_j^\perp\big\rangle + \mathscr{O}(|Z|^3),\\
&\big\langle e_i^\perp, e_j^\perp\big\rangle_Z = \delta_{ij} + \frac{1}{3}\big\langle R^{TB}(\mathcal{R}^\perp, e_i^\perp)\mathcal{R}^\perp, e_j^\perp\big\rangle + \mathscr{O}(|Z|^3).
\end{aligned} \tag{5.38}$$

From (5.24), (5.37) and (5.38), we get

(5.39)

$$\begin{aligned}
&\det g_{ij}(Z) = 1 - 2\big\langle A_{x_0}(e_i^0)e_i^0, \mathcal{R}^\perp\big\rangle + K_2(\mathcal{R}) + \mathscr{O}(|Z|^3),\\
&\kappa^{\frac{1}{2}}(tZ) = (\det g_{ij})^{1/4}(tZ)\\
&\quad = 1 - \frac{t}{2}\big\langle A(e_i^0)e_i^0, \mathcal{R}^\perp\big\rangle - \frac{3t^2}{8}\Big(\sum_i \big\langle A(e_i^0)e_i^0, \mathcal{R}^\perp\big\rangle\Big)^2 + \frac{t^2}{4}K_2(\mathcal{R}) + \mathscr{O}(t^3),\\
&\kappa^{-\frac{1}{2}}(tZ) = 1 + \frac{t}{2}\big\langle A(e_i^0)e_i^0, \mathcal{R}^\perp\big\rangle + \frac{5t^2}{8}\Big(\sum_i \big\langle A(e_i^0)e_i^0, \mathcal{R}^\perp\big\rangle\Big)^2 - \frac{t^2}{4}K_2(\mathcal{R}) + \mathscr{O}(t^3).
\end{aligned}$$

Moreover, as a $2(n-n_0)\times 2(n-n_0)$-matrix, we have

$$\begin{aligned}
\Big((\delta_{ij} - 2\big\langle A_{x_0}(e_i^0)e_j^0, \mathcal{R}^\perp\big\rangle)\Big)^{-1} &= \big(\delta_{ij} + 2\big\langle A_{x_0}(e_i^0)e_j^0, \mathcal{R}^\perp\big\rangle\big)\\
&\quad + 4\Big(\big\langle A_{x_0}(e_i^0)\mathcal{R}^\perp, A_{x_0}(e_j^0)\mathcal{R}^\perp\big\rangle\Big) + \mathscr{O}(|Z|^3).
\end{aligned} \tag{5.40}$$

Note that from (3.9), (5.22),

$$\big[\big\langle A(e_i^0)e_i^0, \mathcal{R}^\perp\big\rangle, \mathscr{L}_2^0\big] = 2\big\langle A(e_i^0)e_i^0, e_k^\perp\big\rangle \nabla_{0,e_k^\perp}. \tag{5.41}$$

Thus from (5.24), (5.27), (5.34), (5.37)-(5.39), the coefficients of t, t^2 in the expansion of $g^{ij}(tZ)t\Gamma_{ij}^k(tZ)\nabla_{t,e_k} = tg^{ij}(tZ)\nabla_{t,(\nabla^{TB}_{e_i}e_j)(tZ)}$ are

$$\begin{aligned}
&\big\langle A(e_i^0)e_i^0, e_k^\perp\big\rangle \nabla_{0,e_k^\perp};\\
&2\big\langle A(e_i^0)e_j^0, \mathcal{R}^\perp\big\rangle \nabla_{A(e_i^0)e_j^0} + \langle \Gamma_{ii}(\mathcal{R}), e_j\rangle \nabla_{0,e_j} + \frac{2}{3}\big\langle R^{TB}(\mathcal{R}^\perp, e_i^\perp)e_i^\perp, e_j\big\rangle \nabla_{0,e_j}\\
&\quad - \Big[\frac{1}{2}\big\langle A(e_l^0)e_l^0, \mathcal{R}^\perp\big\rangle, \nabla_{A(e_i^0)e_i^0}\Big] + \frac{1}{3}(\partial_k R^{L_B})_{x_0} Z_k(\mathcal{R}, A(e_i^0)e_i^0).
\end{aligned} \tag{5.42}$$

By (5.21), (5.27) and (5.37)-(5.42), the coefficient of t in the expansion of $\mathscr{L}_3^t$ is $\mathcal{O}_1'$ in (5.26).

We denote by $[A, B]_+ = AB + BA$.

By (5.21), (5.27), (5.34) and (5.37)-(5.40), the coefficient of t^2 in the expansion of $\mathscr{L}_3^t - (g^{ij}t\Gamma_{ij}^k)(tZ)\nabla_{t,e_k}$ is

$$
\begin{aligned}
(5.43)\quad & I_2 - 2\left\langle A(e_i^0)e_j^0, \mathcal{R}^\perp\right\rangle \Big[\frac{1}{3}\nabla_{0,e_i^0}(\partial_k R^{L_B})_{x_0}(\mathcal{R}, e_j^0)Z_k \\
& + \frac{1}{3}(\partial_k R^{L_B})_{x_0}(\mathcal{R}, e_i^0)Z_k\nabla_{0,e_j^0} - \frac{1}{2}[\left\langle A(e_l^0)e_l^0, \mathcal{R}^\perp\right\rangle, \nabla_{0,e_i^0}\nabla_{0,e_j^0}]\Big] \\
& + I_1 + \Big[\frac{1}{2}\left\langle A(e_l^0)e_l^0, \mathcal{R}^\perp\right\rangle, [\frac{1}{3}(\partial_k R^{L_B})_{x_0}(\mathcal{R}, e_i)Z_k, \nabla_{0,e_i}]_+\Big] \\
& + \Big[\frac{1}{4}K_2(\mathcal{R}) - \frac{3}{8}\Big(\sum_l \left\langle A(e_l^0)e_l^0, \mathcal{R}^\perp\right\rangle\Big)^2, \mathscr{L}_2^0\Big] \\
& - \frac{1}{4}[\left\langle A(e_l^0)e_l^0, \mathcal{R}^\perp\right\rangle, \mathscr{L}_2^0]\left\langle A(e_k^0)e_k^0, \mathcal{R}^\perp\right\rangle - R^{E_B}_{x_0}(\mathcal{R}, e_i)\nabla_{0,e_i} \\
& - \frac{1}{9}\sum_i \Big[\sum_j (\partial_j R^{L_B})_{x_0}(\mathcal{R}, e_i)Z_j\Big]^2 - R^E_{\tau,x_0} + \frac{1}{h_0}(\nabla_{e_j}\nabla_{e_j}h - \nabla_{A(e_i^0)e_i^0}h)_{x_0}.
\end{aligned}
$$

Here I_2 is from the coefficient of t^2 in the expansion of g^{ij}, the second term is the product of the coefficients of t^1 in the expansion of g^{ij} and $\nabla_{t,e_i}\nabla_{t,e_j}$; I_1 is from the coefficient of t^2 in the expansion of R^{L_B}, the fourth term is from the product of the coefficients of t^1 in $\kappa^{1/2}, \kappa^{-1/2}$ and in $\kappa^{-1/2}\nabla_{t,e_i}\nabla_{t,e_i}\kappa^{1/2}$ (cf. (5.27)), the fifth and sixth terms is from the coefficients of t^2 in the expansion of $\kappa^{1/2}, \kappa^{-1/2}$ and the product of the coefficients of t^1 in the expansion of $\kappa^{1/2}$ and $\kappa^{-1/2}$; the seventh term is from R^{E_B}, and the eighth term is from the product of the coefficients of t^1 in the expansion of R^{L_B}.

Certainly,

$$
(5.44)\quad \frac{1}{6}\Big[\left\langle A(e_l^0)e_l^0, \mathcal{R}^\perp\right\rangle, [(\partial_k R^{L_B})_{x_0}(\mathcal{R}, e_i)Z_k, \nabla_{0,e_i}]_+\Big] = -\frac{1}{3}(\partial_k R^{L_B})_{x_0}(\mathcal{R}, A(e_l^0)e_l^0)Z_k.
$$

By (5.41), (5.42), (5.44) and by the fact that $A(e_i^0)e_j^0$ is symmetric on i, j, we see that the coefficient of t^2 in the expansion of $\mathscr{L}_3^t$ is $\mathcal{O}_2'$ in (5.26). □

To simplify the notation, we will often denote by e_i the lift e_i^H of e_i.

Lemma 5.5. *The following identities hold,*

$$(\partial_i R^{L_B})_{x_0}(\mathcal{R}, e_l)Z_i = -3\sqrt{-1}\pi \left\langle JT(\mathcal{R}, e_l) - JT(\mathcal{R}^0, P^{TX_G}e_l), \mathcal{R}^\perp \right\rangle, \tag{5.45a}$$

$$\begin{aligned}\frac{\sqrt{-1}}{\pi}B(Z, e_l^0) &= \frac{1}{6}\left\langle R^{TX_G}(\mathcal{R}^0, J\mathcal{R}^0)\mathcal{R}^0, e_l^0\right\rangle - \frac{5}{4}\left\langle J\mathcal{R}^\perp, \nabla^{TY}_{\mathcal{R}}(T(e_i, e_l^0))Z_i\right\rangle \\ &+ \frac{1}{2}\left\langle 2\nabla^{TX_G}_{\mathcal{R}^0}(A(e_l^0)e_j^\perp)Z_j^\perp + R^{TB}(\mathcal{R}^\perp, e_l^0)\mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)e_l^0, J\mathcal{R}^0\right\rangle \\ &- \frac{1}{2}\left\langle 3\nabla^{TX_G}_{\mathcal{R}^0}(A(e_i^0)e_j^\perp)Z_i^0 Z_j^\perp + 2R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^0, Je_l^0\right\rangle \\ &+ \frac{1}{2}\left\langle J\mathcal{R}^\perp, T(\mathcal{R}^0 - \frac{1}{4}\mathcal{R}, e_i^0)\right\rangle \left\langle J\mathcal{R}^\perp, T(e_i^0, Je_l^0)\right\rangle \\ &+ \frac{1}{8}\left\langle T(\mathcal{R}^0, \mathcal{R}^\perp), T(e_l^0, J\mathcal{R}^0)\right\rangle + \frac{1}{8}\left\langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}^\perp, e_l^0)\right\rangle \\ &- \frac{1}{8}\left\langle T(\mathcal{R}^\perp, J\mathcal{R}^0), T(\mathcal{R}, e_l^0)\right\rangle + \frac{1}{2}\left\langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), T(\mathcal{R}, e_l^0)\right\rangle \\ &- \frac{1}{8}\left\langle T(e_l^0, J\mathcal{R}^0), e_j^\perp\right\rangle \left\langle J\mathcal{R}^\perp, T(\mathcal{R}^\perp, e_j^\perp)\right\rangle.\end{aligned} \tag{5.45b}$$

Proof. By (1.6), (1.14), (1.18) and (2.14),

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}R^{L_B}(e_k, e_l) &= \left\langle Je_k^H, e_l^H\right\rangle + \mu(\Theta)(e_k, e_l) \\ &= \left\langle Je_k^H, e_l^H\right\rangle + \langle\widetilde{\mu}, T(e_k, e_l)\rangle.\end{aligned} \tag{5.46}$$

Thus by (3.33), (5.5a) and (5.6a), we get at x_0 the following formulas which will be used in (5.61),

$$\widetilde{\mu}_{x_0} = 0, \quad (\nabla^{TY}_{\mathcal{R}}\widetilde{\mu})_{x_0} = -J\mathcal{R}^\perp, \quad (\nabla^{TY}_{\cdot}\nabla^{TY}_{\cdot}\widetilde{\mu})_{(\mathcal{R},\mathcal{R})} = T(\mathcal{R}^\perp, J\mathcal{R}^\perp). \tag{5.47}$$

By (3.36) and $\mu = 0$ on P, we have at x_0,

$$\begin{aligned}(\nabla_{e_i}\langle\widetilde{\mu}, T(e_k, e_l)\rangle)_{x_0} &= \left\langle \nabla^{TY}_{e_i^H}\widetilde{\mu}, T(e_k, e_l)\right\rangle + \left\langle\widetilde{\mu}, \nabla^{TY}_{e_i^H}(T(e_k, e_l))\right\rangle \\ &= \langle JT(e_k, e_l), e_i\rangle.\end{aligned} \tag{5.48}$$

By (3.40), (5.6a) and (5.31), we have

$$\begin{aligned}(\nabla_{e_i^H}\left\langle Je_k^H, e_l^H\right\rangle)_{x_0} &= \left\langle J\nabla^{TX}_{e_i^H}e_k^H, e_l^H\right\rangle_{x_0} + \left\langle Je_k^H, \nabla^{TX}_{e_i^H}e_l^H\right\rangle_{x_0} \\ &= -\frac{1}{2}\langle JT(e_i, e_k), e_l\rangle - \frac{1}{2}\langle Je_k, T(e_i, e_l)\rangle \\ &+ \left\langle JA(P^{TX_G}e_i)P^{N_G}e_k + JA(P^{TX_G}e_k)P^{N_G}e_i, P^{TX_G}e_l\right\rangle \\ &+ \left\langle JP^{TX_G}e_k, A(P^{TX_G}e_i)P^{N_G}e_l + A(P^{TX_G}e_l)P^{N_G}e_i\right\rangle.\end{aligned} \tag{5.49}$$

By (5.5a), (5.46), (5.48) and (5.49), for $U \in T_{x_0}B$,

$$\begin{aligned}(5.50)\quad \frac{\sqrt{-1}}{2\pi}(\partial_U R^{L_B})_{x_0}(U,e_l) &= \frac{3}{2}\left\langle JT(U,e_l),U\right\rangle - 2\left\langle A(P^{TX_G}U)P^{N_G}U, JP^{TX_G}e_l\right\rangle \\ &\quad + \left\langle JP^{TX_G}U, A(P^{TX_G}U)P^{N_G}e_l + A(P^{TX_G}e_l)P^{N_G}U\right\rangle \\ &= \frac{3}{2}\left\langle JT(U,e_l) - JT(P^{TX_G}U, P^{TX_G}e_l), U\right\rangle.\end{aligned}$$

Note that $(JTY)_G = N_G$ on X_G, by (5.50), we get (5.45a).

By (5.23) and (5.46), one gets at x_0,

$$(5.51)\qquad \frac{\sqrt{-1}}{\pi}B(Z,e_l) = \frac{1}{2}\Big(\nabla\nabla\left\langle Je_k,e_l\right\rangle + \nabla\nabla\left\langle\widetilde{\mu}, T(e_k,e_l)\right\rangle\Big)_{(\mathcal{R},\mathcal{R})}Z_k.$$

From (5.6a) we have

$$\begin{aligned}(5.52)\quad \Big(\nabla\nabla\left\langle Je_k^H, e_l^H\right\rangle\Big)_{(\mathcal{R},\mathcal{R})}Z_k &= \langle J\mathcal{R}, (\nabla^{TX}\nabla^{TX}e_l^H)_{(\mathcal{R},\mathcal{R})}\rangle \\ &\quad + \langle J(\nabla^{TX}\nabla^{TX}e_k^H)_{(\mathcal{R},\mathcal{R})}, e_l^H\rangle Z_k + 2\langle J\nabla^{TX}_{\mathcal{R}}e_k^H, \nabla^{TX}_{\mathcal{R}}e_l^H\rangle Z_k.\end{aligned}$$

From (1.2), (5.31), one finds at x_0 that

$$\begin{aligned}&J\mathcal{R}^{\perp} \in TY, \quad J\mathcal{R}^0 \in TX_G,\\ &\nabla^{TB}_{\mathcal{R}}e_i^0 = A(e_i^0)\mathcal{R}, \quad \nabla^{TB}_{\mathcal{R}}e_i^{\perp} = A(\mathcal{R}^0)e_i^{\perp},\\ &(\nabla^{T^HX}_{e_j^H}e_i^H)Z_iZ_j = (\nabla^{TB}_{e_j}e_i)^H Z_iZ_j = 2A(\mathcal{R}^0)\mathcal{R}^{\perp} + A(\mathcal{R}^0)\mathcal{R}^0.\end{aligned}\qquad (5.53)$$

Now by (3.40),

$$(5.54)\qquad (\nabla^{TX}_{e_j^H}\nabla^{TX}_{e_i^H}e_k^H)_{x_0} = \nabla^{T^HX}_{e_j^H}\nabla^{T^HX}_{e_i^H}e_k^H - \frac{1}{2}T(e_j^H, \nabla^{T^HX}_{e_i^H}e_k^H) - \frac{1}{2}\nabla^{TX}_{e_j^H}(T(e_i^H,e_k^H)).$$

By (5.33), we get

$$\begin{aligned}(5.55)\quad (\nabla^{TB}\nabla^{TB}e_k)_{(\mathcal{R},\mathcal{R})}Z_k &= \nabla^{TB}_{\mathcal{R}^0}(A(e_j^0)e_i^0)Z_j^0Z_i^0 + 3A(\mathcal{R}^0)A(\mathcal{R}^0)\mathcal{R}^{\perp} \\ &\quad + 3\nabla^{TX_G}_{\mathcal{R}^0}(A(e_i^0)\mathcal{R}^{\perp})Z_i^0 + 2R^{TB}(\mathcal{R}^{\perp},\mathcal{R}^0)\mathcal{R}^{\perp} + R^{TB}(\mathcal{R}^{\perp},\mathcal{R}^0)\mathcal{R}^0.\end{aligned}$$

From (5.33), (5.53), (5.54), (5.55), the anti-symmetric property of the torsion tensor T and the fact that A exchanges TX_G and N_G, we get

$$\begin{aligned}(5.56)\quad \langle J\mathcal{R}, (\nabla^{TX}\nabla^{TX}e_l^{0,H})_{(\mathcal{R},\mathcal{R})}\rangle &= \left\langle \frac{1}{3}R^{TX_G}(\mathcal{R}^0,e_l^0)\mathcal{R}^0 + \nabla^{TB}_{\mathcal{R}^0}(A(e_j^0)e_l^0)Z_j^0, J\mathcal{R}^0\right\rangle \\ &\quad + \left\langle 2\nabla^{TX_G}_{\mathcal{R}^0}(A(e_l^0)e_j^{\perp})Z_j^{\perp} + R^{TB}(\mathcal{R}^{\perp},e_l^0)\mathcal{R}^{\perp} + R^{TB}(\mathcal{R}^{\perp},\mathcal{R}^0)e_l^0, J\mathcal{R}^0\right\rangle \\ &\quad - \frac{1}{2}\left\langle J\mathcal{R}^{\perp}, T(\mathcal{R}, A(e_l^0)\mathcal{R})\right\rangle - \frac{1}{2}\left\langle J\mathcal{R}, \nabla^{TX}_{\mathcal{R}}(T(e_i,e_l^0))Z_i\right\rangle, \\ \langle J(\nabla^{TX}\nabla^{TX}e_k^H)_{(\mathcal{R},\mathcal{R})}, e_l^{0,H}\rangle Z_k &= \left\langle 2JR^{TB}(\mathcal{R}^{\perp},\mathcal{R}^0)\mathcal{R}^{\perp} + JR^{TB}(\mathcal{R}^{\perp},\mathcal{R}^0)\mathcal{R}^0, e_l^0\right\rangle \\ &\quad + \left\langle J\nabla^{TB}_{\mathcal{R}^0}(A(e_j^0)e_i^0)Z_j^0Z_i^0 + 3J\nabla^{TX_G}_{\mathcal{R}^0}(A(e_i^0)e_j^{\perp})Z_i^0Z_j^{\perp}, e_l^0\right\rangle.\end{aligned}$$

Note that from (1.8), (5.3), (5.5a), (5.53) and A exchanges TX_G and N_G,

$$\begin{aligned}(5.57)\qquad \left\langle J\mathcal{R}, \nabla^{TX}_{\mathcal{R}}(T(e_i,e^0_l))Z_i\right\rangle &= \left\langle J\mathcal{R}^\perp, \nabla^{TY}_{\mathcal{R}}(T(e_i,e^0_l))Z_i\right\rangle \\ &\quad + \frac{1}{2}\left\langle T(\mathcal{R},J\mathcal{R}^0), T(\mathcal{R},e^0_l)\right\rangle, \\ \left\langle J\nabla^{TB}_{\mathcal{R}^0}(A(e^0_j)e^0_i)Z^0_jZ^0_i, e^0_l\right\rangle &= -\left\langle A(\mathcal{R}^0)A(\mathcal{R}^0)\mathcal{R}^0, Je^0_l\right\rangle \\ &= -\frac{1}{4}\left\langle T(\mathcal{R}^0,J\mathcal{R}^0), T(\mathcal{R}^0,e^0_l)\right\rangle, \\ \left\langle \nabla^{TB}_{\mathcal{R}^0}(A(e^0_j)e^0_l), J\mathcal{R}^0\right\rangle &= -\left\langle A(e^0_j)e^0_l, A(\mathcal{R}^0)J\mathcal{R}^0\right\rangle = 0.\end{aligned}$$

By (3.40), (5.6a), (5.13), (5.53) and the fact that A exchanges TX_G and N_G, at x_0,

$$\begin{aligned}(5.58)\qquad \left\langle J\nabla^{TX}_{\mathcal{R}}e^H_k, \nabla^{TX}_{\mathcal{R}}e^{0,H}_l\right\rangle Z_k &= \left\langle J\nabla^{TB}_{\mathcal{R}}e_k, A(e^0_l)\mathcal{R} - \frac{1}{2}T(\mathcal{R},e^0_l)\right\rangle Z_k \\ &= \left\langle JA(\mathcal{R}^0)\mathcal{R}^0, -\frac{1}{2}T(\mathcal{R},e^0_l)\right\rangle + 2\left\langle JA(\mathcal{R}^0)\mathcal{R}^\perp, A(e^0_l)\mathcal{R}^\perp\right\rangle \\ &= \frac{1}{4}\left\langle T(\mathcal{R}^0,J\mathcal{R}^0), T(\mathcal{R},e^0_l)\right\rangle + \frac{1}{2}\left\langle J\mathcal{R}^\perp, T(\mathcal{R}^0,e^0_j)\right\rangle\left\langle J\mathcal{R}^\perp, T(e^0_l,Je^0_j)\right\rangle.\end{aligned}$$

By (5.52), (5.56)-(5.58), at x_0,

$$\begin{aligned}(5.59)\quad \left(\nabla\nabla\left\langle Je^H_k, e^{0,H}_l\right\rangle\right)_{(\mathcal{R},\mathcal{R})}Z_k &= \frac{1}{3}\left\langle R^{TX_G}(\mathcal{R}^0,e^0_l)\mathcal{R}^0, J\mathcal{R}^0\right\rangle \\ &\quad + \left\langle 2\nabla^{TX_G}_{\mathcal{R}^0}(A(e^0_l)e^\perp_j)Z^\perp_j + R^{TB}(\mathcal{R}^\perp,e^0_l)\mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)e^0_l, J\mathcal{R}^0\right\rangle \\ &\quad - \left\langle 2R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)\mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)\mathcal{R}^0 + 3\nabla^{TX_G}_{\mathcal{R}^0}(A(e^0_i)e^\perp_j)Z^0_iZ^\perp_j, Je^0_l\right\rangle \\ &\quad - \frac{1}{2}\left\langle J\mathcal{R}^\perp, T(\mathcal{R},A(e^0_l)\mathcal{R}) + \nabla^{TY}_{\mathcal{R}}(T(e_i,e^0_l))Z_i\right\rangle + \frac{1}{4}\left\langle T(\mathcal{R}^0,J\mathcal{R}^0), T(\mathcal{R}^\perp,e^0_l)\right\rangle \\ &\quad - \frac{1}{4}\left\langle T(\mathcal{R}^\perp,J\mathcal{R}^0), T(\mathcal{R},e^0_l)\right\rangle + \left\langle J\mathcal{R}^\perp, T(\mathcal{R}^0,e^0_j)\right\rangle\left\langle J\mathcal{R}^\perp, T(e^0_l,Je^0_j)\right\rangle.\end{aligned}$$

Observe that $A(e^0_i)\mathcal{R}^0 \in N_G$, $A(e^0_i)\mathcal{R}^\perp \in TX_G$. By (5.5a), (5.5b), (5.5d) and (5.13),

$$\begin{aligned}(5.60)\quad \left\langle J\mathcal{R}^\perp, T(\mathcal{R},A(e^0_l)\mathcal{R})\right\rangle &= \langle J\mathcal{R}^\perp, T(\mathcal{R},A(e^0_l)\mathcal{R}^0)\rangle + \langle J\mathcal{R}^\perp, T(\mathcal{R},A(e^0_l)\mathcal{R}^\perp)\rangle \\ &= \frac{1}{2}\left\langle JT(e^0_l,J\mathcal{R}^0), e^\perp_j\right\rangle\left\langle J\mathcal{R}^\perp, T(\mathcal{R},e^\perp_j)\right\rangle + \left\langle J\mathcal{R}^\perp, T(\mathcal{R},A(e^0_l)\mathcal{R}^\perp)\right\rangle \\ &= -\frac{1}{2}\left\langle T(e^0_l,J\mathcal{R}^0), T(\mathcal{R}^0,\mathcal{R}^\perp)\right\rangle + \frac{1}{2}\left\langle J\mathcal{R}^\perp, T(\mathcal{R},e^0_j)\right\rangle\left\langle J\mathcal{R}^\perp, T(e^0_l,Je^0_j)\right\rangle \\ &\quad + \frac{1}{2}\left\langle JT(e^0_l,J\mathcal{R}^0), e^\perp_j\right\rangle\left\langle J\mathcal{R}^\perp, T(\mathcal{R}^\perp,e^\perp_j)\right\rangle.\end{aligned}$$

From (5.47), at x_0,

$$\begin{aligned}(5.61)\quad &\left(\nabla\nabla\left\langle\widetilde{\mu}, T(e_k,e_l)\right\rangle\right)_{(\mathcal{R},\mathcal{R})} \\ &\quad = \left\langle(\nabla^{TY}_{\cdot}\nabla^{TY}_{\cdot}\widetilde{\mu})_{(\mathcal{R},\mathcal{R})}, T(e_k,e_l)\right\rangle + 2\left\langle\nabla^{TY}_{\mathcal{R}}\widetilde{\mu}, \nabla^{TY}_{\mathcal{R}}(T(e_k,e_l))\right\rangle \\ &\quad = \left\langle T(\mathcal{R}^\perp,J\mathcal{R}^\perp), T(e_k,e_l)\right\rangle - 2\left\langle\nabla^{TY}_{\mathcal{R}}(T(e_k,e_l)), J\mathcal{R}^\perp\right\rangle.\end{aligned}$$

Finally, by (5.4), (5.51), (5.59), (5.60) and (5.61), we get (5.45b). □

We now examine the coefficients in the expansion of terms involving the moment map $\widetilde{\mu}$.

