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ABSTRACT

Aims. To understand the fundamental physical processes important for the evolution of solar rotation and distribution of chemical species, we provide theoretical predictions for particle mixing and momentum transport in the stably stratified tachocline.
Methods. By envisioning that turbulence is driven externally in the tachocline (e.g. by plume penetration), we compute the amplitude of turbulent flow, turbulent particle diffusivities, and eddy viscosity, by incorporating the effect of a strong radial differential rotation and stable stratification. We identify the different roles that the shear flow and stable stratification play in turbulence regulation and transport.
Results. Particle transport is found to be severely quenched due to stable stratification as well as radial differential rotation, especially in the radial direction with an effectively more efficient horizontal transport. The eddy viscosity is shown to become negative for parameter values typical of the tachocline, suggesting that turbulence in the stably stratified tachocline leads to a non-uniform radial differential rotation. Similar results also hold in the radiative interiors of stars, in general.

Key words. Turbulence – Sun: interior – Sun: rotation – waves

1. Introduction

Since the formation of the radiative core, which marked the beginning of its journey on the main sequence, the sun has slowed down significantly due to the loss of angular momentum from its surface (e.g. see Stix 1989; Schatzman 1993). The angular momentum transport must have been very efficient during its spin-down in order for the sun to have rotational profile as observed today (see, e.g. Charbonneau et al 1998). Vigorous turbulence in the convection zone can readily provide a mechanism for efficient radial momentum transport, thereby eradicating radial differential rotation therein. Such turbulent transport is however considered to be absent in the stably stratified radiative interior, which has also spun-down during the solar evolution, presently rotating uniformly at the rate roughly the same as the mean average rotation rate on the solar surface. Whichever mechanism is responsible for momentum transport in the interior (which itself is an important problem), it should be closely related to the transport in the tachocline through which the surface spin-down is communicated to the interior. Transport in the tachocline also plays a crucial role in the overall mixing of light elements (lithium, beryllium, etc) (see e.g. Schatzman 1993), thereby determining the level of their surface abundances on the sun. Therefore, it is essential to understand physical mechanisms for transport in the tachocline and then to formulate a consistent theory starting from first principles based on those processes. This is particularly true since virtually all the previous theoretical modelling heavily relies on a crude parameterization of transport process, which is then adjusted to obtain the agreement with observations.

We have initiated the development of a consistent theory of turbulent transport in the tachocline in the previous papers, by taking into account the crucial effect of the large-scale shear flows, provided by a strong radial differential rotation (Kim 2005) as well as latitudinal differential rotation (Leprovost & Kim 2006). By envisioning that the tachocline is perturbed externally, e.g. by plumes penetrating from the convection zone (e.g. see Gilman 2000; Brummell, Clune & Toomre 2002; Rogers & Glazmaier 2005), we demonstrated how turbulence level and transport are affected via shearing in a non-trivial manner (see also Burrell 1997; Kim & Diamond 2003; Kim 2004;2006) in a simplified three dimensional (3D) hydrodynamic turbulence. In particular,

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turbulent transport of chemical species and angular momentum are shown to become strongly anisotropic with effectively much more efficient transport in the horizontal (latitudinal) direction than the vertical (radial) direction due to shear stabilization by strong radial shear. The resulting anisotropic momentum transport was shown to reinforce a strong radial shear (Leprovost & Kim 2006), with a positive feedback on the confinement of the tachocline (Spiegel & Zahn 1992) while chemical species are predicted to have latitudinal dependent mixing due to the variation of radial shear (Kim 2005).

The purpose of this paper is to investigate how a stable stratification in the tachocline modifies the predictions obtained in these studies. We again envision that the tachocline is externally perturbed (e.g. by plumes penetrating from the convection zone above), driving a background turbulence, which gives rise to turbulent transport (affected by shear flow). In the presence of a stable stratification, the turbulence in the tachocline is no longer completely random as the stable stratification provides a restoring force against radial displacement of fluid elements, supporting the propagation of internal gravity waves (Lighthill 1978). These waves tend to increase the memory of otherwise random turbulent fluid motion and can reduce the overall transport due to turbulence. We shall show that the turbulent transport due to shear stabilization found in the previous studies is further enhanced in the presence of stable stratification (gravity waves).

It is important to contrast our approach to those adopted in most previous works, which focus on the momentum transport by gravity waves themselves through dissipative processes (e.g. Plumb 1978; Kim and MacGregor 2001;2003; Talon, Kumar & Zahn 2002). For instance, gravity waves are considered to be generated in the convection zone (Press 1981) and deposit their momentum into background shear flow via radiative damping as they propagate through the tachocline and the interior (Kim and MacGregor 2001;2003). Relying crucially on radiative damping, the momentum transport by gravity waves would occur on a long time scale. [Recall that the transport by waves requires molecular dissipation, and is thus a slow process.] Similarly, weak mixing due to damped gravity waves was also suggested as a mechanism for a modestly enhanced mixing of light elements (lithium) [Press & Rybicki 1981; García López & Spruit 1991]. In these works, a certain level of background turbulence could not be ruled out and was invoked to provide an enhanced viscosity for the evolution of a mean flow (Kim and MacGregor 2001;2003) and the gravity waves (Charbonnel & Talon 2005; Talon, Kumar & Zahn 2002). The value of the effective viscosity is often arbitrarily taken to be a positive constant, being much larger than molecular viscosity, not affected by shear flow nor by gravity waves, or it is simply parameterized.

In this paper, we shall treat turbulence and gravity waves on an equal footing and consistently compute the values of the turbulent eddy viscosity and particle diffusivity, by incorporating the effect of a shear flow (provided by a radial differential rotation) and identify different roles that gravity waves and shear flow play in turbulent transport. Specifically, we shall show that unlike shear flows, which reduce both turbulence level and transport, a stable stratification mainly suppresses turbulent transport without much effect on turbulence level. We shall further demonstrate that the stratification favors a negative eddy viscosity and thus tends to sharpen the gradient of large-scale shear flows rather than smoothing it out. This tendency was also found in Kim & MacGregor (2001;2003). We note that the elucidation of the effects of gravity waves is essential for understanding the momentum transport in the radiative interior as they have often been advocated as a mechanism to explain a uniform rotation in that region (e.g. Charbonnel & Talon 2005). Stratified turbulence with a shear flow is also important for the transport in radiative interiors and/or envelops of stars, in general, as well as in geophysical systems. In particular, it has actively been studied in geophysical systems (e.g. see Jacobitz, Sarkar & van Atta 1997; Stacey, Monismith, & Burau 1999), where stable stratification was shown to inhibit turbulent transport in the direction of a background density gradient, leading to the two dimensional (2D) turbulence.

