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► **To cite this version:**

Vikram Mehta, Christian Pauly. Semistability of Frobenius direct images over curves. 2006. hal-00087308

HAL Id: hal-00087308

<https://hal.science/hal-00087308v1>

Preprint submitted on 22 Jul 2006

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SEMISTABILITY OF FROBENIUS DIRECT IMAGES OVER CURVES

VIKRAM B. MEHTA AND CHRISTIAN PAULY

ABSTRACT. Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$. Given a semistable vector bundle E over X , we show that its direct image F_*E under the Frobenius map F of X is again semistable. We deduce a numerical characterization of the stable rank- p vector bundles F_*L , where L is a line bundle over X .

1. INTRODUCTION

Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$ and let $F : X \rightarrow X_1$ be the relative k -linear Frobenius map. It is by now a well-established fact that on any curve X there exist semistable vector bundles E such that their pull-back under the Frobenius map F^*E is not semistable [LanP], [LasP]. In order to control the degree of instability of the bundle F^*E , one is naturally lead (through adjunction $\mathrm{Hom}_{\mathcal{O}_X}(F^*E, E') = \mathrm{Hom}_{\mathcal{O}_{X_1}}(E, F_*E')$) to ask whether semistability is preserved by direct image under the Frobenius map. The answer is (somewhat surprisingly) yes. In this note we show the following result.

1.1. Theorem. *Assume that $g \geq 2$. If E is a semistable vector bundle over X (of any degree), then F_*E is also semistable.*

Unfortunately we do not know whether also stability is preserved by direct image under Frobenius. It has been shown that F_*L is stable for a line bundle L ([LanP] Proposition 1.2) and that in small characteristics the bundle F_*E is stable for any stable bundle E of small rank [JRXY]. The main ingredient of the proof is Faltings' cohomological criterion of semistability. We also need the fact that the generalized Verschiebung V , defined as the rational map from the moduli space $\mathcal{M}_{X_1}(r)$ of semistable rank- r vector bundles over X_1 with fixed trivial determinant to the moduli space $\mathcal{M}_X(r)$ induced by pull-back under the relative Frobenius map F ,

$$V_r : \mathcal{M}_{X_1}(r) \dashrightarrow \mathcal{M}_X(r), \quad E \longmapsto F^*E$$

is dominant for large r . We actually show a stronger statement for large r .

1.2. Proposition. *If $l \geq g(p-1) + 1$ and l prime, then the generalized Verschiebung V_l is generically étale for any curve X . In particular V_l is separable and dominant.*

As an application of Theorem 1.1 we obtain an upper bound of the rational invariant ν of a vector bundle E , defined as

$$\nu(E) := \mu_{\max}(F^*E) - \mu_{\min}(F^*E),$$

where μ_{\max} (resp. μ_{\min}) denotes the slope of the first (resp. last) piece in the Harder-Narasimhan filtration of F^*E .

1.3. Proposition. *For any semistable rank- r vector bundle E*

$$\nu(E) \leq \min((r-1)(2g-2), (p-1)(2g-2)).$$

We note that the inequality $\nu(E) \leq (r-1)(2g-2)$ was proved in [SB] Corollary 2 and in [S] Theorem 3.1. We suspect that the relationship between both inequalities comes from the conjectural fact that the length (=number of pieces) of the Harder-Narasimhan filtration of F^*E is at most p for semistable E .

Finally we show that direct images of line bundles under Frobenius are characterized by maximality of the invariant ν .

1.4. Proposition. *Let E be a stable rank- p vector bundle over X . Then the following statements are equivalent.*

- (1) *There exists a line bundle L such that $E = F_*L$.*
- (2) $\nu(E) = (p-1)(2g-2)$.

We do not know whether the analogue of this proposition remains true for higher rank.

2. REDUCTION TO THE CASE $\mu(E) = g - 1$.

In this section we show that it is enough to prove Theorem 1.1 for semistable vector bundles E with slope $\mu(E) = g - 1$.

