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## THE ODE METHOD FOR STABILITY OF SKIP-FREE MARKOV CHAINS WITH APPLICATIONS TO MCMC<sup>1</sup>

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Fluid limit techniques have become a central tool to analyze queueing networks over the last decade, with applications to performance analysis, simulation and optimization.

In this paper, some of these techniques are extended to a general class of skip-free Markov chains. As in the case of queueing models, a fluid approximation is obtained by scaling time, space and the initial condition by a large constant. The resulting fluid limit is the solution of an ordinary differential equation (ODE) in “most” of the state space. Stability and finer ergodic properties for the stochastic model then follow from stability of the set of fluid limits. Moreover, similarly to the queueing context where fluid models are routinely used to design control policies, the structure of the limiting ODE in this general setting provides an understanding of the dynamics of the Markov chain. These results are illustrated through application to Markov chain Monte Carlo methods.

The use of ordinary differential equations (ODE) to analyze Markov chains was first suggested by Kurtz (1970). This idea was later refined by Newell (1982), who introduced the so-called *fluid approximations* with applications to queueing networks. Since the 1990s, fluid models have been used to address delay in complex networks [Cruz (1991)] and bottleneck analysis [Chen and Mandelbaum (1991)]. The latter work followed an already extensive research program on diffusion approximations for networks [see Harrison (2000), Whitt (2002), Chen and Yao (2001) and the references therein].

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The purpose of this paper is to extend fluid limit techniques to a general class of discrete-time Markov chains  $\{\Phi_k\}$  on a  $d$ -dimensional Euclidean state space  $\mathsf{X}$ . Recall that a Markov chain is called *skip-free* if the increments  $(\Phi_{k+1} - \Phi_k)$  are uniformly bounded in norm by a deterministic constant for each  $k$  and each initial condition. For example, Markov chain models of queueing systems are typically skip-free. Here, we consider a relaxation of this assumption in which the increments are assumed to be bounded in an  $L^p$ -sense. Consequently, we find that the chain can be represented by the additive noise model

$$(1) \quad \Phi_{k+1} = \Phi_k + \Delta(\Phi_k) + \epsilon_{k+1},$$

where  $\{\epsilon_k\}$  is a martingale increment sequence w.r.t. the natural filtration of the process  $\{\Phi_k\}$  and  $\Delta: \mathsf{X} \rightarrow \mathsf{X}$  is bounded. Associated with this chain, we consider the sequence of continuous-time processes

$$(2) \quad \eta_r^\alpha(t; x) \stackrel{\text{def}}{=} r^{-1} \Phi_{\lfloor tr^{1+\alpha} \rfloor}, \quad \eta_r^\alpha(t; 0) = r^{-1} \Phi_0 = x, \\ r \geq 0, \alpha \geq 0, x \in \mathsf{X},$$

obtained by interpolating and scaling the Markov chain in space and time. A fluid limit is obtained as a subsequential weak limit of a sequence  $\{\eta_{r_n}^\alpha(\cdot; x_n)\}$ , where  $\{r_n\}$  and  $\{x_n\}$  are two sequences such that  $\lim_{n \rightarrow \infty} r_n = \infty$  and  $\lim_{n \rightarrow \infty} x_n = x$ . The set of all such limits is called the *fluid limit model*. In queueing network applications, a fluid limit is easy to interpret in terms of mean flows; in most situations, it is a solution of a deterministic set of equations depending on network characteristics as well as the control policy [see, e.g., [Chen and Mandelbaum \(1991\)](#), [Dai \(1995\)](#), [Dai and Meyn \(1995\)](#), [Chen and Yao \(2001\)](#), [Meyn \(2007\)](#)]. The existence of limits and the continuity of the fluid limit model may be established under general conditions on the increments (see [Theorem 1.2](#)).

The fact that stability of the fluid limit model implies stability of the stochastic network was established in a limited setting in [Malyšev and Menc'šikov \(1979\)](#). This was extended to a very broad class of multiclass networks by [Dai \(1995\)](#). A key step in the proof of these results is a multi-step state-dependent version of Foster's criterion introduced in [Malyšev and Menc'šikov \(1979\)](#) for countable state space models, later extended to general state space models in [Meyn and Tweedie \(1993, 1994\)](#). The main result of [Dai \(1995\)](#) only established positive recurrence. Moments and rates of convergence to stationarity of the Markovian network model were obtained in [Dai and Meyn \(1995\)](#), based on an extension of [Meyn and Tweedie \(1994\)](#) using the subgeometric  $f$ -ergodic theorem in [Tuominen and Tweedie \(1994\)](#) [recently extended and simplified in work of [Douc et al. \(2004\)](#)]. Converse theorems have appeared in [Dai and Weiss \(1996\)](#), [Dai \(1996\)](#), [Meyn \(1995\)](#) that show that,

under rather strong conditions, instability of the fluid model implies transience of the stochastic network. The counterexamples in [Gamarnik and Hasenbein \(2005\)](#), [Dai et al. \(2004\)](#) show that some additional conditions are necessary to obtain a converse.

Under general conditions, including the generalized skip-free assumption, a fluid limit  $\eta$  is a *weak* solution (in a sense given below) to the homogeneous ODE

$$(3) \quad \dot{\mu} = h(\mu).$$

The vector field  $h$  is defined as a radial limit of the function  $\Delta$  appearing in (1) under appropriate renormalization.

Provided that the increments  $\{\epsilon_k\}$  in the decomposition (1) are tight in  $L^p$ , stability of the fluid limit model implies finite moments in steady state, as well as polynomial rates of convergence to stationarity; see [Theorem 1.4](#).

One advantage of the ODE approach over the usual Foster–Lyapunov approach to stability is that the ODE model provides insight into Markov chain dynamics. In the queueing context, the ODE model has many other applications, such as simulation variance reduction [[Henderson et al. \(2003\)](#)] and optimization [[Chen and Meyn \(1999\)](#)].

The remainder of the paper is organized as follows. [Section 1.1](#) contains notation and assumptions, along with a construction of the fluid limit model. The main result is contained in [Section 1.2](#), where it is shown that stability of the fluid limit model implies the existence of polynomial moments as well as polynomial rates of convergence to stationarity [known as *(f, r)-ergodicity*].

Fluid limits are characterized in [Section 1.3](#). [Proposition 1.5](#) provides conditions that guarantee that a fluid limit coincides with the weak solutions of the ODE (3).

These results are applied to establish *(f, r)-ergodicity* of the random walk Metropolis–Hastings algorithm for superexponential densities in [Section 2.1](#) and subexponential densities in [Section 2.2](#). In [Examples 2](#) and [4](#), the fluid limit model is stable and any fluid limit is a weak solution of the ODE (3), yet some fluid limits are nondeterministic.

The conclusions contain proposed extensions, including diffusion limits of the form obtained in [Harrison \(2000\)](#), [Whitt \(2002\)](#), [Chen and Yao \(2001\)](#) and application of ODE methods for variance reduction in simulation and MCMC.

## 1. Assumptions and statement of the results.

1.1. *Fluid limit: definitions.* We consider a Markov chain  $\Phi \stackrel{\text{def}}{=} \{\Phi_k\}_{k \geq 0}$  on a  $d$ -dimensional Euclidean space  $\mathbf{X}$  equipped with its Borel sigma-field  $\mathcal{X}$ . We denote by  $\{\mathcal{F}_k\}_{k \geq 0}$  the natural filtration. The distribution of  $\Phi$  is

specified by its initial state  $\Phi_0 = x \in \mathsf{X}$  and its transition kernel  $P$ . We write  $\mathbb{P}_x$  for the distribution of the chain conditional on the initial state  $\Phi_0 = x$  and  $\mathbb{E}_x$  for the corresponding expectation.

Denote by  $\mathsf{C}(\mathbb{R}^+, \mathsf{X})$  the space of continuous  $\mathsf{X}$ -valued functions on the infinite time interval  $[0, \infty)$ . We equip  $\mathsf{C}(\mathbb{R}^+, \mathsf{X})$  with the local uniform topology. Denote by  $\mathsf{D}(\mathbb{R}^+, \mathsf{X})$  the space of  $\mathsf{X}$ -valued right-continuous functions with left limits on the infinite time interval  $[0, \infty)$ , hereafter *càdlàg functions*. This space is endowed with the Skorokhod topology. For  $0 < T < +\infty$ , denote by  $\mathsf{C}([0, T], \mathsf{X})$  (resp.  $\mathsf{D}([0, T], \mathsf{X})$ ) the space of  $\mathsf{X}$ -valued continuous functions (resp. càdlàg functions) defined on  $[0, T]$ , equipped with the uniform (resp. Skorokhod) topology.