The remainder of the paper is structured as follows. We first investigate the effect of gravity waves on turbulence in Sect. 2. In Sect. 3, we incorporate the effect of strong radial differential rotation and study how the gravity waves modify the overall turbulent transport. Section 4 contains the conclusion and discussions.

2. Internal Gravity Waves

To simplify the problem, we shall consider incompressible fluid with local cartesian coordinates x, y, and z for radial, azimuthal, and latitudinal directions, respectively, and use Boussinesq approximation to capture the effect of stratification. We assume that the fluid is stirred by an external forcing on small scales, giving rise to fluctuations. In the absence of stratification, this fluid forcing will drive turbulence on small scales and maintain it at a level at which the injected energy is balanced by dissipation in the system. As the stratification increases, the forcing will generate not only random turbulent motion but also coherent (gravity) waves. Alternatively, some of turbulent motion will be turned into packets of gravity waves. Therefore, we consider both turbulence and gravity waves as fluctuations on scales much smaller than those associated with mean density or mean background shear flows. Specifically, we express the total mass density $\rho = \rho_0(x) + \rho'$ where $\rho_0(x)$ and ρ' are the background and fluctuating mass densities, respectively; the total velocity $\mathbf{u} = \mathbf{U}_0 + \mathbf{v}$ where $\mathbf{U}_0$ and $\mathbf{v}$ are a large-scale shear flow due to radial differential rotation and small-scale fluctuations; the total particle density of chemical elements $n = n_0(\mathbf{x}) + n'$ where $n_0(\mathbf{x})$ and n' are mean and fluctuating components. Then, the main governing equations for fluctuations $\mathbf{v}$, ρ', and n', involving both turbulence and gravity waves, are as follows (see, e.g., Kim & MacGregor 2003; Moffatt 1978):

$$(\partial_t + \mathbf{U}_0 \cdot \nabla)\mathbf{v} = -\nabla p - g\rho'\hat{x} + \nu\nabla^2\mathbf{v} + \mathbf{f}\,, \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0\,, \tag{2}$$

$$(\partial_t + \mathbf{U}_0 \cdot \nabla)\rho' = \frac{\overline{\rho} N^2}{g} v_x + \mu \nabla^2 \rho' , \tag{3}$$

$$(\partial_t + \mathbf{U}_0 \cdot \nabla) n' = -\partial_x n_0 v_x + D \nabla^2 n' . \tag{4}$$

Here, ν, μ and D are molecular viscosity, thermal diffusivity, and particle diffusivity, respectively; $\mathbf{f}$ in Eq. (1) is the small-scale forcing driving turbulence; $\overline{\rho}$ is the constant background density, and $N = -g(\partial_x \rho_0 + \overline{\rho} g / c_s^2)/\overline{\rho}$ is the Brunt-Väisälä frequency, where c_s is the sound speed. Note that the typical values of ν, μ, and D in the tachocline are 10^2 cm^{2}s^{-1}, 10^7 cm^{2}s^{-1}, and 10^4 cm^{2}s^{-1}, respectively.

In this section, we first ignore a background shear flow and study how the turbulence and transport are modified due to stable stratification in the tachocline. To obtain the overall turbulent transport, we solve coupled equations (1)-(4) in terms of Fourier transform for fluctuating quantities ϕ':

$$\phi'(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \int d^3k \, \tilde{\phi}(\mathbf{k}, t) \exp\{i(k_x x + k_y y + k_z z)\} . \tag{5}$$

In the simplest case of unit Prandtl numbers ($\nu = D = \mu$), solutions can easily be found as:

$$\tilde{v}_x = \frac{1}{\gamma + a^2} \int dt' G(t, t') \cos \overline{N}(t - t') \tilde{h}_1(t') , \tag{6}$$

$$\tilde{v}_y = -\frac{a}{\gamma} \tilde{v}_x - \frac{\beta}{\gamma} \int dt' G(t, t') \tilde{h}_2(t') , \tag{7}$$

$$\tilde{v}_z = -\frac{a\beta}{\gamma} \tilde{v}_x + \frac{1}{\gamma} \int dt' G(t, t') \tilde{h}_2(t') , \tag{8}$$

$$\tilde{\rho} = \frac{N}{g\sqrt{\gamma(\gamma + a^2)}} \int dt' G(t, t') \sin \overline{N}(t - t') \tilde{h}_1(t') . \tag{9}$$

Here, $\overline{N}^2 = \gamma N^2/(\gamma + a^2)$; $G(t, t') = e^{-\nu k^2 (t - t')}$ is the Green's function; $a = k_x/k_y$, $\beta = k_z/k_y$, and $\gamma = 1 + \beta^2$; $\tilde{h}_1$ and $\tilde{h}_2$ are the forcing terms which are related to $\tilde{f}_i$ as

$$\tilde{h}_1 = (1 + \beta^2)\tilde{f}_x - a\tilde{f}_y - a\beta \tilde{f}_z , \tag{10}$$

$$\tilde{h}_2 = -\beta \tilde{f}_y + \tilde{f}_z . \tag{11}$$

For simplicity, we assume the forcing to be homogeneous in space with a short correlation time τ_f:

$$\langle \tilde{f}_i(\mathbf{k}_1, t_1) \tilde{f}_j(\mathbf{k}_2, t_2) \rangle = \tau_f (2\pi)^3 \delta(t_1 - t_2) \delta(\mathbf{k}_1 + \mathbf{k}_2) \psi_{ij}(\mathbf{k}_2) . \tag{12}$$