Set

$$\begin{aligned}\mathcal{O}_2'' = &-\frac{1}{3}\left\langle (\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(\mathcal{R},\mathcal{R})}J\mathcal{R}^{\perp}, J\mathcal{R}^{\perp}\right\rangle + \frac{1}{6}\left\langle \nabla^{TY}_{\mathcal{R}}(T(e_j, J_{x_0}e_i^0)), J\mathcal{R}^{\perp}\right\rangle Z_jZ_i^0 \\ &+ \frac{1}{3}\left\langle \nabla^{N_G}_{\mathcal{R}^0}(A(e_j^0)e_i^0)Z_j^0Z_i^0 + R^{TB}(\mathcal{R}^{\perp},\mathcal{R}^0)\mathcal{R}^0, \mathcal{R}^{\perp}\right\rangle \\ &- \frac{1}{12}\sum_l \left\langle T(\mathcal{R}, e_l), J\mathcal{R}^{\perp}\right\rangle^2 + \frac{1}{4}\left\langle J\mathcal{R}^{\perp}, T(\mathcal{R}^{\perp}, e_l^0)\right\rangle\left\langle J\mathcal{R}^{\perp}, T(\mathcal{R}^0, e_l^0)\right\rangle \\ &+ \frac{7}{12}|T(\mathcal{R}^{\perp}, J\mathcal{R}^{\perp})|^2 + \frac{1}{3}\left\langle T(\mathcal{R}^0, J\mathcal{R}^{\perp}), T(\mathcal{R}^{\perp}, J\mathcal{R}^{\perp})\right\rangle.\end{aligned} \tag{5.62}$$

Lemma 5.6. *For $|t| \leq 1$, we have*

$$\begin{aligned}&|\frac{1}{t}\widetilde{\mu}|^2_{g^{TY}}(tZ) = |Z^{\perp}|^2 - t\left\langle T(\mathcal{R}^{\perp}, J\mathcal{R}^{\perp}), J\mathcal{R}^{\perp}\right\rangle + t^2\mathcal{O}_2'' + \mathscr{O}(t^3), \\ &\left\langle \widetilde{\mu}, \widetilde{\mu}^E\right\rangle_{g^{TY}}(tZ) = -t\left\langle J\mathcal{R}^{\perp}, \widetilde{\mu}^E_{x_0}\right\rangle \\ &\qquad + t^2\Big(\frac{1}{2}\left\langle T(\mathcal{R}^{\perp}, J\mathcal{R}^{\perp}), \widetilde{\mu}^E_{x_0}\right\rangle - \left\langle J\mathcal{R}^{\perp}, \nabla^{TY}_{\mathcal{R}}\widetilde{\mu}^E\right\rangle_{x_0}\Big) + \mathscr{O}(t^3).\end{aligned} \tag{5.63}$$

Proof. By (3.36), (3.38), (3.39), (5.6a), (5.53), $\mathbf{J} = J$ and $\widetilde{\mu} = 0$ on P, we get, at x_0,

$$\begin{aligned}(\nabla^{TY}_{e_k^H}\nabla^{TY}_{e_j^H}\nabla^{TY}_{e_i^H}\widetilde{\mu})_{x_0} = &-P^{TY}J\nabla^{TX}_{e_k^H}\nabla^{TX}_{e_j^H}e_i^H - \frac{1}{2}T(e_k^H, P^{T^HX}J\nabla^{TX}_{e_j^H}e_i^H) \\ &- \frac{1}{2}\nabla^{TY}_{e_k^H}(T(e_j^H, P^{T^HX}Je_i^H)) - \frac{1}{2}(\nabla^{TY}_{e_j^H}\dot{g}^{TY}_{e_i^H})(\nabla^{TY}_{e_k^H}\widetilde{\mu}) \\ &- \frac{1}{2}(\nabla^{TY}_{e_k^H}\dot{g}^{TY}_{e_i^H})(\nabla^{TY}_{e_j^H}\widetilde{\mu}) - \frac{1}{2}\dot{g}^{TY}_{e_i^H}(\nabla^{TY}_{e_k^H}\nabla^{TY}_{e_j^H}\widetilde{\mu}).\end{aligned} \tag{5.64}$$

From (3.40), (5.47), (5.53), (5.54), (5.55) and (5.64), we have

$$\begin{aligned}&(\nabla^{TY}_{\cdot}\nabla^{TY}_{\cdot}\nabla^{TY}_{\cdot}\widetilde{\mu})_{(\mathcal{R},\mathcal{R},\mathcal{R})} := (\nabla^{TY}_{e_k^H}\nabla^{TY}_{e_j^H}\nabla^{TY}_{e_i^H}\widetilde{\mu})_{x_0}Z_kZ_jZ_i \\ &= -J\nabla^{N_G}_{\mathcal{R}^0}(A(e_j^0)e_i^0)Z_j^0Z_i^0 - 3JA(\mathcal{R}^0)A(\mathcal{R}^0)\mathcal{R}^{\perp} - 2P^{TY}JR^{TB}(\mathcal{R}^{\perp},\mathcal{R}^0)\mathcal{R}^{\perp} \\ &\qquad - P^{TY}JR^{TB}(\mathcal{R}^{\perp},\mathcal{R}^0)\mathcal{R}^0 - T(\mathcal{R}, JA(\mathcal{R}^0)\mathcal{R}^{\perp}) \\ &- \frac{1}{2}\nabla^{TY}_{\mathcal{R}}(T(e_j^H, P^{T^HX}Je_i^H))Z_jZ_i + (\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(\mathcal{R},\mathcal{R})}J\mathcal{R}^{\perp} - \frac{1}{2}\dot{g}^{TY}_{\mathcal{R}}(T(\mathcal{R}^{\perp}, J\mathcal{R}^{\perp})).\end{aligned} \tag{5.65}$$

Now by (3.50), (5.47), (5.65), and $\widetilde{\mu} = 0$ on P, we have

$$\begin{aligned}|\frac{1}{t}\widetilde{\mu}|^2_{g^{TY}}(tZ) &= \sum_{k=2}^{4}\frac{1}{k!}\frac{\partial^k}{\partial t^k}\Big(|\widetilde{\mu}|^2_{g^{TY}}(tZ)\Big)|_{t=0}\,t^{k-2} + \mathscr{O}(t^3) \\ &= |\nabla^{TY}_{\mathcal{R}}\widetilde{\mu}|^2_{x_0} + t\left\langle (\nabla^{TY}_{\cdot}\nabla^{TY}_{\cdot}\widetilde{\mu})_{(\mathcal{R},\mathcal{R})}, \nabla^{TY}_{\mathcal{R}}\widetilde{\mu}\right\rangle_{x_0} \\ &+ \frac{t^2}{4!}\Big(8\left\langle (\nabla^{TY}_{\cdot}\nabla^{TY}_{\cdot}\nabla^{TY}_{\cdot}\widetilde{\mu})_{(\mathcal{R},\mathcal{R},\mathcal{R})}, \nabla^{TY}_{\mathcal{R}}\widetilde{\mu}\right\rangle_{x_0} + 6|(\nabla^{TY}_{\cdot}\nabla^{TY}_{\cdot}\widetilde{\mu})_{(\mathcal{R},\mathcal{R})}|^2_{x_0}\Big) + \mathscr{O}(t^3).\end{aligned} \tag{5.66}$$

By (5.5c),

$$T(\mathcal{R}^0, J\mathcal{R}^{\perp}) = \frac{1}{2}T(\mathcal{R}^{\perp}, J\mathcal{R}^0). \tag{5.67}$$

From (1.6), (5.13), (5.47), (5.65), (5.66) and (5.67), we get the coefficients of t^0, t^1 in the expansion of $|\frac{1}{t}\widetilde{\mu}|^2_{g^{TY}}(tZ)$ in (5.63), and the coefficients of t^2 is

$$\begin{aligned}
&\frac{1}{3}\left\langle J\nabla^{N_G}_{\mathcal{R}^0}(A(e^0_j)e^0_i)Z^0_jZ^0_i + 3JA(\mathcal{R}^0)A(\mathcal{R}^0)\mathcal{R}^{\perp} + JR^{TB}(\mathcal{R}^{\perp},\mathcal{R}^0)\mathcal{R}^0, J\mathcal{R}^{\perp}\right\rangle \\
&\qquad + \frac{1}{3}\left\langle 2JR^{TB}(\mathcal{R}^{\perp},\mathcal{R}^0)\mathcal{R}^{\perp} + T(\mathcal{R}, JA(\mathcal{R}^0)\mathcal{R}^{\perp}), J\mathcal{R}^{\perp}\right\rangle \\
&\quad - \frac{1}{3}\left\langle (\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(\mathcal{R},\mathcal{R})}J\mathcal{R}^{\perp}, J\mathcal{R}^{\perp}\right\rangle + \frac{1}{6}\left\langle \nabla^{TY}_{\mathcal{R}}(T(e^H_j, P^{T^HX}Je^H_i))Z_jZ_i, J\mathcal{R}^{\perp}\right\rangle \\
&\qquad + \frac{1}{3}\left\langle T(\mathcal{R}, J\mathcal{R}^{\perp}), T(\mathcal{R}^{\perp}, J\mathcal{R}^{\perp})\right\rangle + \frac{1}{4}\left|T(\mathcal{R}^{\perp}, J\mathcal{R}^{\perp})\right|^2 \\
&= -\frac{1}{3}\left\langle (\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(\mathcal{R},\mathcal{R})}J\mathcal{R}^{\perp}, J\mathcal{R}^{\perp}\right\rangle + \frac{1}{6}\left\langle \nabla^{TY}_{\mathcal{R}}(T(e^H_j, P^{T^HX}Je^H_i))Z_jZ_i, J\mathcal{R}^{\perp}\right\rangle \\
&\qquad + \frac{1}{3}\left\langle \nabla^{N_G}_{\mathcal{R}^0}(A(e^0_j)e^0_i)Z^0_jZ^0_i + R^{TB}(\mathcal{R}^{\perp},\mathcal{R}^0)\mathcal{R}^0, \mathcal{R}^{\perp}\right\rangle \\
&\quad - \frac{1}{4}\sum_j \left\langle T(\mathcal{R}^0, e^0_j), J\mathcal{R}^{\perp}\right\rangle^2 + \frac{1}{6}\left\langle T(\mathcal{R}, e^0_j), J\mathcal{R}^{\perp}\right\rangle\left\langle T(\mathcal{R}^0, e^0_j), J\mathcal{R}^{\perp}\right\rangle \\
&\qquad + \frac{7}{12}\left|T(\mathcal{R}^{\perp}, J\mathcal{R}^{\perp})\right|^2 + \frac{1}{3}\left\langle T(\mathcal{R}^0, J\mathcal{R}^{\perp}), T(\mathcal{R}^{\perp}, J\mathcal{R}^{\perp})\right\rangle.
\end{aligned} \tag{5.68}$$

To get (5.63) from (5.68), we need to compute $\nabla^{TY}_{e^H_k}(T(e^H_j, P^{T^HX}Je^H_i))$.

For W a section of TX, U a section of TB, we have

$$\begin{aligned}
\left\langle \nabla^{T^HX}_{e^H_k}P^{T^HX}W, U^H\right\rangle &= e^H_k\left\langle W, U^H\right\rangle - \left\langle P^{T^HX}W, \nabla^{TX}_{e^H_k}U^H\right\rangle \\
&= \left\langle P^{T^HX}\nabla^{TX}_{e^H_k}W, U^H\right\rangle + \left\langle P^{TY}W, \nabla^{TX}_{e^H_k}U^H\right\rangle.
\end{aligned} \tag{5.69}$$

From (1.7), (5.69), we get at x_0,

$$\nabla^{T^HX}_{e^H_k}P^{T^HX}W = P^{T^HX}\nabla^{TX}_{e^H_k}W - \frac{1}{2}\left\langle T(e^H_k, e^H_l), P^{TY}W\right\rangle e^H_l. \tag{5.70}$$

Remark that $Je^{\perp,H}_i \in TY, Je^0_i \in T^HX$ only hold on P.

From (3.40), (5.5b), (5.6a), (5.13), (5.31) and (5.70),

$$\begin{aligned}
(\nabla^{T^HX}_{e^H_k}P^{T^HX}Je^{\perp,H}_i)_{x_0} &= JA(P^{TX_G}e_k)e^{\perp}_i - \frac{1}{2}JT(e_k, e^{\perp}_i) - \frac{1}{2}\left\langle T(e_k, e_l), Je^{\perp}_i\right\rangle e_l \\
&= -\frac{1}{2}JT(e_k, e^{\perp}_i) - \frac{1}{2}\left\langle T(e_k, e_l) - T(P^{TX_G}e_k, P^{TX_G}e_l), Je^{\perp}_i\right\rangle e_l,
\end{aligned}$$

$$\begin{aligned}
(\nabla^{T^HX}_{e_k}P^{T^HX}Je^0_i)_{x_0} &= P^{T^HX}J\nabla^{TX}_{e^H_k}e^{0,H}_i = JA(e^0_i)P^{N_G}e_k - \frac{1}{2}JT(e_k, e^0_i) \\
&= -\frac{1}{2}JT(e_k, e^0_i) + \frac{1}{2}\left\langle JP^{N_G}e_k, T(e^0_i, e^0_l)\right\rangle e^0_l,
\end{aligned} \tag{5.71}$$

$$(\nabla^{TB}_{e_k}J_{x_0}e^0_i)_{x_0} = A(J_{x_0}e^0_i)e_k = -\frac{1}{2}JT(P^{TX_G}e_k, e^0_i) + \frac{1}{2}\left\langle JP^{N_G}e_k, T(e^0_i, e^0_l)\right\rangle e^0_l.$$

From (5.71), we get at x_0 that

$$
\begin{aligned}
(5.72)\quad &\left\langle \nabla^{TY}_{\mathcal{R}}(T(e_j^H, P^{T^HX}Je_i^H))Z_jZ_i, J\mathcal{R}^\perp\right\rangle - \left\langle \nabla^{TY}_{\mathcal{R}}(T(e_j, J_{x_0}e_i^0))Z_jZ_i^0, J\mathcal{R}^\perp\right\rangle \\
&= \left\langle T(e_j, \nabla^{T^HX}_{\mathcal{R}}P^{T^HX}Je_i^H - \nabla^{T^HX}_{\mathcal{R}}(J_{x_0}P^{T^HX}e_i)^H)Z_jZ_i, J\mathcal{R}^\perp\right\rangle \\
&= \left\langle T\left(\mathcal{R}, -\frac{1}{2}JT(\mathcal{R},\mathcal{R}^\perp) - \frac{1}{2}\left\langle T(\mathcal{R},e_l) - T(\mathcal{R}^0, P^{TX_G}e_l), J\mathcal{R}^\perp\right\rangle e_l\right), J\mathcal{R}^\perp\right\rangle \\
&\quad + \left\langle T\left(e_j, -\frac{1}{2}JT(e_k,e_i^0) + \frac{1}{2}JT(P^{TX_G}e_k, e_i^0)\right)Z_kZ_jZ_i^0, J\mathcal{R}^\perp\right\rangle \\
&\quad = -\frac{1}{2}\sum_l \left\langle T(\mathcal{R},e_l), J\mathcal{R}^\perp\right\rangle^2 + \frac{1}{2}\left\langle T(\mathcal{R},e_l^0), J\mathcal{R}^\perp\right\rangle\left\langle T(\mathcal{R}^0,e_l^0), J\mathcal{R}^\perp\right\rangle.
\end{aligned}
$$

From (5.68) and (5.72), $\mathcal{O}_2''$ is the coefficient of t^2 in the expansion of $|\frac{1}{t}\widetilde{\mu}|^2_{g^{TY}}(tZ)$. By (5.47), we get also the second equation of (5.63).
The proof of Lemma 5.6 is completed. □

The following is the main result of this Subsection.

Theorem 5.7. *The following identities hold,*

$$
\begin{aligned}
\mathcal{O}_1 =& 2\pi\sqrt{-1}\left\langle JT(\mathcal{R}^\perp, e_i^0), \mathcal{R}^\perp\right\rangle \nabla_{0,e_i^0} + 2\pi\sqrt{-1}\left\langle JT(\mathcal{R}, e_i^\perp), \mathcal{R}^\perp\right\rangle \nabla_{0,e_i^\perp} \\
&+ \pi\sqrt{-1}\left\langle JT(\mathcal{R}^0, e_i^\perp), e_i^\perp\right\rangle - \left\langle JT(e_i^0, Je_j^0), \mathcal{R}^\perp\right\rangle \nabla_{0,e_i^0}\nabla_{0,e_j^0} \\
&+ 4\pi^2\left\langle JT(\mathcal{R}^\perp, J\mathcal{R}^\perp), \mathcal{R}^\perp\right\rangle + 4\pi\sqrt{-1}\left\langle J\mathcal{R}^\perp, \widetilde{\mu}^E_{x_0}\right\rangle, \qquad (5.73)\\
\mathcal{O}_2 =& \mathcal{O}_2' + 4\pi^2\mathcal{O}_2'' - 4\pi\sqrt{-1}\Big(\frac{1}{2}\left\langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), \widetilde{\mu}^E_{x_0}\right\rangle - \left\langle J\mathcal{R}^\perp, \nabla^{TY}_{\mathcal{R}}\widetilde{\mu}^E\right\rangle\Big) \\
&- \left\langle \widetilde{\mu}^E_{x_0}, \widetilde{\mu}^E_{x_0}\right\rangle_{g^{TY}}.
\end{aligned}
$$

Proof. By (5.5e),

$$
\langle JT(\mathcal{R}, e_i), e_i\rangle = \left\langle JT(\mathcal{R}^0, e_i^\perp), e_i^\perp\right\rangle. \qquad (5.74)
$$

By (5.45a), (5.50) and (5.74),

$$
\begin{aligned}
&-\frac{2}{3}(\partial_{\mathcal{R}}R^{L_B})_{x_0}(\mathcal{R}, e_i)\nabla_{0,e_i} \\
&\qquad = 2\pi\sqrt{-1}\Big(\left\langle JT(\mathcal{R}^\perp, e_i^0), \mathcal{R}^\perp\right\rangle \nabla_{0,e_i^0} + \left\langle JT(\mathcal{R}, e_i^\perp), \mathcal{R}^\perp\right\rangle \nabla_{0,e_i^\perp}\Big), \qquad (5.75)\\
&-\frac{1}{3}(\partial_i R^{L_B})_{x_0}(\mathcal{R}, e_i) = \pi\sqrt{-1}\left\langle JT(\mathcal{R}^0, e_i^\perp), e_i^\perp\right\rangle.
\end{aligned}
$$

From (5.22), (5.26), (5.63) and (5.75), we get (5.73). □

5.3. **Computation of the coefficient Φ_1.** Recall that the operator $\mathscr{L}_2^0$ is defined in (5.22), $P_{\mathscr{L}^\perp}$ is the orthogonal projection from $L^2(\mathbb{R}^{n_0})$ onto $\operatorname{Ker}\mathscr{L}^\perp$ and $P_{\mathscr{L}}$ is the orthogonal projection from $L^2(\mathbb{R}^{2n-2n_0})$ onto $\operatorname{Ker}\mathscr{L}$ as in (3.19).

For $Z^{\perp} \in \mathbb{R}^{n_0}$, set

$$\begin{aligned}
\Psi_{1,1}(Z^{\perp}) &= \left((\mathscr{L}_2^0)^{-1}P^{N^{\perp}}\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^{\perp}}\mathcal{O}_1P^N\right)\left((0,Z^{\perp}),(0,Z^{\perp})\right),\\
\Psi_{1,2}(Z^{\perp}) &= -\left((\mathscr{L}_2^0)^{-1}P^{N^{\perp}}\mathcal{O}_2P^N\right)\left((0,Z^{\perp}),(0,Z^{\perp})\right),\\
\Psi_{1,3}(Z^{\perp}) &= \left((\mathscr{L}_2^0)^{-1}P^{N^{\perp}}\mathcal{O}_1P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^{\perp}}\right)\left((0,Z^{\perp}),(0,Z^{\perp})\right),\\
\Psi_{1,4}(Z^{\perp}) &= \left(P^N\mathcal{O}_1(\mathscr{L}_2^0)^{-2}P^{N^{\perp}}\mathcal{O}_1P^N\right)\left((0,Z^{\perp}),(0,Z^{\perp})\right),\\
\widetilde{\Psi}_{1,1}(Z^{\perp}) &= \left((\mathscr{L}_2^0)^{-1}P^{N^{\perp}}P_{\mathscr{L}^{\perp}}\mathcal{O}_1(\mathscr{L}_2^0)^{-1}P^{N^{\perp}}\mathcal{O}_1P^N\right)\left((0,Z^{\perp}),(0,Z^{\perp})\right),\\
\widetilde{\Psi}_{1,2}(Z^{\perp}) &= -\left((\mathscr{L}_2^0)^{-1}P^{N^{\perp}}P_{\mathscr{L}^{\perp}}\mathcal{O}_2P^N\right)\left((0,Z^{\perp}),(0,Z^{\perp})\right),\\
\Phi_{1,i} &= \int_{\mathbb{R}^{n_0}}\Psi_{1,i}(Z^{\perp})dv_{N_G}(Z^{\perp}),\quad \text{for } i=1,2,3,4.
\end{aligned} \tag{5.76}$$

Proposition 5.8. *The following two identities hold for $i = 1, 2$,*

$$\int_{\mathbb{R}^{n_0}}\widetilde{\Psi}_{1,i}(Z^{\perp})dv_{N_G}(Z^{\perp}) = \Phi_{1,i}. \tag{5.77}$$

Proof. In fact, in our case, by (3.21), $P^N = P_{\mathscr{L}} \otimes P_{\mathscr{L}^{\perp}} \otimes \mathrm{Id}_E$.

By (3.18), (3.19),

$$\left((\mathscr{L}_2^0)^{-1}P^{N^{\perp}}\mathcal{O}_2P^N\right)(Z,(0,Z'^{\perp})) = \left((\mathscr{L}_2^0)^{-1}P^{N^{\perp}}\mathcal{O}_2P_{\mathscr{L}}(\cdot,0)G^{\perp}\right)(Z)G^{\perp}(Z'^{\perp}). \tag{5.78}$$

From Theorem 3.1, (5.78),

$$\begin{aligned}
\Phi_{1,2} &= \left\langle\left(-(\mathscr{L}_2^0)^{-1}P^{N^{\perp}}\mathcal{O}_2P_{\mathscr{L}}(\cdot,0)G^{\perp}\right)(0,Z^{\perp}),G^{\perp}(Z^{\perp})\right\rangle_{L^2(\mathbb{R}^{n_0})}\\
&= \left\langle\left(-(\mathscr{L}_2^0)^{-1}P^{N^{\perp}}P_{\mathscr{L}^{\perp}}\mathcal{O}_2P_{\mathscr{L}}(\cdot,0)G^{\perp}\right)(0,Z^{\perp}),G^{\perp}(Z^{\perp})\right\rangle_{L^2(\mathbb{R}^{n_0})}\\
&= \int_{\mathbb{R}^{n_0}}\widetilde{\Psi}_{1,2}(Z^{\perp})dv_{N_G}(Z^{\perp}).
\end{aligned} \tag{5.79}$$

In the same way, we get (5.77) for $i = 1$. □

Note that the restriction of $\|\cdot\|_{t,0}$ in (2.115) on $\mathscr{C}^{\infty}(\mathbb{R}^{2n-n_0}, E_{G,x_0})$ does not depend on t and we denote it by $\|\cdot\|_0$.

Since $\mathscr{L}_2^t$ in (5.22) is a self-adjoint elliptic operator with respect to $\|\cdot\|_0$ as we conjugated the operator with $\kappa^{1/2}$, $\mathscr{L}_2^0$ and $\mathcal{O}_r$ are also formally self-adjoint with respect to $\|\cdot\|_0$. Thus in the right hand side of (3.62), the third and fourth terms are the adjoints of the first two terms.

From (3.62), (5.1) and (5.76), we get

$$\Phi_1 = \Phi_{1,1} + \Phi_{1,2} + (\Phi_{1,1} + \Phi_{1,2})^* + \Phi_{1,3} - \Phi_{1,4}. \tag{5.80}$$

From (5.76), (5.77), (5.80), we learn that in order to compute Φ_1, we only need to evaluate $\Psi_{1,i}$ and $\widetilde{\Psi}_{1,i}$ ($i \in \{1,2,3,4\}$).