Let E be a semistable vector bundle over X of rank r and let s be the integer defined by the equality

$$\mu(E) = g - 1 + \frac{s}{r}.$$

Applying the Grothendieck-Riemann-Roch theorem to the Frobenius map $F : X \rightarrow X_1$, we obtain

$$\mu(F_*E) = g - 1 + \frac{s}{pr}.$$

Let $\pi : \tilde{X} \rightarrow X$ be a connected étale covering of degree n and let $\pi_1 : \tilde{X}_1 \rightarrow X_1$ denote its twist by the Frobenius of k (see [R] section 4). The diagram

$$(2.1) \quad \begin{array}{ccc} \tilde{X} & \xrightarrow{F} & \tilde{X}_1 \\ \pi \downarrow & & \downarrow \pi_1 \\ X & \xrightarrow{F} & X_1 \end{array}$$

is Cartesian and we have an isomorphism

$$\pi_1^*(F_*E) \cong F_*(\pi^*E).$$

Since semistability is preserved under pull-back by a separable morphism of curves, we see that π^*E is semistable. Moreover if $F_*(\pi^*E)$ is semistable, then F_*E is also semistable.

Let L be a degree d line bundle over \tilde{X}_1 . The projection formula

$$F_*(\pi^*E \otimes F^*L) = F_*(\pi^*E) \otimes L$$

shows that semistability of $F_*(\pi^*E)$ is equivalent to semistability of $F_*(\pi^*E \otimes F^*L)$.

Let \tilde{g} denote the genus of \tilde{X} . By the Riemann-Hurwitz formula $\tilde{g} - 1 = n(g - 1)$. We compute

$$\mu(\pi^*E \otimes F^*L) = n(g - 1) + n\frac{s}{r} + pd = \tilde{g} - 1 + n\frac{s}{r} + pd,$$

which gives

$$\mu(F_*(\pi^*E \otimes F^*L)) = \tilde{g} - 1 + n\frac{s}{pr} + d.$$

2.1. Lemma. *For any integer m there exists a connected étale covering $\pi : \tilde{X} \rightarrow X$ of degree $n = p^k m$ for some k .*

Proof. If the p -rank of X is nonzero, the statement is clear. If the p -rank is zero, we know by Corollaire 4.3.4 [R] that there exist connected étale coverings $Y \rightarrow X$ of degree p^t for infinitely many integers t (more precisely for all t of the form $(l-1)(g-1)$ where l is a large prime). Now we decompose $m = p^s u$ with p not dividing u . We then take a covering $Y \rightarrow X$ of degree p^t with $t \geq s$ and a covering $\tilde{X} \rightarrow Y$ of degree u . \square

Now the lemma applied to the integer $m = pr$ shows existence of a connected étale covering $\pi : \tilde{X} \rightarrow X$ of degree $n = p^k m$. Hence $n \frac{s}{pr}$ is an integer and we can take d such that $n \frac{s}{pr} + d = 0$.

To summarize, we have shown that for any semistable E over X there exists a covering $\pi : \tilde{X} \rightarrow X$ and a line bundle L over \tilde{X}_1 such that the vector bundle $\tilde{E} := \pi^* E \otimes F^* L$ is semistable with $\mu(\tilde{E}) = \tilde{g} - 1$ and such that semistability of $F_* \tilde{E}$ implies semistability of $F_* E$.

3. PROOF OF THEOREM 1.1

In order to prove semistability of $F_* E$ we shall use the cohomological criterion of semistability due to Faltings [F].

3.1. Proposition ([L] Théorème 2.4). *Let E be a rank- r vector bundle over X with $\mu(E) = g - 1$ and l an integer $> \frac{r^2}{4}(g-1)$. Then E is semistable if and only if there exists a rank- l vector bundle G with trivial determinant such that*

$$h^0(X, E \otimes G) = h^1(X, E \otimes G) = 0.$$

Moreover if the previous condition holds for one bundle G , it holds for a general bundle by upper semicontinuity of the function $G \mapsto h^0(X, E \otimes G)$.

Remark. The proof of this proposition (see [L] section 2.4) works over any algebraically closed field k .

By Proposition 1.2 (proved in section 4) we know that V_l is dominant when l is a large prime number. Hence a general vector bundle $G \in \mathcal{M}_X(l)$ is of the form $F^* G'$ for some $G' \in \mathcal{M}_{X_1}(l)$. Consider a semistable E with $\mu(E) = g - 1$. Then by Proposition 3.1 $h^0(X, E \otimes G) = 0$ for general $G \in \mathcal{M}_X(l)$. Assuming G general, we can write $G = F^* G'$ and we obtain by adjunction

$$h^0(X, E \otimes F^* G') = h^0(X_1, F_* E \otimes G') = 0.$$

This shows that $F_* E$ is semistable by Proposition 3.1.