A straightforward algebra by using Eqs. (6)–(12) then gives us the following results on turbulence level and particle diffusivities defined by $\langle n' v_i \rangle = -D_T^{ij} \partial_j n_0$:

$$\langle v_x^2 \rangle = \tau_f \int \frac{d^3k}{(2\pi)^3} \frac{\phi_{11}(\mathbf{k})}{2(\gamma + a^2)^2} \sqrt{W_3} , \tag{13}$$

$$\langle v_y^2 \rangle = \tau_f \int \frac{d^3k}{(2\pi)^3} \left[\frac{a^2 \phi_{11}(\mathbf{k})}{2\gamma^2(\gamma + a^2)^2} + \frac{\beta^2 \phi_{22}(\mathbf{k})}{\gamma^2} \right] \sqrt{W_3} , \tag{14}$$

$$\langle v_z^2 \rangle = \tau_f \int \frac{d^3k}{(2\pi)^3} \left[\frac{a^2\beta^2 \phi_{11}(\mathbf{k})}{2\gamma^2(\gamma + a^2)^2} + \frac{\phi_{22}(\mathbf{k})}{\gamma^2} \right] \sqrt{W_3} , \tag{15}$$

$$D_T^{xx} = \frac{\tau_f}{N^2} \int \frac{d^3k}{(2\pi)^3} \frac{\phi_{11}(\mathbf{k})}{4\gamma(\gamma + a^2)} \left[1 + W_1 (2\nu k^2)^2 \right] , \tag{16}$$

$$D_T^{yy} = \tau_f \int \frac{d^3k}{(2\pi)^3 \gamma^2} \left[\frac{W_1 a^2 \phi_{11}(\mathbf{k})}{(\gamma + a^2)^2} + \frac{W_2 a\beta \phi_{12}(\mathbf{k})}{(\gamma + a^2)} + W_3 \beta^2 \phi_{22}(\mathbf{k}) \right] , \tag{17}$$

$$D_T^{zz} = \tau_f \int \frac{d^3k}{(2\pi)^3 \gamma^2} \left[\frac{W_1 a^2 \beta^2 \phi_{11}(\mathbf{k})}{(\gamma + a^2)^2} - \frac{W_2 a\beta \phi_{12}(\mathbf{k})}{(\gamma + a^2)} + W_3 \phi_{22}(\mathbf{k}) \right] . \tag{18}$$

Here, $W_1 = 1/((2\nu k^2)^2 + (2\overline{N})^2)$, $W_2 = 2(2\nu k^2)^2/[(2\nu k^2)^2 + \overline{N}^2]$, and $W_3 = 1/(2\nu k^2)^2$; $\overline{N} \gg \nu k^2$ was used to obtain Eqs. (13)–(15); $\mathbf{k}$ inside the integrals in Eqs. (13)–(18) is the wavenumber of the forcing; $\phi_{ij}(\mathbf{k})$ ($i = 1, 2$) is the power spectrum of the forcing defined by:

$$\langle \tilde{h}_i(\mathbf{k}_1, t_1) \tilde{h}_j(\mathbf{k}_2, t_2) \rangle = \tau_f (2\pi)^3 \delta(t_1 - t_2) \delta(\mathbf{k}_1 + \mathbf{k}_2) \phi_{ij}(\mathbf{k}_2) . \tag{19}$$

In this paper, we shall consider an incompressible forcing ($\nabla \cdot \mathbf{f} = 0$) for simplicity, in which case ϕ_{ij} in Eq. (19) is related to ψ_{ij} in Eq. (12) as:

$$\phi_{11} = \frac{k^4}{k_y^4}\psi_{11}\,, \phi_{12} = \frac{1}{k_y^4}(k^2 k_x k_z \psi_{11} + k^2 k_H^2 \psi_{13})\,,$$
$$\phi_{22} = \frac{1}{k_y^4}(k_x^2 k_z^2 \psi_{11} + k_H^4 \psi_{33} + 2k_x k_z k_H^2 \psi_{13})\,. \quad (20)$$

Note that ϕ_{11} and ϕ_{22} can signify strong radial forcing (e.g. by plumes) and horizontal forcing, respectively. For instance, in the case of strong radial forcing by plumes ($\psi_{11} \gg \psi_{33}$) with $k_H \gg k_x$ [see Fig. 2 in Leprovost & Kim (2006)], ϕ_{11} will dominate over ϕ_{22}. Note further that in the 2D limit with $k_z = 0$ and $\psi_{33} = 0$, ϕ_{22} vanishes.

We discuss some of the important implications of Eqs. (13)–(18). First, all three components of fluctuating velocity amplitude in Eqs. (13)-(15) are independent of N, clearly showing that the amplitude of the turbulent flow is not influenced by stable stratification. The exact level of fluctuations is determined by the characteristics of the external forcing (ϕ_{ij}). We examine this in the simple 2D limit where $k_z = 0$ ($\gamma = 1$) and $\psi_{33} = 0$. In this limit, $\phi_{22} = 0$ in Eqs. (13)–(15) (due to incompressibility), leading to $\langle v_x^2\rangle/\langle v_y^2\rangle \sim 1/a^2 = k_y^2/k_x^2$ and $\langle v_z^2\rangle = 0$, as should be the case for the incompressible fluid. If the forcing $\mathbf{f}$ contains only gravity modes, the wave number of the forcing would satisfy the local dispersion relation $k_x = \pm k_y\sqrt{N^2/(\omega - U_0 k_y)^2 - 1}$ (here, ω is the frequency of the gravity waves). Furthermore, if these gravity modes are generated by the overturning of fluids in the convection zone (e.g. Press 1981) with $U_0 \sim 0$, they would have a strong power at low frequencies $\omega \ll N$, thereby giving $k_x/k_y \sim N/\omega \gg 1$, and thus $\langle v_x^2\rangle/\langle v_y^2\rangle \ll 1$. Note that this is the situation normally considered in the previous works (e.g. Kim & MacGregor 2001;2003; Talon, Kumar & Zahn 2002), where the main focus was on the momentum deposition by such gravity waves due to radiative damping after entering the tachocline from the bottom of the convection zone. In contrast, in this paper, the forcing $\mathbf{f}$ is not restricted to gravity modes, but is taken to be general, including any form of perturbation with arbitrary values of k_x/k_y. For instance, in the case of a strong radial forcing due to plumes with $\phi_{11} \gg \phi_{22}$, it is plausible that $a = k_x/k_y \ll 1$ [see Fig 2 in Leprovost & Kim (2006)], thereby leading to a stronger radial fluctuation than horizontal one, i.e. $\langle v_x^2\rangle/\langle v_y^2\rangle \gg 1$.