Lemma 5.9. *The following identity holds,*

$$\widetilde{\Psi}_{1,1}(Z^{\perp}) = -\frac{1}{8\pi}\Big|T(\tfrac{\partial}{\partial\overline{z}_j^0}, e_k^{\perp})\Big|^2 P_{\mathscr{L}^{\perp}}(Z^{\perp}, Z^{\perp}). \tag{5.81}$$

Proof. Recall that the operators b_i, b_i^+, $b_j^{\perp}$ and $b_j^{\perp+}$ have been defined in (3.8). In particular, by (5.15), one has

$$4\pi Z_j^{\perp} = b_j^{\perp} + b_j^{\perp+}, \quad \nabla_{0,e_j^{\perp}} = \tfrac{\partial}{\partial Z_j^{\perp}} = \tfrac{1}{2}(b_j^{\perp+} - b_j^{\perp}). \tag{5.82}$$

By (3.8), (3.9) and (5.82), set

$$\begin{aligned} B_{jk}^{\perp} &= (4\pi)^2 Z_j^{\perp} Z_k^{\perp} = b_j^{\perp+} b_k^{\perp+} + b_k^{\perp} b_j^{\perp+} + b_j^{\perp} b_k^{\perp+} + b_j^{\perp} b_k^{\perp} + 4\pi\delta_{jk}, \\ B_{ijk}^{\perp} &= b_i^{\perp} b_j^{\perp} b_k^{\perp} + 3b_i^{\perp} b_j^{\perp} b_k^{\perp+} + 3b_i^{\perp} b_j^{\perp+} b_k^{\perp+} + b_i^{\perp+} b_j^{\perp+} b_k^{\perp+}. \end{aligned} \tag{5.83}$$

If a_{ijk} is symmetric on i, j, k, then by (3.8), (3.9), (5.82) and (5.83), one verifies

$$a_{ijk}(4\pi)^3 Z_i^{\perp} Z_j^{\perp} Z_k^{\perp} = a_{ijk} B_{ijk}^{\perp} + 12\pi a_{ijj}(b_i^{\perp} + b_i^{\perp+}). \tag{5.84}$$

By (3.9), (5.5e), (5.14), (5.82), (5.83) and the fact that $T(\,,\,)$ is anti-symmetric, we get

$$\begin{aligned} 2\pi \left\langle JT(\mathcal{R}^{\perp}, e_i^{\perp}), \mathcal{R}^{\perp} \right\rangle \nabla_{0,e_i^{\perp}} &= \frac{1}{16\pi}\widetilde{\mathcal{T}}_{jik} B_{jk}^{\perp}(b_i^{\perp+} - b_i^{\perp}) \\ &= \frac{1}{16\pi}\widetilde{\mathcal{T}}_{jik}\left[(b_j^{\perp} b_k^{\perp+} + b_j^{\perp} b_k^{\perp}) b_i^{\perp+} - (b_j^{\perp+} b_k^{\perp+} + b_k^{\perp} b_j^{\perp+} + b_j^{\perp} b_k^{\perp+}) b_i^{\perp}\right] \\ &= -\frac{1}{8\pi}\widetilde{\mathcal{T}}_{ijk}(b_j^{\perp} b_k^{\perp+} + b_j^{\perp} b_k^{\perp}) b_i^{\perp+}. \end{aligned} \tag{5.85}$$

By Theorem 5.1, Remark 5.2, (3.9), (3.12), (5.14), (5.73), (5.83)-(5.85), we can reformulate $\mathcal{O}_1$ as follows by using the creation and annihilation operators introduced in (3.8),

$$\begin{aligned} \mathcal{O}_1 = &-\frac{\sqrt{-1}}{8\pi}\left\langle JT(\tfrac{\partial}{\partial z_i^0}, e_j^{\perp}), e_k^{\perp}\right\rangle B_{jk}^{\perp} b_i^{+} + b_i \frac{\sqrt{-1}}{8\pi}\left\langle JT(\tfrac{\partial}{\partial \overline{z}_i^0}, e_j^{\perp}), e_k^{\perp}\right\rangle B_{jk}^{\perp} \\ &+ \frac{\sqrt{-1}}{4}\left\langle JT(\mathcal{R}^0, e_i^{\perp}), e_j^{\perp}\right\rangle (b_i^{\perp+} b_j^{\perp+} - b_i^{\perp} b_j^{\perp}) - \frac{\sqrt{-1}}{8\pi}\widetilde{\mathcal{T}}_{ijk}(b_j^{\perp} b_k^{\perp+} + b_j^{\perp} b_k^{\perp}) b_i^{\perp+} \\ &- \frac{\sqrt{-1}}{4\pi}\left\langle JT(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0}), e_k^{\perp}\right\rangle (b_k^{\perp+} + b_k^{\perp})(2b_j b_i^{+} + 4\pi\delta_{ij}) + \sqrt{-1}\left\langle Je_j^{\perp}, \widetilde{\mu}_{x_0}^E\right\rangle (b_j^{\perp+} + b_j^{\perp}) \\ &+ \frac{1}{16\pi}\left\langle JT(e_i^{\perp}, Je_j^{\perp}), e_k^{\perp}\right\rangle [B_{ijk}^{\perp} + 12\pi\delta_{ik}(b_j^{\perp+} + b_j^{\perp})] \\ = &-\frac{\sqrt{-1}}{8\pi}\mathcal{T}_{jk}(\tfrac{\partial}{\partial z_i^0}) B_{jk}^{\perp} b_i^{+} + \frac{\sqrt{-1}}{8\pi}\mathcal{T}_{jk}(\tfrac{\partial}{\partial \overline{z}_i^0}) b_i B_{jk}^{\perp} + \frac{\sqrt{-1}}{4}\mathcal{T}_{ij}(\mathcal{R}^0)(b_i^{\perp+} b_j^{\perp+} - b_i^{\perp} b_j^{\perp}) \\ &+ \sqrt{-1}\left\langle Je_j^{\perp}, \widetilde{\mu}_{x_0}^E\right\rangle (b_j^{\perp+} + b_j^{\perp}) - \frac{\sqrt{-1}}{4\pi}\left\langle JT(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0}), e_k^{\perp}\right\rangle (b_k^{\perp+} + b_k^{\perp})(2b_j b_i^{+} + 4\pi\delta_{ij}) \\ &- \frac{\sqrt{-1}}{8\pi}\widetilde{\mathcal{T}}_{ijk}(b_j^{\perp} b_k^{\perp+} + b_j^{\perp} b_k^{\perp}) b_i^{\perp+} + \frac{1}{16\pi}\mathcal{T}_{ijk}[B_{ijk}^{\perp} + 12\pi\delta_{ik}(b_j^{\perp+} + b_j^{\perp})]. \end{aligned} \tag{5.86}$$

From Theorem 3.1, (3.54), (5.83), (5.86) and $a_i = a_i^+ = 2\pi$, we get

$$\begin{aligned}(5.87)\quad \big((\mathscr{L}_2^0)^{-1}\mathcal{O}_1 P^N\big)(Z,Z') = \sqrt{-1}\Big\{ &\frac{b_l}{8\pi}\mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_l^0}) + \big\langle Je_k^\perp, \widetilde{\mu}_{x_0}^E\big\rangle \frac{b_k^\perp}{4\pi} \\ &- \Big\langle JT(\tfrac{\partial}{\partial z_l^0}, \tfrac{\partial}{\partial\overline{z}_l^0}), e_k^\perp\Big\rangle \frac{b_k^\perp}{4\pi} - \frac{b_l^\perp b_k^\perp}{32\pi}\mathcal{T}_{kl}(z^0 + \overline{z}'^0) \\ &- \frac{\sqrt{-1}}{16\pi}\mathcal{T}_{klm}\Big[\frac{b_m^\perp b_l^\perp b_k^\perp}{12\pi} + 3b_k^\perp\delta_{lm}\Big]\Big\} P^N(Z,Z').\end{aligned}$$

By Theorem 3.1, (3.55), (5.83) and (5.86),

$$\begin{aligned}(5.88)\quad P^{N^\perp}P_{\mathscr{L}^\perp}\mathcal{O}_1 = \sqrt{-1}P^{N^\perp}P_{\mathscr{L}^\perp}\Big\{ &-\frac{1}{2}\mathcal{T}_{jj}(\tfrac{\partial}{\partial z_i^0})b_i^+ + \frac{1}{2}\mathcal{T}_{jj}(\tfrac{\partial}{\partial\overline{z}_i^0})b_i + \big\langle Je_j^\perp, \widetilde{\mu}_{x_0}^E\big\rangle b_j^{\perp+} \\ &- \frac{1}{4}\Big\langle JT(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial\overline{z}_j^0}), e_{j'}^\perp\Big\rangle b_{j'}^{\perp+}(2b_j b_i^+ + 4\pi\delta_{ij}) \\ &+ \frac{1}{4}\Big(\mathcal{T}_{jj'}(\mathcal{R}^0) - \mathcal{T}_{jj'}(\tfrac{\partial}{\partial z_i^0})\frac{b_i^+}{2\pi} + \mathcal{T}_{jj'}(\tfrac{\partial}{\partial\overline{z}_i^0})\frac{b_i}{2\pi}\Big)b_j^{\perp+}b_{j'}^{\perp+} \\ &- \frac{\sqrt{-1}}{16\pi}\mathcal{T}_{ijj'}[b_i^{\perp+}b_j^{\perp+}b_{j'}^{\perp+} + 12\pi\delta_{ij'}b_j^{\perp+}].\end{aligned}$$

In the following equation, by (3.9), (3.54), (3.55), we only need to pair the terms in (5.87) and (5.88) which have the same length on $b_j^{\perp+}$ and $b_j^\perp$, and the total degree on $b_i, b_i^+, z^0, \overline{z}^0$ should not be zero. Thus by (3.9), (3.54), (5.87) and (5.88),

$$\begin{aligned}(5.89)\quad \Big(P^{N^\perp}P_{\mathscr{L}^\perp}\mathcal{O}_1(\mathscr{L}_2^0)^{-1}\mathcal{O}_1 P^N\Big)(Z,(0,Z'^\perp)) = \Big\{ &P^{N^\perp}\Big[-\frac{1}{16\pi}\Big(\sum_{ij} b_i\mathcal{T}_{jj}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big)^2 \\ &+ \frac{1}{128\pi}\Big(\mathcal{T}_{jj'}(\mathcal{R}^0) + \frac{b_i}{2\pi}\mathcal{T}_{jj'}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big)b_j^{\perp+}b_{j'}^{\perp+}\cdot b_l^\perp b_k^\perp\mathcal{T}_{kl}(z^0)\Big]P^N\Big\}(Z,(0,Z'^\perp)).\end{aligned}$$

From (3.9), (3.54), (5.5d), (5.14), (5.89) and $a_i = a_i^+ = 2\pi$, one gets

$$\begin{aligned}(5.90)\quad \Big(P^{N^\perp}P_{\mathscr{L}^\perp}\mathcal{O}_1(\mathscr{L}_2^0)^{-1}\mathcal{O}_1 P^N\Big)(Z,(0,Z'^\perp)) = \Big\{ &P^{N^\perp}\Big[-\frac{1}{16\pi}\Big(\sum_{ij} b_i\mathcal{T}_{jj}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big)^2 \\ &+ \frac{1}{8}\Big\langle 2\pi JT(\mathcal{R}^0, e_l^\perp) + b_i JT(\tfrac{\partial}{\partial\overline{z}_i^0}, e_l^\perp), JT(z^0, e_l^\perp)\Big\rangle\Big]P^N\Big\}(Z,(0,Z'^\perp)).\end{aligned}$$

Set $P_{\mathscr{L}}^\perp = \mathrm{Id}_{L^2(\mathbb{R}^{2n-2n_0})} - P_{\mathscr{L}}$.

Let $h_i(Z^0)$ (resp. $F(Z^0)$) be polynomials in Z^0 with degree 1 (resp. 2) and $a_{ij}\in\mathbb{C}$. By Theorem 3.1, (3.9) and (3.54),

$$(5.91)\quad \big(F(Z^0)P_{\mathscr{L}}\big)(Z^0,0) = \Big(\frac{1}{2}\frac{\partial^2 F}{\partial z_i^0\partial z_j^0}z_i^0 z_j^0 + \frac{\partial^2 F}{\partial z_i^0\partial\overline{z}_j^0}z_i^0\frac{b_j}{a_j} + \frac{1}{2}\frac{\partial^2 F}{\partial\overline{z}_i^0\partial\overline{z}_j^0}\frac{b_i b_j}{a_i a_j}\Big)P_{\mathscr{L}}(Z^0,0).$$

By Theorem 3.1, (3.8), (3.9), (3.19), (3.54), (5.91) and $a_j = 2\pi$, we have

$$
\begin{aligned}
&(P^{\perp}_{\mathscr{L}} F P_{\mathscr{L}})(0,0) = -\frac{1}{\pi}\frac{\partial^2 F}{\partial z_i^0 \partial \overline{z}_i^0},\\
&\left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} a_{ij} b_i b_j P_{\mathscr{L}}\right)(0,0) = \left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} h_i P_{\mathscr{L}}\right)(0,0) = 0,\\
&\left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} h_i b_i P_{\mathscr{L}}\right)(0,0) = \left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} b_i h_i P_{\mathscr{L}}\right)(0,0) = -\frac{1}{2\pi}\frac{\partial h_i}{\partial z_i^0},\\
&\left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} F P_{\mathscr{L}}\right)(0,0) = -\frac{1}{4\pi^2}\frac{\partial^2 F}{\partial z_i^0 \partial \overline{z}_i^0},\\
&\left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} b_i F b_j P_{\mathscr{L}}\right)(0,0) = -\left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} b_i b_j F P_{\mathscr{L}}\right)(0,0) = -\frac{1}{2\pi}\frac{\partial^2 F}{\partial z_i^0 \partial z_j^0},\\
&\left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} F b_i b_j P_{\mathscr{L}}\right)(0,0) = -\frac{3}{2\pi}\frac{\partial^2 F}{\partial z_i^0 \partial z_j^0},\\
&\left(\mathscr{L}^{-1} P^{\perp}_{\mathscr{L}} \Big(\sum_i b_i h_i\Big)^2 P_{\mathscr{L}}\right)(0,0) = -\frac{1}{2\pi}\Big(\frac{\partial h_i}{\partial z_j^0}\frac{\partial h_j}{\partial z_i^0} - \Big(\sum_i \frac{\partial h_i}{\partial z_i^0}\Big)^2\Big).
\end{aligned}
\tag{5.92}
$$

Finally by (5.77), (5.90), (5.92) and $\mathscr{L}_2^0 = \mathscr{L} + \mathscr{L}^{\perp}$, we get (5.81). □

Lemma 5.10. *The following identity holds,*

$$
\Phi_{1,3} = \Phi_{1,4}. \tag{5.93}
$$

Proof. Let $\mathcal{F}_2 \in T^*_{x_0} X_G$ with values in real polynomials on $Z^{\perp}$ with even degree, $\mathcal{F}_1 \in N^*_{G,x_0}$, $\mathcal{F}_3(Z^{\perp})$ a polynomial on $Z^{\perp}$ with odd degree, be defined by

$$
\begin{aligned}
&\mathcal{F}_1(e_k^{\perp}) = \sqrt{-1}\left\langle J e_k^{\perp}, \widetilde{\mu}^E_{x_0}\right\rangle - \sqrt{-1}\left\langle JT(\tfrac{\partial}{\partial z_l^0}, \tfrac{\partial}{\partial \overline{z}_l^0}), e_k^{\perp}\right\rangle + \frac{3}{4}\mathcal{T}_{llk},\\
&\mathcal{F}_2(\cdot, Z^{\perp}) P^N(Z, Z') = \Big(\mathcal{T}_{kl}(\cdot)\frac{b_l^{\perp} b_k^{\perp}}{32\pi} P^N\Big)(Z, Z'),\\
&\mathcal{F}_3(Z^{\perp}) P^N(Z, Z') = \frac{1}{16\pi}\Big(\mathcal{T}_{klm}\frac{b_m^{\perp} b_l^{\perp} b_k^{\perp}}{12\pi} P^N\Big)(Z, Z').
\end{aligned}
\tag{5.94}
$$

Then from (3.54), (5.87) and (5.94),

$$
\begin{aligned}
\left((\mathscr{L}_2^0)^{-1}\mathcal{O}_1 P^N\right)(Z, Z') = \Big(&\frac{\sqrt{-1}}{4}\mathcal{T}_{kk}(\overline{z}^0 - \overline{z}'^0) - \sqrt{-1}\mathcal{F}_2(z^0 + \overline{z}'^0, Z^{\perp})\\
&+ \Big(\mathcal{F}_1 + \mathcal{F}_3\Big)(Z^{\perp})\Big) P^N(Z, Z').
\end{aligned}
\tag{5.95}
$$

Observe that $\mathcal{F}_i(Z^{\perp})^* = \mathcal{F}_i(Z^{\perp})$ for $i = 1, 3$, thus from (5.87) and (5.95),

$$
\begin{aligned}
&\left(P^N \mathcal{O}_1 (\mathscr{L}_2^0)^{-1}\right)(Z', Z) = \left(\left((\mathscr{L}_2^0)^{-1}\mathcal{O}_1 P^N\right)(Z, Z')\right)^*\\
&= \Big(-\frac{\sqrt{-1}}{4}\mathcal{T}_{kk}(z^0 - z'^0) + \sqrt{-1}\mathcal{F}_2(\overline{z}^0 + z'^0, Z^{\perp}) + \Big(\mathcal{F}_1 + \mathcal{F}_3\Big)(Z^{\perp})\Big) P^N(Z', Z).
\end{aligned}
\tag{5.96}
$$

For $h_1(z^0), h_2(\overline{z}^0)$ two linear functions on $z^0, \overline{z}^0$, by Theorem 3.1, (3.54),

$$
(P_{\mathscr{L}} h_1(z^0) h_2(\overline{z}^0) P_{\mathscr{L}})(0,0) = \Big(P_{\mathscr{L}} h_1(z^0)\frac{\partial h_2}{\partial \overline{z}_i^0}\frac{b_i}{2\pi} P_{\mathscr{L}}\Big)(0,0) = \frac{1}{\pi}\frac{\partial h_1}{\partial z_i^0}\frac{\partial h_2}{\partial \overline{z}_i^0}. \tag{5.97}
$$

From (5.76), (5.95)-(5.97),

$$\Psi_{1,3}(Z^{\perp})=\Big[\Big((\mathcal{F}_1+\mathcal{F}_3)(Z^{\perp})\Big)^2+\frac{1}{\pi}\Big|\frac{1}{4}\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_i^0})+\mathcal{F}_2(\tfrac{\partial}{\partial\overline{z}_i^0},Z^{\perp})\Big|^2\Big]G^{\perp}(Z^{\perp})^2. \tag{5.98}$$

By Theorem 3.1, (3.18), (5.93), $\mathcal{F}_jG^{\perp}$, $(j=1,3)$, $\mathcal{F}_2(\frac{\partial}{\partial\overline{z}_i^0},\cdot)G^{\perp}$ are eigenfunctions of $\mathscr{L}^{\perp}$ with eigenvalues $4\pi j$, 8π, thus they are orthogonal to each other.

From (5.76), (5.95)-(5.97), we have

$$\begin{aligned}\Psi_{1,4}(Z^{\perp})=G^{\perp}(Z^{\perp})^2\int_{\mathbb{R}^{n_0}}\Big\{&\Big((\mathcal{F}_1G^{\perp})(Z'^{\perp})\Big)^2+\big((\mathcal{F}_3G^{\perp})(Z'^{\perp})\big)^2\\&+\frac{1}{16\pi}\Big|\sum_k\mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_i^0})G^{\perp}\Big|^2(Z'^{\perp})+\frac{1}{\pi}\Big|\mathcal{F}_2(\tfrac{\partial}{\partial\overline{z}_i^0},\cdot)G^{\perp}\Big|^2(Z'^{\perp})\Big\}dv_{N_G}(Z'^{\perp}).\end{aligned} \tag{5.99}$$

From (3.18), (5.76), (5.98), (5.99) and the above discussion, we get (5.93). □

Now we need to compute the contribution from $-(\mathscr{L}_2^0)^{-1}P^{N^{\perp}}\mathcal{O}_2P^N$.

Lemma 5.11. *The following identity holds,*

$$\begin{aligned}\widetilde{\Psi}_{1,2}(Z^{\perp})=\Big\{&\frac{1}{2\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_i^0})\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle+\frac{1}{48\pi}\Big\langle R^{TB}(e_k^{\perp},\tfrac{\partial}{\partial z_j^0})e_k^{\perp},\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle\\&+\frac{1}{96\pi}\Big|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2-\frac{\sqrt{-1}}{16\pi}\Big\langle T(e_k^{\perp},Je_k^{\perp}),T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big\rangle+\frac{13}{192\pi}\Big|T(e_k^{\perp},\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2\\&+\frac{\sqrt{-1}}{96\pi}\Big\langle 11\,\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^{\perp},\tfrac{\partial}{\partial\overline{z}_j^0}))+4\,\nabla^{TY}_{\frac{\partial}{\partial\overline{z}_j^0}}(T(e_k^{\perp},\tfrac{\partial}{\partial z_j^0}))+7\,\nabla^{TY}_{e_k^{\perp}}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})),Je_k^{\perp}\Big\rangle\\&-\frac{2}{3\pi}\nabla_{\frac{\partial}{\partial z_j^0}}\nabla_{\frac{\partial}{\partial\overline{z}_j^0}}\log h+\frac{1}{2\pi}R^{E_B}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big\}P_{\mathscr{L}^{\perp}}(Z^{\perp},Z^{\perp}).\end{aligned} \tag{5.100}$$

Proof. By (3.9), (3.12), (3.54), (5.23) and (5.82),

$$I_1P^N=\Big\{\frac{1}{2}b_i^{\perp}B(Z,\tfrac{\partial}{\partial Z_i^{\perp}})+b_jB(Z,\tfrac{\partial}{\partial\overline{z}_j^0})+\tfrac{\partial}{\partial z_j^0}\Big(B(Z,\tfrac{\partial}{\partial\overline{z}_j^0})\Big)-\tfrac{\partial}{\partial\overline{z}_j^0}\Big(B(Z,\tfrac{\partial}{\partial z_j^0})\Big)\Big\}P^N. \tag{5.101}$$

By (3.55) and (5.101),

$$P_{\mathscr{L}^{\perp}}I_1P^N=P_{\mathscr{L}^{\perp}}\Big\{b_jB(Z,\tfrac{\partial}{\partial\overline{z}_j^0})+\tfrac{\partial}{\partial z_j^0}\Big(B(Z,\tfrac{\partial}{\partial\overline{z}_j^0})\Big)-\tfrac{\partial}{\partial\overline{z}_j^0}\Big(B(Z,\tfrac{\partial}{\partial z_j^0})\Big)\Big\}P^N. \tag{5.102}$$

By (5.45b), and observe that from Theorem 3.1, only the monomials which have even degree on $Z^{\perp}$ and $\nabla_{e_j^{\perp}}$, and which have also strictly positive degree on Z^0 and ∇_{0,e_j^0}, have contributions in $P^{N^{\perp}}P_{\mathscr{L}^{\perp}}I_1P^N$.

By Remark 5.2, (3.55) and (5.45b),

$$\begin{aligned}&P^{N^{\perp}}P_{\mathscr{L}^{\perp}}\Big(\tfrac{\partial}{\partial z_j^0}\Big(B(Z,\tfrac{\partial}{\partial\overline{z}_j^0})\Big)-\tfrac{\partial}{\partial\overline{z}_j^0}\Big(B(Z,\tfrac{\partial}{\partial z_j^0})\Big)\Big)P^N=-\pi\sqrt{-1}P^{N^{\perp}}P_{\mathscr{L}^{\perp}}\\&\quad\frac{1}{6}\Big\{\tfrac{\partial}{\partial z_j^0}\Big\langle R^{TX_G}(\mathcal{R}^0,J\mathcal{R}^0)\mathcal{R}^0,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle-\tfrac{\partial}{\partial z_j^0}\Big\langle R^{TX_G}(\mathcal{R}^0,J\mathcal{R}^0)\mathcal{R}^0,\tfrac{\partial}{\partial z_j^0}\Big\rangle\Big\}P^N\\&=-\frac{\pi}{3}P^{N^{\perp}}\Big\langle 2R^{TX_G}(z^0,\overline{z}^0)\tfrac{\partial}{\partial z_j^0}+R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\mathcal{R}^0)z^0+R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\overline{z}^0)\mathcal{R}^0,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle P^N.\end{aligned} \tag{5.103}$$

By (5.22), (5.92) and (5.103),

(5.104)
$$-\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}P_{\mathscr{L}^\perp}\Big(\tfrac{\partial}{\partial z_j^0}\Big(B(Z,\tfrac{\partial}{\partial \overline{z}_j^0})\Big)-\tfrac{\partial}{\partial \overline{z}_j^0}\Big(B(Z,\tfrac{\partial}{\partial z_j^0})\Big)\Big)P^N\Big)((0,Z^\perp),(0,Z^\perp))$$
$$=-\frac{1}{6\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_j^0}+R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle P_{\mathscr{L}^\perp}(Z^\perp,Z^\perp).$$

Observe that if Q is an odd degree monomial on $b_j, b_j^+, z_j^0, \overline{z}_j^0$, then

$$\big(QP^N\big)\big((0,Z^\perp),(0,Z'^\perp)\big)=0. \tag{5.105}$$

By using this observation and (5.45b), we get

(5.106)
$$\begin{aligned}
&-\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}b_jB(Z,\tfrac{\partial}{\partial \overline{z}_j^0})P^N\Big)((0,Z^\perp),(0,Z'^\perp))\\
&\quad=\pi\sqrt{-1}\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}b_j\Big[\frac{1}{6}\Big\langle R^{TX_G}(\mathcal{R}^0,J\mathcal{R}^0)\mathcal{R}^0,\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle\\
&\quad-\frac{5}{4}\Big\langle\nabla^{TY}_{\mathcal{R}^0}(T(e_k^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}))Z_k^\perp+\nabla^{TY}_{\mathcal{R}^\perp}(T(e_k^0,\tfrac{\partial}{\partial \overline{z}_j^0}))Z_k^0,J\mathcal{R}^\perp\Big\rangle\\
&\quad+\Big\langle\frac{1}{2}R^{TB}(\mathcal{R}^\perp,J\mathcal{R}^0)\mathcal{R}^\perp+\sqrt{-1}R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)\mathcal{R}^\perp,\tfrac{\partial}{\partial \overline{z}_j^0}\Big\rangle\\
&\quad-\frac{3}{8}\sqrt{-1}\big\langle J\mathcal{R}^\perp,T(\mathcal{R}^0,e_i^0)\big\rangle\Big\langle J\mathcal{R}^\perp,T(e_i^0,\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle\\
&\quad-\frac{1}{8}\Big\langle T(\mathcal{R}^\perp,J\mathcal{R}^0),T(\mathcal{R}^\perp,\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle+\frac{1}{2}\Big\langle T(\mathcal{R}^\perp,J\mathcal{R}^\perp),T(\mathcal{R}^0,\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle\\
&\quad-\frac{1}{8}\Big\langle JT(\tfrac{\partial}{\partial \overline{z}_j^0},J\mathcal{R}^0),e_j^\perp\Big\rangle\big\langle J\mathcal{R}^\perp,T(\mathcal{R}^\perp,e_j^\perp)\big\rangle\Big]P^N\Big\}((0,Z^\perp),(0,Z'^\perp)).
\end{aligned}$$

From (3.6), (3.54), (5.5b) and (5.83), we have

$$\Big\langle T(\tfrac{\partial}{\partial z_j^0},e_i^0),T(e_i^0,\tfrac{\partial}{\partial \overline{z}_j^0})\Big\rangle=-2\Big|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2, \tag{5.107a}$$

$$P_{\mathscr{L}^\perp}Z_k^\perp Z_l^\perp P_{\mathscr{L}^\perp}=\frac{\delta_{kl}}{4\pi}P_{\mathscr{L}^\perp}. \tag{5.107b}$$