For  $x \in \mathsf{X}$ ,  $\alpha \geq 0$  and  $r > 0$ , consider the interpolated process

$$(4) \quad \eta_r^\alpha(t; x) \stackrel{\text{def}}{=} r^{-1} \Phi_{\lfloor tr^{1+\alpha} \rfloor}, \quad \eta_r^\alpha(t; 0) = r^{-1} \Phi_0 = x,$$

where  $\lfloor \cdot \rfloor$  stands for the lower integer part. Denote by  $\mathbb{Q}_{r;x}^\alpha$  the image probability on  $\mathsf{D}(\mathbb{R}^+, \mathsf{X})$  of  $\mathbb{P}_x$  by  $\eta_r^\alpha(\cdot; x)$ . In words, the renormalized process is obtained by scaling the Markov chain in space, time and initial condition. This is made precise in the following definition.

**DEFINITION 1.1** ( *$\alpha$ -fluid limit*). Let  $\alpha \geq 0$  and  $x \in \mathsf{X}$ . A probability measure  $\mathbb{Q}_x^\alpha$  on  $\mathsf{D}(\mathbb{R}^+, \mathsf{X})$  is said to be an  *$\alpha$ -fluid limit* if there exist sequences of scaling factors  $\{r_n\} \subset \mathbb{R}_+$  and initial states  $\{x_n\} \subset \mathsf{X}$  satisfying  $\lim_{n \rightarrow \infty} r_n = +\infty$  and  $\lim_{n \rightarrow \infty} x_n = x$  such that  $\{\mathbb{Q}_{r_n; x_n}^\alpha\}$  converges weakly to  $\mathbb{Q}_x^\alpha$  on  $\mathsf{D}(\mathbb{R}^+, \mathsf{X})$  (denoted  $\mathbb{Q}_{r_n; x_n}^\alpha \Rightarrow \mathbb{Q}_x^\alpha$ ).

The set  $\{\mathbb{Q}_x^\alpha, x \in \mathsf{X}\}$  of all such limits is referred to as the  *$\alpha$ -fluid limit model*. An  $\alpha$ -fluid limit  $\mathbb{Q}_x^\alpha$  is said to be *deterministic* if there exists a function  $g \in \mathsf{D}(\mathbb{R}^+, \mathsf{X})$  such that  $\mathbb{Q}_x^\alpha = \delta_g$ , the Dirac mass at  $g$ .

Assume that  $\mathbb{E}_x[|\Phi_1|] < \infty$  for all  $x \in \mathsf{X}$ , where  $|\cdot|$  denotes the Euclidean norm, and consider the decomposition

$$(5) \quad \Phi_k = \Phi_{k-1} + \Delta(\Phi_{k-1}) + \epsilon_k, \quad k \geq 1,$$

where

$$(6) \quad \Delta(x) \stackrel{\text{def}}{=} \mathbb{E}_x[\Phi_1 - \Phi_0] = \mathbb{E}_x[\Phi_1] - x \quad \text{for all } x \in \mathsf{X},$$

$$(7) \quad \epsilon_k \stackrel{\text{def}}{=} \Phi_k - \mathbb{E}[\Phi_k | \mathcal{F}_{k-1}] \quad \text{for all } k \geq 1.$$

In the sequel, we assume the following.

- B1. There exists  $p > 1$  such that  $\lim_{K \rightarrow \infty} \sup_{x \in \mathsf{X}} \mathbb{E}_x[|\epsilon_1|^p \mathbb{1}\{|\epsilon_1| \geq K\}] = 0$ .
- B2. There exists  $\beta \in [0, 1 \wedge (p-1))$  such that  $N(\beta, \Delta) \stackrel{\text{def}}{=} \sup_{x \in \mathsf{X}} \{(1 + |x|^\beta) \times |\Delta(x)|\} < \infty$ .

**THEOREM 1.2.** *Assume B1 and B2. Then, for all  $0 \leq \alpha \leq \beta$  and any sequences  $\{r_n\} \subset \mathbb{R}_+$  and  $\{x_n\} \subset \mathsf{X}$  such that  $\lim_{n \rightarrow \infty} r_n = +\infty$  and  $\lim_{n \rightarrow \infty} x_n = x$ , there exists a probability measure  $\mathbb{Q}_x^\alpha$  on  $\mathcal{C}(\mathbb{R}_+, \mathsf{X})$  and subsequences  $\{r_{n_j}\} \subseteq \{r_n\}$  and  $\{x_{n_j}\} \subseteq \{x_n\}$  such that  $\mathbb{Q}_{r_{n_j}; x_{n_j}}^\alpha \Rightarrow \mathbb{Q}_x^\alpha$ . Furthermore, for all  $0 \leq \alpha < \beta$ , the  $\alpha$ -fluid limits are trivial in the sense that  $\mathbb{Q}_x^\alpha = \delta_g$  with  $g(t) \equiv x$ .*

Note that for any  $x \in \mathsf{X}$  and  $0 \leq \alpha \leq \beta$ , we have  $\mathbb{Q}_x^\alpha(\eta, \eta(0) = x) = 1$ , showing that  $x$  is the initial point of the fluid limit.

**1.2. Stability of fluid limits and Markov chain stability.** There are several notions of stability that have appeared in the literature [see [Meyn \(2001\)](#), Theorem 3] and the surrounding discussion. We adopt the notion of stability introduced in [Stolyar \(1995\)](#).

**DEFINITION 1.3 (Stability).** The  $\alpha$ -fluid limit model is said to be *stable* if there exist  $T > 0$  and  $\rho < 1$  such that for any  $x \in \mathsf{X}$  with  $|x| = 1$ ,

$$(8) \quad \mathbb{Q}_x^\alpha \left( \eta \in \mathcal{D}(\mathbb{R}_+, \mathsf{X}), \inf_{0 \leq t \leq T} |\eta(t)| \leq \rho \right) = 1.$$

Let  $f: \mathsf{X} \rightarrow [1, \infty)$  and  $L_\infty^f$  denote the vector space of all measurable functions  $g$  on  $\mathsf{X}$  such that  $\sup_{x \in \mathsf{X}} |g(x)|/f(x)$  is finite.  $L_\infty^f$  equipped with the norm  $|g|_f \stackrel{\text{def}}{=} \sup_{x \in \mathsf{X}} |g(x)|/f(x)$  is a Banach space.

Denote by  $\|\cdot\|_f$  the  $f$ -total variation norm, defined for any finite signed measure  $\nu$  as  $\|\nu\|_f = \sup_{|g| \leq f} |\nu(g)|$ .

We recall some basic definitions related to Markov chains on general state space; see [Meyn and Tweedie \(1993\)](#) for an in-depth presentation. A chain is said to be *phi-irreducible* if there exists a  $\sigma$ -finite measure  $\phi$  such that  $\sum_{n \geq 0} P^n(x, A) > 0$  for all  $x \in \mathsf{X}$  whenever  $\phi(A) > 0$ . A set  $C \in \mathcal{X}$  is  $\nu_m$ -small if there exist a nontrivial measure  $\nu_m$  and a positive integer  $m$  such that such that  $P^m(x, \cdot) \geq \mathbb{1}_C(x) \nu_m(\cdot)$ . *Petite* sets are a generalization of small sets: a set  $C$  is said to be petite if there exists a distribution  $a$  on the positive integers and a distribution  $\nu$  such that  $\sum_{n \geq 0} a(n) P^n(x, \cdot) \geq \mathbb{1}_C(x) \nu(\cdot)$ . Finally, an *aperiodic chain* is a chain such that the greatest common divisor of the set

$$\{m, C \text{ is } \nu_m\text{-small and } \nu_m = \delta_m \nu \text{ for some } \delta_m > 0\},$$

is one, for some small set  $C$ . For a phi-irreducible aperiodic chain, the petite sets are small [[Meyn and Tweedie \(1993\)](#), Proposition 5.5.7].