Second, turbulent transport in Eqs. (16)–(18) are strongly affected by stratification as N^2 increases, in contrast to fluctuation levels in Eqs. (13)–(15). Since $N \gg \nu k^2$ on reasonable scales $l \gg 10^2$ cm, the vertical (radial) mixing is severely quenched as the stratification increases (i.e. as N^{-2}). In comparison, the part of the horizontal (latitudinal) mixing due to the radial forcing ϕ_{11} is reduced proportional to N^{-2} while the one due to the horizontal forcing ϕ_{22} is independent of N. The comparison of these two contributions (or alternatively D_{xx} and D_{yy}) in the case of $\phi_{11} \sim \phi_{22}$ gives us a cut-off scale $l_c \sim \sqrt{\nu/N}$ above which vertical (radial) mixing is strongly reduced compared to the horizontal (latitudinal) mixing. That is, in the presence of both radial and horizontal forcings of comparable strength, stable stratification mainly reduces the vertical transport without much effect on horizontal transport on scales $l > l_c$. For parameter values typical of the tachocline $\nu \sim 10^2$ cm^{3}s^{-1} and $N \sim 3\times 10^{-3}$s^{-1}, $l_c \sim 10^2$cm. Thus, the stratification is very likely to play an important role over a broad range of physically reasonable scales. Finally, in the limit of strong radial forcing with $\phi_{11} \gg \phi_{22}$ and $a \sim 0$, the stratification influences both the radial and horizontal transports to the same degree while the incompressibility renders $D_T^{xx}/D_T^{yy} \sim D_T^{xx}/D_T^{zz} \sim 1/a^2 \sim \langle v_x^2\rangle/\langle v_y^2\rangle \gg 1$, with an effectively more efficient radial transport.

The results discussed above are obtained in the case of $\nu \sim D \sim \mu$. However, in the solar tachocline, the thermal diffusivity is much larger than viscosity ($\mu \sim 10^5\nu$). In fact, this large thermal diffusivity is often considered to reduce the stabilizing effect of stable stratification via the weakening of the buoyancy restoring force. To elucidate this effect, it is illuminating to consider the limit of strong thermal diffusion in which case density fluctuation becomes stationary with $\partial_t\rho' = N^2 v_x/g - \mu k^2\rho' = 0$ in Eq. (3). In this limit, by following a similar analysis as previously, we can easily obtain the vertical and horizontal particle diffusivities as follows:

$$D_{xx} \sim \frac{\tau_f}{2\alpha^2}\int \frac{d^3k}{(2\pi)^3}\frac{\phi_{11}(\mathbf{k})}{(\gamma + a^2)^2}\,, \quad (21)$$
$$D_{yy} \sim \tau_f \int \frac{d^3k}{(2\pi)^3\gamma^2}\left[\frac{a^2\phi_{11}(\mathbf{k})}{\gamma^2(\gamma + a^2)^2}\frac{1}{2\alpha^2} + \frac{\beta^2\phi_{22}(\mathbf{k})}{(2\nu k^2)^2}\right], \quad (22)$$

where $\alpha = \gamma N^2/\mu k^2(\gamma + a^2)$. Equations (21)-(22) show that the vertical mixing is reduced more than horizontal mixing by a factor of $(\nu k^2)^2/\alpha^2$, being severely inhibited on scales larger than the critical scale $l_{c\mu} = (\mu\nu)^{1/4}/N^{1/2} \sim 10 l_c$. Here, $l_c = \sqrt{\nu/N}$ is the critical scale in the case of $\nu = \mu$. Therefore, a strong thermal diffusion weakens the buoyancy effect and makes the effect of stratification become important on larger scales, compared to the case of a weak thermal diffusion. This result is thus consistent with the expectation employed in previous works.

To summarize, a stable stratification can dramatically quench turbulent transport with a more effective mixing in the horizontal directions orthogonal to the background density gradient. It does not however affect the amplitude of the turbulent flow, which is found to be of comparable strength in both vertical and horizontal directions (for a temporally short-correlated forcing).

3. Stratification and shearing

In this section, we study how turbulence and transport are affected by stable stratification in the background of a shear flow by incorporating a radial differential rotation in the tachocline. The shear flow introduces a non-trivial effect not only on turbulence, as shown in the previous works (Kim 2005; Leprovost & Kim 2006), but also on gravity waves. Again, we envision that turbulence is maintained in the tachocline by an external forcing while gravity waves are excited due to this external forcing in the stably stratified tachocline. We shall then compute the overall turbulent transport consistently by taking into account the interaction among turbulence, shear flow and gravity waves, instead of simply assuming a (large) constant value of turbulent viscosity for mean shear flow (and gravity waves). Note that this treatment is essential when there is no clear scale separation between gravity waves and turbulence, in which case turbulence cannot be considered to give an enhanced value of viscosity for gravity waves (c.f. Charbonnel & Talon 2005).

For the evolution of fluctuations, we approximate the radial differential rotation by a linear shear flow with $\mathbf{U}_0 = -x\mathcal{A}\hat{y}$. Here, $\mathcal{A}$ is shearing rate which we assume to be positive without loss of generality. As done in the previous papers (Kim 2005; Leprovost & Kim 2006), to capture the effect of shearing due to radial differential rotation ($\mathcal{A} \sim 3\times10^{-6}$ s^{-1} for the tachocline), which can easily dominate over that of dissipation ($\mathcal{A} > Dk^2$ for $k < 10^{-4}$ cm$^{-1} \sim 10^6/H_0$, where H_0 is the pressure scale height $\sim 6\times10^9$ cm), we use the special Fourier transform for fluctuating quantities ϕ':

$$\phi'(\mathbf{x},t) = \frac{1}{(2\pi)^3}\int d^3k\tilde{\phi}(\mathbf{k},t)\exp\left\{i(k_x(t)x + k_y y + k_z z)\right\}. \tag{23}$$

Here, k_x satisfies an eikonal equation

$$\partial_t k_x(t) = k_y\mathcal{A}\,. \tag{24}$$

Equation (24) implies that k_x linearly increases in time as $k_x(t) = k_x(0) + k_y\mathcal{A}t$, manifesting the main effect of shearing by a shear flow $U_0(x)\hat{y}$, i.e., generation of fine scales in the x direction due to tilting and distortion of fluid eddies (e.g. see Burrell 1997; Kim 2004). It is interesting to note that a gravity wave with an initially positive value of $k_x(0)$ can change its sign after the time interval $k_x(0)/k_y\mathcal{A}$ (for $k_y > 0$) due to shearing. Since the local (radial) group velocity of gravity waves is given by $v_{gx} = -k_y^2k_xN^2/k^4(\omega - U_0k_y)$ (e.g., see Kim & MacGregor 2003), the gravity wave thus alters its propagation direction as k_x flips its sign. Therefore, a gravity wave which initially propagates downward to the interior from the convection with a negative vertical group velocity (i.e. $v_{gx} < 0$) can propagate upwards when the vertical group velocity becomes positive ($v_{gx} > 0$) due to shearing.