By (3.54), (5.5e), (5.92), (5.106) and (5.107a),

$$
\begin{aligned}
(5.108)\quad & -\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}P_{\mathscr{L}^\perp}b_jB(Z,\tfrac{\partial}{\partial\overline{z}_j^0})P^N\Big)\left((0,Z^\perp),(0,Z^\perp)\right)\\
&=\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}b_j\Big[\frac{\pi}{3}\left\langle R^{TX_G}(z^0,\overline{z}^0)\mathcal{R}^0,\tfrac{\partial}{\partial\overline{z}_j^0}\right\rangle\\
&\quad-\frac{5\sqrt{-1}}{16}\left\langle\nabla^{TY}_{\mathcal{R}^0}(T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}))+\nabla^{TY}_{e_k^\perp}(T(e_i^0,\tfrac{\partial}{\partial\overline{z}_j^0}))Z_i^0,Je_k^\perp\right\rangle\\
&\quad+\frac{1}{8}\left\langle\sqrt{-1}R^{TB}(e_k^\perp,J\mathcal{R}^0)e_k^\perp-2R^{TB}(e_k^\perp,\mathcal{R}^0)e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}\right\rangle\\
&\quad+\frac{3}{32}\left\langle T(\mathcal{R}^0,e_i^0),T(e_i^0,\tfrac{\partial}{\partial\overline{z}_j^0})\right\rangle-\frac{\sqrt{-1}}{32}\left\langle T(e_k^\perp,J\mathcal{R}^0),T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})\right\rangle\\
&\quad+\frac{\sqrt{-1}}{8}\left\langle T(e_k^\perp,Je_k^\perp),T(\mathcal{R}^0,\tfrac{\partial}{\partial\overline{z}_j^0})\right\rangle\Big]P^N\Big\}\left((0,Z^\perp),(0,Z^\perp)\right)\\
&=\Big\{-\frac{1}{12\pi}\left\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_i^0})\tfrac{\partial}{\partial z_i^0}+R^{TX_G}(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_i^0})\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0}\right\rangle\\
&\quad+\frac{5\sqrt{-1}}{32\pi}\left\langle\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}))+\nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})),Je_k^\perp\right\rangle\\
&\quad+\frac{3}{16\pi}\left\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}\right\rangle+\frac{3}{32\pi}\left|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})\right|^2\\
&\quad-\frac{1}{64\pi}\left|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})\right|^2-\frac{\sqrt{-1}}{16\pi}\left\langle T(e_k^\perp,Je_k^\perp),T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})\right\rangle\Big\}P_{\mathscr{L}^\perp}(Z^\perp,Z^\perp).
\end{aligned}
$$

For $G_1(Z)$ (resp. $G_2(Z)$) polynomials on Z with degree 1 (resp. 2) and $F\in T^*_{x_0}X_G\otimes T^*_{x_0}X_G$, by Theorem 3.1, (3.9), (3.12), (3.19), (3.54) and (3.55), for any $k,l,\,k',l'$,

$$
\begin{aligned}
&\nabla_{0,e_j^\perp}P^N=-2\pi Z_j^\perp P^N,\\
&P^{N^\perp}P_{\mathscr{L}^\perp}(G_1(Z)b_k^\perp+G_2(Z)b_k^\perp b_l^\perp+Z_{k'}^\perp b_{l'})P^N=0,\\
(5.109)\quad&\frac{1}{3}\left\langle R^{TB}(\mathcal{R}^\perp,e_i^\perp)\mathcal{R}^\perp,e_j^\perp\right\rangle\nabla_{0,e_i^\perp}\nabla_{0,e_j^\perp}P^N=-\frac{2\pi}{3}\left\langle R^{TB}(\mathcal{R}^\perp,e_j^\perp)\mathcal{R}^\perp,e_j^\perp\right\rangle P^N,\\
&F(e_i^0,e_j^0)\nabla_{0,e_i^0}\nabla_{0,e_j^0}P^N=\Big[F(\tfrac{\partial}{\partial\overline{z}_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})b_ib_j-4\pi F(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big]P^N.
\end{aligned}
$$

By (5.23), (5.109), we get

$$
\begin{aligned}
(5.110)\quad I_2P^N=&\Big\{\Big(\Big\langle\frac{1}{3}R^{TX_G}(\mathcal{R}^0,\tfrac{\partial}{\partial\overline{z}_i^0})\mathcal{R}^0+R^{TB}(\mathcal{R}^\perp,\tfrac{\partial}{\partial\overline{z}_i^0})\mathcal{R}^\perp+\nabla^{TX_G}_{\mathcal{R}^0}(A(\tfrac{\partial}{\partial\overline{z}_i^0})\mathcal{R}^\perp),\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle\\
&-3\left\langle A(\tfrac{\partial}{\partial\overline{z}_i^0})\mathcal{R}^\perp,A(\tfrac{\partial}{\partial\overline{z}_j^0})\mathcal{R}^\perp\right\rangle+\left\langle\tfrac{\partial}{\partial\overline{z}_i^0},\nabla^{TX_G}_{\mathcal{R}^0}(A(\tfrac{\partial}{\partial\overline{z}_j^0})\mathcal{R}^\perp)\right\rangle\Big)b_ib_j\\
&-4\pi\Big\langle\frac{1}{3}R^{TX_G}(\mathcal{R}^0,\tfrac{\partial}{\partial z_j^0})\mathcal{R}^0+R^{TB}(\mathcal{R}^\perp,\tfrac{\partial}{\partial z_j^0})\mathcal{R}^\perp+\nabla^{TX_G}_{\mathcal{R}^0}(A(\tfrac{\partial}{\partial z_j^0})\mathcal{R}^\perp),\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle\\
&+12\pi|A(\tfrac{\partial}{\partial\overline{z}_j^0})\mathcal{R}^\perp|^2-4\pi\Big\langle\tfrac{\partial}{\partial z_j^0},\nabla^{TX_G}_{\mathcal{R}^0}(A(\tfrac{\partial}{\partial\overline{z}_j^0})\mathcal{R}^\perp)\Big\rangle-\frac{2\pi}{3}\left\langle R^{TB}(\mathcal{R}^\perp,e_j^\perp)\mathcal{R}^\perp,e_j^\perp\right\rangle\Big\}P^N.
\end{aligned}
$$

Observe that as $A(e_i^0)e_i^0\in N_G$, we have at x_0,

$$
(5.111)\qquad\left\langle\nabla^{TB}_{\mathcal{R}^0}(A(e_i^0)e_i^0),e_j^0\right\rangle=\left\langle A(\mathcal{R}^0)A(e_i^0)e_i^0,e_j^0\right\rangle.
$$

Thus by (3.12), (3.54), (3.55), (5.24), (5.107b), (5.109)-(5.111), $a_j = a_j^{\perp} = 2\pi$, and the arguments above (5.103),

$$P^{N^{\perp}} P_{\mathscr{L}^{\perp}} \langle \Gamma_{ii}(\mathcal{R}), e_l \rangle \nabla_{0,e_l} P^N = -\frac{2}{3} P^{N^{\perp}} \left\langle R^{TX_G}(\mathcal{R}^0, e_i^0) e_i^0, \tfrac{\partial}{\partial \overline{z}_j^0} \right\rangle b_j P^N, \tag{5.112a}$$

$$\begin{aligned} P^{N^{\perp}} P_{\mathscr{L}^{\perp}} I_2 P^N = P^{N^{\perp}} \Big\{ \Big(\Big\langle \frac{1}{3} R^{TX_G}(\mathcal{R}^0, \tfrac{\partial}{\partial \overline{z}_i^0}) \mathcal{R}^0 + \frac{1}{4\pi} R^{TB}(e_k^{\perp}, \tfrac{\partial}{\partial \overline{z}_i^0}) e_k^{\perp}, \tfrac{\partial}{\partial \overline{z}_j^0} \Big\rangle \\ - \frac{3}{4\pi} \left\langle A(\tfrac{\partial}{\partial \overline{z}_i^0}) e_k^{\perp}, A(\tfrac{\partial}{\partial \overline{z}_j^0}) e_k^{\perp} \right\rangle \Big) b_i b_j - \frac{4\pi}{3} \left\langle R^{TX_G}(\mathcal{R}^0, \tfrac{\partial}{\partial z_j^0}) \mathcal{R}^0, \tfrac{\partial}{\partial \overline{z}_j^0} \right\rangle \Big\} P^N. \end{aligned} \tag{5.112b}$$

By (3.6), (5.4), (5.92), (5.112a), (5.112b) and the fact that $R^{TX_G}(\,,\,)$ is $(1,1)$-form, we get

$$\begin{aligned} - \Big((\mathscr{L}_2^0)^{-1} P^{N^{\perp}} P_{\mathscr{L}^{\perp}} (I_2 + \langle \Gamma_{ii}(\mathcal{R}), e_l \rangle \nabla_{0,e_l}) P^N \Big) ((0, Z^{\perp}), (0, Z^{\perp})) \\ = \frac{1}{6\pi} \Big\{ 3 \left\langle R^{TX_G}(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_i^0}) \tfrac{\partial}{\partial z_j^0} + R^{TX_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_i^0}) \tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0} \right\rangle \\ - 2 \left\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0}, e_i^0) e_i^0 + R^{TX_G}(\tfrac{\partial}{\partial \overline{z}_i^0}, \tfrac{\partial}{\partial z_j^0}) \tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0} \right\rangle \Big\} P_{\mathscr{L}^{\perp}}(Z^{\perp}, Z^{\perp}) \\ = \frac{2}{3\pi} \left\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_i^0}) \tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0} \right\rangle P_{\mathscr{L}^{\perp}}(Z^{\perp}, Z^{\perp}). \end{aligned} \tag{5.113}$$

Now by (5.45a), (5.83), (5.107b) and (5.109),

$$\begin{aligned} - P^{N^{\perp}} P_{\mathscr{L}^{\perp}} \frac{1}{9} \sum_i \Big[\sum_j (\partial_j R^{L_B})_{x_0}(\mathcal{R}, e_i) Z_j \Big]^2 P^N = \frac{\pi}{4} P^{N^{\perp}} \Big| T(\mathcal{R}^0, e_i^{\perp}) \Big|^2 P^N, \\ - \frac{1}{4} (\mathscr{L}_2^0)^{-1} P^{N^{\perp}} P_{\mathscr{L}^{\perp}} [K_2(\mathcal{R}), \mathscr{L}_2^0] P^N = \frac{1}{4} P^{N^{\perp}} P_{\mathscr{L}^{\perp}} K_2(\mathcal{R}) P^N \\ = \frac{1}{12} P^{N^{\perp}} \left\langle R^{TX_G}(\mathcal{R}^0, e_i^0) \mathcal{R}^0, e_i^0 \right\rangle P^N. \end{aligned} \tag{5.114}$$

By (5.13), (5.46), (5.48) and (5.49), we get

$$\frac{\sqrt{-1}}{2\pi} (\partial_j^0 R^{L_B})_{x_0}(\mathcal{R}, e_i^0) = -\frac{1}{2} \left\langle J\mathcal{R}^{\perp}, T(e_j^0, e_i^0) \right\rangle + \left\langle JA(e_j^0)\mathcal{R}^{\perp}, e_i^0 \right\rangle = 0. \tag{5.115}$$

Thus by (3.9), (5.26), (5.45a), (5.114) and (5.115), we get

$$\begin{aligned} - P^{N^{\perp}} P_{\mathscr{L}^{\perp}} \mathcal{O}_2' P^N = P^{N^{\perp}} P_{\mathscr{L}^{\perp}} \Big\{ - I_1 - (I_2 + \langle \Gamma_{ii}(\mathcal{R}), e_l \rangle \nabla_{0,e_l}) \\ - \frac{1}{4} [K_2(\mathcal{R}), \mathscr{L}_2^0] - R^{E_B}(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_j^0}) b_j - \frac{\pi}{4} \Big| T(\mathcal{R}^0, e_i^{\perp}) \Big|^2 \Big\} P^N. \end{aligned} \tag{5.116}$$

Note that $R^{TX_G}(\cdot,\cdot)$ is a $(1,1)$-form, by (3.54), (5.4), (5.92), (5.102), (5.104), (5.108), (5.113) and (5.116),

$$
\begin{aligned}
(5.117)\quad & -\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}P_{\mathscr{L}^\perp}\mathcal{O}_2'P^N\Big)\big((0,Z^\perp),(0,Z^\perp)\big)\\
&=-\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}P_{\mathscr{L}^\perp}(I_1+I_2+\langle\Gamma_{ii}(\mathcal{R}),e_l\rangle\,\nabla_{0,e_l})P^N\Big)\big((0,Z^\perp),(0,Z^\perp)\big)\\
&+\frac{1}{2\pi}\Big\{R^{E_B}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})+\frac{1}{3}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},e_i^0)e_i^0,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle+\frac{1}{4}\Big|T(\tfrac{\partial}{\partial\overline{z}_j^0},e_i^\perp)\Big|^2\Big\}P_{\mathscr{L}^\perp}(Z^\perp,Z^\perp)\\
&=\Big\{\frac{1}{2\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_i^0})\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle+\frac{3}{16\pi}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle\\
&+\frac{3}{32\pi}\Big|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2-\frac{\sqrt{-1}}{16\pi}\Big\langle T(e_k^\perp,Je_k^\perp),T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big\rangle+\frac{7}{64\pi}\Big|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2\\
&+\frac{5\sqrt{-1}}{32\pi}\Big\langle\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}))+\nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})),Je_k^\perp\Big\rangle+\frac{1}{2\pi}R^{E_B}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big\}P_{\mathscr{L}^\perp}(Z^\perp,Z^\perp).
\end{aligned}
$$

By (3.54), (5.62), (5.83), (5.107b), (5.109) and the arguments above (5.103),

$$
\begin{aligned}
(5.118)\quad 4\pi^2P^{N^\perp}P_{\mathscr{L}^\perp}\mathcal{O}_2''P^N&=4\pi^2P^{N^\perp}P_{\mathscr{L}^\perp}\Big\{-\frac{1}{3}\big\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(\mathcal{R}^0,\mathcal{R}^0)}J\mathcal{R}^\perp,J\mathcal{R}^\perp\big\rangle\\
&+\frac{1}{6}\big\langle\nabla^{TY}_{\mathcal{R}^0}(T(e_j^\perp,J_{x_0}e_i^0))Z_j^\perp Z_i^0+\nabla^{TY}_{\mathcal{R}^\perp}(T(e_j^0,J_{x_0}e_i^0))Z_j^0Z_i^0,J\mathcal{R}^\perp\big\rangle\\
&+\frac{1}{3}\big\langle R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)\mathcal{R}^0,\mathcal{R}^\perp\big\rangle-\frac{1}{12}\sum_l\big\langle T(\mathcal{R}^0,e_l),J\mathcal{R}^\perp\big\rangle^2\Big\}P^N\\
&=\frac{\pi}{3}P^{N^\perp}\Big\{\frac{1}{2}\Big\langle\nabla^{TY}_{\mathcal{R}^0}(T(e_k^\perp,J_{x_0}e_i^0))Z_i^0+\nabla^{TY}_{e_k^\perp}(T(e_j^0,J_{x_0}e_i^0))Z_j^0Z_i^0,Je_k^\perp\Big\rangle\\
&-\big\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(\mathcal{R}^0,\mathcal{R}^0)}Je_k^\perp,Je_k^\perp\big\rangle+\big\langle R^{TB}(e_k^\perp,\mathcal{R}^0)\mathcal{R}^0,e_k^\perp\big\rangle-\frac{1}{4}|T(\mathcal{R}^0,e_l)|^2\Big\}P^N.
\end{aligned}
$$

Let $\{f_l\}$ be an orthonormal frame of TY on X.
As ∇^{TY} preserves the metric g^{TY}, by (1.4), (1.24),

$$
(5.119)\qquad \Big\langle(\nabla^{TY}_{e_i^0}\dot{g}^{TY}_{e_j^0})f_l,f_l\Big\rangle=\nabla_{e_i^0}\Big\langle\dot{g}^{TY}_{e_j^0}f_l,f_l\Big\rangle=4\nabla_{e_i^0}\nabla_{e_j^0}\log h.
$$

Now $\{Je_k^\perp\}$ is an orthonormal basis of TY along the fiber Y_{x_0} and $\{e_l\}=\{e_i^0\}\cup\{e_k^\perp\}$.
By (3.54), (5.92), (5.107a), (5.118) and (5.119),

$$
\begin{aligned}
(5.120)\quad & -4\pi^2\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}P_{\mathscr{L}^\perp}\mathcal{O}_2''P^N\Big)\big((0,Z^\perp),(0,Z^\perp)\big)\\
&=\frac{1}{4\pi}\Big\{\frac{\sqrt{-1}}{6}\Big\langle-\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}))+\nabla^{TY}_{\frac{\partial}{\partial\overline{z}_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial z_j^0}))-2\nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})),Je_k^\perp\Big\rangle\\
&-\frac{8}{3}\nabla_{\frac{\partial}{\partial z_j^0}}\nabla_{\frac{\partial}{\partial\overline{z}_j^0}}\log h-\frac{1}{3}\Big|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2-\frac{1}{6}\Big|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2\\
&-\frac{2}{3}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle\Big\}P_{\mathscr{L}^\perp}(Z^\perp,Z^\perp).
\end{aligned}
$$

By (5.73), (5.76), (5.117) and (5.120), we get (5.100). The proof of Lemma 5.11 is complete. □

5.4. **Final computations: the proof of Theorem 0.6.** By (3.40), (5.3), (5.5a), (5.6a) and (5.31), as $Je_k^{\perp} \in TY$ on P, we get at x_0,

$$\begin{aligned}
&\nabla^{TY}_{e_i^0} Je_k^{\perp} = P^{TY}\nabla^{TX}_{e_i^0} Je_k^{\perp} = P^{TY} J\nabla^{TX}_{e_i^0} e_k^{\perp} = 0,\\
&\nabla^{TB}_{e_i^0} Je_j^0 = \nabla^{TX_G}_{e_i^0} Je_j^0 + A(e_i^0)Je_j^0 = -\frac{1}{2}JT(e_i^0,e_j^0) = \nabla^{TB}_{e_i^0}(J_{x_0}e_j^0).
\end{aligned} \tag{5.121}$$

By (1.6), (1.24), (5.5c) and (5.121), as in (5.119), at x_0,

$$\begin{aligned}
\left\langle \nabla^{TY}_{e_i^0}(T(e_k^{\perp},e_j^0)), Je_k^{\perp}\right\rangle_{x_0} &= -2\left\langle \nabla^{TY}_{e_i^0}(T(Je_j^0, Je_k^{\perp})), Je_k^{\perp}\right\rangle\\
&= -\left\langle (\nabla^{TY}_{e_i^0}\dot{g}^{TY}_{Je_j^0})Je_k^{\perp}, Je_k^{\perp}\right\rangle = -4\nabla_{e_i^0}\nabla_{J_{x_0}e_j^0}\log h.
\end{aligned} \tag{5.122}$$

By (1.21) and (5.122), we get

$$\begin{aligned}
\sqrt{-1}\left\langle \nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^{\perp},\tfrac{\partial}{\partial \overline{z}_j^0})), Je_k^{\perp}\right\rangle &= -4\nabla_{\frac{\partial}{\partial z_j^0}}\nabla_{\frac{\partial}{\partial \overline{z}_j^0}}\log h = \Delta_{X_G}\log h,\\
\sqrt{-1}\left\langle \nabla^{TY}_{\frac{\partial}{\partial \overline{z}_j^0}}(T(e_k^{\perp},\tfrac{\partial}{\partial z_j^0})), Je_k^{\perp}\right\rangle &= -\Delta_{X_G}\log h.
\end{aligned} \tag{5.123}$$

Note that $T(e_i,e_j) = -[e_i^H, e_j^H]$, as $[e_i,e_j]=0$. By (1.4) and (1.6),

$$\begin{aligned}
\nabla^{TY}_{e_k^{\perp,H}}(T(e_i^{0,H},e_j^{0,H})) &= -\Big[e_k^{\perp,H},[e_i^{0,H},e_j^{0,H}]\Big] + T(e_k^{\perp,H},T(e_i^{0,H},e_j^{0,H}))\\
&= L_{e_i^{0,H}}(T(e_k^{\perp,H},e_j^{0,H})) - L_{e_j^{0,H}}(T(e_k^{\perp,H},e_i^{0,H})) + T(e_k^{\perp,H},T(e_i^{0,H},e_j^{0,H}))\\
&= \nabla^{TY}_{e_i^{0,H}}(T(e_k^{\perp,H},e_j^{0,H})) - \nabla^{TY}_{e_j^{0,H}}(T(e_k^{\perp,H},e_i^{0,H})) - T(e_i^{0,H},T(e_k^{\perp,H},e_j^{0,H}))\\
&\quad + T(e_j^{0,H},T(e_k^{\perp,H},e_i^{0,H})) + T(e_k^{\perp,H},T(e_i^{0,H},e_j^{0,H})).
\end{aligned} \tag{5.124}$$

Thus by Theorem 5.1, (5.123) and (5.124),

$$\begin{aligned}
&\sqrt{-1}\left\langle \nabla^{TY}_{e_k^{\perp}}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})), Je_k^{\perp}\right\rangle = \sqrt{-1}\Big\{2\Big\langle \nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^{\perp},\tfrac{\partial}{\partial \overline{z}_j^0})), Je_k^{\perp}\Big\rangle\\
&-\left\langle T(\tfrac{\partial}{\partial z_j^0}, Je_k^{\perp}), T(e_k^{\perp},\tfrac{\partial}{\partial \overline{z}_j^0})\right\rangle + \left\langle T(\tfrac{\partial}{\partial \overline{z}_j^0}, Je_k^{\perp}), T(e_k^{\perp},\tfrac{\partial}{\partial z_j^0})\right\rangle + \left\langle T(e_k^{\perp}, Je_k^{\perp}), T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})\right\rangle\Big\}\\
&\qquad = 2\Delta_{X_G}\log h + |T(e_k^{\perp},\tfrac{\partial}{\partial \overline{z}_j^0})|^2 + \sqrt{-1}\left\langle T(e_k^{\perp}, Je_k^{\perp}), T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})\right\rangle.
\end{aligned} \tag{5.125}$$

By $T(e_i,e_j) = -[e_i^H,e_j^H]$, (3.40) and (5.54), we have

$$\begin{aligned}
R^{TX}(e_k^H,e_j^H)e_i^H &= \nabla^{TX}_{e_k^H}\nabla^{TX}_{e_j^H}e_i^H - \nabla^{TX}_{e_j^H}\nabla^{TX}_{e_k^H}e_i^H - \nabla^{TX}_{[e_k^H,e_j^H]}e_i^H\\
&= R^{TB}(e_k,e_j)e_i - \frac{1}{2}T(e_k,\nabla^{TB}_{e_j}e_i) + \frac{1}{2}T(e_j,\nabla^{TB}_{e_k}e_i)\\
&\quad - \frac{1}{2}\nabla^{TX}_{e_k^H}(T(e_j,e_i)) + \frac{1}{2}\nabla^{TX}_{e_j^H}(T(e_k,e_i)) + \nabla^{TX}_{T(e_k^H,e_j^H)}e_i^H,\\
\left\langle R^{TX}(e_k^{\perp,H},e_j^{0,H})(J_{x_0}e_j^0)^H, J_{x_0}e_k^{\perp,H}\right\rangle &= \left\langle R^{TX}(e_k^{\perp,H},e_j^{0,H})e_j^{0,H}, e_k^{\perp,H}\right\rangle.
\end{aligned} \tag{5.126}$$

By (5.5a), (5.6a), (5.13), (5.121) and $T(e_k^\perp, e_i^0) \in TY$, at x_0, we get

$$\begin{aligned}(5.127)\qquad &\nabla^{TB}_{e_j^0}(J_{x_0}e_j^0) = 0, \quad \nabla^{TB}_{e_k^\perp}(J_{x_0}e_j^0) = \frac{1}{2}\left\langle T(e_j^0, e_l^0), Je_k^\perp\right\rangle e_l^0,\\ &\left\langle \nabla^{TX}_{T(e_k^\perp, e_i^0)}(J_{x_0}e_i^0)^H, J_{x_0}e_k^\perp\right\rangle = \left\langle \nabla^{TX}_{T(e_k^\perp, e_i^0)}e_i^{0,H}, e_k^\perp\right\rangle.\end{aligned}$$

We apply now the first equation of (5.126) into the second equation of (5.126), by using (1.8) and (5.133) and $T(\ ,\ )$ is $(1,1)$-form, we get at x_0,

$$\begin{aligned}(5.128)\quad &\frac{1}{4}|T(e_j^0, e_l^0)|^2 - \frac{1}{2}\left\langle \nabla^{TY}_{e_k^\perp}(T(e_j^0, J_{x_0}e_j^0)), Je_k^\perp\right\rangle + \frac{1}{2}\left\langle \nabla^{TY}_{e_j^0}(T(e_k^\perp, J_{x_0}e_j^0)), Je_k^\perp\right\rangle\\ &= \left\langle R^{TB}(e_k^\perp, e_j^0)e_j^0, e_k^\perp\right\rangle + \frac{1}{2}\left\langle \nabla^{TX}_{e_j^0}(T(e_k^\perp, e_j^0)), e_k^\perp\right\rangle\\ &= \left\langle R^{TB}(e_k^\perp, e_j^0)e_j^0, e_k^\perp\right\rangle - \frac{1}{4}|T(e_k^\perp, e_l^0)|^2.\end{aligned}$$

Finally from (3.6), (5.123), (5.125) and (5.128) and $T(\ ,\ )$ is $(1,1)$-form, we get

$$\begin{aligned}(5.129)\quad &2\left\langle R^{TB}(e_k^\perp, \tfrac{\partial}{\partial z_j^0})\tfrac{\partial}{\partial \overline{z}_j^0}, e_k^\perp\right\rangle = \sqrt{-1}\left\langle \nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0})), Je_k^\perp\right\rangle\\ &- \sqrt{-1}\left\langle \nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0})), Je_k^\perp\right\rangle + \frac{1}{2}|T(e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0})|^2 + \left|T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0})\right|^2\\ &= \Delta_{X_G}\log h + \frac{3}{2}|T(e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0})|^2 + \left|T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0})\right|^2 + \sqrt{-1}\left\langle T(e_k^\perp, Je_k^\perp), T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0})\right\rangle.\end{aligned}$$

From (5.123)-(5.129),

$$\begin{aligned}(5.130)\quad &\frac{\sqrt{-1}}{96\pi}\left\langle 11\,\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0})) + 4\,\nabla^{TY}_{\frac{\partial}{\partial \overline{z}_j^0}}(T(e_k^\perp, \tfrac{\partial}{\partial z_j^0})) + 7\,\nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0})), Je_k^\perp\right\rangle\\ &+ \frac{1}{48\pi}\left\langle R^{TB}(e_k^\perp, \tfrac{\partial}{\partial z_j^0})e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0}\right\rangle - \frac{\sqrt{-1}}{16\pi}\left\langle T(e_k^\perp, Je_k^\perp), T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0})\right\rangle\\ &= \frac{5}{24\pi}\Delta_{X_G}\log h + \frac{11}{192\pi}\left|T(e_k^\perp, \tfrac{\partial}{\partial \overline{z}_j^0})\right|^2 - \frac{1}{96\pi}\left|T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0})\right|^2.\end{aligned}$$

By (3.19), (5.76), (5.81), (5.100) and (5.130),

$$\begin{aligned}(5.131)\quad \Phi_{1,1} + \Phi_{1,2} &= \frac{1}{2\pi}\left\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_i^0})\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial \overline{z}_j^0}\right\rangle + \frac{3}{8\pi}\Delta_{X_G}\log h + \frac{1}{2\pi}R^{E_B}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0})\\ &= \frac{1}{16\pi}r^{X_G}_{x_0} + \frac{3}{8\pi}\Delta_{X_G}\log h + \frac{1}{4\pi}R^{E_G}_{x_0}(w_j^0, \overline{w}_j^0).\end{aligned}$$

From Lemma 5.10, (5.80) and (5.131), we get (0.25).