Let  $\{r(n)\}_{n \in \mathbb{N}}$  be a sequence of positive real numbers. An aperiodic phi-irreducible positive Harris chain with stationary distribution  $\pi$  is called *(f, r)-ergodic* if

$$\lim_{n \rightarrow \infty} r(n) \|P^n(x, \cdot) - \pi\|_f = 0$$

for all  $x \in \mathsf{X}$ . If  $P$  is positive Harris recurrent with invariant probability  $\pi$ , then the fundamental kernel  $Z$  is defined as  $Z \stackrel{\text{def}}{=} (\text{Id} - P + \Pi)^{-1}$ , where the kernel  $\Pi$  is  $\Pi(x, \cdot) \equiv \pi(\cdot)$  for all  $x \in \mathsf{X}$  and  $\text{Id}$  is the identity kernel. For any measurable function  $g$  on  $\mathsf{X}$ , the function  $\hat{g} = Zg$  is a solution to the Poisson equation, whenever the inverse is well defined [see [Meyn and Tweedie \(1993\)](#)].

The following theorem may be seen as an extension of [[Dai and Meyn \(1995\)](#), Theorem 5.5], which relates the stability of the fluid limit to the  $(f, r)$ -ergodicity of the original chain.

**THEOREM 1.4.** *Let  $\{\Phi_k\}_{k \in \mathbb{N}}$  be a phi-irreducible and aperiodic Markov chain such that compact sets are petite. Assume B1 and B2 and that the  $\beta$ -fluid limit model is stable. Then, for any  $1 \leq q \leq (1 + \beta)^{-1}p$ ,*

- (i) *the Markov chain  $\{\Phi_k\}_{k \in \mathbb{N}}$  is  $(f^{(q)}, r^{(q)})$ -ergodic with  $f^{(q)}(x) \stackrel{\text{def}}{=} 1 + |x|^{p-q(1+\beta)}$  and  $r^{(q)}(n) = n^{q-1}$ ;*
- (ii) *the fundamental kernel  $Z$  is a bounded linear transformation from  $L_\infty^{f^{(q)}}$  to  $L_\infty^{f^{(q-1)}}$ .*

**1.3. Characterization of the fluid limits.** Theorem 1.4 relates the ergodicity of the Markov chain to the stability of the fluid limit and raises the question: *how can we determine if the  $\beta$ -fluid model is stable?* To answer this question, we first characterize the set of fluid limits.

In addition to assumptions B1–B2, we require conditions on the limiting behavior of the function  $\Delta$ .

- B3.** There exist an open cone  $\mathsf{O} \subseteq \mathsf{X} \setminus \{0\}$  and a continuous function  $\Delta_\infty : \mathsf{O} \rightarrow \mathsf{X}$  such that, for any compact subset  $\mathsf{H} \subseteq \mathsf{O}$ ,

$$\lim_{r \rightarrow +\infty} \sup_{x \in \mathsf{H}} |r^\beta |x|^\beta \Delta(rx) - \Delta_\infty(x)| = 0,$$

where  $\beta$  is given by B2.

The easy situation is when  $\mathsf{O} = \mathsf{X} \setminus \{0\}$ , in which case the radial limit  $\lim_{r \rightarrow \infty} r^\beta |x|^\beta \Delta(rx)$  exists for  $x \neq 0$ . Though this condition is met in examples of interest, there are several situations for which the radial limits do not exist for directions belonging to some low-dimensional manifolds of the unit sphere. Let  $h$  be given by

$$(9) \quad h(x) \stackrel{\text{def}}{=} |x|^{-\beta} \Delta_\infty(x).$$

A function  $\mu : I \rightarrow \mathsf{X}$  (where  $I \subset \mathbb{R}^+$  is an interval which can be open or closed, bounded or unbounded) is said to be a *solution of the ODE (3)* on  $I$  with initial condition  $x$  if  $\mu$  is continuously differentiable on  $I$  for all  $t \in I$ ,  $\mu(t) \in \mathsf{O}$ ,  $\mu(0) = x$  and  $\dot{\mu}(t) = h \circ \mu(t)$ . The following theorem shows that the

fluid limits restricted to  $\mathbf{O}$  evolve deterministically and, more precisely, that their supports on  $\mathbf{O}$  belong to the flow of the ODE.

PROPOSITION 1.5. *Assume B1, B2 and B3. For any  $0 \leq s \leq t$ , define*

$$(10) \quad \mathbf{A}(s, t) \stackrel{\text{def}}{=} \{\eta \in \mathcal{C}(\mathbb{R}^+, \mathbf{X}) : \eta(u) \in \mathbf{O} \text{ for all } u \in [s, t]\}.$$

*Then, for any  $x \in \mathbf{X}$  and any  $\beta$ -fluid limit  $\mathbb{Q}_x^\beta$ , on  $\mathbf{A}(s, t)$ ,*

$$\sup_{s \leq u \leq t} \left| \eta(u) - \eta(s) - \int_s^u h \circ \eta(v) dv \right| = 0, \quad \mathbb{Q}_x^\beta\text{-a.s.}$$

Under very weak additional conditions, one may assume that the solutions of the ODE (3) with initial condition  $x \in \mathbf{O}$  exist and are unique on a nonvanishing interval  $[0, T_x]$ . In such a case, Proposition 1.5 provides a handy description of the fluid limit.

B4. Assume that for all  $x \in \mathbf{O}$ , there exists  $T_x > 0$  such that the ODE (3) with initial condition  $x$  has a *unique* solution, denoted  $\mu(\cdot; x)$  on an interval  $[0, T_x]$ .

Assumption B4 is satisfied if  $\Delta_\infty$  is locally Lipschitz on  $\mathbf{O}$ ; in such a case,  $h$  is locally Lipschitz on  $\mathbf{O}$  and it then follows from classical results on the existence of solutions of the ODE [see, e.g., Verhulst (1996)] that for any  $x \in \mathbf{O}$ , there exists  $T_x > 0$  such that, on the interval  $[0, T_x]$ , the ODE (3) has a unique solution  $\mu$  with initial condition  $\mu(0) = x$ . In addition, if the ODE (3) has two solutions,  $\mu_1$  and  $\mu_2$ , on an interval  $I$  which satisfy  $\mu_1(t_0) = \mu_2(t_0) = x_0$  for some  $t_0 \in I$ , then  $\mu_1(t) = \mu_2(t)$  for any  $t \in I$ .

An elementary application of Proposition 1.5 shows that under this additional assumption, a fluid limit starting at  $x_0 \in \mathbf{O}$  coincides with the solution of the ODE (3) with initial condition  $x_0$  on a nonvanishing interval.

THEOREM 1.6. *Assume B1–B4. Let  $x \in \mathbf{O}$ . There then exists  $T_x > 0$  such that  $\mathbb{Q}_x^\beta = \delta_{\mu(\cdot; x)}$  on  $\mathcal{D}([0, T_x], \mathbf{X})$ .*

As a corollary of Theorem 1.6, we have the following.

COROLLARY 1.7. *Assume that  $\mathbf{O} = \mathbf{X} \setminus \{0\}$  in B3. Then all  $\beta$ -fluid limits are deterministic and solve the ODE (3). Furthermore, for any  $\epsilon > 0$  and  $x \in \mathbf{X}$ , and any sequences  $\{r_n\} \subset \mathbb{R}_+$  and  $\{x_n\} \subset \mathbf{X}$  such that  $\lim_{n \rightarrow \infty} r_n = +\infty$  and  $\lim_{n \rightarrow \infty} x_n = x$ ,*

$$\lim_n \mathbb{P}_{r_n x_n} \left( \sup_{0 \leq t \leq T_x} |\eta_{r_n}^\beta(t; x_n) - \mu(t; x)| \geq \epsilon \right) = 0.$$



Hence, the fluid limit depends only on the initial value  $x$  and does not depend upon the choice of the sequences  $\{r_n\}$  and  $\{x_n\}$ .

The last step is to relate the stability of the fluid limit [see (8)] to the behavior of the solutions of the ODE, when such solutions are well defined. From the discussion above, we may deduce a first elementary stability condition. Assume that B3 holds with  $\mathbf{O} = \mathbf{X} \setminus \{0\}$ . In this case, the fluid limit model is stable if there exist  $\rho < 1$  and  $T < \infty$  such that, for any  $|x| = 1$ ,  $\inf_{[0, T]} |\mu(\cdot; x)| < \rho$ , that is, the solutions of the ODE enter a sphere of radius  $\rho < 1$  before a given time  $T$ .