For parameter values typical of the tachocline $N \sim 10^3$ s^{-1} and $\mathcal{A} \sim 3\times10^{-6}$ s^{-1}, $Ri = N^2/\mathcal{A}^2 \gg Ri_c \simeq 1/4$, satisfying stability criterion (Lighthill 1978). We thus assume $Ri = N^2/\mathcal{A}^2 \gg 1$ in the remainder of the paper. In this case, the solutions to Eqs. (1)-(4) can easily be obtained (see Appendix A for details) in the case of unity Prandtl numbers $\nu = \mu = D$. We then use these solutions to obtain turbulence level and transport coefficient defined by $\langle n'v_i\rangle = -D_T^{ij}\partial_j n_0$ for $i = 1, 2$ and 3, and momentum flux $\langle v_xv_y\rangle = -\nu_T\partial_xU_0$ by using the correlation functions of forcing defined in Eq. (19) (see Appendix A for details):

$$\langle v_x^2\rangle = \frac{\tau_f}{\mathcal{A}}\int\frac{d^3k}{(2\pi)^3}\frac{\phi_{11}(\mathbf{k})}{2\gamma^{3/2}}, \tag{25}$$

$$\langle v_y^2\rangle = \frac{\tau_f}{\mathcal{A}}\int\frac{d^3k}{(2\pi)^3}\left[\frac{\phi_{11}(\mathbf{k})}{2\gamma^{5/2}}\ln\xi + \frac{\beta^2\phi_{22}(\mathbf{k})}{\gamma^2}G_0\right], \tag{26}$$

$$\langle v_z^2\rangle = \frac{\tau_f}{\mathcal{A}}\int\frac{d^3k}{(2\pi)^3}\left[\frac{\beta^2\phi_{11}(\mathbf{k})}{2\gamma^{5/2}}\ln\xi + \frac{\phi_{22}(\mathbf{k})}{\gamma^2}G_0\right], \tag{27}$$

$$D_T^{xx} = \frac{\tau_f}{2N^2}\int\frac{d^3k}{(2\pi)^3}\frac{\phi_{11}(\mathbf{k})}{\gamma\sqrt{\gamma+a^2}}\xi G_1\,, \tag{28}$$

$$D_T^{yy} = \frac{\tau_f}{2}\int\frac{d^3k}{(2\pi)^3\gamma^2}\left[\frac{\phi_{11}(\mathbf{k})\Gamma(\frac{4}{3})}{2N^2\gamma\sqrt{\gamma+a^2}}\left(\frac{3}{2\xi}\right)^{\frac{1}{3}} + \frac{\beta^2\phi_{22}(\mathbf{k})\Gamma(\frac{5}{3})}{\mathcal{A}^2}\left(\frac{3}{2\xi}\right)^{\frac{2}{3}}\right], \tag{29}$$

$$\nu_T = -\frac{\tau_f}{2}\int\frac{d^3k}{(2\pi)^3}\frac{\phi_{11}(\mathbf{k})}{\gamma(\gamma+a^2)}\left[\frac{1}{\mathcal{A}^2} + \frac{1}{12N^2(\gamma+a^2)}\right], \tag{30}$$

where $\xi = \nu k_y^2/\mathcal{A}$; $G_0 = \frac{1}{3}\Gamma(1/3)(3/2\xi)^{1/3}$ and $G_1 = \frac{1}{3}\Gamma(2/3)(3/2\xi)^{2/3}$. The spectrum ϕ_{ij} in Eqs. (25)–(30) are given by Eqs. (19), which are related to power spectrum of forcing ψ_{ij} in Eq. (12) in the incompressible case.

Equations (25)-(30) reveal the following interesting features. Turbulence levels given in Eqs. (25)-(27) are again independent of stratification, similarly to the case without a shear flow [Eqs. (13)–(15)] while they are reduced for strong shear $\mathcal{A}$. This indicates that waves and shear flows play different roles in turbulence regulation – waves do not quench fluctuation levels while shear flows can reduce them through enhanced dissipation via shearing. The turbulence regulation by shearing gives us horizontal velocity fluctuations in Eqs. (26) and (27) which are effectively higher than vertical one in (25). While a similar tendency was

also found in the absence of gravity waves (Kim 2005), the exact value of the radio of vertical to horizontal turbulence levels is not the same. For example, $\langle v_x^2\rangle/\langle v_y^2\rangle > (\ln \xi)^{-1}$ due to ϕ_{11}, which is larger than that $\xi^{1/3}$ in the unstratified medium (Kim 2005). This is probably because gravity waves reduce the degree of anisotropy in turbulence level, reminiscent of the reduced anisotropy due to stratification found in the numerical simulation by Jacobitz, Sarkar & Atta (1997).