Recall that we compute everything on $\mathscr{C}^\infty(X, L^p \otimes E)$.

From (5.18), (5.21), (5.22), comparing to (2.110), we know that in (0.20), $\Phi_r(x_0) \in \mathrm{End}(E_G)_{x_0}$, and the term r^X, $R^{\det}$ will not appear here, and $\tau = 2\pi n$, thus we get the remainder part of Theorem 0.6 from Corollary 0.4.

The proof of Theorem 0.6 is complete.

5.5. **Coefficient $P^{(2)}(0,0)$.** As in (5.80), we have

$$P^{(2)}(0,0) = (\Psi_{1,1}+\Psi_{1,2})(0) + (\Psi_{1,1}+\Psi_{1,2})^*(0) + (\Psi_{1,3}-\Psi_{1,4})(0). \tag{5.132}$$

For $k \in \mathbb{N}$, let $H_k(x)$ be the Hermite polynomial,

$$H_k(x) = \sum_{j=0}^{[k/2]} (-1)^j \frac{k!\,(2x)^{k-2j}}{j!\,(k-2j)!}. \tag{5.133}$$

By [38, §8.6], (3.8) and $a_l^\perp = 2\pi$, we have

$$(b_l^\perp)^k e^{-\pi|Z_l^\perp|^2} = (2\pi)^{k/2} H_k(\sqrt{2\pi} Z_l^\perp) e^{-\pi|Z_l^\perp|^2}. \tag{5.134}$$

Especially, for l fixed, $i \in \mathbb{N}$,

$$\begin{aligned} &((b_l^\perp)^{2i+1} e^{-\pi|Z_l^\perp|^2})(0) = 0, \\ &((b_l^\perp)^2 e^{-\pi|Z_l^\perp|^2})(0) = -4\pi, \quad ((b_l^\perp)^4 e^{-\pi|Z_l^\perp|^2})(0) = 3\cdot(4\pi)^2, \\ &((b_l^\perp)^6 e^{-\pi|Z_l^\perp|^2})(0) = 15\cdot(-4\pi)^3. \end{aligned} \tag{5.135}$$

Recall that when we meet the operation $|\quad|^2$, we will first do this operation, then take the sum of the indices. Thus $|\mathcal{T}_{ijk}|^2$ means $\sum_{ijk} |\mathcal{T}_{ijk}|^2$, etc.

By (3.22), (5.94) and (5.135),

$$\mathcal{F}_2(\cdot,0) = -\frac{1}{8}\mathcal{T}_{kk}; \quad P^N(0,0) = 2^{n_0/2}. \tag{5.136}$$

By (5.98), (5.135) and (5.136), we know

$$\Psi_{1,3}(0) = \frac{2^{n_0/2}}{\pi}\Big|\frac{1}{4}\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_i^0}) + \mathcal{F}_2(\tfrac{\partial}{\partial\overline{z}_i^0},0)\Big|^2 = \frac{2^{n_0/2}}{64\pi}\Big|\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2. \tag{5.137}$$

and from (3.17), (3.18), (3.54), (5.99), (5.136) and $a_i^\perp = 2\pi$,

$$\begin{aligned} \Psi_{1,4}(0) &= G^\perp(0)^2\Big\{\frac{1}{4\pi}\sum_k \mathcal{F}_1(e_k^\perp)^2 + \frac{6\cdot(4\pi)^3}{(192\pi^2)^2}|\mathcal{T}_{klm}|^2 \\ &\qquad + \frac{1}{16\pi}\Big|\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2 + \frac{2\cdot(4\pi)^2}{\pi\cdot(32\pi)^2}\Big|\mathcal{T}_{kl}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2\Big\} \\ &= \frac{2^{n_0/2}}{4\pi}\Big\{\sum_k \mathcal{F}_1(e_k^\perp)^2 + \frac{1}{24}|\mathcal{T}_{klm}|^2 + \frac{1}{4}\Big|\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2 + \frac{1}{8}\Big|\mathcal{T}_{kl}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2\Big\}. \end{aligned} \tag{5.138}$$

Lemma 5.12. *The following identity holds,*

$$\begin{aligned} \Psi_{1,1}(0) = \Big\{ &-\frac{19}{2^6\cdot 3\pi}|\mathcal{T}_{jj'}(\tfrac{\partial}{\partial\overline{z}_i^0})|^2 - \frac{11}{2^7\cdot 3\pi}\mathcal{T}_{klm}^2 + \frac{1}{2^8\pi}\mathcal{T}_{kkm}\mathcal{T}_{llm} \\ &- \frac{5}{2^7\pi}\mathcal{T}_{jj}(\tfrac{\partial}{\partial z_l^0})\mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_l^0}) - \frac{1}{8\pi}\sum_k \mathcal{F}_1(e_k^\perp)^2 - \frac{1}{16\pi}\mathcal{F}_1(e_k^\perp)\mathcal{T}_{kll}\Big\} P^N(0,0). \end{aligned} \tag{5.139}$$

Proof. Set

$$\begin{aligned}
\mathcal{I}_1 &= -\sqrt{-1}\Big(\mathcal{T}_{jj'}(\tfrac{\partial}{\partial z_i^0})\frac{b_i^+}{8\pi} + \frac{1}{4}\mathcal{T}_{jj'}(z^0)\Big)b_j^\perp b_{j'}^\perp\frac{\sqrt{-1}}{8\pi}b_l\mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_l^0}) \\
\mathcal{I}_2 &= \sqrt{-1}\Big(\mathcal{T}_{jj'}(\tfrac{\partial}{\partial\overline{z}_i^0})b_i\frac{B_{jj'}^\perp}{8\pi} + \frac{1}{4}\mathcal{T}_{jj'}(\overline{z}^0)(b_j^{\perp+}b_{j'}^{\perp+} - b_j^\perp b_{j'}^\perp)\Big)\frac{-\sqrt{-1}}{32\pi}\mathcal{T}_{kl}(z^0)b_k^\perp b_l^\perp, \\
\mathcal{I}_3 &= -\frac{\sqrt{-1}}{8\pi}\widetilde{\mathcal{T}}_{ijj'}(b_j^\perp b_{j'}^{\perp+} + b_j^\perp b_{j'}^\perp)b_i^{\perp+}\Big(\mathcal{F}_1(e_k^\perp)\frac{b_k^\perp}{4\pi} + \mathcal{T}_{klm}\frac{b_k^\perp b_l^\perp b_m^\perp}{192\pi^2}\Big).
\end{aligned} \tag{5.140}$$

Observe that by (5.92), when we evaluate $\Psi_{1,1}$ in (5.76), in each monomial, if the total degree of b_l, $\overline{z}^0$ is not as same as the total degree of b_l^+, z^0, then the contribution of this term is 0. Thus by (3.9), (3.54), (5.76), (5.83), (5.86), (5.87), (5.94) and (5.140),

$$\begin{aligned}
\Psi_{1,1}(Z^\perp) = \Big\{&(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big[\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \\
&+ \Big(\mathcal{F}_1(e_j^\perp)(b_j^{\perp+} + b_j^\perp) + \mathcal{T}_{ijj'}\frac{B_{ijj'}^\perp}{16\pi}\Big)\Big(\mathcal{F}_1(e_k^\perp)\frac{b_k^\perp}{4\pi} + \mathcal{T}_{klm}\frac{b_k^\perp b_l^\perp b_m^\perp}{192\pi^2}\Big)\Big]P^N\Big\}(Z^\perp, Z^\perp).
\end{aligned} \tag{5.141}$$

By (3.8), (3.19) and (5.135),

$$(b_j z_i^0 P^N)(0,0) = -2\delta_{ij}P^N(0,0), \quad (b_k^\perp b_l^\perp b_j z_i^0 P^N)(0,0) = 8\pi\delta_{ij}\delta_{kl}P^N(0,0). \tag{5.142}$$

From Theorem 3.1, (3.9), (3.54), (5.135), (5.140) and (5.142),

$$\begin{aligned}
&((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{I}_1P^N)(0,0) \\
&= \frac{1}{32\pi}\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_l^0})\Big(4\mathcal{T}_{jj'}(\tfrac{\partial}{\partial z_l^0})b_j^\perp b_{j'}^\perp + b_l b_j^\perp b_{j'}^\perp\mathcal{T}_{jj'}(z^0)\Big)P^N\Big\}(0,0) \\
&= \frac{1}{32\pi}\mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_l^0})\Big\{\Big(\mathcal{T}_{jj'}(\tfrac{\partial}{\partial z_l^0})\frac{b_j^\perp b_{j'}^\perp}{2\pi} + \frac{b_l b_j^\perp b_{j'}^\perp}{12\pi}\mathcal{T}_{jj'}(z^0)\Big)P^N\Big\}(0,0) \\
&= -\frac{1}{24\pi}\mathcal{T}_{jj}(\tfrac{\partial}{\partial z_l^0})\mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_l^0})P^N(0,0).
\end{aligned} \tag{5.143}$$

By (3.9), (3.54), (5.83) and (5.140),

$$\begin{aligned}
(P^{N^\perp}\mathcal{I}_2P^N)(Z,(0,Z'^\perp)) &= \frac{1}{2^8\pi^2}\Big\{P^{N^\perp}\mathcal{T}_{jj'}(\tfrac{\partial}{\partial\overline{z}_i^0}) \\
&\quad\Big[b_i\mathcal{T}_{kl}(z^0)B_{jj'}^\perp + (b_j^{\perp+}b_{j'}^{\perp+} - b_j^\perp b_{j'}^\perp)\mathcal{T}_{kl}(z^0)b_i\Big]b_k^\perp b_l^\perp P^N\Big\}(Z,(0,Z'^\perp)) \\
&= \frac{1}{2^8\pi^2}\mathcal{T}_{jj'}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big\{P^{N^\perp}\Big[b_i\mathcal{T}_{kl}(z^0)(2b_j^{\perp+}b_{j'}^{\perp+} + 2b_j^\perp b_{j'}^{\perp+} + 4\pi\delta_{jj'}) \\
&\quad + 2\mathcal{T}_{kl}(\tfrac{\partial}{\partial z_i^0})(b_j^{\perp+}b_{j'}^{\perp+} - b_j^\perp b_{j'}^\perp)\Big]b_k^\perp b_l^\perp P^N\Big\}(Z,(0,Z'^\perp)) \\
&= \frac{1}{2^8\pi^2}\mathcal{T}_{jj'}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big\{b_i\Big(64\pi^2\mathcal{T}_{jj'}(z^0) + 16\pi\mathcal{T}_{kj'}(z^0)b_j^\perp b_k^\perp + 4\pi\delta_{jj'}\mathcal{T}_{kl}(z^0)b_k^\perp b_l^\perp\Big) \\
&\quad - 2\mathcal{T}_{kl}(\tfrac{\partial}{\partial z_i^0})b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp P^N\Big\}(Z,(0,Z'^\perp)).
\end{aligned} \tag{5.144}$$

Thus by Theorem 3.1, (3.8), (5.135), (5.142), (5.144) and use a similar equation as (5.152) for $\mathcal{T}_{jj'}(\frac{\partial}{\partial\overline{z}_i^0})\mathcal{T}_{kl}(\frac{\partial}{\partial z_i^0})b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp$, we get

$$
\begin{aligned}
(5.145)\quad ((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{I}_2P^N)(0,0) &= \frac{1}{2^8\pi^2}\mathcal{T}_{jj'}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big[\Big(16\pi b_i\mathcal{T}_{jj'}(z^0)\\
&+\frac{4}{3}b_i\mathcal{T}_{kj'}(z^0)b_j^\perp b_k^\perp+\frac{1}{3}\delta_{jj'}b_i\mathcal{T}_{kl}(z^0)b_k^\perp b_l^\perp-\frac{1}{8\pi}\mathcal{T}_{kl}(\tfrac{\partial}{\partial z_i^0})b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp\Big)P^N\Big](0,0)\\
&=\frac{1}{2^8\pi^2}\Big[-\frac{64\pi}{3}|\mathcal{T}_{jj'}(\tfrac{\partial}{\partial\overline{z}_i^0})|^2+\frac{8\pi}{3}\mathcal{T}_{jj}(\tfrac{\partial}{\partial\overline{z}_i^0})\mathcal{T}_{kk}(\tfrac{\partial}{\partial z_i^0})\\
&-2\pi\Big(2|\mathcal{T}_{jj'}(\tfrac{\partial}{\partial\overline{z}_i^0})|^2+\mathcal{T}_{jj}(\tfrac{\partial}{\partial\overline{z}_i^0})\mathcal{T}_{kk}(\tfrac{\partial}{\partial z_i^0})\Big)\Big]P^N(0,0)\\
&=\frac{1}{2^8\cdot 3\pi}\Big[-76|\mathcal{T}_{jj'}(\tfrac{\partial}{\partial\overline{z}_i^0})|^2+2\mathcal{T}_{jj}(\tfrac{\partial}{\partial\overline{z}_i^0})\mathcal{T}_{kk}(\tfrac{\partial}{\partial z_i^0})\Big]P^N(0,0).
\end{aligned}
$$

By (3.9), (3.54), (5.140), we get

$$
(5.146)\qquad \mathcal{I}_3P^N=-\frac{\sqrt{-1}}{8\pi}\widetilde{\mathcal{T}}_{ijj'}\Big[b_j^\perp b_{j'}^\perp\mathcal{F}_1(e_i^\perp)+\mathcal{T}_{ilm}b_j^\perp b_{j'}^\perp\frac{b_l^\perp b_m^\perp}{16\pi}+\frac{1}{2}\mathcal{T}_{ilj'}b_j^\perp b_l^\perp\Big]P^N.
$$

By (5.5e), (5.14), (5.135), (5.146) and a similar equation as (5.152) for $\widetilde{\mathcal{T}}_{jij'}\mathcal{T}_{kli}b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp$, we get

$$
(5.147)\qquad \Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{I}_3P^N\Big)(0,0)=\frac{\sqrt{-1}}{64\pi}\widetilde{\mathcal{T}}_{ijj'}\mathcal{T}_{ijj'}P^N(0,0)=0,
$$

as $\widetilde{\mathcal{T}}_{ijj'}$ is anti-symmetric on i,j and $\mathcal{T}_{ijj'}$ is symmetric on i,j.

By Theorem 3.1, (3.9), (3.54) and (5.135),

$$
\begin{aligned}
(5.148)\quad &\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{F}_1(e_j^\perp)(b_j^{\perp+}+b_j^\perp)\mathcal{F}_1(e_k^\perp)\frac{b_k^\perp}{4\pi}P^N\Big)(0,0)\\
&=\frac{1}{32\pi^2}\Big(\mathcal{F}_1(e_j^\perp)^2(b_j^\perp)^2P^N\Big)(0,0)=-\frac{1}{8\pi}\sum_j\mathcal{F}_1(e_j^\perp)^2P^N(0,0).
\end{aligned}
$$

Recall that $\mathcal{T}_{klm}$ is symmetric on k,l,m.

By Theorem 3.1, (3.9), (3.54), (5.83) and (5.135),

$$
\begin{aligned}
(5.149)\quad &\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big(\mathcal{F}_1(e_j^\perp)(b_j^{\perp+}+b_j^\perp)\mathcal{T}_{klm}\frac{b_m^\perp b_l^\perp b_k^\perp}{192\pi^2}+\mathcal{T}_{ijj'}\frac{B_{ijj'}^\perp}{64\pi^2}\mathcal{F}_1(e_k^\perp)b_k^\perp\Big)P^N\Big\}(0,0)\\
&=\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{F}_1(e_j^\perp)\Big(b_j^\perp\mathcal{T}_{klm}\frac{b_k^\perp b_l^\perp b_m^\perp}{48\pi^2}+\mathcal{T}_{jlm}\frac{b_l^\perp b_m^\perp}{4\pi}\Big)P^N\Big\}(0,0)\\
&=\frac{1}{32\pi^2}\Big\{\mathcal{F}_1(e_j^\perp)\Big(\mathcal{T}_{klm}\frac{b_j^\perp b_k^\perp b_l^\perp b_m^\perp}{24\pi}+\mathcal{T}_{jlm}b_l^\perp b_m^\perp\Big)P^N\Big\}(0,0)\\
&=\frac{1}{32\pi^2}\Big\{\mathcal{F}_1(e_j^\perp)\Big(\sum_{l\neq j}\mathcal{T}_{jll}\frac{(b_j^\perp)^2(b_l^\perp)^2}{8\pi}+\mathcal{T}_{jjj}\frac{(b_j^\perp)^4}{24\pi}+\mathcal{T}_{jll}(b_l^\perp)^2\Big)P^N\Big\}(0,0)\\
&=-\frac{1}{16\pi}\mathcal{F}_1(e_j^\perp)\mathcal{T}_{jll}P^N(0,0).
\end{aligned}
$$

As $\mathcal{T}_{klm}$ is symmetric on k,l,m, we know that

$$\mathcal{T}_{klm}^2 = 6\sum_{k<l<m}\mathcal{T}_{klm}^2 + 3\sum_{k\neq m}\mathcal{T}_{kkm}^2 + \mathcal{T}_{mmm}^2, \tag{5.150}$$
$$\mathcal{T}_{kkm}\mathcal{T}_{llm} = \sum_{k\neq l\neq m\neq k}\mathcal{T}_{kkm}\mathcal{T}_{llm} + \sum_{k\neq m}(2\mathcal{T}_{kkm}\mathcal{T}_{mmm} + \mathcal{T}_{kkm}^2) + \mathcal{T}_{mmm}^2.$$

From (5.135), (5.150),

$$\begin{aligned}\Big(\mathcal{T}_{ijj'}\mathcal{T}_{klm}b_i^\perp b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp b_m^\perp P^N\Big)(0,0) &= \Bigg\{\Big(36\sum_{k<l<m}\mathcal{T}_{klm}^2(b_k^\perp)^2(b_l^\perp)^2(b_m^\perp)^2 \\ &+ 9\sum_{k\neq l\neq m\neq k}\mathcal{T}_{kkm}\mathcal{T}_{llm}(b_k^\perp)^2(b_l^\perp)^2(b_m^\perp)^2 + 6\sum_{k\neq m}\mathcal{T}_{kkm}\mathcal{T}_{mmm}(b_k^\perp)^2(b_m^\perp)^4 \\ &+9\sum_{k\neq m}\mathcal{T}_{mmk}\mathcal{T}_{mmk}(b_k^\perp)^2(b_m^\perp)^4 + \mathcal{T}_{mmm}^2(b_m^\perp)^6\Big)P^N\Bigg\}(0,0) \\ &= (-4\pi)^3\Big(36\sum_{k<l<m}\mathcal{T}_{klm}^2 + 9\sum_{k\neq l\neq m\neq k}\mathcal{T}_{kkm}\mathcal{T}_{llm} \\ &+ 3\sum_{k\neq m}(6\mathcal{T}_{kkm}\mathcal{T}_{mmm} + 9\mathcal{T}_{mmk}\mathcal{T}_{mmk}) + 15\mathcal{T}_{mmm}^2\Big)P^N(0,0) \\ &= (-4\pi)^3\cdot 3\Big(2\mathcal{T}_{klm}^2 + 3\mathcal{T}_{kkm}\mathcal{T}_{llm}\Big)P^N(0,0).\end{aligned} \tag{5.151}$$

By (5.135),

$$\begin{aligned}&\Big(\mathcal{T}_{ijm}\mathcal{T}_{klm}b_i^\perp b_j^\perp b_k^\perp b_l^\perp P^N\Big)(0,0) \\ &= \Bigg\{\Bigg[\sum_{k\neq l}\Big(2\mathcal{T}_{klm}^2 + \mathcal{T}_{kkm}\mathcal{T}_{llm}\Big)(b_k^\perp)^2(b_l^\perp)^2 + \mathcal{T}_{llm}^2(b_l^\perp)^4\Bigg]P^N\Bigg\}(0,0) \\ &= (4\pi)^2\Big(\sum_{k\neq l}(2\mathcal{T}_{klm}^2 + \mathcal{T}_{kkm}\mathcal{T}_{llm}) + 3\mathcal{T}_{llm}^2\Big)P^N(0,0) \\ &= (4\pi)^2(2\mathcal{T}_{klm}^2 + \mathcal{T}_{kkm}\mathcal{T}_{llm})P^N(0,0).\end{aligned} \tag{5.152}$$

By (3.9), (3.54) and (5.83), we have also

$$\begin{aligned}P^{N^\perp}\mathcal{T}_{ijj'}B_{ijj'}^\perp\mathcal{T}_{klm}b_k^\perp b_l^\perp b_m^\perp P^N &= \Big(\mathcal{T}_{ijj'}\mathcal{T}_{klm}b_i^\perp b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp b_m^\perp \\ &+ 36\pi\mathcal{T}_{ijm}\mathcal{T}_{klm}b_i^\perp b_j^\perp b_k^\perp b_l^\perp + 36\pi\cdot 8\pi\mathcal{T}_{ilm}\mathcal{T}_{klm}b_i^\perp b_k^\perp\Big)P^N.\end{aligned} \tag{5.153}$$

Thus from Theorem 3.1, (5.151)-(5.153),

$$\begin{aligned}(5.154)\quad &\left\{\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\frac{1}{16\pi}\mathcal{T}_{ijj'}B^\perp_{ijj'}\mathcal{T}_{klm}\frac{b_k^\perp b_l^\perp b_m^\perp}{192\pi^2}\Big)P^N\right\}(0,0)\\ &=\frac{1}{2^{10}\cdot 3\pi^3}\left\{\Big(\frac{1}{24\pi}\mathcal{T}_{ijj'}\mathcal{T}_{klm}b_i^\perp b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp b_m^\perp+\frac{9}{4}\mathcal{T}_{ijm}\mathcal{T}_{klm}b_i^\perp b_j^\perp b_k^\perp b_l^\perp\right.\\ &\qquad\left.+36\pi\mathcal{T}_{ilm}\mathcal{T}_{klm}b_i^\perp b_k^\perp\Big)P^N\right\}(0,0)\\ &=\frac{1}{2^{10}\cdot 3\pi}\left\{-8(2\mathcal{T}_{klm}^2+3\mathcal{T}_{kkm}\mathcal{T}_{llm})+36(2\mathcal{T}_{klm}^2+\mathcal{T}_{kkm}\mathcal{T}_{llm})-144\mathcal{T}_{klm}^2\right\}P^N(0,0)\\ &=\frac{1}{2^8\cdot 3\pi}\left(-22\mathcal{T}_{klm}^2+3\mathcal{T}_{kkm}\mathcal{T}_{llm}\right)P^N(0,0).\end{aligned}$$

From (5.141), (5.143), (5.145), (5.147), (5.148), (5.149) and (5.154), we get

$$\begin{aligned}(5.155)\quad \Psi_{1,1}(0)=&\Big\{\frac{1}{2^8\cdot 3\pi}\Big[-76|\mathcal{T}_{jj'}(\tfrac{\partial}{\partial\overline{z}_i^0})|^2+2\mathcal{T}_{jj}(\tfrac{\partial}{\partial\overline{z}_i^0})\mathcal{T}_{kk}(\tfrac{\partial}{\partial z_i^0})-22\mathcal{T}_{klm}^2+3\mathcal{T}_{kkm}\mathcal{T}_{llm}\Big]\\ &-\frac{1}{24\pi}\mathcal{T}_{jj}(\tfrac{\partial}{\partial z_l^0})\mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_l^0})-\frac{1}{8\pi}\sum_j\mathcal{F}_1(e_j^\perp)^2-\frac{1}{16\pi}\mathcal{F}_1(e_j^\perp)\mathcal{T}_{jll}\Big\}P^N(0,0).\end{aligned}$$

From (5.155) we get (5.139). □

Recall that $B(Z,e_l^\perp)$ was defined in (5.23).