**THEOREM 1.8.** *Let  $\{\Phi_k\}_{k \in \mathbb{N}}$  be a phi-irreducible and aperiodic Markov chain such that compact sets are petite. Let  $\rho$ ,  $0 < \rho < 1$  and  $T > 0$ . Assume that B1–B4 hold with  $\mathbf{O} = \mathbf{X} \setminus \{0\}$ . Assume, in addition, that for any  $x$  satisfying  $|x| = 1$ , the solution  $\mu(\cdot; x)$  is such that  $\inf_{[0, T \wedge T_x]} |\mu(\cdot; x)| \leq \rho$ . Then, the  $\beta$ -fluid limit model is stable and the conclusions of Theorem 1.4 hold.*

When B3 holds for a strict subset of the state space  $\mathbf{O} \subsetneq \mathbf{X} \setminus \{0\}$ , the situation is more difficult because some fluid limits are not solutions of the ODE. Regardless, under general assumptions, stability of the ODE implies stability of the fluid limit model.

**THEOREM 1.9.** *Let  $\{\Phi_k\}_{k \in \mathbb{N}}$  be a phi-irreducible and aperiodic Markov chain such that compact sets are petite. Assume that B1–B4 hold with  $\mathbf{O} \subsetneq \mathbf{X} \setminus \{0\}$ . Assume, in addition, that:*

(i) *there exists  $T_0 > 0$  such that for any  $x$ ,  $|x| = 1$ , and for any  $\beta$ -fluid limit  $\mathbb{Q}_x^\beta$ ,*

$$(11) \quad \mathbb{Q}_x^\beta(\eta : \eta([0, T_0]) \cap \mathbf{O} \neq \emptyset) = 1;$$

(ii) *for any  $K > 0$ , there exist  $T_K > 0$  and  $0 < \rho_K < 1$  such that for any  $x \in \mathbf{O}$ ,  $|x| \leq K$ ,*

$$(12) \quad \inf_{[0, T_K \wedge T_x]} |\mu(\cdot; x)| \leq \rho_K;$$

(iii) *for any compact set  $\mathbf{H} \subset \mathbf{O}$  and any  $K$ ,*

$$\Omega_{\mathbf{H}} \stackrel{\text{def}}{=} \{\mu(t; x) : x \in \mathbf{H}, t \in [0, T_x \wedge T_K]\}$$

*is a compact subset of  $\mathbf{O}$ .*

*Then, the  $\beta$ -fluid model is stable and the conclusions of Theorem 1.4 hold.*

Condition (i) implies that each  $\beta$ -fluid limit reaches the set  $\mathcal{O}$  in a finite time. When the initial condition  $x \neq 0$  does belongs to  $\mathcal{O}$ , this condition is automatically fulfilled. When  $x$  does not belong to  $\mathcal{O}$ , this condition typically requires that there is a force driving the chain into  $\mathcal{O}$ . The verification of this property generally requires some problem-dependent and sometimes intricate constructions (see, e.g., Example 2). Condition (ii) implies that the solution  $\mu(\cdot; x)$  of the ODE with initial point  $x \in \mathcal{O}$  reaches a ball inside the unit sphere before approaching the singularity. This also means that the singular set is repulsive for the solution of the ODE.

**2. The ODE method for the Metropolis–Hastings algorithm.** The Metropolis–Hastings (MH) algorithm [see Robert and Casella (2004) and the references therein] is a popular computational method for generating samples from virtually any distribution  $\pi$ . In particular, there is no need for the normalizing constant to be known and the space  $\mathsf{X} = \mathbb{R}^d$  (for some integer  $d$ ) on which it is defined can be high-dimensional. The method consists of simulating an ergodic Markov chain  $\{\Phi_k\}_{k \geq 0}$  on  $\mathsf{X}$  with transition probability  $P$  such that  $\pi$  is the stationary distribution for this chain, that is,  $\pi P = \pi$ .

The MH algorithm requires the choice of a *proposal kernel*  $q$ . In order to simplify the discussion, we will here assume that  $\pi$  and  $q$  admit densities with respect to the Lebesgue measure  $\lambda^{\text{Leb}}$ , denoted (with an abuse of notation)  $\pi$  and  $q$  hereafter. We denote by  $\mathbb{Q}$  the probability defined by  $\mathbb{Q}(A) = \int_A q(y) \lambda^{\text{Leb}}(dy)$ . The role of the kernel  $q$  consists of proposing potential transitions for the Markov chain  $\{\Phi_k\}$ . Given that the chain is currently at  $x$ , a candidate  $y$  is accepted with probability  $\alpha(x, y)$ , defined as  $\alpha(x, y) = 1 \wedge \frac{\pi(y) q(y, x)}{\pi(x) q(x, y)}$ . Otherwise it is rejected and the Markov chain stays at its current location  $x$ . The transition kernel  $P$  of this Markov chain takes the form, for  $x \in \mathsf{X}$  and  $A \in \mathcal{B}(\mathsf{X})$ ,

$$(13) \quad \begin{aligned} P(x, A) &= \int_{A-x} \alpha(x, x+y) q(x, x+y) \lambda^{\text{Leb}}(dy) \\ &\quad + \mathbb{1}_A(x) \int_{\mathsf{X}-x} \{1 - \alpha(x, x+y)\} q(x, x+y) \lambda^{\text{Leb}}(dy), \end{aligned}$$

where  $A - x \stackrel{\text{def}}{=} \{y \in \mathsf{X}, x + y \in A\}$ . The Markov chain  $P$  is reversible with respect to  $\pi$  and therefore admits  $\pi$  as invariant distribution. For the purpose of illustration, we focus on the symmetric increments random walk MH algorithm (hereafter SRWM), in which  $q(x, y) = q(y - x)$  for some symmetric distribution  $q$  on  $\mathsf{X}$ . Under these assumptions, the acceptance probability simplifies to  $\alpha(x, y) = 1 \wedge [\pi(y)/\pi(x)]$ . For any measurable function  $W : \mathsf{X} \rightarrow \mathbb{R}$ ,

$$\mathbb{E}_x[W(\Phi_1)] - W(x) = \int_{A_x} \{W(x+y) - W(x)\} q(y) \lambda^{\text{Leb}}(dy)$$

$$+ \int_{\mathbb{R}_x} \{W(x+y) - W(x)\} \frac{\pi(x+y)}{\pi(x)} q(y) \lambda^{\text{Leb}}(dy),$$

where  $\mathbb{A}_x \stackrel{\text{def}}{=} \{y \in \mathbb{X}, \pi(x+y) \geq \pi(x)\}$  is the acceptance region (moves toward  $x + \mathbb{A}_x$  are accepted with probability one) and  $\mathbb{R}_x \stackrel{\text{def}}{=} \mathbb{X} \setminus \mathbb{A}_x$  is the potential rejection region. From [Roberts and Tweedie \(1996\)](#), Theorem 2.2, we obtain the following basic result.

**THEOREM 2.1.** *Suppose that the target density  $\pi$  is positive and continuous and that  $q$  is bounded away from zero, that is, there exist  $\delta_q > 0$  and  $\epsilon_q > 0$  such that  $q(x) \geq \epsilon_q$  for  $|x| \leq \delta_q$ . Then, the random-walk-based Metropolis algorithm on  $\{\mathbb{X}, \mathcal{X}\}$  is  $\lambda^{\text{Leb}}$ -irreducible, aperiodic and every nonempty bounded set is small.*

In the sequel, we assume that  $q$  has a moment of order  $p > 1$ . To apply the results presented in Section 1, we must first compute  $\Delta(x) = \mathbb{E}_x[\Phi_1] - x$ , that is, to set  $W(x) = x$  in the previous formula. Since  $q$  is symmetric and therefore zero-mean, the previous reduces to

$$(14) \quad \Delta(x) = \int_{\mathbb{R}_x} y \left( \frac{\pi(x+y)}{\pi(x)} - 1 \right) q(y) \lambda^{\text{Leb}}(dy).$$

Note that, for any  $x \in \mathbb{X}$ ,  $|\epsilon_1| \leq |\Phi_1 - \Phi_0| + m \mathbb{P}_x$ -a.s., where  $m = \int |y| q(y) \times \lambda^{\text{Leb}}(dy)$ . Therefore, for any  $K > 0$ ,

$$\begin{aligned} \mathbb{E}_x[|\epsilon_1|^p \mathbb{1}\{|\epsilon_1| \geq K\}] &\leq 2^p \mathbb{E}_x[(|\Phi_1 - \Phi_0|^p + m^p) \mathbb{1}\{|\Phi_1 - \Phi_0| \geq K - m\}] \\ &\leq 2^p \int |y|^p \mathbb{1}\{|y| \geq K - m\} q(y) \lambda^{\text{Leb}}(dy), \end{aligned}$$

showing that assumption B1 is satisfied as soon as the increment distribution has a bounded  $p$ th moment. Because, on the set  $\mathbb{R}_x$ ,  $\pi(x+y) \leq \pi(x)$ , we similarly have  $|\Delta(x)| \leq \int |y| q(y) \lambda^{\text{Leb}}(dy)$  showing, that B2 is satisfied with  $\beta = 0$ ; nevertheless, in some examples, for  $\beta = 0$ ,  $\Delta_\infty$  can be zero and the fluid limit model is unstable. In these cases, it is necessary to use larger  $\beta$  (see Section 2.2).