4. PROOF OF PROPOSITION 1.2

According to [MS] section 2 it will be enough to prove the existence of a stable vector bundle $E \in \mathcal{M}_{X_1}(l)$ satisfying $F^* E$ stable and

$$h^0(X_1, B \otimes \text{End}_0(E)) = 0,$$

because the vector space $H^0(X_1, B \otimes \text{End}_0(E))$ can be identified with the kernel of the differential of V_l at the point $E \in \mathcal{M}_{X_1}(l)$. Here B denotes the sheaf of locally exact differentials over X_1 (see [R] section 4).

Let $l \neq p$ be a prime number and let $\alpha \in JX_1[l] \cong JX[l]$ be a nonzero l -torsion point. We denote by

$$\pi : \tilde{X} \rightarrow X \quad \text{and} \quad \pi_1 : \tilde{X}_1 \rightarrow X_1$$

the associated cyclic étale cover of X and X_1 and by σ a generator of the Galois group $\text{Gal}(\tilde{X}/X) = \mathbb{Z}/l\mathbb{Z}$. We recall that the kernel of the Norm map

$$\text{Nm} : J\tilde{X} \longrightarrow JX$$

has l connected components and we denote by

$$i : P := \text{Prym}(\tilde{X}/X) \subset J\tilde{X}$$

the associated Prym variety, i.e., the connected component containing the origin. Then we have an isogeny

$$\pi^* \times i : JX \times P \longrightarrow J\tilde{X}$$

and taking direct image under π induces a morphism

$$P \longrightarrow \mathcal{M}_X(l), \quad L \longmapsto \pi_* L.$$

Similarly we define the Prym variety $P_1 \subset JX_1$ and the morphism $P_1 \rightarrow \mathcal{M}_{X_1}(l)$ (obtained by twisting with the Frobenius of k). Note that $\pi_{1*}L$ is semistable for any $L \in P_1$ and stable for general $L \in P_1$ (see e.g. [B]). Since $F^*(\pi_{1*}L) \cong \pi_*(F^*L)$ — see diagram (2.1) — and since F^* induces the Verschiebung $V_P : P_1 \rightarrow P$, which is surjective, we obtain that $\pi_{1*}L$ and $F^*(\pi_{1*}L)$ are stable for general $L \in P_1$.

Therefore Proposition 1.2 will immediately follow from the next Proposition.

4.1. Proposition. *If $l \geq g(p-1) + 1$ then there exists a cyclic degree l étale cover $\pi_1 : \tilde{X}_1 \rightarrow X_1$ with the property that*

$$h^0(X_1, B \otimes \text{End}_0(\pi_{1*}L)) = 0$$

for general $L \in P_1$.

Proof. By relative duality for the étale map π_1 we have $(\pi_{1*}L)^* \cong \pi_{1*}L^{-1}$. Therefore

$$\text{End}(\pi_{1*}L) \cong \pi_{1*}L \otimes \pi_{1*}L^{-1} \cong \pi_{1*}(L^{-1} \otimes \pi_1^* \pi_{1*}L)$$

by the projection formula. Moreover since π_1 is Galois étale we have a direct sum decomposition

$$\pi_1^* \pi_{1*}L \cong \bigoplus_{i=0}^{l-1} (\sigma^i)^* L.$$

Putting these isomorphisms together we find that

$$\begin{aligned} H^0(X_1, B \otimes \text{End}(\pi_{1*}L)) &= H^0(X_1, B \otimes \pi_{1*}(\bigoplus_{i=0}^{l-1} L^{-1} \otimes (\sigma^i)^* L)) \\ &= \bigoplus_{i=0}^{l-1} H^0(X_1, B \otimes \pi_{1*}(L^{-1} \otimes (\sigma^i)^* L)) \\ &= H^0(X_1, B \otimes \pi_{1*} \mathcal{O}_{\tilde{X}_1}) \oplus \bigoplus_{i=1}^{l-1} H^0(X_1, B \otimes \pi_{1*}(L^{-1} \otimes (\sigma^i)^* L)). \end{aligned}$$