Transport properties in Eqs. (28)–(30) however exhibit a very different behaviour, with both vertical and horizontal mixing being inhibited in a non-trivial manner by strong stratification as well as by shearing. First, the vertical transport is reduced as $D_T^{xx} \propto \xi^{1/3}N^{-2}$, becoming small as either stratification or shearing increases. Compared to the case with $N = 0$ where $D_T^{xx} \propto \mathcal{A}^{-2}$, the vertical mixing is much more quenched by a factor of $\xi^{1/3}(\mathcal{A}/N)^2 \ll 1$ (recall $\mathcal{A}/N < 1$ by assumption). This is because shear flow, orthogonal to radial (density gradient) direction, further inhibits the radial transport. Second, D_T^{yy} is less affected by stratification since it involves the two parts – the one from ϕ_{11} is proportional to $\xi^{-1/3}N^{-2}$ while the other from ϕ_{22} is independent of N as $\mathcal{A}^{-2}\xi^{-2/3}$. In the simplest case of a strong radial forcing with $\phi_{22} = 0$, $D_T^{xx}/D_T^{yy} \sim \xi^{2/3} \ll 1$, similarly to the case without stratification. However, in the general case where $\phi_{11} \neq 0$ and $\phi_{22} \neq 0$, D_T^{yy} exhibits a non-trivial, rather complex interplay among stratification and shear flow in determining the overall transport. To appreciate this, we compare Eq. (29) with the result obtained without shear flow (17) to find that shear flow enhances the contribution from ϕ_{11} by a factor $\xi^{-1/3} \gg 1$ while it reduces the one from ϕ_{22} by a factor of $\xi^{4/3} \ll 1$. The enhancement of horizontal transport by shear flow (due to ϕ_{11}) is probably because of the enhanced dissipation via distortion and breakup of eddies, which effectively increase the irreversibility, thereby helping the transport due to gravity waves. In other words, shear flow can interfere with stratification in transport reduction, effectively helping horizontal transport (by a factor of $\xi^{-1/3}$ in this case). On the other hand, shear flow alone reduces the horizontal transport that is not affected by stratification (i.e. the part from ϕ_{22}) by a factor proportional to $\mathcal{A}^{-2}\xi^{-2/3}$. The end result is the enhancement of the effect of stratification. To see this, we compare the two contributions from ϕ_{11} and ϕ_{22} to D_T^{yy} and obtain a characteristic scale $l_* = (\nu/N)^{1/2}(\mathcal{A}/N)^{5/2} < l_c$ above which the stratification is important, being mainly responsible for quenching transport. Since l_* is smaller than that $l_c = (\nu/N)^{1/2}$ obtained without a shear flow, it indicates that the effect of stratification becomes more important compared to the case without shear flows. It is also worth comparing these two parts contributing D_T^{yy} separately to those in the case of $N = 0$, where $D_T^{yy} \propto \mathcal{A}^{-2}\xi^{-2/3}$ from both ϕ_{11} and ϕ_{22}. That is, the contribution from ϕ_{11} is further reduced by a factor of $(\mathcal{A}/N)^2\xi^{1/3}$ by stratification, while the one from ϕ_{22} is not affected.

We now examine how the stratification affects the anisotropy in transport. Equations (28) and (29) for a strong radial forcing with $\phi_{11} \gg \phi_{22}$ and $a \sim 0$ gives $D_T^{xx}/D_T^{yy} \sim \xi^{2/3} \ll 1$, which is similar to the result obtained in the unstratified case (Kim 2005). In this case, the anisotropy in the transport is solely caused by shear stabilization. In general, with both forcings with $\phi_{11} \sim \phi_{22}$ and $a \sim 1$ (i.e. an isotropic forcing), $D_T^{xx}/D_T^{yy} \sim (\mathcal{A}/N)^2\xi \ll 1$, depending on stratification as well as shear. Since $D_T^{xx}/D_T^{yy} \sim (\mathcal{A}/N)^2\xi \ll 1$ is much smaller than $\xi^{2/3}$ obtained in the unstratified case (Kim 2005), the anisotropy in transport can further be enhanced due to stratification. This should be contrasted to the anisotropy in turbulence level.

Finally, Eq. (30) demonstrates one of the most important effects of a stable stratification, which is to drive a system away from a uniform rotation with a negative eddy viscosity. Recall that in the absence of stratification the eddy viscosity is positive in 3D while negative in 2D limit (e.g. see Kim 2005). In contrast, the eddy viscosity in Eq. (30) is negative even in 2D limit ($\beta = 0$ and $\gamma = 1$). This behavior was also found in recent numerical simulation by Miesch (2003) of a stably stratified turbulence with an imposed shear driven by penetrative convection, and is a generic feature of a strongly stratified medium (i.e. a geophysical system). This result is also consistent with the previous work on momentum transport due to radiatively damped gravity waves (e.g. Kim and MacGregor 2001;2003).

4. Discussion and conclusions

We have studied turbulent transport in the stably stratified tachocline with a strong radial differential rotation, when turbulence is driven and maintained by an external forcing (e.g. due to plumes penetrating from the convection zone). Unlike a shear flow which regulates both turbulence level and transport, a stable stratification is shown to mainly inhibit turbulent transport, leading to a further reduction in transport compared to the unstratified case (Kim 2005). Specifically, for parameter values typical of the tachocline ($N/\mathcal{A} \gg 1$), particle transport due to a strong radial forcing is reduced as $\xi^{1/3}N^{-2}$, becoming much smaller than horizontal transport ($\propto \xi^{-1/3}N^{-2}$) by a factor of $\xi^{2/3} \ll 1$. Here, $\xi = \nu k_y^2/\mathcal{A} \ll 1$ is the small parameter characterizing a strong shear limit; N and $\mathcal{A}$ are the Brunt-Väisälä frequency and shearing rate, respectively. Note that in this case $D_T^{xx}/D_T^{yy} \propto \xi^{2/3}$ depends only on shearing but not on stratification, indicating that the anisotropy in particle transport is mainly governed by radial differential rotation. The same scaling was also found in the unstratified case (Kim 2005). However, in the case of an isotropic forcing with $\phi_{11} \sim \phi_{22}$ and $k_x/k_y \sim 1$, the horizontal mixing is less reduced with $D_T^{yy} \sim \mathcal{A}^{-2}\xi^{-2/3}$ (with no effect of stratification), leading to a stronger anisotropy in transport with $D_T^{xx}/D_T^{yy} \sim (\mathcal{A}/N)^2\xi \ll \xi^{2/3} \ll 1$. Furthermore, the vertical momentum transport was shown to be anti-diffusive with a negative eddy viscosity. That is, small-scale turbulence influenced by gravity waves accentuates the gradient in a radial differential rotation rather than makes it uniform. This is similar to the tendency obtained in the case of momentum deposition by gravity waves due to radiative damping (Kim & MacGregor 2001;2003).