**2.1. Superexponential target densities.** In this section, we focus on target densities  $\pi$  on  $\mathbb{X}$  which are *superexponential*. Define  $n(x) \stackrel{\text{def}}{=} x/|x|$ .

**DEFINITION 2.2** (*Superexponential p.d.f.*). A probability density function  $\pi$  is said to be *superexponential* if  $\pi$  is positive, has continuous first derivatives and  $\lim_{|x| \rightarrow \infty} \langle n(x), \ell(x) \rangle = -\infty$ , where  $\ell(x) \stackrel{\text{def}}{=} \nabla \log \pi(x)$ .

The condition implies that for any  $H > 0$ , there exists  $R > 0$  such that

$$(15) \quad \frac{\pi(x + an(x))}{\pi(x)} \leq \exp(-aH) \quad \text{for } |x| \geq R, a \geq 0,$$

that is,  $\pi(x)$  is at least exponentially decaying along any ray with the rate  $H$  tending to infinity as  $|x|$  goes to infinity. It also implies that for  $x$  large enough, the contour manifold  $C_x \stackrel{\text{def}}{=} \{y \in X, \pi(x + y) = \pi(x)\}$  can be parameterized by the unit sphere  $S$  since each ray meets  $C_x$  at exactly one point. In addition, for sufficiently large  $|x|$ , the acceptance region  $A_x$  is the set enclosed by the contour manifold  $C_x$  (see Figure 1). Denote by  $A \ominus B$  the symmetric difference of the sets  $A$  and  $B$ .

**DEFINITION 2.3 (*q-radial limit*).** We say that the family of rejection regions  $\{R_{rx}, r \geq 0, x \in O\}$  has *q-radial limits* over the open cone  $O \subseteq X \setminus \{0\}$  if there exists a collection of sets  $\{R_{\infty,x}, x \in O\}$  such that, for any compact subset  $H \subseteq O$ ,  $\lim_{r \rightarrow \infty} \sup_{x \in H} Q(R_{rx} \ominus R_{\infty,x}) = 0$ .

**PROPOSITION 2.4.** Assume that the target density  $\pi$  is super-exponential. Assume, in addition, that the family  $\{R_{rx}, r \geq 0, x \in O\}$  has a q-radial limit over an open cone  $O \subseteq X \setminus \{0\}$ . Then, for any compact set  $H \subset O$ ,  $\lim_{r \rightarrow \infty} \sup_{x \in H} |\Delta(rx) - \Delta_\infty(x)| = 0$ , where  $\Delta_\infty(x) \stackrel{\text{def}}{=} -\int_{R_{\infty,x}} yq(y)\lambda^{\text{Leb}}(dy)$ .

The proof is postponed to Section 5.1. The definition of the limiting field  $\Delta_\infty$  becomes simple when the rejection region radially converges to a half-space.

**DEFINITION 2.5 (*q-regularity in the tails*).** We say that the target density  $\pi$  is *q-regular in the tails* over  $O$  if the family  $\{R_{rx}, r \geq 0, x \in O\}$  has

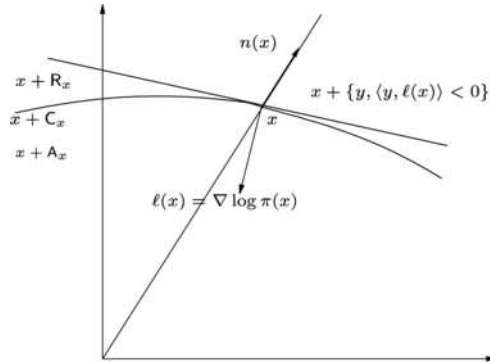


FIG. 1.

$q$ -radial limits over an open cone  $\mathbf{O} \subseteq \mathbf{X} \setminus \{0\}$  and there exists a continuous function  $\ell_\infty: \mathbf{X} \setminus \{0\} \rightarrow \mathbf{X}$  such that, for all  $x \in \mathbf{O}$ ,

$$(16) \quad \mathbf{Q}(R_{\infty,x} \ominus \{y \in \mathbf{X}, \langle y, \ell_\infty(x) \rangle < 0\}) = 0.$$

Regularity in the tails holds with  $\ell_\infty(x) = \lim_{r \rightarrow \infty} n(\ell(rx))$  when the curvature at 0 of the contour manifold  $C_{rx}$  goes to zero as  $r \rightarrow \infty$ ; nevertheless, this condition may still hold in situations where there exists a sequence  $\{x_n\}$  with  $\lim |x_n| = \infty$  such that the curvature of the contour manifolds  $C_{x_n}$  at zero can grow to infinity (see Examples 1 and 2). Assume that

$$(17) \quad q(x) = \det^{-1/2}(\Sigma) q_0(\Sigma^{-1/2}x),$$

where  $\Sigma$  is a positive definite matrix and  $q_0$  is a *rotationally invariant distribution*, that is,  $q_0(Ux) = q_0(x)$  for any unitary matrix  $U$ , and is such that

$$\int_{\mathbf{X}} y_1^2 q_0(y) \lambda^{\text{Leb}}(dy) < \infty.$$

**PROPOSITION 2.6.** *Assume that the target density  $\pi$  is super-exponential and  $q$ -regular in the tails over the open cone  $\mathbf{O} \subseteq \mathbf{X} \setminus \{0\}$ . Then, the SRWM algorithm with proposal  $q$  given in (17) satisfies assumption B3 on  $\mathbf{O}$  with*

$$(18) \quad \Delta_\infty(x) = m_1(q_0) \frac{\Sigma \ell_\infty(x)}{|\sqrt{\Sigma} \ell_\infty(x)|},$$

where  $\ell_\infty$  is defined in (16) and  $m_1(q_0) \stackrel{\text{def}}{=} \int_{\mathbf{X}} y_1 \mathbf{1}_{\{y_1 \geq 0\}} q_0(y) \lambda^{\text{Leb}}(dy) > 0$ , where  $y = (y_1, \dots, y_d)$ .

The proof is given in Section 5.1. If  $\Sigma = \text{Id}$  and  $\ell_\infty(x) = \lim_{r \rightarrow \infty} n(\ell(rx))$ , then the ODE may be seen as a version of steepest ascent algorithm to maximize  $\log \pi$ . It may appear that convergence would be faster if  $m_1(q_0)$  is increased. While it is true for the ODE, we cannot reach such a positive conclusion for the algorithm itself because we do not control the fluctuation of the algorithm around its limit.

**2.1.1. Regular case.** The tail regularity condition and the definition of the ODE limit are more transparent in a class of models which are very natural in many statistical contexts, namely, the exponential family. Following Roberts and Tweedie (1996), define the class  $\mathcal{P}$  as consisting of those everywhere positive densities with continuous second derivatives  $\pi$  satisfying

$$(19) \quad \pi(x) \propto g(x) \exp\{-p(x)\},$$

where:

- $g$  is a positive function slowly varying at infinity, that is, for any  $K > 0$ ,

$$(20) \quad \limsup_{|x| \rightarrow \infty} \inf_{|y| \leq K} \frac{g(x+y)}{g(x)} = \limsup_{|x| \rightarrow \infty} \sup_{|y| \leq K} \frac{g(x+y)}{g(x)} = 1;$$

- $p$  is a positive polynomial in  $\mathbb{X}$  of even order  $m$  and  $\lim_{|x| \rightarrow \infty} p_m(x) = +\infty$ , where  $p_m$  denotes the polynomial consisting only of the  $p$ 's  $m$ th order terms.