Moreover $\pi_* \mathcal{O}_{\tilde{X}_1} = \bigoplus_{i=0}^{l-1} \alpha^i$, which implies that

$$(4.1) \quad H^0(X_1, B \otimes \text{End}_0(\pi_{1*}L)) = \bigoplus_{i=1}^{l-1} H^0(X_1, B \otimes \alpha^i) \oplus \bigoplus_{i=1}^{l-1} H^0(X_1, B \otimes \pi_{1*}(L^{-1} \otimes (\sigma^i)^* L)).$$

Let us denote for $i = 1, \dots, l-1$ by ϕ_i the isogeny

$$\phi_i : P_1 \longrightarrow P_1, \quad L \longmapsto L^{-1} \otimes (\sigma^i)^* L.$$

Since the function $L \mapsto h^0(X_1, B \otimes \text{End}_0(\pi_{1*}L))$ is upper semicontinuous, it will be enough to show the existence of a cover $\pi_1 : \tilde{X}_1 \rightarrow X_1$ satisfying

- (1) for $i = 1, \dots, l-1$, $h^0(X_1, B \otimes \alpha^i) = 0$ (or equivalently, P is an ordinary abelian variety).
- (2) for M general in P , $h^0(X_1, B \otimes \pi_{1*}M) = 0$.

Note that these two conditions imply that the vector space (4.1) equals $\{0\}$ for general $L \in P_1$, because the ϕ_i 's are surjective.

We recall that $\ker(\pi_1^* : JX_1 \rightarrow J\tilde{X}_1) = \langle \alpha \rangle \cong \mathbb{Z}/l\mathbb{Z}$ and that

$$P_1[l] = P_1 \cap \pi_1^*(JX_1) \cong \alpha^\perp / \langle \alpha \rangle$$

where $\alpha^\perp = \{\beta \in JX_1[l] \mid \omega(\alpha, \beta) = 1\}$ and $\omega : JX_1[l] \times JX_1[l] \rightarrow \mu_l$ denotes the symplectic Weil form. Consider a $\beta \in \alpha^\perp \setminus \langle \alpha \rangle$. Then $\pi_1^* \beta \in P_1[l]$ and

$$\pi_{1*} \pi_1^* \beta = \bigoplus_{i=0}^{l-1} \beta \otimes \alpha^i.$$

Again by upper semicontinuity of the function $M \mapsto h^0(X_1, B \otimes \pi_{1*}M)$ one observes that the conditions (1) and (2) are satisfied because of the following lemma (take $M = \pi_{1*}\beta$).

4.2. Lemma. *If $l \geq g(p-1) + 1$ then there exists a pair $(\alpha, \beta) \in JX_1[l] \times JX_1[l]$ satisfying*

- (1) $\alpha \neq 0$ and $\beta \in \alpha^\perp \setminus \langle \alpha \rangle$,
- (2) for $i = 1, \dots, l-1$ $h^0(X_1, B \otimes \alpha^i) = 0$,
- (3) for $i = 0, \dots, l-1$ $h^0(X_1, B \otimes \beta \otimes \alpha^i) = 0$.

Proof. We adapt the proof of [R] Lemme 4.3.5. We denote by \mathbb{F}_l the finite field $\mathbb{Z}/l\mathbb{Z}$. Then there exists a symplectic isomorphism $JX_1[l] \cong \mathbb{F}_l^g \times \mathbb{F}_l^g$, where the latter space is endowed with the standard symplectic form. Note that composition is written multiplicatively in $JX_1[l]$ and additively in \mathbb{F}_l^{2g} . A quick computation shows that the number of isotropic 2-planes in $\mathbb{F}_l^g \times \mathbb{F}_l^g$ equals

$$N(l) = \frac{(l^{2g} - 1)(l^{2g-2} - 1)}{(l^2 - 1)(l - 1)}.$$

Let $\Theta_B \subset JX_1$ denote the theta divisor associated to B . Then by [R] Lemma 4.3.5 the cardinality $A(l)$ of the finite set $\Sigma(l) := JX_1[l] \cap \Theta_B$ satisfies