Lemma 5.13. *The following identity holds,*

$$\begin{aligned}(5.156)\quad \frac{\sqrt{-1}}{\pi}B(Z,e_l^\perp)=&\frac{1}{2}\left\langle R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)e_l^\perp,J\mathcal{R}^0\right\rangle-\frac{5}{4}\left\langle\nabla^{TY}_{\mathcal{R}}(T(e_k,e_l^\perp)),J\mathcal{R}^\perp\right\rangle Z_k\\ &+\frac{1}{2}\left\langle\frac{1}{3}R^{TB}(\mathcal{R}^\perp,e_l^\perp)\mathcal{R}^\perp+\nabla^{TX_G}_{\mathcal{R}^0}(A(e_k^0)e_l^\perp)Z_k^0,J\mathcal{R}^0\right\rangle\\ &+\frac{1}{8}\left\langle T(\mathcal{R}^0,e_j^0),Je_l^\perp\right\rangle\left\langle T(\mathcal{R}^\perp-\mathcal{R}^0,Je_j^0),J\mathcal{R}^\perp\right\rangle\\ &+\frac{1}{4}\left\langle T(\mathcal{R}^\perp,e_j^0),Je_l^\perp\right\rangle\left\langle T(\mathcal{R}^0,Je_j^0),J\mathcal{R}^\perp\right\rangle\\ &+\frac{1}{8}\left\langle T(\mathcal{R}^0,J\mathcal{R}^0),T(\mathcal{R}^\perp,e_l^\perp)\right\rangle-\frac{1}{8}\left\langle T(\mathcal{R},e_l^\perp),T(\mathcal{R}^\perp,J\mathcal{R}^0)\right\rangle\\ &+\frac{1}{8}\left\langle T(\mathcal{R}^\perp,JT(\mathcal{R}^0,J\mathcal{R}^0)),Je_l^\perp\right\rangle+\frac{1}{2}\left\langle T(\mathcal{R}^\perp,J\mathcal{R}^\perp),T(\mathcal{R},e_l^\perp)\right\rangle.\end{aligned}$$

Proof. By (5.33), (5.54) and $A(\mathcal{R}^0)A(\mathcal{R}^0)e_l^\perp\in N_G$, as A exchanges TX_G and N_G, we get

$$\begin{aligned}(5.157)\quad \left\langle J\mathcal{R},(\nabla^{TX}\nabla^{TX}e_l^{\perp,H})_{(\mathcal{R},\mathcal{R})}\right\rangle=&-\frac{1}{2}\left\langle J\mathcal{R},T(\mathcal{R},\nabla^{TB}_{\mathcal{R}}e_l^\perp)+\nabla^{TX}_{\mathcal{R}}(T(e_i^H,e_l^\perp))Z_i\right\rangle\\ &+\left\langle J\mathcal{R}^0,\frac{1}{3}R^{TB}(\mathcal{R}^\perp,e_l^\perp)\mathcal{R}^\perp+R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)e_l^\perp+\nabla^{TX_G}_{\mathcal{R}^0}(A(e_k^0)e_l^\perp)Z_k^0\right\rangle.\end{aligned}$$

By (1.8), (5.13), (5.53), we have at x_0,

$$-\frac{1}{2}\left\langle J\mathcal{R}^{\perp}, T(\mathcal{R}, \nabla^{TB}_{\mathcal{R}} e_l^{\perp})\right\rangle = \frac{1}{4}\left\langle Je_l^{\perp}, T(\mathcal{R}^0, e_j^0)\right\rangle \left\langle J\mathcal{R}^{\perp}, T(\mathcal{R}, Je_j^0)\right\rangle, \tag{5.158}$$

$$-\frac{1}{2}\left\langle J\mathcal{R}^0, \nabla^{TX}_{\mathcal{R}}(T(e_i^H, e_l^{\perp,H}))Z_i\right\rangle = -\frac{1}{4}\left\langle T(\mathcal{R}, e_l^{\perp}), T(\mathcal{R}, J\mathcal{R}^0)\right\rangle.$$

By (5.5a), (5.5d), (5.13), (5.53), (5.54) and $\nabla^{TX}_{\mathcal{R}}(T(e_i^H, e_k^H))Z_iZ_k = 0$, we have

$$\begin{aligned}\langle J(\nabla^{TX}\nabla^{TX}e_k^H)_{(\mathcal{R},\mathcal{R})}, e_l^{\perp}\rangle Z_k &= \frac{1}{2}\left\langle T(\mathcal{R}, \nabla^{TB}_{\mathcal{R}} e_k), Je_l^{\perp}\right\rangle Z_k \\ &= \frac{1}{2}\left\langle T(\mathcal{R}, 2A(\mathcal{R}^0)\mathcal{R}^{\perp} + A(\mathcal{R}^0)\mathcal{R}^0), Je_l^{\perp}\right\rangle \\ &= \frac{1}{2}\left\langle T(\mathcal{R}, e_j^0), Je_l^{\perp}\right\rangle \left\langle T(\mathcal{R}^0, Je_j^0), J\mathcal{R}^{\perp}\right\rangle \\ &\quad -\frac{1}{4}\left\langle T(\mathcal{R}^0, e_l^{\perp}), T(\mathcal{R}^0, J\mathcal{R}^0)\right\rangle + \frac{1}{4}\left\langle T(\mathcal{R}^{\perp}, JT(\mathcal{R}^0, J\mathcal{R}^0)), Je_l^{\perp}\right\rangle.\end{aligned} \tag{5.159}$$

From (3.40), (5.5a), (5.13), (5.53) and the fact that A exchanges TX_G and N_G, we get

$$\begin{aligned}\left\langle J\nabla^{TX}_{\mathcal{R}} e_k^H, \nabla^{TX}_{\mathcal{R}} e_l^{\perp,H}\right\rangle Z_k &= \left\langle J\nabla^{TB}_{\mathcal{R}} e_k, A(\mathcal{R}^0)e_l^{\perp} - \frac{1}{2}T(\mathcal{R}, e_l^{\perp})\right\rangle Z_k \\ &= \left\langle JA(\mathcal{R}^0)\mathcal{R}^0, -\frac{1}{2}T(\mathcal{R}, e_l^{\perp})\right\rangle + 2\left\langle JA(\mathcal{R}^0)\mathcal{R}^{\perp}, A(\mathcal{R}^0)e_l^{\perp}\right\rangle, \\ &= \frac{1}{4}\left\langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}, e_l^{\perp})\right\rangle - \frac{1}{2}\left\langle Je_l^{\perp}, T(\mathcal{R}^0, e_j^0)\right\rangle \left\langle J\mathcal{R}^{\perp}, T(\mathcal{R}^0, Je_j^0)\right\rangle.\end{aligned} \tag{5.160}$$

From (5.51), (5.52), (5.61), (5.157)-(5.160), we get

$$\begin{aligned}\frac{\sqrt{-1}}{\pi}B(Z, e_l^{\perp}) &= \frac{1}{8}\left\langle Je_l^{\perp}, T(\mathcal{R}^0, e_j^0)\right\rangle \left\langle J\mathcal{R}^{\perp}, T(\mathcal{R}, Je_j^0)\right\rangle \\ &\quad -\frac{1}{4}\left\langle J\mathcal{R}^{\perp}, \nabla^{TY}_{\mathcal{R}}(T(e_i, e_l^{\perp}))Z_i\right\rangle - \frac{1}{8}\left\langle T(\mathcal{R}, e_l^{\perp}), T(\mathcal{R}, J\mathcal{R}^0)\right\rangle \\ &\quad +\frac{1}{2}\left\langle J\mathcal{R}^0, \frac{1}{3}R^{TB}(\mathcal{R}^{\perp}, e_l^{\perp})\mathcal{R}^{\perp} + R^{TB}(\mathcal{R}^{\perp}, \mathcal{R}^0)e_l^{\perp} + \nabla^{TX_G}_{\mathcal{R}^0}(A(e_k^0)e_l^{\perp})Z_k^0\right\rangle \\ &\quad +\frac{1}{4}\left\langle T(\mathcal{R}, e_j^0), Je_l^{\perp}\right\rangle \left\langle T(\mathcal{R}^0, Je_j^0), J\mathcal{R}^{\perp}\right\rangle - \frac{1}{8}\left\langle T(\mathcal{R}^0, e_l^{\perp}), T(\mathcal{R}^0, J\mathcal{R}^0)\right\rangle \\ &\quad +\frac{1}{8}\left\langle T(\mathcal{R}^{\perp}, JT(\mathcal{R}^0, J\mathcal{R}^0)), Je_l^{\perp}\right\rangle + \frac{1}{4}\left\langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}, e_l^{\perp})\right\rangle \\ &\quad -\frac{1}{2}\left\langle Je_l^{\perp}, T(\mathcal{R}^0, e_j^0)\right\rangle \left\langle J\mathcal{R}^{\perp}, T(\mathcal{R}^0, Je_j^0)\right\rangle \\ &\quad +\frac{1}{2}\left\langle T(\mathcal{R}^{\perp}, J\mathcal{R}^{\perp}), T(\mathcal{R}, e_l^{\perp})\right\rangle - \left\langle \nabla^{TY}_{\mathcal{R}}(T(e_k, e_l^{\perp})), J\mathcal{R}^{\perp}\right\rangle Z_k.\end{aligned} \tag{5.161}$$

From (5.161) we get (5.156). □

Now we need to compute the contribution from $-(\mathscr{L}_2^0)^{-1}P^{N^{\perp}}\mathcal{O}_2P^N$.

Lemma 5.14. *The following identity holds,*

$$\begin{aligned}(5.162)\quad \Psi_{1,2}(0) &= \Big\{\frac{1}{16\pi}r^{X_G}_{x_0}+\frac{1}{2\pi}R^{E_G}(\tfrac{\partial}{\partial z^0_j},\tfrac{\partial}{\partial \overline{z}^0_j})+\frac{1}{2\pi}\Delta_{X_G}\log h+\frac{29}{2^5\cdot 3\pi}|T(e^\perp_k,\tfrac{\partial}{\partial\overline{z}^0_j})|^2\\ &+\frac{\sqrt{-1}}{16\pi}\Big\langle T(e^\perp_k,Je^\perp_k),T(\tfrac{\partial}{\partial z^0_j},\tfrac{\partial}{\partial\overline{z}^0_j})\Big\rangle+\frac{1}{4\pi}\Big|T(\tfrac{\partial}{\partial z^0_i},\tfrac{\partial}{\partial\overline{z}^0_j})\Big|^2+\frac{1}{32\pi}\Big|\sum_j\mathcal{T}_{jj}(\tfrac{\partial}{\partial\overline{z}^0_i})\Big|^2\\ &-\frac{1}{2^6\pi}\Big\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(e^\perp_j,e^\perp_j)}Je^\perp_k,Je^\perp_k\Big\rangle-\frac{1}{2^5\pi}\Big\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(e^\perp_j,e^\perp_k)}Je^\perp_j,Je^\perp_k\Big\rangle\\ &+\frac{1}{2^7\pi}\widetilde{\mathcal{T}}_{ijk}(\widetilde{\mathcal{T}}_{kji}+\widetilde{\mathcal{T}}_{ijk})+\frac{7}{2^8\pi}\Big(2\mathcal{T}^2_{jkm}+\mathcal{T}_{jjm}\mathcal{T}_{kkm}\Big)\\ &-\frac{\sqrt{-1}}{16\pi}\Big(\big\langle T(e^\perp_j,Je^\perp_j),\widetilde{\mu}^E\big\rangle-2\Big\langle Je^\perp_j,\nabla^{TY}_{e^\perp_j}\widetilde{\mu}^E\Big\rangle\Big)\Big\}P^N(0,0).\end{aligned}$$

Proof. From Theorem 3.1, (3.9), (3.54), (5.15) and (5.135),

$$\begin{aligned}(5.163)\quad 4\pi\Big((\mathscr{L}^0_2)^{-1}P^{N^\perp}Z^\perp_kZ^\perp_lP^N\Big)(0,0)&=\Big((\mathscr{L}^0_2)^{-1}P^{N^\perp}Z^\perp_lb^\perp_kP^N\Big)(0,0)\\ &=\big((\mathscr{L}^0_2)^{-1}b^\perp_kZ^\perp_lP^N\big)(0,0)=\Big(\frac{b^\perp_kb^\perp_l}{32\pi^2}P^N\Big)(0,0)=-\frac{\delta_{kl}}{8\pi}P^N(0,0).\end{aligned}$$

Set

$$(5.164)\qquad \mathcal{I}_4=-\Big\{(\mathscr{L}^0_2)^{-1}P^{N^\perp}\Big(\tfrac{\partial}{\partial z^0_j}\Big(B(Z,\tfrac{\partial}{\partial\overline{z}^0_j})\Big)-\tfrac{\partial}{\partial\overline{z}^0_j}\Big(B(Z,\tfrac{\partial}{\partial z^0_j})\Big)\Big)P^N\Big\}(0,0).$$

At first, if Q is a monomial on $b_i,b^+_i,\ b^\perp_j,b^{\perp+}_j,Z_i$ and the total degree of b_i,b^+_i,Z^0_i or $b^\perp_j,b^{\perp+}_j,Z^\perp_j$ is odd, then by Theorem 3.1,

$$(5.165)\qquad \Big((\mathscr{L}^0_2)^{-1}P^{N^\perp}QP^N\Big)(0,0)=0.$$

By (5.165), only the monomials of $B(Z,e^0_l)$ with odd degree on Z^0 have contributions for $\mathcal{I}_4$.

If we denote by $\widetilde{B}_Z(e^0_l)$ the odd degree component on Z^0 of the difference of $B(Z,e^0_l)$ and of the sum the the first two and the last terms of $B(Z,e^0_l)$ in (5.45b), then by (5.45b) we know that $\widetilde{B}_Z(e^0_l)$ is a linear function on Z^0 and $\frac{\partial}{\partial z^0_j}\Big(\widetilde{B}_Z(\frac{\partial}{\partial\overline{z}^0_j})\Big)$ and $-\frac{\partial}{\partial\overline{z}^0_j}\Big(\widetilde{B}_Z(\frac{\partial}{\partial z^0_j})\Big)$ are equal.

Moreover, by (5.5e), (5.163), we know the contribution of the last term of $B(Z,e^0_l)$ in (5.45b) is zero in $\mathcal{I}_4$.

Thus by Remark 5.2, (5.45b) and (5.164),

$$\begin{aligned}(5.166)\quad \mathcal{I}_4 = \pi\sqrt{-1}\Big\{&(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big[\frac{1}{6}\tfrac{\partial}{\partial z_j^0}\Big\langle R^{TX_G}(\mathcal{R}^0,J\mathcal{R}^0)\mathcal{R}^0,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle\\ &-\frac{1}{6}\tfrac{\partial}{\partial\overline{z}_j^0}\Big\langle R^{TX_G}(\mathcal{R}^0,J\mathcal{R}^0)\mathcal{R}^0,\tfrac{\partial}{\partial z_j^0}\Big\rangle\\ &-\frac{5}{4}\Big\langle J\mathcal{R}^\perp,2\nabla^{TY}_{\mathcal{R}^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0}))+\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_i^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}))Z_i^\perp-\nabla^{TY}_{\frac{\partial}{\partial\overline{z}_j^0}}(T(e_i^\perp,\tfrac{\partial}{\partial z_j^0}))Z_i^\perp\Big\rangle\\ &+3\sqrt{-1}\Big\langle R^{TB}(\mathcal{R}^\perp,\tfrac{\partial}{\partial z_j^0})\mathcal{R}^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle-\frac{3\sqrt{-1}}{4}\Big\langle J\mathcal{R}^\perp,T(\tfrac{\partial}{\partial z_j^0},e_i^0)\Big\rangle\Big\langle J\mathcal{R}^\perp,T(e_i^0,\tfrac{\partial}{\partial\overline{z}_j^0})\Big\rangle\\ &-\frac{\sqrt{-1}}{4}\Big\langle T(\mathcal{R}^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}),T(\mathcal{R}^\perp,\tfrac{\partial}{\partial z_j^0})\Big\rangle+\Big\langle T(\mathcal{R}^\perp,J\mathcal{R}^\perp),T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big\rangle\Big]P^N\Big\}(0,0).\end{aligned}$$

By (5.92), (5.107a), (5.163) and (5.166), comparing with (5.103) and (5.104), we get

$$\begin{aligned}(5.167)\quad \mathcal{I}_4 = \Big\{&-\frac{1}{6\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_i^0})\tfrac{\partial}{\partial z_j^0}+R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_i^0})\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle\\ &+\frac{5\sqrt{-1}}{2^7\pi}\Big\langle Je_k^\perp,2\nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0}))+\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}))-\nabla^{TY}_{\frac{\partial}{\partial\overline{z}_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial z_j^0}))\Big\rangle\\ &+\frac{3}{32\pi}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle+\frac{3}{64\pi}|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})|^2\\ &-\frac{1}{2^7\pi}|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})|^2-\frac{\sqrt{-1}}{32\pi}\Big\langle T(e_k^\perp,Je_k^\perp),T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big\rangle\Big\}P^N(0,0).\end{aligned}$$

By (3.9), (3.54) and (5.83),

$$\begin{aligned}&(z_i^0\overline{z}_j^0P^N)(Z,0)=(z_i^0\frac{b_j}{2\pi}P^N)(Z,0)=\frac{1}{2\pi}((b_jz_i^0+2\delta_{ij})P^N)(Z,0),\\ &Z_k^\perp Z_l^\perp P^N=\frac{1}{16\pi^2}(b_k^\perp b_l^\perp+4\pi\delta_{kl})P^N,\\ &(4\pi)^4(Z_k^\perp)^4P^N=\big((b_k^\perp)^4+24\pi(b_k^\perp)^2+3\cdot(4\pi)^2\big)P^N.\end{aligned}\qquad(5.168)$$

From Theorem 3.1, (3.9), (3.54), (5.92), (5.135), (5.142) and (5.168),

$$
\begin{aligned}
&(P^{N^\perp}Z_k^\perp Z_l^\perp P^N)(0,0)=\frac{1}{16\pi^2}(b_k^\perp b_l^\perp P^N)(0,0)=-\frac{\delta_{kl}}{4\pi},\\
&\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}b_j z_i^0 Z_k^\perp Z_l^\perp P^N\Big)(0,0)\\
&\quad=\frac{1}{16\pi^2}\Big\{\Big(\frac{1}{12\pi}b_k^\perp b_l^\perp b_j z_i^0+\delta_{kl}b_j z_i^0\Big)P^N\Big\}(0,0)=-\frac{1}{12\pi^2}\delta_{ij}\delta_{kl}P^N(0,0),\\
&\big((\mathscr{L}_2^0)^{-1}b_l^\perp Z_k^\perp z_i^0\overline{z}_j^0 P^N\big)(0,0)=\frac{1}{8\pi^2}\Big\{b_l^\perp b_k^\perp\Big(\frac{b_j}{12\pi}z_i^0+\frac{2}{8\pi}\delta_{ij}\Big)P^N\Big\}(0,0)\\
&\quad=-\frac{1}{24\pi^2}\delta_{ij}\delta_{kl}P^N(0,0),\\
&\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}Z_l^\perp Z_k^\perp z_i^0\overline{z}_j^0 P^N\Big)(0,0)\\
&\quad=\frac{1}{4\pi}\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}(b_l^\perp Z_k^\perp+\delta_{kl})z_i^0\overline{z}_j^0 P^N\Big\}(0,0)=\frac{-7}{96\pi^3}\delta_{ij}\delta_{kl}P^N(0,0).
\end{aligned}
\tag{5.169}
$$

By (5.5e), (5.106), (5.107a), (5.169) and comparing with (5.108), we get

$$
\begin{aligned}
&-\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}b_j B(Z,\tfrac{\partial}{\partial\overline{z}_j^0})P^N\Big)(0,0)\\
&\quad=\Big\{-\frac{1}{12\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_i^0})\tfrac{\partial}{\partial z_i^0}+R^{TX_G}(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_i^0})\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle\\
&\qquad+\frac{5\sqrt{-1}}{48\pi}\Big\langle\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}))+\nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})),Je_k^\perp\Big\rangle\\
&\qquad+\frac{1}{8\pi}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle+\frac{1}{16\pi}\Big|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2\\
&\qquad-\frac{1}{96\pi}\Big|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2-\frac{\sqrt{-1}}{24\pi}\Big\langle T(e_k^\perp,Je_k^\perp),T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big\rangle\Big\}P(0,0).
\end{aligned}
\tag{5.170}
$$

From (5.156) and (5.165),

$$
\begin{aligned}
&\big((\mathscr{L}_2^0)^{-1}b_l^\perp B(Z,e_l^\perp)P^N\big)(0,0)=-\pi\sqrt{-1}\big\{(\mathscr{L}_2^0)^{-1}b_l^\perp\\
&\qquad\Big[\frac{1}{2}\big\langle R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)e_l^\perp,J\mathcal{R}^0\big\rangle-\frac{5}{4}\big\langle\nabla^{TY}_{\mathcal{R}^\perp}(T(e_k^\perp,e_l^\perp)),J\mathcal{R}^\perp\big\rangle Z_k^\perp\\
&-\frac{5}{4}\big\langle\nabla^{TY}_{\mathcal{R}^0}(T(e_k^0,e_l^\perp)),J\mathcal{R}^\perp\big\rangle Z_k^0-\frac{1}{8}\big\langle T(\mathcal{R}^0,e_j^0),Je_l^\perp\big\rangle\big\langle T(\mathcal{R}^0,Je_j^0),J\mathcal{R}^\perp\big\rangle\\
&\qquad+\frac{1}{8}\big\langle T(\mathcal{R}^0,J\mathcal{R}^0),T(\mathcal{R}^\perp,e_l^\perp)\big\rangle-\frac{1}{8}\big\langle T(\mathcal{R}^0,e_l^\perp),T(\mathcal{R}^\perp,J\mathcal{R}^0)\big\rangle\\
&+\frac{1}{8}\big\langle T(\mathcal{R}^\perp,JT(\mathcal{R}^0,J\mathcal{R}^0)),Je_l^\perp\big\rangle+\frac{1}{2}\big\langle T(\mathcal{R}^\perp,J\mathcal{R}^\perp),T(\mathcal{R}^\perp,e_l^\perp)\big\rangle\Big]P^N\Big\}(0,0).
\end{aligned}
\tag{5.171}
$$

As T is anti-symmetric, from (3.9), (3.54), we get

$$
\begin{aligned}
&b_l^\perp\big\langle\nabla^{TY}_{\mathcal{R}^\perp}(T(e_k^\perp,e_l^\perp)),J\mathcal{R}^\perp\big\rangle Z_k^\perp P^N=-\Big(\tfrac{\partial}{\partial Z_l^\perp}\big\langle\nabla^{TY}_{\mathcal{R}^\perp}(T(e_k^\perp,e_l^\perp)),J\mathcal{R}^\perp\big\rangle\Big)Z_k^\perp P^N,\\
&b_l^\perp\big\langle T(\mathcal{R}^\perp,J\mathcal{R}^\perp),T(\mathcal{R}^\perp,e_l^\perp)\big\rangle P^N=-\big\langle T(\mathcal{R}^\perp,Je_l^\perp)+T(e_l^\perp,J\mathcal{R}^\perp),T(\mathcal{R}^\perp,e_l^\perp)\big\rangle P^N.
\end{aligned}
\tag{5.172}
$$

From (5.5e), (5.123), (5.163), (5.169), (5.171), (5.172) and the anti-symmetric property of T, we get

$$
\begin{aligned}
(5.173)\quad & -\frac{1}{2}\left((\mathscr{L}_2^0)^{-1}b_l^{\perp}B(Z,e_l^{\perp})P^N\right)(0,0)\\
&=\frac{\sqrt{-1}}{2\pi}\left\{-\frac{5}{2^7}\Big(\Big\langle\nabla^{TY}_{e_k^{\perp}}(T(e_k^{\perp},e_l^{\perp})),Je_l^{\perp}\Big\rangle+\Big\langle\nabla^{TY}_{e_l^{\perp}}(T(e_k^{\perp},e_l^{\perp})),Je_k^{\perp}\Big\rangle\Big)\right.\\
&\qquad+\frac{5}{96}\Big\langle\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(\tfrac{\partial}{\partial\overline{z}_j^0},e_l^{\perp}))+\nabla^{TY}_{\frac{\partial}{\partial\overline{z}_j^0}}(T(\tfrac{\partial}{\partial z_j^0},e_l^{\perp})),Je_l^{\perp}\Big\rangle\\
&\qquad\left.+\frac{1}{2^6}\left\langle T(e_k^{\perp},Je_l^{\perp})+T(e_l^{\perp},Je_k^{\perp}),T(e_k^{\perp},e_l^{\perp})\right\rangle\right\}P^N(0,0)=0.
\end{aligned}
$$

By (5.101), (5.123), (5.164), (5.167), (5.170), (5.173) and since $R^{TX_G}(\cdot,\cdot)$ is a $(1,1)$-form, comparing with (5.104) and (5.108), we get

$$
\begin{aligned}
(5.174)\quad & -\left((\mathscr{L}_2^0)^{-1}P^{N^{\perp}}I_1P^N\right)(0,0)=\left\{-\frac{1}{2\pi}\left\langle R^{TX_G}(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_i^0})\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0}\right\rangle\right.\\
&\quad+\frac{7}{6}\left[\frac{5\sqrt{-1}}{2^5\pi}\left\langle Je_k^{\perp},\nabla^{TY}_{e_k^{\perp}}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0}))+\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^{\perp},\tfrac{\partial}{\partial\overline{z}_j^0}))\right\rangle\right.\\
&\qquad+\frac{3}{16\pi}\left\langle R^{TB}(e_k^{\perp},\tfrac{\partial}{\partial z_j^0})e_k^{\perp},\tfrac{\partial}{\partial\overline{z}_j^0}\right\rangle+\frac{3}{32\pi}|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})|^2\\
&\quad\left.\left.-\frac{1}{2^6\pi}|T(e_k^{\perp},\tfrac{\partial}{\partial\overline{z}_j^0})|^2-\frac{\sqrt{-1}}{16\pi}\left\langle T(e_k^{\perp},Je_k^{\perp}),T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})\right\rangle\right]\right\}P^N(0,0).
\end{aligned}
$$

Recall that from (3.6), (5.5a), (5.5b) and (5.13),

$$
\begin{aligned}
(5.175)\quad & |A(e_i^0)e_k^{\perp}|^2=4|A(\tfrac{\partial}{\partial z_i^0})e_k^{\perp}|^2=|T(\tfrac{\partial}{\partial z_i^0},Je_j^0)|^2=2|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})|^2,\\
& \left\langle A(e_i^0)e_i^0,A(e_j^0)e_j^0\right\rangle=4\Big|\sum_i T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2,\\
& |A(e_i^0)e_j^0|^2=\frac{1}{4}|T(e_i^0,Je_j^0)|^2=|T(\tfrac{\partial}{\partial z_i^0},Je_j^0)|^2=2|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})|^2.
\end{aligned}
$$

From (5.92), (5.110), (5.163), (5.175) and since $R^{TX_G}(\cdot,\cdot)$ is a $(1,1)$-form (comparing with (5.112b), (5.113)), we get

$$
\begin{aligned}
(5.176)\quad & -\left((\mathscr{L}_2^0)^{-1}P^{N^{\perp}}I_2P^N\right)(0,0)=\left\{\frac{4}{3\pi}\left\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_i^0})\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0}\right\rangle\right.\\
&\left.-\frac{1}{8\pi}\left\langle R^{TB}(e_k^{\perp},\tfrac{\partial}{\partial z_j^0})e_k^{\perp},\tfrac{\partial}{\partial\overline{z}_j^0}\right\rangle-\frac{1}{48\pi}\left\langle R^{TB}(e_k^{\perp},e_j^{\perp})e_k^{\perp},e_j^{\perp}\right\rangle+\frac{3}{16\pi}|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})|^2\right\}P^N(0,0).
\end{aligned}
$$

By (3.6), (3.54), (5.24), (5.82), (5.92), (5.111), (5.163), (5.175) and since $R^{TX_G}(\cdot,\cdot)$ is a $(1,1)$-form (comparing with (5.112a)), we get

$$
\begin{aligned}
(5.177)\quad & -\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\left\langle\Gamma_{ii}(\mathcal{R}),e_l\right\rangle\nabla_{0,e_l}P^N\Big)(0,0)\\
&=\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big(\frac{2}{3}\Big\langle R^{TX_G}(\mathcal{R}^0,e_i^0)e_i^0,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle b_j\\
&\quad+\frac{1}{2}\left\langle R^{TB}(\mathcal{R}^\perp,e_i^0)e_i^0+A(e_i^0)A(e_i^0)\mathcal{R}^\perp,e_k^\perp\right\rangle b_k^\perp\Big)P^N\Big\}(0,0)\\
&=\Big\{-\frac{1}{3\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},e_i^0)e_i^0,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle-\frac{1}{16\pi}\left\langle R^{TB}(e_k^\perp,e_i^0)e_i^0,e_k^\perp\right\rangle+\frac{1}{16\pi}|A(e_i^0)e_k^\perp|^2\Big\}P^N(0,0)\\
&=\Big\{\Big\langle-\frac{2}{3\pi}R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_i^0})\tfrac{\partial}{\partial z_i^0}+\frac{1}{4\pi}R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle+\frac{1}{8\pi}|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})|^2\Big\}P^N(0,0).
\end{aligned}
$$