Even in the stably stratified radiative interiors of stars as well as in the tachocline, background turbulence has often been assumed to be present to enhance the value of effective viscosity. While the clarification of the source of turbulence responsible for such an enhanced eddy viscosity is an interesting problem, our results demonstrate how this residual turbulence interacts

with gravity waves and shear flow, providing the prediction for the values of eddy viscosity as well as the particle diffusivity which depend on physical quantities like $\mathcal{A}$ and N. In particular, the radial particle transport $D_T^{xx} \sim \xi^{1/3}N^{-2}$ indicates that turbulent transport of particles can be inhibited due to stable stratification as well as shear flow. This finding can have interesting implications for the surface depletion of lithium in the Sun and other stars (i.e., Pinsonneault 1997), and will be studied in a future publication. Furthermore, if a similar physical process operates in the bulk of the radiative interior of the sun, a negative viscosity that we obtain implies that a radial differential rotation which is created during its spin-down would not be eliminated, in contrast to previous suggestions (e.g. Charbonnel & Talon 2005). Note that in these works, turbulence was envisioned to give an enhanced viscous dissipation of gravity waves as well as mean shear flow, which is valid only when turbulence operates on characteristic time and spatial scales which are much smaller than those for gravity waves and shear flow. We emphasize again that in the case where there is no clear scale separation between turbulence and gravity waves, a consistent treatment of interaction among gravity waves, turbulence and shear flow becomes crucial.

The tachocline is believed to possess a strong toroidal magnetic fields of strength $10^4 \sim 10^5$ G, which could have an important influence on turbulent transport in that region. This is an important problem since the presence of a weak poloidal magnetic field in the radiative interior together with a strong toroidal magnetic field in the tachocline (acting as a boundary layer between the radiative interior and convection zone) could offer a mechanism for a uniform rotation in the interior as well as for the tachocline confinement (Rüdiger & Kichatinov 1996; MacGregor & Charbonneauu 1997; Gough & McIntyre 1998). In this case, the values of effective diffusivity of magnetic fields as well as eddy viscosity play a crucial role in determining the thickness of the tachocline. Therefore, a consistent computation of magnetic diffusivity and eddy viscosity with their dependences on physical quantities such as the strength of magnetic fields, the Brunt-Väisälä frequency, and shearing rate, would be of primary interest. Furthermore, it is an interesting question, in general, whether a negative eddy viscosity favored in a stably stratified medium remains as a robust feature in the presence of magnetic field in view of the forward energy cascade (i.e. positive eddy viscosity) in MHD turbulence. The work addressing this problem is in progress and will be published in a future paper.

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Appendix A:

In this Appendix, we show how to derive Eqs. (25)–(30). To this end, we use $U_0(x)\hat{y} = -x\mathcal{A}\hat{y}$ and $\nu = \mu = D$ in Eqs. (1)–(4), and introduce $\hat{w}$ for $w = v_i, n, p$ and f defined by

$$\hat{w} \equiv \tilde{w} \exp D(k_x^3/3k_y\mathcal{A} + k_H^2 t)\,, \tag{A.1}$$

where $k_H^2 = k_y^2 + k_z^2$. We then use a new time variable $\tau = k_x/k_y + \mathcal{A}t$ to rewrite Eqs. (1)–(4) as:

$$\begin{aligned}
\mathcal{A}\partial_\tau \hat{v}_x &= -i\tau k_y \hat{p} - g\hat{\rho} + \hat{f}_x\,, && \text{(A.2)}\\
\mathcal{A}\partial_\tau \hat{v}_y - \mathcal{A}\hat{v}_x &= -ik_y\hat{p} + \hat{f}_y\,, && \text{(A.3)}\\
\mathcal{A}\partial_\tau \hat{v}_z &= -ik_z\hat{p} + \hat{f}_z\,, && \text{(A.4)}\\
0 &= \tau\hat{v}_x + \hat{v}_y + \frac{k_z}{k_y}\hat{v}_z\,, && \text{(A.5)}\\
\mathcal{A}\partial_\tau\hat{\rho} &= \frac{N^2}{g}\hat{v}_x\,, && \text{(A.6)}\\
\mathcal{A}\partial_\tau\hat{n} &= -(\partial_x n_0)\hat{v}_x\,. && \text{(A.7)}
\end{aligned}$$

Equations (A.2)–(A.6) can easily be combined to yield the equation for $\hat{\rho}$ as

$$\partial_\tau[(\gamma+\tau^2)\partial_\tau\hat{\rho}] + \frac{\gamma N^2}{\mathcal{A}^2}\hat{\rho} = \frac{N^2}{g\mathcal{A}^2}\hat{h}_1(\tau)\,, \tag{A.8}$$

where $\hat{h}_1(\tau) = \gamma\hat{f}_x - \tau\hat{f}_y - \tau\beta\hat{f}_z$; $\gamma = 1+\beta^2$ and $\beta = k_z/k_y$. In the limit of $N^2/\mathcal{A}^2 \gg 1$, Eq. (A.8) can be solved with the following solutions valid up to $O(\overline{N}^{-2})$:

$$\begin{aligned}
\hat{\rho}(\tau) &= \frac{N_*^2}{\overline{N}\mathcal{A}}\frac{1}{(\gamma+\tau^2)^{1/4}}\int^\tau \frac{d\tau_1}{(\gamma+\tau_1^2)^{1/4}}\sin[\varphi(\tau)-\varphi(\tau_1)]\hat{h}_1(\tau_1)\,,\\
\hat{v}_x(\tau) &= \frac{1}{\overline{N}\mathcal{A}}\frac{1}{(\gamma+\tau^2)^{5/4}}\int^\tau \frac{d\tau_1}{\varpi(\tau_1)(\gamma+\tau_1^2)^{1/4}}\Big[-\frac{\tau}{2}\sin[\varphi(\tau)-\varphi(\tau_1)]\\
&\quad +\overline{N}\sqrt{\gamma+\tau^2}\varpi(\tau)\cos[\varphi(\tau)-\varphi(\tau_1)]\Big]\hat{h}_1(\tau_1)\,,
\end{aligned}$$

$$\hat{v}_y(\tau) = -\frac{\tau}{\gamma}\hat{v}_x(\tau) + \frac{\beta^2}{\mathcal{A}\gamma}\hat{\rho}(\tau) - \frac{\beta}{\gamma\mathcal{A}}\int^{\tau} d\tau_1 \hat{h}_2(\tau_1)\,,$$
$$\hat{v}_z(\tau) = -\frac{\tau\beta}{\gamma}\hat{v}_x(\tau) - \frac{\beta}{\mathcal{A}\gamma}\hat{\rho}(\tau) + \frac{1}{\gamma\mathcal{A}}\int^{\tau} d\tau_1 \hat{h}_2(\tau_1)\,. \tag{A.9}$$