$$A(l) \leq l^{2g-2}g(p-1).$$

Suppose that there exists an isotropic 2-plane $\Pi \subset \mathbb{F}_l^g \times \mathbb{F}_l^g$ which contains $\leq l-2$ points of $\Sigma(l)$. Then we can find a pair (α, β) satisfying the 3 properties of the Lemma as follows: any nonzero point $x \in \Pi$ determines a line ($=\mathbb{F}_l$ -vector space of dimension 1). Since a line contains $l-1$ nonzero points, we obtain at most $(l-1)(l-2)$ nonzero points lying on lines generated by $\Sigma(l) \cap \Pi$. Since $(l-1)(l-2) < l^2 - 1$ there exists a nonzero α in the complement of these lines. Now we note that there are $l-1$ affine lines parallel to the line generated by α and the l points on any of these affine lines are of the form $\beta\alpha^i$ for $i = 0, \dots, l-1$ for some $\beta \in \alpha^\perp \setminus \langle \alpha \rangle$. The points $\Sigma(l) \cap \Pi$ lie on at most $l-2$ such affine lines, hence there exists at least one affine line parallel to $\langle \alpha \rangle$ avoiding $\Sigma(l)$. This gives β .

Finally let us suppose that any isotropic 2-plane contains $\geq l-1$ points of $\Sigma(l)$. Then we will arrive at a contradiction as follows: we introduce the set

$$S = \{(x, \Pi) \mid x \in \Pi \cap \Sigma(l) \text{ and } \Pi \text{ isotropic 2-plane}\}.$$

with cardinality $|S|$. Then by our assumption we have

$$(4.2) \quad |S| \geq (l-1)N(l).$$

On the other hand, since any nonzero $x \in \mathbb{F}_l^g \times \mathbb{F}_l^g$ is contained in $\frac{l^{2g-2}-1}{l-1}$ isotropic 2-planes, we obtain

$$(4.3) \quad |S| \leq \frac{l^{2g-2} - 1}{l - 1} A(l).$$

Putting (4.2) and (4.3) together, we obtain

$$A(l) \geq \frac{l^{2g} - 1}{l + 1}.$$

But this contradicts the inequality $A(l) \leq l^{2g-2}g(p-1)$ if $l \geq g(p-1) + 1$. □

This completes the proof of Proposition 4.1. □

Remark. It has been shown [O] Theorem A.6 that V_r is dominant for any rank r and any curve X , by using a versal deformation of a direct sum of r line bundles.

Remark. We note that V_r is not separable when p divides the rank r and X is non-ordinary. In that case the Zariski tangent space at a stable bundle $E \in \mathcal{M}_{X_1}(r)$ identifies with the quotient

$H^1(X_1, \text{End}_0(E))/\langle e \rangle$ where e denotes the nonzero extension class of $\text{End}_0(E)$ by \mathcal{O}_{X_1} given by $\text{End}(E)$. Then the inclusion of homotheties $\mathcal{O}_{X_1} \hookrightarrow \text{End}_0(E)$ induces an inclusion $H^1(X_1, \mathcal{O}_{X_1}) \subset H^1(X_1, \text{End}_0(E))/\langle e \rangle$ and the restriction of the differential of V_r at the point E to $H^1(X_1, \mathcal{O}_{X_1})$ coincides with the non-injective Hasse-Witt map.

5. PROOF OF PROPOSITION 1.3

Since we already know that $\nu(E) \leq (r-1)(2g-2)$ ([SB], [S]) it suffices to show that $\nu(E) \leq (p-1)(2g-2)$.

We consider the quotient $F^*E \rightarrow Q$ with minimal slope, i.e., $\mu(Q) = \mu_{\min}(F^*E)$ and Q semistable. By adjunction we obtain a nonzero morphism $E \rightarrow F_*(Q)$, from which we deduce (using Theorem 1.1) that

$$\mu(E) \leq \mu(F_*Q) = \frac{1}{p}(\mu_{\min}(F^*E) + (p-1)(g-1))$$

hence

$$\mu(F^*E) \leq \mu_{\min}(F^*E) + (p-1)(g-1).$$

Similarly we consider the subbundle $S \hookrightarrow F^*E$ with maximal slope, i.e., $\mu(S) = \mu_{\max}(F^*E)$ and S semistable. Taking the dual and proceeding as above, we obtain that

$$\mu(F^*E) \geq \mu_{\max}(F^*E) - (p-1)(g-1).$$

Now we combine both inequalities and we are done.