PROPOSITION 2.7. *Assume that  $\pi \in \mathcal{P}$  and let  $q$  be given by (17). Then,  $\pi$  is super-exponential,  $q$ -regular in the tails over  $\mathbb{X} \setminus \{0\}$  with  $\ell_\infty(x) = -n[\nabla p_m(n(x))]$ . For any  $x \in \mathbb{X} \setminus \{0\}$ , there exists  $T_x > 0$  such that the ODE  $\dot{\mu} = \Delta_\infty(\mu)$  with initial condition  $x$  and  $\Delta_\infty$  given by (18) has a unique solution on  $[0, T_x]$  and  $\lim_{t \rightarrow T_x^-} \mu(t; x) = 0$ . In addition, the fluid limit  $\mathbb{Q}_x^0$  is deterministic on  $\mathcal{D}([0, T_x], \mathbb{X})$ , with support function  $\mu(\cdot; x)$ .*

The proof is skipped for brevity [see Fort et al. (2006)]. Because all the solutions of the initial value problem  $\dot{\mu} = -m_1(q_0)\sqrt{\Sigma}n[\sqrt{\Sigma}\nabla p_m(n(\mu))]$ ,  $\mu(0) = x$  are zero after a fixed amount of time  $T$  for any initial condition on the unit sphere, we may apply Theorem 1.8. We have, from Theorem 2.1 and Theorem 1.8, the following.

THEOREM 2.8. *Consider the SRWM Markov chain with target distribution  $\pi \in \mathcal{P}$  and increment distribution  $q$  having a moment of order  $p > 1$  and satisfying (17). Then, for any  $1 \leq u \leq p$ , the SRWM Markov chain is  $(f_u, r_u)$ -ergodic with*

$$f_u(x) = 1 + |x|^{p-u}, \quad r_u(t) \sim t^{u-1}.$$

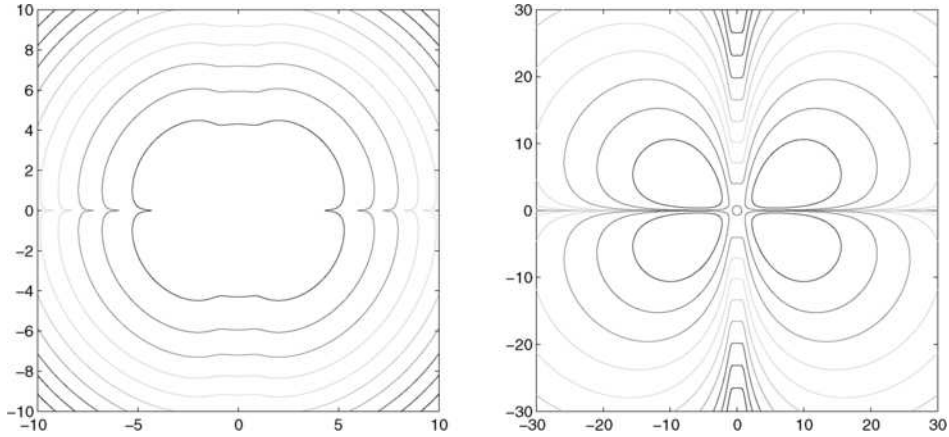


FIG. 2. Contour curves of the target densities (21) (left panel) and (26) with  $\delta = 0.4$  (right panel).

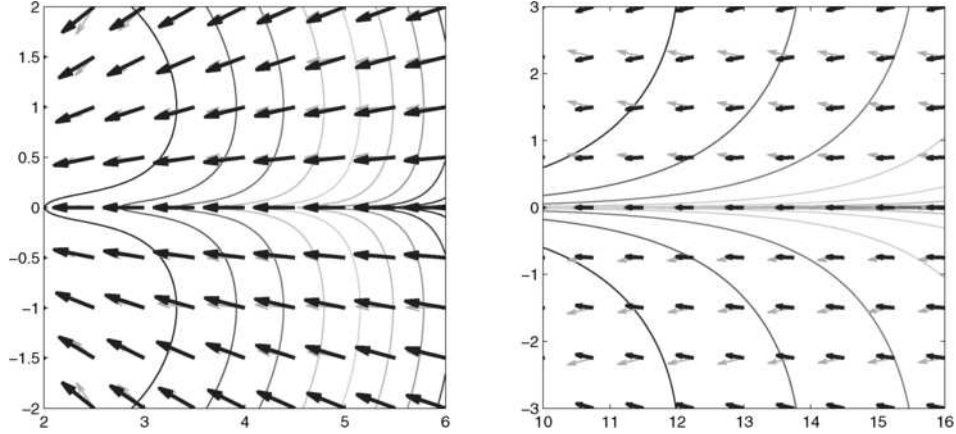


FIG. 3. Grey lines:  $\Delta$ ; Black lines:  $\Delta_\infty$  for the target densities (21) (left panel) and (26) with  $\delta = 0.4$  (right panel).

EXAMPLE 1. To illustrate our findings, consider the target density, borrowed from [Jarner and Hansen \(2000\)](#), Example 5.3,

$$(21) \quad \pi(x_1, x_2) \propto (1 + x_1^2 + x_2^2 + x_1^8 x_2^2) \exp(-(x_1^2 + x_2^2)).$$

The contour curves are illustrated in Figure 2. They are almost circular except from some small wedges by the  $x$ -axis. Due to the wedges, the curvature of the contour manifold at  $(x, 0)$  is  $(x^6 - 1)/x$  and therefore tends to infinity along the  $x$ -axis [[Jarner and Hansen \(2000\)](#)]. Since  $\pi \in$

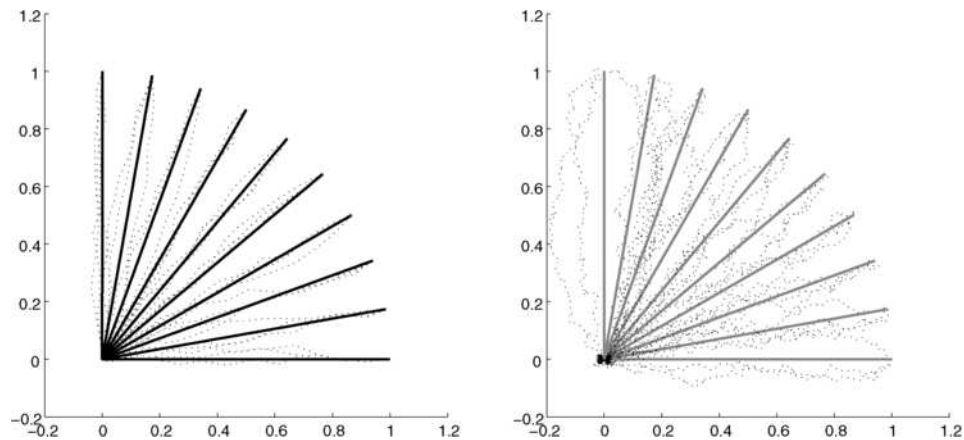


FIG. 4. Dotted lines: trajectories of the interpolated process (2) for the random walk Metropolis-Hastings (SRWM) algorithm for a set of initial conditions on the unit sphere in  $(0, \pi/2)$  for the target densities (21) (left panel) and (26) (right panel); Solid lines: flow of the associated ODE.



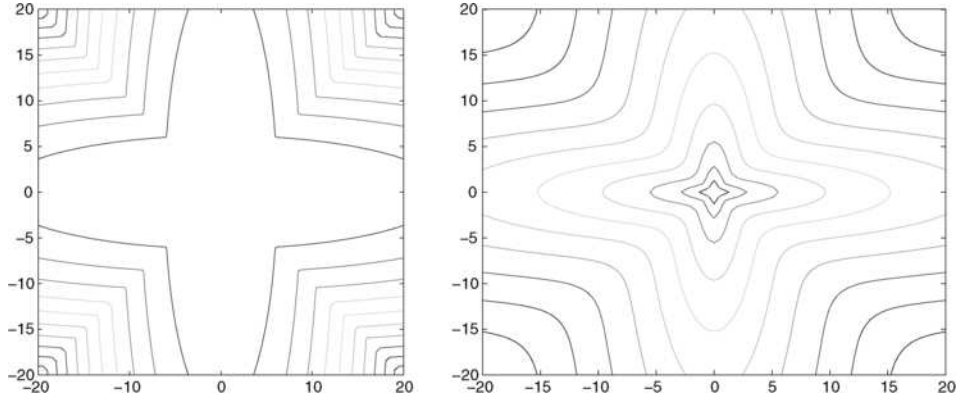


FIG. 5. Contour plot of the target densities (22) (left panel) and (27) (right panel).