By $\mathscr{L}_2^0P^N=0$, (5.24), (5.92), (5.169), (5.175) and since $R^{TX_G}(\cdot,\cdot)$ is a $(1,1)$-form (comparing with (5.114)), we get

$$
\begin{aligned}
(5.178)\quad & -\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big[\frac{1}{4}K_2(\mathcal{R})-\frac{3}{8}\Big(\sum_l\left\langle A(e_l^0)e_l^0,\mathcal{R}^\perp\right\rangle\Big)^2,\mathscr{L}_2^0\Big]P^N\Big\}(0,0)\\
&=\Big\{P^{N^\perp}\Big[\frac{1}{4}K_2(\mathcal{R})-\frac{3}{8}\Big(\sum_l\left\langle A(e_l^0)e_l^0,\mathcal{R}^\perp\right\rangle\Big)^2\Big]P^N\Big\}(0,0)\\
&=\frac{1}{4}\Big\{P^{N^\perp}\Big[\Big\langle\frac{1}{3}R^{TX_G}(\mathcal{R}^0,e_i^0)\mathcal{R}^0+R^{TB}(\mathcal{R}^\perp,e_i^0)\mathcal{R}^\perp,e_i^0\Big\rangle\\
&\quad+\frac{1}{3}\left\langle R^{TB}(\mathcal{R}^\perp,e_i^\perp)\mathcal{R}^\perp,e_i^\perp\right\rangle+\frac{1}{2}\Big(\sum_i\left\langle A(e_i^0)e_i^0,\mathcal{R}^\perp\right\rangle\Big)^2-|A(e_i^0)\mathcal{R}^\perp|^2\Big]P^N\Big\}(0,0)\\
&=\Big(\frac{1}{6\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},e_i^0)e_i^0,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle-\frac{1}{16\pi}\left\langle R^{TB}(e_k^\perp,e_i^0)e_k^\perp,e_i^0\right\rangle\\
&\quad-\frac{1}{48\pi}\left\langle R^{TB}(e_k^\perp,e_i^\perp)e_k^\perp,e_i^\perp\right\rangle-\frac{1}{32\pi}\Big|\sum_i A(e_i^0)e_i^0\Big|^2+\frac{1}{16\pi}|A(e_i^0)e_k^\perp|^2\Big)P^N(0,0),\\
&=\Big(\frac{1}{3\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_i^0})\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle-\frac{1}{4\pi}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle\\
&\quad-\frac{1}{8\pi}\Big|\sum_i T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2+\frac{1}{8\pi}|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})|^2-\frac{1}{48\pi}\left\langle R^{TB}(e_k^\perp,e_j^\perp)e_k^\perp,e_j^\perp\right\rangle\Big)P^N(0,0).
\end{aligned}
$$

By (3.12), (3.54), (5.82), (5.92), (5.163) and (5.175),

(5.179)
$$\begin{aligned}
&-\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big(-\frac{1}{2}\left\langle A(e_l^0)e_l^0,\mathcal{R}^\perp\right\rangle\nabla_{A(e_k^0)e_k^0}+2\left\langle A(e_i^0)e_j^0,\mathcal{R}^\perp\right\rangle\nabla_{A(e_i^0)e_j^0}\\
&\qquad\qquad+\frac{2}{3}\left\langle R^{TB}(\mathcal{R}^\perp,e_i^\perp)e_i^\perp,e_j\right\rangle\nabla_{0,e_j}\Big)P^N\Big\}(0,0)\\
&=-\frac{1}{16\pi}\Big(-\frac{1}{2}\Big|\sum_l A(e_l^0)e_l^0\Big|^2+2|A(e_i^0)e_j^0|^2+\frac{2}{3}\left\langle R^{TB}(e_j^\perp,e_i^\perp)e_i^\perp,e_j^\perp\right\rangle\Big)P^N(0,0),\\
&=\Big(\frac{1}{8\pi}\Big|\sum_i T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2-\frac{1}{4\pi}\Big|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2+\frac{1}{24\pi}\left\langle R^{TB}(e_k^\perp,e_j^\perp)e_k^\perp,e_j^\perp\right\rangle\Big)P^N(0,0),\\
&-\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}(-R^{E_B}(\mathcal{R},e_i))\nabla_{0,e_i}P^N\Big\}(0,0)=\frac{1}{2\pi}R^{E_B}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})P^N(0,0).
\end{aligned}$$

For $F_{ij;kl}\in\mathbb{C}$, from Theorem 3.1, (5.15), (5.135), (5.168) and comparing with (5.152), we get

(5.180)
$$\begin{aligned}
&\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}F_{ij;kl}Z_i^\perp Z_j^\perp Z_k^\perp Z_l^\perp P^N\Big\}(0,0)\\
&=\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}\Big[\sum_{j\neq k}(F_{jj;kk}+F_{kj;kj}+F_{kj;jk})(Z_j^\perp)^2(Z_k^\perp)^2+F_{kk;kk}(Z_k^\perp)^4\Big]P^N\Big\}(0,0)\\
&\quad=\frac{1}{2^8\pi^4}\Big\{P^{N^\perp}\Big[\sum_{j\neq k}(F_{jj;kk}+F_{kj;kj}+F_{kj;jk})\Big(\frac{(b_j^\perp)^2(b_k^\perp)^2}{16\pi}+\frac{1}{2}((b_j^\perp)^2+(b_k^\perp)^2)\Big)\\
&\qquad\qquad+F_{kk;kk}\Big(\frac{(b_k^\perp)^4}{16\pi}+3(b_k^\perp)^2\Big)\Big]P^N\Big\}(0,0)\\
&\qquad\qquad=\frac{-3}{2^8\pi^3}(F_{jj;kk}+F_{kj;kj}+F_{kj;jk})P^N(0,0).
\end{aligned}$$

By (5.45a), (5.74),

(5.181)
$$\begin{aligned}
\frac{1}{9}\sum_i\Big[(\partial_{\mathcal{R}}R^{L_B})_{x_0}(\mathcal{R},e_i)\Big]^2=&-\pi^2\sum_i\left\langle JT(\mathcal{R}^\perp,e_i^0),\mathcal{R}^\perp\right\rangle^2\\
&-\pi^2\sum_j\left\langle JT(\mathcal{R},e_j^\perp),\mathcal{R}^\perp\right\rangle^2.
\end{aligned}$$

By (3.6), (5.14), (5.180) and $\mathcal{T}_{kl}(e_i^0)$ is symmetric on k,l, we get

(5.182)
$$\begin{aligned}
&-\pi^2\sum_i\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\left\langle JT(\mathcal{R}^\perp,e_i^0),\mathcal{R}^\perp\right\rangle^2P^N\Big)(0,0)\\
&\quad=-\pi^2\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{T}_{jj'}(e_i^0)\mathcal{T}_{kl}(e_i^0)Z_j^\perp Z_{j'}^\perp Z_k^\perp Z_l^\perp P^N\Big)(0,0)\\
&\quad=\frac{3}{2^8\pi}\Big(2\mathcal{T}_{jk}(e_i^0)^2+\mathcal{T}_{jj}(e_i^0)\mathcal{T}_{kk}(e_i^0)\Big)P^N(0,0)\\
&\quad=\frac{3}{2^6\pi}\Big(2|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})|^2+\Big|\sum_j\mathcal{T}_{jj}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2\Big)P^N(0,0).
\end{aligned}$$

In the same way, by (5.5e), (5.14), (5.180), we get

$$(5.183)\quad -\pi^2\sum_j\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\left\langle JT(\mathcal{R}^\perp,e_j^\perp),\mathcal{R}^\perp\right\rangle^2P^N\Big)(0,0)$$
$$=\frac{3}{2^8\pi}\widetilde{\mathcal{T}}_{ijk}(\widetilde{\mathcal{T}}_{ijk}+\widetilde{\mathcal{T}}_{kji})P^N(0,0).$$

By (5.14), (5.169),

$$(5.184)\quad -\pi^2\sum_j\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\left\langle JT(\mathcal{R}^0,e_j^\perp),\mathcal{R}^\perp\right\rangle^2P^N\Big)(0,0)$$
$$=\frac{7}{48\pi}|\mathcal{T}_{jk}(\tfrac{\partial}{\partial\overline{z}_i^0})|^2P^N(0,0)=\frac{7}{48\pi}|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})|^2P^N(0,0).$$

By (5.45a) and (5.115), the total degree of Z^0, ∇_{0,e_i^0} in the fourth term of $\mathcal{O}_2'$ in (5.26) is 1, thus the contribution of the fourth term of $\mathcal{O}_2'$ in (5.26) for $-((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_2'P^N)(0,0)$ is zero. By (5.26), (5.174), (5.176)-(5.179) and (5.181)-(5.184), comparing with (5.117), we get

$$(5.185)\quad -\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_2'P^N\Big)(0,0)=\Big\{\frac{1}{2\pi}\left\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_i^0})\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0}\right\rangle$$
$$+\frac{7}{6}\Bigg[\frac{5\sqrt{-1}}{2^5\pi}\left\langle Je_k^\perp,\nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0}))+\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}))\right\rangle+\frac{3}{32\pi}|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})|^2$$
$$+\frac{3}{16\pi}\left\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}\right\rangle+\frac{7}{2^6\pi}|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})|^2-\frac{\sqrt{-1}}{16\pi}\left\langle T(e_k^\perp,Je_k^\perp),T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})\right\rangle\Bigg]$$
$$-\frac{1}{8\pi}\left\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}\right\rangle+\frac{3}{16\pi}|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})|^2+\frac{3}{32\pi}|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})|^2$$
$$+\frac{3}{64\pi}\Big|\sum_j\mathcal{T}_{jj}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2+\frac{3}{2^8\pi}\widetilde{\mathcal{T}}_{ijk}(\widetilde{\mathcal{T}}_{ijk}+\widetilde{\mathcal{T}}_{kji})+\frac{1}{2\pi}R^{E_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big\}P^N(0,0).$$

By (5.62) and (5.165),

$$(5.186)\quad -4\pi^2\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_2''P^N\Big)(0,0)=-4\pi^2\Big\{(\mathscr{L}_2^0)^{-1}P^{N^\perp}$$
$$\Big[-\frac{1}{3}\left\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(\mathcal{R}^0,\mathcal{R}^0)}J\mathcal{R}^\perp+(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(\mathcal{R}^\perp,\mathcal{R}^\perp)}J\mathcal{R}^\perp,J\mathcal{R}^\perp\right\rangle$$
$$+\frac{1}{6}\left\langle\nabla^{TY}_{\mathcal{R}^0}(T(e_j^\perp,J_{x_0}e_i^0))Z_j^\perp Z_i^0+\nabla^{TY}_{\mathcal{R}^\perp}(T(e_j^0,J_{x_0}e_i^0))Z_j^0Z_i^0,J\mathcal{R}^\perp\right\rangle$$
$$+\frac{1}{3}\left\langle R^{TB}(\mathcal{R}^\perp,\mathcal{R}^0)\mathcal{R}^0,\mathcal{R}^\perp\right\rangle-\frac{1}{12}\sum_l\left\langle T(\mathcal{R}^0,e_l),J\mathcal{R}^\perp\right\rangle^2$$
$$-\frac{1}{12}\sum_l\left\langle T(\mathcal{R}^\perp,e_l),J\mathcal{R}^\perp\right\rangle^2+\frac{7}{12}|T(\mathcal{R}^\perp,J\mathcal{R}^\perp)|^2\Big]P^N\Big\}(0,0).$$

Now $\{e_l\}=\{e_i^0\}\cup\{e_k^\perp\}$.

By (5.107a), (5.119), (5.123), (5.169), (5.180), (5.182), (5.183), (5.186) and comparing with (5.120),

$$
\begin{aligned}
(5.187)\quad -4\pi^2\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathcal{O}_2''P^N\Big)(0,0) &= \Big\{\frac{7}{24\pi}\Big[-\frac{8}{3}\nabla_{\frac{\partial}{\partial z_j^0}}\nabla_{\frac{\partial}{\partial \overline{z}_j^0}}\log h \\
&\quad+\frac{\sqrt{-1}}{3}\Big\langle -\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}))-\nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})),Je_k^\perp\Big\rangle \\
&\quad-\frac{1}{3}\Big|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2-\frac{1}{6}\Big|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2-\frac{2}{3}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle\Big] \\
&\quad-\frac{1}{2^6\pi}\Big\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(e_j^\perp,e_j^\perp)}Je_k^\perp,Je_k^\perp\Big\rangle-\frac{1}{2^5\pi}\Big\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(e_j^\perp,e_k^\perp)}Je_j^\perp,Je_k^\perp\Big\rangle \\
&\quad-\frac{1}{2^8\pi}\Big(8|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})|^2+4\Big|\sum_j\mathcal{T}_{jj}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2+\widetilde{\mathcal{T}}_{ijk}(\widetilde{\mathcal{T}}_{ijk}+\widetilde{\mathcal{T}}_{kji})\Big) \\
&\quad+\frac{7}{2^8\pi}\Big(2\mathcal{T}_{jkm}^2+\mathcal{T}_{jjm}\mathcal{T}_{kkm}\Big)\Big\}P^N(0,0).
\end{aligned}
$$

By (5.73), (5.76), (5.163), (5.185) and (5.187), comparing with (5.100), we have

$$
\begin{aligned}
(5.188)\quad \Psi_{1,2}(0) &= -\Big((\mathscr{L}_2^0)^{-1}P^{N^\perp}(\mathcal{O}_2'+4\pi^2\mathcal{O}_2'')P^N\Big)(0,0) \\
&\quad-\frac{\sqrt{-1}}{16\pi}\Big(\big\langle T(e_j^\perp,Je_j^\perp),\widetilde{\mu}^E\big\rangle-2\Big\langle Je_j^\perp,\nabla^{TY}_{e_j^\perp}\widetilde{\mu}^E\Big\rangle\Big)P^N(0,0) \\
&\quad=\Big\{\frac{1}{2\pi}\Big\langle R^{TX_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_i^0})\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle+\frac{1}{2\pi}R^{E_G}(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0}) \\
&\quad+\frac{7}{6}\Big[\frac{1}{6\pi}\Delta_{X_G}\log h+\frac{1}{48\pi}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle+\frac{1}{96\pi}\Big|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2 \\
&\quad-\frac{\sqrt{-1}}{16\pi}\Big\langle T(e_k^\perp,Je_k^\perp),T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})\Big\rangle+\frac{13}{192\pi}\Big|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2 \\
&\quad+\frac{7\sqrt{-1}}{96\pi}\Big\langle\nabla^{TY}_{\frac{\partial}{\partial z_j^0}}(T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}))+\nabla^{TY}_{e_k^\perp}(T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial\overline{z}_j^0})),Je_k^\perp\Big\rangle\Big] \\
&\quad-\frac{1}{8\pi}\Big\langle R^{TB}(e_k^\perp,\tfrac{\partial}{\partial z_j^0})e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0}\Big\rangle+\frac{3}{16\pi}|T(\tfrac{\partial}{\partial z_i^0},\tfrac{\partial}{\partial\overline{z}_j^0})|^2+\frac{1}{16\pi}|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})|^2 \\
&\quad+\frac{1}{32\pi}\Big|\sum_j\mathcal{T}_{jj}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2+\frac{1}{2^7\pi}\widetilde{\mathcal{T}}_{ijk}(\widetilde{\mathcal{T}}_{ijk}+\widetilde{\mathcal{T}}_{kji})+\frac{7}{2^8\pi}\Big(2\mathcal{T}_{jkm}^2+\mathcal{T}_{jjm}\mathcal{T}_{kkm}\Big) \\
&\quad-\frac{1}{2^6\pi}\Big\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(e_j^\perp,e_j^\perp)}Je_k^\perp,Je_k^\perp\Big\rangle-\frac{1}{2^5\pi}\Big\langle(\nabla^{TY}_{\cdot}\dot{g}^{TY}_{\cdot})_{(e_j^\perp,e_k^\perp)}Je_j^\perp,Je_k^\perp\Big\rangle \\
&\quad-\frac{\sqrt{-1}}{16\pi}\Big(\big\langle T(e_j^\perp,Je_j^\perp),\widetilde{\mu}^E\big\rangle-2\Big\langle Je_j^\perp,\nabla^{TY}_{e_j^\perp}\widetilde{\mu}^E\Big\rangle\Big)\Big\}P^N(0,0).
\end{aligned}
$$

By (5.123), (5.130), the term $\frac{7}{6}[\cdots]$ in (5.188) is $\frac{7}{6}\Big(\frac{3}{8\pi}\Delta_{X_G}\log h+\frac{1}{8\pi}|T(e_k^\perp,\frac{\partial}{\partial\overline{z}_j^0})|^2\Big)$.
By (5.129) and (5.188), we get (5.162).
The proof of Lemma 5.14 is complete. □

Lemma 5.15. *The following identity holds,*

$$\begin{aligned}
&\Big\langle (\nabla^{TY}_{e_k^\perp}\dot{g}^{TY}_{e_k^\perp})Je_l^\perp, Je_l^\perp\Big\rangle = 4\nabla_{e_k^\perp}\nabla_{e_k^\perp}\log h,\\
&\Big\langle (\nabla^{TY}_{e_k^\perp}\dot{g}^{TY}_{e_l^\perp})Je_l^\perp, Je_k^\perp\Big\rangle = 4\nabla_{e_k^\perp}\nabla_{e_k^\perp}\log h + 2\Big|\sum_l \mathcal{T}_{ll}(\tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2\\
&\qquad\qquad - 2|T(e_k^\perp,\tfrac{\partial}{\partial\overline{z}_j^0})|^2 + \frac{1}{2}(\widetilde{\mathcal{T}}_{jki}+\widetilde{\mathcal{T}}_{ijk})\widetilde{\mathcal{T}}_{ijk}.
\end{aligned} \tag{5.189}$$

Proof. By using the same argument as in (5.119), we get the first equation of (5.189). Recall that P^{T^HX}, P^{TY} are the projections from $TX = T^HX \oplus TY$ onto T^HX,TY. By (1.3), (1.7), (3.1), (3.40) and (3.41) (cf. also (5.31)),

$$(P^{T^HX}Je_l^{\perp,H})|_{\mu^{-1}(0)} = 0, \quad (Je_l^{\perp,H})_{x_0} \in TY, \tag{5.190a}$$

$$(\nabla^{TX}_{e_k^{\perp,H}}e_l^{\perp,H})_{x_0} = -\frac{1}{2}T(e_k^\perp, e_l^\perp), \tag{5.190b}$$

$$(\nabla^{TX}_{e_j^0}e_l^{\perp,H})_{x_0} = (A(e_j^0)e_l^\perp)^H - \frac{1}{2}T(e_j^0, e_l^\perp),$$

$$(\nabla^{TX}_{Je_l^{\perp,H}}e_k^H)_{x_0} = \frac{1}{2}\left\langle T(e_k,e_j), Je_l^\perp\right\rangle e_j^H + \left\langle T(e_k, Je_l^\perp), Je_j^\perp\right\rangle Je_j^\perp. \tag{5.190c}$$

From (5.190a), we get

$$\nabla^{TX}_{e_i^0}P^{T^HX}Je_l^{\perp,H} = \nabla^{TX}_{Je_k^{\perp,H}}P^{T^HX}Je_l^{\perp,H} = 0. \tag{5.191}$$

By (3.40), (5.14), (5.71) and (5.190b), we get at x_0,

$$\begin{aligned}
\nabla^{TX}_{e_k^{\perp,H}}P^{T^HX}Je_l^{\perp,H} &= \nabla^{T^HX}_{e_k^{\perp,H}}P^{T^HX}Je_l^{\perp,H}\\
&= -\frac{1}{2}JT(e_k^\perp,e_l^\perp) + \frac{1}{2}\left\langle JT(e_k^\perp,e_j), e_l^\perp\right\rangle e_j\\
&= -\frac{1}{2}(\widetilde{\mathcal{T}}_{klj}-\widetilde{\mathcal{T}}_{kjl})e_j^\perp + \frac{1}{2}\left\langle JT(e_k^\perp,e_j^0), e_l^\perp\right\rangle e_j^0.
\end{aligned} \tag{5.192}$$

By (5.6a), (5.190b) and (5.190c), at x_0,

$$\begin{aligned}
[e_k^{\perp,H}, Je_l^{\perp,H}] &= \nabla^{TX}_{e_k^{\perp,H}}Je_l^{\perp,H} - \nabla^{TX}_{Je_l^{\perp,H}}e_k^{\perp,H} = J\nabla^{TX}_{e_k^{\perp,H}}e_l^{\perp,H} - \nabla^{TX}_{Je_l^{\perp,H}}e_k^{\perp,H}\\
&= -\frac{1}{2}JT(e_k^\perp,e_l^\perp) + \frac{1}{2}\left\langle JT(e_k^\perp,e_j), e_l^\perp\right\rangle e_j - \left\langle T(e_k^\perp, Je_j^\perp), Je_l^\perp\right\rangle Je_j^{\perp,H}.
\end{aligned} \tag{5.193}$$

By (5.14), (5.191), (5.192) and (5.193), we get at x_0,

$$\begin{aligned}
&\Big\langle \nabla^{TX}_{[e_k^{\perp,H}, Je_l^{\perp,H}]}P^{T^HX}Je_l^{\perp,H}, e_k^{\perp,H}\Big\rangle\\
&\quad = -\frac{1}{2}\Big\langle \nabla^{TX}_{JT(e_k^\perp,e_l^\perp)-\left\langle JT(e_k^\perp,e_j^\perp),e_l^\perp\right\rangle e_j^\perp}P^{T^HX}Je_l^{\perp,H}, e_k^{\perp,H}\Big\rangle\\
&\quad = \frac{1}{4}(\widetilde{\mathcal{T}}_{klj}-\widetilde{\mathcal{T}}_{kjl})(\widetilde{\mathcal{T}}_{jlk}-\widetilde{\mathcal{T}}_{jkl}).
\end{aligned} \tag{5.194}$$

By (5.6a), (5.190b), (5.191) and (5.192), at x_0,

$$\begin{aligned}
&\nabla^{TX}_{e_j^{0,H}} P^{TY} J e_l^{\perp,H} = J\nabla^{TX}_{e_j^{0,H}} e_l^{\perp,H} = JA(e_j^0)e_l^\perp - \frac{1}{2}JT(e_j^0, e_l^\perp),\\
&\nabla^{TX}_{e_k^{\perp,H}} P^{TY} J e_l^{\perp,H} = J\nabla^{TX}_{e_k^{\perp,H}} e_l^{\perp,H} - \nabla^{TX}_{e_k^{\perp,H}} P^{T^HX} J e_l^{\perp,H}\\
&\qquad = -\frac{1}{2}\left\langle JT(e_k^\perp, e_j), e_l^\perp\right\rangle e_j = -\frac{1}{2}\widetilde{\mathcal{T}}_{kjl} e_j^\perp - \frac{1}{2}\left\langle JT(e_k^\perp, e_j^0), e_l^\perp\right\rangle e_j^0.
\end{aligned} \tag{5.195}$$

Thus by (5.195), at x_0,

$$\nabla^{TY}_{e_k^{\perp,H}} P^{TY} J e_l^{\perp,H} = P^{TY}\nabla^{TX}_{e_k^{\perp,H}} P^{TY} J e_l^{\perp,H} = 0. \tag{5.196}$$

By (1.3), (1.6), (1.7) and (5.196), at x_0,

$$\begin{aligned}
&\left\langle (\nabla^{TY}_{e_k^\perp}\dot{g}^{TY}_{e_l^\perp}) J e_l^\perp, J e_k^\perp\right\rangle = e_k^\perp\left\langle \dot{g}^{TY}_{e_l^\perp} P^{TY} J e_l^\perp, P^{TY} J e_k^\perp\right\rangle = 2e_k^\perp\left\langle \nabla^{TX}_{P^{TY} J e_l^\perp} e_l^\perp, P^{TY} J e_k^\perp\right\rangle\\
&= 2e_k^\perp\left\langle \nabla^{TX}_{J e_l^\perp} e_l^\perp, J e_k^\perp\right\rangle - 2e_k^\perp\left\langle \nabla^{TX}_{P^{T^HX} J e_l^\perp} e_l^\perp, J e_k^\perp\right\rangle - 2e_k^\perp\left\langle \nabla^{TX}_{P^{TY} J e_l^\perp} e_l^\perp, P^{T^HX} J e_k^\perp\right\rangle.
\end{aligned} \tag{5.197}$$

By (5.5d), (5.14), (5.190a), (5.190b) and (5.192), at x_0

$$\begin{aligned}
&-2e_k^\perp\left\langle \nabla^{TX}_{P^{T^HX} J e_l^\perp} e_l^\perp, J e_k^\perp\right\rangle = -2\Big\langle \nabla^{TX}_{\nabla^{TX}_{e_k^\perp} P^{T^HX} J e_l^\perp} e_l^\perp, J e_k^\perp\Big\rangle\\
&\quad = \left\langle T\Big(-\frac{1}{2}(\widetilde{\mathcal{T}}_{klj} - \widetilde{\mathcal{T}}_{kjl})e_j^\perp + \frac{1}{2}\left\langle JT(e_k^\perp, e_j^0), e_l^\perp\right\rangle e_j^0, e_l^\perp\Big), J e_k^\perp\right\rangle\\
&\quad = \frac{1}{2}(\widetilde{\mathcal{T}}_{klj} - \widetilde{\mathcal{T}}_{kjl})\widetilde{\mathcal{T}}_{jlk} + \frac{1}{2}\left\langle T(e_j^0, e_l^\perp), J e_k^\perp\right\rangle\left\langle JT(e_k^\perp, e_j^0), e_l^\perp\right\rangle\\
&\quad = \frac{1}{2}(\widetilde{\mathcal{T}}_{klj} - \widetilde{\mathcal{T}}_{kjl})\widetilde{\mathcal{T}}_{jlk} + \frac{1}{2}|T(e_k^\perp, e_j^0)|^2.
\end{aligned} \tag{5.198}$$

By (5.5e), (5.14), (5.190a), (5.190b) and (5.192), at x_0, we have

$$\begin{aligned}
&-2e_k^\perp\left\langle \nabla^{TX}_{P^{TY} J e_l^\perp} e_l^\perp, P^{T^HX} J e_k^\perp\right\rangle = -2\left\langle \nabla^{TX}_{P^{TY} J e_l^\perp} e_l^\perp, \nabla^{TX}_{e_k^\perp} P^{T^HX} J e_k^\perp\right\rangle\\
&\qquad = -\frac{1}{2}\left\langle JT(e_k^\perp, e_j), e_k^\perp\right\rangle\left\langle T(e_l^\perp, e_j), J e_l^\perp\right\rangle = \frac{1}{2}\mathcal{T}_{ll}(e_j^0)\mathcal{T}_{kk}(e_j^0).
\end{aligned} \tag{5.199}$$