Remark. We note that the inequality of Proposition 1.3 is sharp. The maximum $(p-1)(2g-2)$ is obtained for the bundles $E = F_*E'$ (see [JRXY] Theorem 5.3).

6. CHARACTERIZATION OF DIRECT IMAGES

Consider a line bundle L over X . Then the direct image F_*L is stable ([LanP] Proposition 1.2) and the Harder-Narasimhan filtration of F^*F_*L is of the form (see [JRXY])

$$0 = V_0 \subset V_1 \subset \dots \subset V_{p-1} \subset V_p = F^*F_*L, \quad \text{with} \quad V_i/V_{i-1} \cong L \otimes \omega_X^{p-i}.$$

In particular $\nu(F_*L) = (p-1)(2g-2)$. In this section we will show a converse statement.

More generally let E be a stable rank- rp vector bundle with $\mu(E) = g-1 + \frac{d}{rp}$ for some integer d and satisfying

- (1) the Harder-Narasimhan filtration of F^*E has l terms.
- (2) $\nu(E) = (p-1)(2g-2)$.

Questions. Do we have $l \leq p$? Is E of the form $E = F_*G$ for some rank- r vector bundle G ? We will give a positive answer in the case $r = 1$ (Proposition 6.1).

Let us denote the Harder-Narasimhan filtration by

$$0 = V_0 \subset V_1 \subset \dots \subset V_{l-1} \subset V_l = F^*E, \quad V_i/V_{i-1} = M_i.$$

satisfying the inequalities

$$\mu_{\max}(F^*E) = \mu(M_1) > \mu(M_2) > \dots > \mu(M_l) = \mu_{\min}(F^*E).$$

The quotient $F^*E \rightarrow M_l$ gives via adjunction a nonzero map $E \rightarrow F_*M_l$. Since F_*M_l is semistable, we obtain that $\mu(E) \leq \mu(F_*M_l)$. This implies that $\mu(M_l) \geq g-1 + \frac{d}{r}$. Similarly taking the dual of the inclusion $M_1 \subset F^*E$ gives a map $F^*(E^*) \rightarrow M_1^*$ and by adjunction $E^* \rightarrow F_*(M_1^*)$. Let us denote $\mu(M_1^*) = g-1 + \delta$, so that $\mu(F_*(M_1^*)) = g-1 + \frac{\delta}{p}$. Because of semistability of $F_*(M_1^*)$, we obtain $-(g-1 + \frac{d}{rp}) = \mu(E^*) \leq \mu(F_*(M_1^*))$, hence $\delta \geq -2p(g-1) - \frac{d}{r}$. This implies that

$\mu(M_1) \leq (2p-1)(g-1) + \frac{d}{r}$. Combining this inequality with $\mu(M_l) \geq g-1 + \frac{d}{r}$ and the assumption $\mu(M_1) - \mu(M_l) = (p-1)(2g-2)$, we obtain that

$$\mu(M_1) = (2p-1)(g-1) + \frac{d}{r}, \quad \mu(M_l) = g-1 + \frac{d}{r}.$$

Let us denote by r_i the rank of the semistable bundle M_i . We have the equality

$$(6.1) \quad \sum_{i=1}^l r_i = rp.$$

Since E is stable and $F_*(M_l)$ is semistable and since these bundles have the same slope, we deduce that $r_l \geq r$. Similarly we obtain that $r_1 \geq r$.

Note that it is enough to show that $r_l = r$. Since E is stable and F_*M_l semistable and since the two bundles have the same slope and rank, they will be isomorphic.

We introduce the integers for $i = 1, \dots, l-1$

$$\delta_i = \mu(M_{i+1}) - \mu(M_i) + 2(g-1) = \mu(M_{i+1} \otimes \omega) - \mu(M_i).$$

Then we have the equality

$$(6.2) \quad \sum_{i=1}^{l-1} \delta_i = \mu(M_l) - \mu(M_1) + 2(l-1)(g-1) = 2(l-p)(g-1).$$

We note that if $\delta_i < 0$, then $\text{Hom}(M_i, M_{i+1} \otimes \omega) = 0$.