Here, again $\beta = k_z/k_y$, $\gamma = 1+\beta^2$; $N_*^2 = N^2/g\mathcal{A}$, $\overline{N}^2 = N^2\gamma/\mathcal{A}^2$;

$$\hat{h}_1 = (1+\beta^2)\hat{f}_x - \tau_1\hat{f}_y - \tau_1\beta\hat{f}_z\,,$$
$$\hat{h}_2 = -\beta\hat{f}_y + \hat{f}_z\,,$$
$$\varphi(\tau) = \frac{\overline{N}\alpha}{2}\ln\frac{\sqrt{\gamma+\tau^2}+\tau}{\sqrt{\gamma+\tau^2}-\tau} - \frac{1}{8\overline{N}}\frac{\tau}{\sqrt{\gamma+\tau^2}}\,,$$
$$\varpi(\tau) = \alpha - \frac{1}{8\overline{N}^2}\frac{\gamma}{\gamma+\tau^2}\,,$$
$$\alpha = 1 - \frac{1}{8\overline{N}^2}\,. \tag{A.10}$$

By following a similar algebra used in Kim (2005), with the help of correlation function of the forcing given in Eq. (19), we can obtain the following correlation fictions $\langle v_i v_j\rangle$ to the leading orders in $1/\overline{N}^2$ and $\xi = \nu k_y^2/\mathcal{A}$:

$$\langle v_x^2\rangle \sim \frac{\tau_f}{2\mathcal{A}}\int\frac{d^3k d\tau}{(2\pi)^3}\frac{\phi_{11}(\mathbf{k})}{\sqrt{\gamma+a^2}}\frac{e^{-2\xi Q(\tau,a)}}{(\gamma+\tau^2)^{3/2}}\left[1+\cos\left[2(\varphi(\tau)-\varphi(a))\right]\right],$$
$$\langle v_y^2\rangle \sim \frac{\tau_f}{\mathcal{A}}\int\frac{d^3k d\tau}{(2\pi)^3}e^{-2\xi Q(\tau,a)}\left[\frac{\phi_{11}(\mathbf{k})}{2\sqrt{\gamma+a^2}}\frac{\tau^2}{\gamma^2(\gamma+\tau^2)^{3/2}} + \frac{\beta^2\phi_{22}(\mathbf{k})}{\gamma^2}\right],$$
$$\langle v_z^2\rangle \sim \frac{\tau_f}{\mathcal{A}}\int\frac{d^3k d\tau}{(2\pi)^3}e^{-2\xi Q(\tau,a)}\left[\frac{\phi_{11}(\mathbf{k})}{2\sqrt{\gamma+a^2}}\frac{\tau^2\beta^2}{\gamma^2(\gamma+\tau^2)^{3/2}} + \frac{\phi_{22}(\mathbf{k})}{\gamma^2}\right],$$
$$\langle v_x v_y\rangle \sim -\frac{\tau_f}{\mathcal{A}\gamma\overline{N}^2}\int\frac{d^3k d\tau}{(2\pi)^3}\frac{\phi_{11}(\mathbf{k})}{\sqrt{\gamma+a^2}\varpi(a)^2}\frac{e^{-2\xi Q(\tau,a)}\tau}{(\gamma+\tau^2)^{5/2}}$$
$$\times\left[-\frac{\tau}{2}\sin\left[\varphi(\tau)-\varphi(a)\right] + \overline{N}\sqrt{\gamma+\tau^2}\varpi(\tau)\cos\left[\varphi(\tau)-\varphi(a)\right]\right]^2. \tag{A.11}$$

Here, $a = k_x/k_y$, $Q(\tau,a) = (\gamma\tau+\tau^3/3) - (\gamma a + a^3/3)$; special symmetries $\psi_{ij}(k_y) = \psi_{ij}(-k_y)$ and $\psi_{ij}(k_z) = \psi_{ij}(-k_z)$ were used. We then compute τ integrals in Eq. (A.11) in the strong shear limit $\xi \ll 1$ and use $\langle v_x v_y\rangle = -\nu_T^{xx}\partial_x U_0 = \nu_T^{xx}\mathcal{A}$ to obtain Eqs. (25)-(27) and (30).

Next, to compute particle transport, we integrate Eq. (A.7) in time to obtain

$$\tilde{n}(\mathbf{k}(t),t) = -\partial_x n_0\int dt_1 d^3k_1 \hat{g}(\mathbf{k},t;\mathbf{k}_1,t_1)e^{-DQ(t,t_1)}\tilde{v}_x(\mathbf{k}_1,x,t_1)\,. \tag{A.12}$$

Here, $Q(t,t_1) = \int_{t_1}^t dt'[k_x^2(t')+k_H^2] = [k_x^3 - k_{1x}^3]/3k_y\mathcal{A} + k_H^2(t-t_1)$; $k_H^2 = k_y^2 + k_z^2$ is the amplitude of wave number in the horizontal plane; $k^2 = k_H^2 + k_x^2$; $\hat{g}$ is the Green's function given by

$$\hat{g}(\mathbf{k},t;\mathbf{k}_1,t_1) = \delta(k_y-k_{1y})\delta(k_z-k_{1z})\delta\left[k_x - k_{1x} - k_{1y}(t-t_1)\mathcal{A}\right]. \tag{A.13}$$

A similar analysis using Eqs. (A.9), (A.13), (10)–(12), and (19) and $\langle n v_i\rangle = -D_{ij}\partial_j n_0$ gives

$$D_T^{xx} = \frac{\tau_f}{2(2\pi)^3\mathcal{A}^2\overline{N}^2}\int d^3k d\tau\frac{\xi\phi_{11}(\mathbf{k})e^{-2\xi Q(\tau,a)}}{\sqrt{\gamma+a^2}}(\gamma+\tau^2)^{1/2}\,,$$
$$D_T^{yy} = \frac{\tau_f}{(2\pi)^3\mathcal{A}^2}\int d^3k d\tau\frac{\xi e^{-2\xi Q(\tau,a)}}{\gamma^2}\left[\frac{\phi_{11}(\mathbf{k})\tau^2\sqrt{\gamma+\tau^2}}{2\overline{N}^2\sqrt{\gamma+a^2}} + \beta^2\phi_{22}(\tau-a)^2(\gamma+\tau^2)\right]. \tag{A.14}$$

Finally, the evaluation of τ integrals in Eq. (A.14) gives us Eqs. (28)–(29) in the main text.

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