$\mathcal{P}$ , Proposition 2.7 shows that  $\pi$  is super-exponential, regular in the tails and  $\ell_\infty(x) = -n(x)$ . Taking  $q \sim \mathcal{N}(0, \sigma^2 \text{Id})$ ,  $\Delta_\infty(x) = -\sigma n(x)/\sqrt{2\pi}$  and the (Carathéodory) solution of the initial value problem  $\dot{\mu} = \Delta_\infty(\mu)$ ,  $\mu(0) = x$  is given by  $\mu(t; x) = (|x| - \sigma t/\sqrt{2\pi})\mathbb{1}\{\sigma t \leq \sqrt{2\pi}|x|\}x/|x|$ . Along the sequence  $\{x_k \stackrel{\text{def}}{=} (k, \pm k^{-4})\}_{k \geq 1}$ , the normed gradient  $n[\ell(x_k)]$  converges to  $(0, \pm 1)$ , showing that whereas  $\ell_\infty$  is the radial limit of the normed gradient  $n[\ell]$  (i.e., for any  $u \in \mathbb{S}$ ,  $\lim_{\lambda \rightarrow \infty} n[\ell(\lambda u)] = \ell_\infty(u)$ ),  $\limsup_{|x| \rightarrow \infty} |n[\ell(x)] - \ell_\infty(x)| = 2$ . Therefore, the normed gradient  $n[\ell(x)]$  does not have a limit as  $|x| \rightarrow \infty$  along the  $x$ -axis. Nevertheless, the fluid limit exists and is extremely simple to determine. Hence, the ergodicity of the SRWM sampler with target distribution (21) may be established [note that for this example, the theory developed in Roberts and Tweedie (1996) and in Jarner and Hansen (2000) does not apply]. The functions  $\Delta$  and  $\Delta_\infty$  are displayed in Figure 3. The flow of the initial value problem  $\dot{\mu} = \Delta_\infty(\mu)$  for a set of initial conditions on the unit sphere in  $(0, \pi/2)$  is displayed in Figure 4.

2.1.2. *Irregular case.* We give an example for which, in Proposition 2.4,  $\mathcal{O} \subsetneq \mathbb{X} \setminus \{0\}$ .

EXAMPLE 2. In this example [also borrowed from Jarner and Hansen (2000)], we consider the mixture of two Gaussian distributions on  $\mathbb{R}^2$ . For some  $a^2 > 1$  and  $0 < \alpha < 1$ , set

$$(22) \quad \pi(x) \propto \alpha \exp(-(1/2)x' \Gamma_1^{-1} x) + (1 - \alpha) \exp(-(1/2)x' \Gamma_2^{-1} x),$$

where  $\Gamma_1^{-1} \stackrel{\text{def}}{=} \text{diag}(a^2, 1)$  and  $\Gamma_2^{-1} \stackrel{\text{def}}{=} \text{diag}(1, a^2)$ . The contour curves for  $\pi$  with  $a = 4$  are illustrated in Figure 5. We see that the contour curves have some sharp bends along the diagonals that do not disappear in the limit,



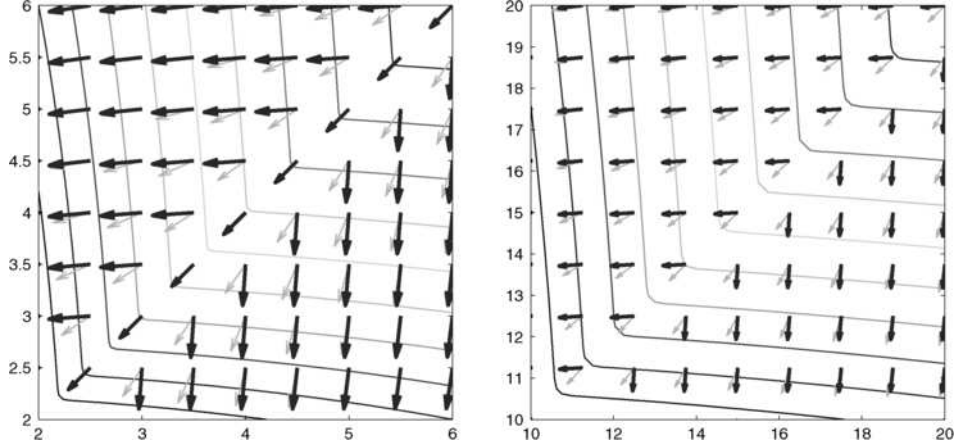


FIG. 6. Grey lines:  $\Delta$ ; Black lines:  $\Delta_\infty$  for the target density (22) (left panel) and (27) (right panel).

even though the contour curves of the two components of the mixtures are smooth ellipses. Equation (51) of [Jarner and Hansen \(2000\)](#), indeed shows that the curvature of the contour curve on the diagonal tends to infinity. As shown in the following lemma, however, this target density is regular in the tails over  $O = X \setminus \{x = (x_1, x_2) \in \mathbb{R}^2, |x_1| = |x_2|\}$  (and not over  $X \setminus \{0\}$ ). More precisely, we have the following.

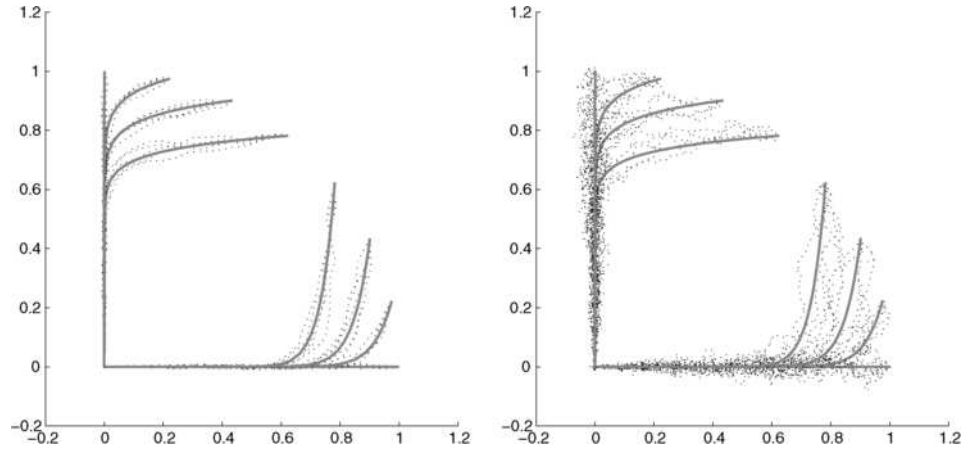


FIG. 7. Dotted lines: interpolated process for a set of initial conditions on the unit sphere for the target density (22) (left panel) and (27) (right panel); Solid lines: flow of the initial value problem  $\dot{\mu} = h(\mu)$  with  $h(x) = |x|^{-\beta} \Delta_\infty(x)$ ;  $\beta = 0$  and  $\Delta_\infty$  are given by Lemma 2.9 (left panel) and  $\beta, \Delta_\infty$  are given by Lemma 2.16 (right panel).

LEMMA 2.9. *For any  $\varepsilon > 0$ , there exist  $M$  and  $K$  such that*

$$(23) \quad \sup_{|x| \geq K, ||x_1| - |x_2|| \geq M} |\Delta(x) - \Delta_\infty(x)| \leq \varepsilon,$$

where  $\Delta_\infty(x) \stackrel{\text{def}}{=} -\int \mathbb{1}_{R_{\infty,x}}(y) y q(y) \lambda^{\text{Leb}}(dy)$  with  $R_{\infty,x} \stackrel{\text{def}}{=} \{y, \langle y, \Gamma_2^{-1}x \rangle \geq 0\}$  if  $|x_1| > |x_2|$  and  $R_{\infty,x} \stackrel{\text{def}}{=} \{y, \langle y, \Gamma_1^{-1}x \rangle \geq 0\}$  otherwise.

The proof is postponed to Section 5.2. Since  $q$  satisfies (17), when  $\Sigma = \text{Id}$ , for any  $x \in \mathbb{O}$ , we have either  $\Delta_\infty(x) = -c_q n(\Gamma_2^{-1}x)$  if  $|x_1| > |x_2|$  or  $\Delta_\infty(x) = -c_q n(\Gamma_1^{-1}x)$  if  $|x_1| < |x_2|$ , where  $c_q$  is a constant depending on the increment distribution  $q$ . This is illustrated in Figure 6, which displays the functions  $\Delta$  and  $\Delta_\infty$  and shows that these two functions are asymptotically close outside a band along the main diagonal. The flows of the initial value problem  $\dot{\mu} = \Delta_\infty(\mu)$  for a set of initial conditions in  $(0, \pi/2)$  are displayed in Figure 7.

We now prove that Theorem 1.9 applies. Conditions B1–B2 hold, as discussed above. Condition B3 results from Lemma 2.9. It remains to prove that B4 and conditions (i)–(iii) are verified. The proof of condition (i) is certainly the most difficult to check in this example.