6.1. Proposition. *Let E be stable rank- p vector bundle with $\mu(E) = g-1 + \frac{d}{p}$ and $\nu(E) = (p-1)(2g-2)$. Then $E = F_*L$ for some line bundle L of degree $g-1+d$.*

Proof. Let us first show that $l = p$. We suppose that $l < p$. Then $\sum_{i=1}^{l-1} \delta_i = 2(l-p)(g-1) < 0$ so that there exists a $k \leq l-1$ such that $\delta_k < 0$. We may choose k minimal, i.e., $\delta_i \geq 0$ for $i < k$. Then we have

$$(6.3) \quad \mu(M_k) > \mu(M_i) + 2(g-1) \quad \text{for } i > k.$$

We recall that $\mu(M_i) \leq \mu(M_{k+1})$ for $i > k$. The Harder-Narasimhan filtration of V_k is given by the first k terms of the Harder-Narasimhan filtration of F^*E . Hence $\mu_{\min}(V_k) = \mu(M_k)$.

Consider now the canonical connection ∇ on F^*E and its first fundamental form

$$\phi_k : V_k \hookrightarrow F^*E \xrightarrow{\nabla} F^*E \otimes \omega_X \longrightarrow (F^*E/V_k) \otimes \omega_X.$$

Since $\mu_{\min}(V_k) > \mu(M_i \otimes \omega)$ for $i > k$ we obtain $\phi_k = 0$. Hence ∇ preserves V_k and since ∇ has zero p -curvature, there exists a subbundle $E_k \subset E$ such that $F^*E_k = V_k$.

We now evaluate $\mu(E_k)$. By assumption $\delta_i \geq 0$ for $i < k$. Hence

$$\mu(M_i) \geq \mu(M_1) - 2(i-1)(g-1) \quad \text{for } i \leq k,$$

which implies that

$$\deg(V_k) = \sum_{i=1}^k r_i \mu(M_i) \geq \text{rk}(V_k) \mu(M_1) - 2(g-1) \sum_{i=1}^k r_i (i-1).$$

Hence we obtain

$$p\mu(E_k) = \mu(V_k) \geq \mu(M_1) - 2(g-1)C,$$

where C denotes the fraction $\frac{\sum_{i=1}^k r_i (i-1)}{\text{rk}(V_k)}$. We will prove in a moment that $C \leq \frac{p-1}{2}$, so that we obtain by substitution

$$p\mu(E_k) \geq (2p-1)(g-1) + d - (g-1)(p-1) = p(g-1) + d = p\mu(E),$$

contradicting stability of E . Now let us show that $C \leq \frac{p-1}{2}$ or equivalently

$$\sum_{i=1}^k ir_i \leq \frac{p+1}{2} \sum_{i=1}^k r_i.$$

But that is obvious if $k \leq \frac{p-1}{2}$. Now if $k > \frac{p-1}{2}$ we note that passing from E to E^* reverses the order of the δ_i 's, so that the index k^* for E^* satisfies $k^* \leq \frac{p-1}{2}$. This proves that $l = p$.

Because of (6.1) we obtain $r_i = 1$ for all i and therefore $E = F_*M_p$. □

7. STABILITY OF F_*E ?

Is stability also preserved by F_* ?

We show the following result in that direction.

7.1. Proposition. *Let E be a stable vector bundle over X . Then F_*E is simple.*

Proof. Using relative duality $(F_*E)^* \cong F_*(E^* \otimes \omega_X^{1-p})$ we obtain

$$H^0(X_1, \text{End}(F_*E)) = H^0(X, F^*F_*E \otimes E^* \otimes \omega_X^{1-p}).$$

Moreover the Harder-Narasimhan filtration of F^*F_*E is of the form (see [JRXY])

$$0 = V_0 \subset V_1 \subset \dots \subset V_{p-1} \subset V_p = F^*F_*E, \quad \text{with} \quad V_i/V_{i-1} \cong E \otimes \omega_X^{p-i}.$$

We deduce that

$$H^0(X, F^*F_*E \otimes E^* \otimes \omega_X^{1-p}) = H^0(X, V_1 \otimes E^* \otimes \omega_X^{1-p}) = H^0(X, \text{End}(E)),$$

and we are done. □

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