Now by (5.6a),

$$\begin{aligned}
&e_k^\perp\left\langle \nabla^{TX}_{J e_l^\perp} e_l^\perp, J e_k^\perp\right\rangle = -e_k^\perp\left\langle \nabla^{TX}_{J e_l^\perp} J e_l^\perp, e_k^\perp\right\rangle\\
&\qquad = -e_k^\perp\left\langle \nabla^{TX}_{P^{TY} J e_l^\perp} P^{TY} J e_l^\perp + \nabla^{TX}_{P^{T^HX} J e_l^\perp} P^{TY} J e_l^\perp + \nabla^{TX}_{J e_l^\perp} P^{T^HX} J e_l^\perp, e_k^\perp\right\rangle.
\end{aligned} \tag{5.200}$$

By Theorem 5.1, (1.7), (5.190a) and (5.192), at x_0,

$$
\begin{aligned}
(5.201)\quad & -2e_k^{\perp}\left\langle \nabla^{TX}_{P^{TH}X\,Je_l^{\perp}}P^{TY}Je_l^{\perp},e_k^{\perp}\right\rangle = -2\Big\langle \nabla^{TX}_{\nabla^{TX}_{e_k^{\perp}}P^{TH}X\,Je_l^{\perp}}P^{TY}Je_l^{\perp},e_k^{\perp}\Big\rangle \\
&= -\left\langle T\Big(-\frac{1}{2}(\widetilde{\mathcal{T}}_{klj}-\widetilde{\mathcal{T}}_{kjl})e_j^{\perp}+\frac{1}{2}\left\langle JT(e_k^{\perp},e_j^0),e_l^{\perp}\right\rangle e_j^0,e_k^{\perp}\Big),Je_l^{\perp}\right\rangle \\
&= -\frac{1}{2}(\widetilde{\mathcal{T}}_{klj}-\widetilde{\mathcal{T}}_{kjl})\widetilde{\mathcal{T}}_{jkl}-\frac{1}{2}|T(e_k^{\perp},e_j^0)|^2.
\end{aligned}
$$

And by (5.5d), (5.190a)-(5.190c), (5.191), (5.192), (5.194) and (5.201), at x_0,

$$
\begin{aligned}
(5.202)\quad & -2e_k^{\perp}\left\langle \nabla^{TX}_{Je_l^{\perp}}P^{TH}XJe_l^{\perp},e_k^{\perp}\right\rangle = -2\left\langle \Big(\nabla^{TX}_{Je_l^{\perp}}\nabla^{TX}_{e_k^{\perp}}+\nabla^{TX}_{[e_k^{\perp,H},Je_l^{\perp,H}]}\Big)P^{TH}XJe_l^{\perp,H},e_k^{\perp}\right\rangle \\
&= -\left\langle T\Big(-\frac{1}{2}(\widetilde{\mathcal{T}}_{klj}-\widetilde{\mathcal{T}}_{kjl})e_j^{\perp}+\frac{1}{2}\left\langle JT(e_k^{\perp},e_j),e_l^{\perp}\right\rangle e_j,e_k^{\perp}\Big),Je_l^{\perp}\right\rangle -\frac{1}{2}(\widetilde{\mathcal{T}}_{klj}-\widetilde{\mathcal{T}}_{kjl})(\widetilde{\mathcal{T}}_{jlk}-\widetilde{\mathcal{T}}_{jkl}) \\
&= -\frac{1}{2}(\widetilde{\mathcal{T}}_{klj}-\widetilde{\mathcal{T}}_{kjl})\widetilde{\mathcal{T}}_{jlk}-\frac{1}{2}|T(e_k^{\perp},e_j^0)|^2.
\end{aligned}
$$

Finally, by (1.4), (1.7), (1.24) and (5.196), as in (5.119),

$$
\begin{aligned}
(5.203)\quad & -2e_k^{\perp}\left\langle \nabla^{TX}_{P^{TY}Je_l^{\perp}}P^{TY}Je_l^{\perp},e_k^{\perp}\right\rangle = 2e_k^{\perp}\left\langle T(e_k^{\perp},P^{TY}Je_l^{\perp}),P^{TY}Je_l^{\perp}\right\rangle \\
&= \left\langle (\nabla^{TY}_{e_k^{\perp}}\dot{g}^{TY}_{e_k^{\perp}})Je_l^{\perp},Je_l^{\perp}\right\rangle = 4\nabla_{e_k^{\perp}}\nabla_{e_k^{\perp}}\log h.
\end{aligned}
$$

Thus by (5.197)-(5.203),

$$
\begin{aligned}
(5.204)\quad \left\langle (\nabla^{TY}_{e_k^{\perp}}\dot{g}^{TY}_{e_l^{\perp}})Je_l^{\perp},Je_k^{\perp}\right\rangle &= 4\nabla_{e_k^{\perp}}\nabla_{e_k^{\perp}}\log h-\frac{1}{2}|T(e_k^{\perp},e_j^0)|^2 \\
&\quad +\frac{1}{2}\mathcal{T}_{ll}(e_j^0)\mathcal{T}_{kk}(e_j^0)-\frac{1}{2}(\widetilde{\mathcal{T}}_{klj}-\widetilde{\mathcal{T}}_{kjl})\widetilde{\mathcal{T}}_{jkl}.
\end{aligned}
$$

From (3.6) and (5.204), we get (5.189). □

Proof of Theorem 0.7. By (5.14), (5.94),

$$
\begin{aligned}
(5.205)\quad \sum_k \mathcal{F}_1(e_k^{\perp})^2 &= -\left\langle \widetilde{\mu}^E_{x_0},\widetilde{\mu}^E_{x_0}\right\rangle_{g^{TY}}-\left\langle \widetilde{\mu}^E,\frac{3}{2}\sqrt{-1}T(e_l^{\perp},Je_l^{\perp})+2T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})\right\rangle \\
&\quad +\Big|\sum_j T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})\Big|^2+\frac{9}{16}\mathcal{T}_{llm}\mathcal{T}_{kkm}-\frac{3\sqrt{-1}}{2}\left\langle T(e_l^{\perp},Je_l^{\perp}),T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})\right\rangle, \\
\mathcal{F}_1(e_k^{\perp})\mathcal{T}_{kll} &= -\sqrt{-1}\left\langle T(e_l^{\perp},Je_l^{\perp}),\widetilde{\mu}^E+T(\tfrac{\partial}{\partial z_j^0},\tfrac{\partial}{\partial \overline{z}_j^0})\right\rangle+\frac{3}{4}\mathcal{T}_{llm}\mathcal{T}_{kkm}.
\end{aligned}
$$

By (5.137), (5.138), (5.139) and (5.205), we have

$$
\begin{aligned}
(5.206)\quad (\Psi_{1,1}+\Psi_{1,1}^*+\Psi_{1,3}-\Psi_{1,4})(0) &= \Big\{ -\frac{1}{2\pi}\sum_k \mathcal{F}_1(e_k^\perp)^2 - \frac{1}{8\pi}\mathcal{F}_1(e_k^\perp)\mathcal{T}_{kll} \\
&- \frac{13}{2^6\cdot 3\pi}\mathcal{T}_{klm}^2 + \frac{1}{2^7\pi}\mathcal{T}_{kkm}\mathcal{T}_{llm} - \frac{11}{48\pi}\Big|\mathcal{T}_{kl}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2 - \frac{1}{8\pi}\Big|\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2 \Big\} P^N(0,0) \\
&= \Big\{ \frac{1}{2\pi}\left\langle \widetilde{\mu}_{x_0}^E, \widetilde{\mu}_{x_0}^E\right\rangle_{g^{TY}} + \frac{1}{\pi}\Big\langle \widetilde{\mu}^E, \frac{7}{8}\sqrt{-1}T(e_l^\perp, Je_l^\perp) + T(\tfrac{\partial}{\partial z_l^0}, \tfrac{\partial}{\partial\overline{z}_l^0})\Big\rangle \\
&- \frac{1}{2\pi}\Big|\sum_j T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2 + \frac{7\sqrt{-1}}{8\pi}\Big\langle T(e_l^\perp, Je_l^\perp), T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial\overline{z}_j^0})\Big\rangle - \frac{47}{2^7\pi}\mathcal{T}_{kkm}\mathcal{T}_{llm} \\
&- \frac{13}{2^6\cdot 3\pi}\mathcal{T}_{klm}^2 - \frac{11}{48\pi}\Big|T(e_k^\perp, \tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2 - \frac{1}{8\pi}\Big|\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2 \Big\} P^N(0,0).
\end{aligned}
$$

By (5.162) and (5.189), we get

$$
\begin{aligned}
(5.207)\quad \Psi_{1,2}(0)+\Psi_{1,2}(0)^* &= \Big\{ \frac{1}{8\pi}r_{x_0}^{X_G} + \frac{1}{\pi}R^{E_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial\overline{z}_j^0}) + \frac{1}{\pi}\Delta_{X_G}\log h \\
&- \frac{3}{8\pi}\nabla_{e_k^\perp}\nabla_{e_k^\perp}\log h + \frac{35}{48\pi}|T(e_k^\perp, \tfrac{\partial}{\partial\overline{z}_j^0})|^2 + \frac{\sqrt{-1}}{8\pi}\Big\langle T(e_l^\perp, Je_l^\perp), T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial\overline{z}_j^0})\Big\rangle \\
&+ \frac{1}{2\pi}\Big|T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2 - \frac{1}{16\pi}\Big|\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2 + \frac{1}{2^6\pi}\Big[\widetilde{\mathcal{T}}_{ijk}(\widetilde{\mathcal{T}}_{kji}+\widetilde{\mathcal{T}}_{ijk}) - 2(\widetilde{\mathcal{T}}_{jki}+\widetilde{\mathcal{T}}_{ijk})\widetilde{\mathcal{T}}_{ijk}\Big] \\
&+ \frac{7}{2^7\pi}\Big(2\mathcal{T}_{jkm}^2 + \mathcal{T}_{jjm}\mathcal{T}_{kkm}\Big) - \frac{\sqrt{-1}}{8\pi}\Big(\left\langle T(e_l^\perp, Je_l^\perp), \widetilde{\mu}^E\right\rangle - 2\left\langle Je_k^\perp, \nabla_{e_k^\perp}^{TY}\widetilde{\mu}^E\right\rangle\Big)\Big\} P^N(0,0).
\end{aligned}
$$

Thus by (5.132), (5.206) and (5.207), as $\widetilde{\mathcal{T}}_{ijk}$ is anti-symmetric on i,j, we get

$$
\begin{aligned}
(5.208)\quad P^{(2)}(0,0) &= \Big\{ \frac{1}{8\pi}r_{x_0}^{X_G} + \frac{1}{\pi}R^{E_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial\overline{z}_j^0}) + \frac{1}{\pi}\Delta_{X_G}\log h - \frac{3}{8\pi}\nabla_{e_k^\perp}\nabla_{e_k^\perp}\log h \\
&+ \frac{1}{2\pi}|T(e_k^\perp, \tfrac{\partial}{\partial\overline{z}_j^0})|^2 + \frac{1}{2\pi}\Big|T(\tfrac{\partial}{\partial z_i^0}, \tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2 - \frac{1}{2\pi}\Big|\sum_j T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial\overline{z}_j^0})\Big|^2 \\
&+ \frac{\sqrt{-1}}{\pi}\Big\langle T(e_l^\perp, Je_l^\perp), T(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial\overline{z}_j^0})\Big\rangle - \frac{3}{16\pi}\Big|\sum_k \mathcal{T}_{kk}(\tfrac{\partial}{\partial\overline{z}_i^0})\Big|^2 + \frac{1}{24\pi}\mathcal{T}_{klm}^2 \\
&- \frac{5}{16\pi}\mathcal{T}_{kkm}\mathcal{T}_{llm} + \frac{1}{2^6\pi}\widetilde{\mathcal{T}}_{ijk}(3\widetilde{\mathcal{T}}_{kji} - \widetilde{\mathcal{T}}_{ijk}) + \frac{1}{2\pi}\left\langle \widetilde{\mu}_{x_0}^E, \widetilde{\mu}_{x_0}^E\right\rangle_{g^{TY}} \\
&+ \frac{1}{\pi}\Big\langle \widetilde{\mu}^E, \frac{3}{4}\sqrt{-1}T(e_l^\perp, Je_l^\perp) + T(\tfrac{\partial}{\partial z_l^0}, \tfrac{\partial}{\partial\overline{z}_l^0})\Big\rangle + \frac{\sqrt{-1}}{4\pi}\Big\langle Je_k^\perp, \nabla_{e_k^\perp}^{TY}\widetilde{\mu}^E\Big\rangle\Big\} P^N(0,0).
\end{aligned}
$$

By Theorem 5.1, (1.4), (1.24), (5.5c) and (5.14), as same as in (5.119), we get for $U\in T_{x_0}X_G$,

$$
\begin{aligned}
(5.209)\quad &\mathcal{T}_{llm} = \left\langle T(e_m^\perp, Je_l^\perp), Je_l^\perp\right\rangle = 2\nabla_{e_m^\perp}\log h, \quad T(e_l^\perp, Je_l^\perp) = 2(\nabla_{e_k^\perp}\log h)Je_k^\perp, \\
&\mathcal{T}_{kk}(U) = -2\left\langle T(JU, Je_k^\perp), Je_k^\perp\right\rangle = -\left\langle \dot{g}_{JU}^{TY}Je_k^\perp, Je_k^\perp\right\rangle = -4\nabla_{JU}\log h.
\end{aligned}
$$

By (5.136), (5.208) and (5.209), we get Theorem 0.7. □

5.6. **Coefficient Φ_1: general case.** We use the general assumption at the beginning of this Section, but we do not suppose that $\mathbf{J} = J$ in (0.2).

Let $\overline{\partial}^{L^p\otimes E,*}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p\otimes E}$ on the Dolbeault complex $\Omega^{0,\bullet}(X, L^p \otimes E)$ with the scalar product $\langle \quad \rangle$ induced by g^{TX}, h^L, h^E as in Section 2.2.

Set

$$D_p = \sqrt{2}\left(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*}\right) \tag{5.210}$$

Then

$$D_p^2 = 2\left(\overline{\partial}^{L^p\otimes E}\overline{\partial}^{L^p\otimes E,*} + \overline{\partial}^{L^p\otimes E,*}\overline{\partial}^{L^p\otimes E}\right) \tag{5.211}$$

preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(X, L^p \otimes E)$.

For p large enough,

$$\operatorname{Ker} D_p = \operatorname{Ker} D_p^2 = H^0(X, L^p \otimes E). \tag{5.212}$$

Here D_p need not be a spin^c Dirac operator on $\Omega^{0,\bullet}(X, L^p \otimes E)$.

Let $P_p^G(x, x')$ $(x, x' \in X)$ be the smooth kernel of the orthogonal projection P_p^G from $(\mathscr{C}^\infty(X, L^p \otimes E), \langle \quad \rangle)$ onto $(\operatorname{Ker} D_p^2)^G$ with respect to the Riemannian volume form $dv_X(x')$ for p large enough.

We explain now how to reduce the study of the asymptotic expansion of $P_p^G(x, x')$ to the $\mathbf{J} = J$ case.

Let $g_\omega^{TX}(\cdot,\cdot) := \omega(\cdot, J\cdot)$ be the metric on TX induced by ω, J. We will use a subscript ω to indicate the objects corresponding to g_ω^{TX}, especially r_ω^X is the scalar curvature of (TX, g_ω^{TX}), and $\Delta_{X_G,\omega}$ is the Bochner-Laplace operator on X_G as in (1.21) associated to $g_\omega^{TX_G}$.

Let $\det_{\mathbb{C}}$ denote the determinant function on the complex bundle $T^{(1,0)}X$, and $|\mathbf{J}| = (-\mathbf{J}^2)^{-1/2}$.

Let $h_\omega^E := (\det_{\mathbb{C}}|\mathbf{J}|)^{-1}h^E$ define a metric on E. Let R_ω^E be the curvature associated to the holomorphic Hermitian connection on (E, h_ω^E).

Let $\langle \quad \rangle_\omega$ be the Hermitian product on $\mathscr{C}^\infty(X, L^p \otimes E)$ induced by $g_\omega^{TX}, h^L, h_\omega^E$ as in (1.19), then

$$(\mathscr{C}^\infty(X, L^p \otimes E), \langle \quad \rangle_\omega) = (\mathscr{C}^\infty(X, L^p \otimes E), \langle \quad \rangle), \quad dv_{X,\omega} = (\det_{\mathbb{C}}|\mathbf{J}|)dv_X. \tag{5.213}$$

Observe that $H^0(X, L^p \otimes E)$ does not depend on g^{TX}, h^L, h^E.

Let $P_{\omega,p}^G(x, x')$ $(x, x' \in X)$ be the smooth kernel of the orthogonal projection $P_{\omega,p}^G$ from $(\mathscr{C}^\infty(X, L^p \otimes E), \langle \quad \rangle_\omega)$ onto $H^0(X, L^p \otimes E)^G$ with respect to $dv_{X,\omega}(x)$.

By (5.213),

$$P_p^G(x, x') = (\det_{\mathbb{C}}|\mathbf{J}|)(x')P_{\omega,p}^G(x, x'). \tag{5.214}$$

We will use the trivialization in Introduction corresponding to g_ω^{TX}.

Since $g_\omega^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is a Kähler metric on TX, $D_{\omega,p}$ is a Dirac operator (cf. Def. 2.1). Thus Theorems 0.1, 0.2 hold for $P_{\omega,p}^G(x, x')$.

Let dv_B be the volume form on B induced by g^{TX} as in Introduction.

As in (0.11), let $\widetilde{\kappa} \in \mathscr{C}^\infty(TB|_{X_G}, \mathbb{R})$ be defined by for $Z \in T_{x_0}B$, $x_0 \in X_G$,

$$dv_B(x_0, Z) = \widetilde{\kappa}(x_0, Z) dv_{X_G,\omega}(x_0) dv_{N_{G,\omega,x_0}}. \tag{5.215}$$

As in (0.17), we introduce $\mathscr{I}_p(x_0)$ a section of $\mathrm{End}(E_G)$ on X_G,

$$\mathscr{I}_p(x_0) = \int_{Z \in N_G, |Z| \leq \varepsilon_0} h^2(x_0, Z) P_p^G \circ \Psi((x_0, Z), (x_0, Z)) \widetilde{\kappa}(x_0, Z) dv_{N_{G,\omega,x_0}}. \tag{5.216}$$

Then (0.18) still holds.

Summarizes, we have the following result,

Theorem 5.16. *The smooth kernel $P_p^G(x, x')$ has a full off–diagonal asymptotic expansion analogous to* (0.14) *with $\mathcal{Q}_0 = (\det_{\mathbb{C}} |\mathbf{J}|) \mathrm{Id}_{E_G}$ as $p \to \infty$. There exist $\Phi_r(x_0) \in \mathrm{End}(E_G)_{x_0}$ polynomials in A_ω, R^{TB}_ω, R^{E_B}, μ^E, R^E (resp. h_ω, R^{L_B}; resp. μ) and their derivatives at x_0 to order $2r-1$ (resp. $2r$, resp. $2r+1$), and $\Phi_0 = \mathrm{Id}_{E_G}$ such that* (0.25) *holds for $\mathscr{I}_p$. Moreover*

(5.217)

$$\Phi_1(x_0) = \frac{1}{8\pi}\Big[r^{X_G}_\omega + 6\Delta_{X_G,\omega} \log\left(h_\omega|_{X_G}\right) - 2\Delta_{X_G,\omega}\Big(\log(\det_{\mathbb{C}}|\mathbf{J}|)\Big) + 4R^{E_G}(w^0_{\omega,j}, \overline{w}^0_{\omega,j})\Big].$$

Here $\{w_{\omega,j}\}$ is an orthogonal basis of $(T^{(1,0)}X_G, g^{TX_G}_\omega)$.

Proof. By (5.213)-(5.216),

$$\mathscr{I}_p(x_0) = \int_{Z \in N_G, |Z| \leq \varepsilon_0} h^2_\omega(x_0, Z) P^G_{\omega,p} \circ \Psi((x_0, Z), (x_0, Z)) \kappa_\omega(x_0, Z) dv_{N_G,\omega}(Z). \tag{5.218}$$

From the above discussion, only (5.217) reminds to be proved. But

$$R^{E_G}_\omega = R^{E_G} - \overline{\partial}\partial \log(\det_{\mathbb{C}}|\mathbf{J}|), \tag{5.219}$$

Thus

$$2R^{E_G}_\omega(w^0_{\omega,j}, \overline{w}^0_{\omega,j}) = 2R^{E_G}(w^0_{\omega,j}, \overline{w}^0_{\omega,j}) - \Delta_{X_G,\omega} \log(\det_{\mathbb{C}}|\mathbf{J}|), \tag{5.220}$$

and (5.217) is from (0.7) and (5.218). □

6. Bergman kernel and geometric quantization

In this Section, we prove Theorems 0.10, 0.11.

Proof of Theorem 0.10. We use the notation in Section 4.4.

By Theorem 4.4 and Lemma 4.6, we know that $p^{-\frac{n_0}{4}}(\sigma_p \circ \sigma_p^*)^{\frac{1}{2}}$ is a Toeplitz operator with principal symbol $(2^{\frac{n_0}{4}}/\widetilde{h}(x_0))\,\mathrm{Id}_{E_G}$ in the sense of Def. 4.3, and its kernel has an expansion analogous to (4.45) and $Q_{0,0}$ therein is $2^{\frac{n_0}{4}}/\widetilde{h}(x_0)$.

We claim that

$$\mathbb{T}_p = p^{-\frac{n_0}{2}}(\sigma_p \circ \sigma_p^*)^{\frac{1}{2}} \widetilde{h}^2 (\sigma_p \circ \sigma_p^*)^{\frac{1}{2}} \tag{6.1}$$

is a Toeplitz operator with principal symbol $2^{\frac{n_0}{2}}\,\mathrm{Id}_{E_G}$.

Indeed, when $E = \mathbb{C}$, this is a consequence of [9] on the composition of the Toeplitz operators.

To get the above claim for general E, we need just keep in mind that the kernel $\mathbb{T}_p(x_0, x_0')$ of $\mathbb{T}_p$ with respect $dv_{X_G}(x_0')$ has the expansion analogous to (4.45) and $Q_{0,0}$ therein is $2^{\frac{n_0}{2}}\,\mathrm{Id}_{E_G}$.

Our claim then follows from the composition of the expansion of the kernel of $p^{-\frac{n_0}{4}}(\sigma_p \circ \sigma_p^*)^{\frac{1}{2}}$, as well as the Taylor expansion of $\widetilde{h}^2$ (cf. also the recent book [28] for a more detailed proof).

Now we still denote by $\langle\,,\,\rangle$ the L^2-scalar product on $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$ induced by $h^{L_G^p}$, h^{E_G}, g^{TX_G} as in (1.19).

Let $\{s_i^p\}$ be an orthonormal basis of $(H^0(X, L^p \otimes E)^G, \langle\,,\,\rangle)$, then $\varphi_i^p = (\sigma_p \circ \sigma_p^*)^{-\frac{1}{2}}\sigma_p s_i^p$ is an orthonormal basis of $(H^0(X_G, L_G^p \otimes E_G), \langle\,,\,\rangle)$.

From Def. 4.3, (0.28), (1.19) and (6.1), we get

$$(6.2)\quad (2p)^{-\frac{n_0}{2}}\left\langle \sigma_p s_i^p, \sigma_p s_j^p\right\rangle_{\widetilde{h}} = (2p)^{-\frac{n_0}{2}}\left\langle (\sigma_p \circ \sigma_p^*)^{\frac{1}{2}}\varphi_i^p, (\sigma_p \circ \sigma_p^*)^{\frac{1}{2}}\varphi_j^p\right\rangle_{\widetilde{h}} = 2^{-\frac{n_0}{2}}\left\langle \mathbb{T}_p\varphi_i^p, \varphi_j^p\right\rangle = \delta_{ij} + \mathscr{O}\left(\frac{1}{p}\right).$$

The proof of Theorem 0.10 is complete. □

Proof of Theorem 0.11. Set

$$\widetilde{h}^{E_G} = \widetilde{h}^2 h^{E_G}.$$

Then $\widetilde{P}_p^{X_G}$ is the orthogonal projection from $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$ onto $H^0(X, L_G^p \otimes E_G)$, associated to the Hermitian product on $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$ induced by the metrics h^{L_G}, $\widetilde{h}^{E_G}$, g^{TX_G} as in (1.19).

Let $\widetilde{P}_{p,\omega}^{X_G}(x_0, x_0')$ be the smooth kernel of $\widetilde{P}_p^{X_G}$ with respect to $dv_{X_G}(x_0')$. Then

$$(6.3)\qquad \widetilde{P}_{p,\omega}^{X_G}(x_0, x_0') = \widetilde{h}^2(x_0')\widetilde{P}_p^{X_G}(x_0, x_0').$$

Let $\widetilde{\nabla}^{E_G}$ be the Hermitian holomorphic connection on $(E_G, \widetilde{h}^{E_G})$ with curvature $\widetilde{R}^{E_G}$. Then

$$(6.4)\qquad \widetilde{\nabla}^{E_G} = \nabla^{E_G} + \partial\log(\widetilde{h}^2), \quad \widetilde{R}^{E_G} = R^{E_G} + 2\overline{\partial}\partial\log\widetilde{h}.$$

Thus from (6.4),

$$(6.5)\qquad \widetilde{R}^{E_G}(w_j^0, \overline{w}_j^0) = 2\widetilde{R}^{E_G}(\tfrac{\partial}{\partial z_j^0}, \tfrac{\partial}{\partial \overline{z}_j^0}) = R^{E_G}(w_j^0, \overline{w}_j^0) + \Delta_{X_G}\log\widetilde{h}.$$

By (5.18), (6.3) and (6.5), Theorem 0.11 is a direct consequence of [17, Theorem 1.3] (or Theorem 0.6 with $G = \{1\}$) for $\widetilde{P}_{p,\omega}^{X_G}(x_0, x_0)$. □

Acknowledgments. We thank Professor Jean-Michel Bismut for useful conversations. The work of the second author was partially supported by MOEC, MOSTC and NNSFC. Part of work was done while the second author was visiting IHES during February and March, 2005. He would like to thank Professor Jean-Pierre Bourguignon for the hospitality.

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