PROPOSITION 2.10. *Consider the SRWM Markov chain with target distribution given by (22). Assume that  $q$  is rotationally invariant and with compact support. Then, B4 as well as conditions (i), (ii) and (iii) of Theorem 1.9 hold.*

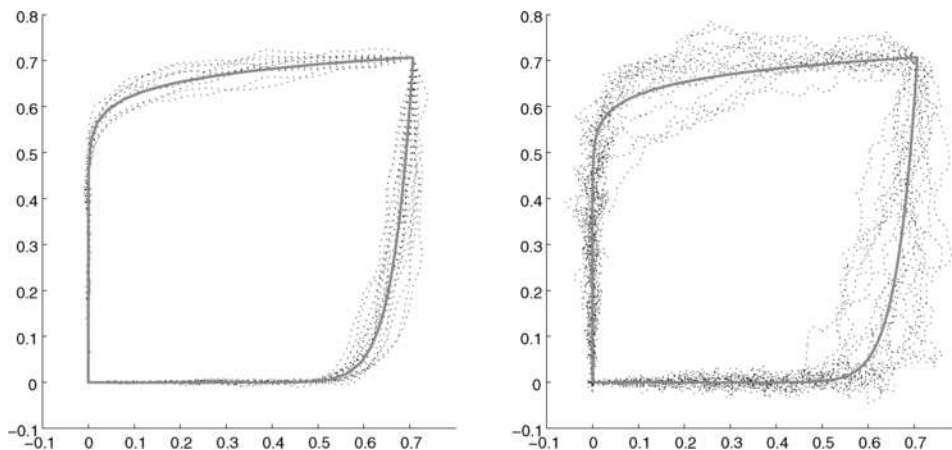


FIG. 8. *Dotted lines: trajectories of the interpolated process (2) for the SRWM with target density (22) (left panel) and (27) (right panel) and initial condition  $(1/\sqrt{2}, 1/\sqrt{2})$ ; Solid lines: flow of the associated ODE.*

A detailed proof is provided in Section 5.2. Note that the fluid limit model is not deterministic in this example: for  $x$  on the diagonal in  $\mathsf{X}$ , the support of the fluid limit  $\mathbb{Q}_x^0$  consists of two trajectories, each of which are solutions of the ODE. This is illustrated in Figure 8. By Theorem 1.9 and the discussion above, we may conclude that if the increment distribution  $q$  is compactly supported, then the SRWM Markov chain with target distribution  $\pi$  given by (22) is  $(f_u, r_s)$ -ergodic with  $f_u(x) = 1 + |x|^u$  and  $r_s(t) \sim t^s$  for any  $u \geq 0$  and  $s \geq 0$ .

**2.2. Subexponential density.** In this section, we focus on target densities  $\pi$  on  $\mathsf{X}$  which are *subexponential*. We assume that  $q$  satisfies (17) and has moment of order  $p \geq 2$ . This section is organized as above: we start with the regular case (Example 3) and then consider the irregular case (Example 4).

**DEFINITION 2.11** (Subexponential p.d.f.). A probability density function  $\pi$  is said to be *subexponential* if  $\pi$  is positive with continuous first derivatives,  $\langle n(x), n(\ell(x)) \rangle < 0$  for all sufficiently large  $x$  and  $\lim_{|x| \rightarrow \infty} |\ell(x)| = 0$ .

The condition implies that for any  $R < \infty$ ,  $\lim_{|x| \rightarrow \infty} \sup_{|y| \leq R} \pi(x+y)/\pi(x) = 1$ , which implies that  $\lim_{|x| \rightarrow \infty} |\Delta(x)| = 0$ . Subexponential target densities provide examples that require the use of positive  $\beta$  in the normalization to obtain a nontrivial fluid limit model.

The condition  $\langle n(x), n(\ell(x)) \rangle < 0$  for all sufficiently large  $|x|$  implies that for  $\epsilon$  small enough, the contour manifold  $\mathsf{C}_\epsilon$  can be parameterized by the unit sphere (see the discussion above) and that for sufficiently large  $|x|$ , the acceptance region  $\mathsf{A}_x$  is the set enclosed by the contour manifold  $\mathsf{C}_x$  (see Figure 1).

**DEFINITION 2.12** [ $q$ -regularity in the tails (subexponential)]. We say that  $\pi$  is  *$q$ -regular in the tails* over an open cone  $\mathsf{O} \subseteq \mathsf{X} \setminus \{0\}$  if there exists a continuous function  $\ell_\infty : \mathsf{O} \rightarrow \mathsf{X}$  and  $\beta \in (0, 1)$  such that, for any compact set  $\mathsf{H} \subset \mathsf{O}$  and any  $K > 0$ ,

$$\begin{aligned} \lim_{r \rightarrow \infty} \sup_{x \in \mathsf{H}} \int_{\mathsf{R}_{rx} \cap \{|y| \leq K\}} \left| r^\beta |x|^\beta \left\{ \frac{\pi(rx+y)}{\pi(rx)} - 1 \right\} - \langle \ell_\infty(x), y \rangle \right| q(y) \lambda^{\text{Leb}}(dy) = 0, \\ \lim_{r \rightarrow \infty} \sup_{x \in \mathsf{H}} \mathbb{Q}(\mathsf{R}_{rx} \ominus \{y, \langle \ell_\infty(x), y \rangle \geq 0\}) = 0. \end{aligned}$$

**PROPOSITION 2.13.** *Assume that the target density  $\pi$  is subexponential and  $q$ -regular in the tails over an open cone  $\mathsf{O} \subseteq \mathsf{X} \setminus \{0\}$  and that  $q$  satisfies (17). Then, for any compact set  $\mathsf{H} \subset \mathsf{O}$ ,  $\lim_{r \rightarrow \infty} \sup_{x \in \mathsf{H}} |r^\beta |x|^\beta \Delta(rx) - \Delta_\infty(x)| = 0$  with*

$$\Delta_\infty(x) \stackrel{\text{def}}{=} \int_{\{y, \langle \ell_\infty(x), y \rangle \geq 0\}} y \langle \ell_\infty(x), y \rangle q(y) \lambda^{\text{Leb}}(dy) = m_2(q_0) \Sigma \ell_\infty(x),$$

where  $m_2(q_0) \stackrel{\text{def}}{=} \int_{\mathbf{X}} y_1^2 \mathbb{1}_{\{y_1 \geq 0\}} q_0(y) \lambda^{\text{Leb}}(dy) > 0$ .

The proof is similar to Proposition 2.4 and is omitted for brevity. Once again, if the curvature of the contour curve goes to zero at infinity, then  $\ell_\infty(x)$  is, for large  $x$ , asymptotically colinear to  $n[\nabla \log \pi(x)]$ . However, whereas  $|\nabla \log \pi(x)| \rightarrow 0$  as  $|x| \rightarrow \infty$ , the renormalization prevents  $\ell_\infty(x)$  from vanishing at  $\infty$ ; on the contrary, it converges radially to a constant along each ray. As above, the tail regularity condition may still hold, even when the curvature goes to infinity; see Example 3. As above, the subexponential tail regularity condition and the definition of the ODE limit are more transparent in the Weibullian family. Mimicking the construction above, define, for  $\delta > 0$ , the class  $\mathcal{P}_\delta$  as consisting of those everywhere positive densities with continuous second derivatives  $\pi$  satisfying

$$(24) \quad \pi(x) \propto g(x) \exp\{-p^\delta(x)\},$$

where  $g$  is a positive function slowly varying at infinity [see (20)] and  $p$  is a positive polynomial in  $\mathbf{X}$  of even order  $m$  with  $\lim_{|x| \rightarrow \infty} p_m(x) = +\infty$ .

PROPOSITION 2.14. *Assume that  $\pi \in \mathcal{P}_\delta$  for some  $0 < \delta < 1/m$  and let  $q$  be given by (17). Then,  $\pi$  is subexponential and  $q$ -regular in the tails with  $\beta = 1 - m\delta$  and  $\ell_\infty(x) = -\delta p_m^{\delta-1}(n(x)) \nabla p_m(n(x))$ . For any  $x \in \mathbf{X} \setminus \{0\}$ , there exists  $T_x > 0$  such that the ODE  $\dot{\mu} = h(\mu)$  with initial condition  $x$  and  $h$  given by*

$$(25) \quad h(x) = -\delta |x|^{-(1-m\delta)} m_2(q_0) p_m^{\delta-1}(n(x)) \Sigma \nabla p_m(n(x))$$

has a unique solution on  $[0, T_x)$  and  $\lim_{t \rightarrow T_x^-} \mu(t; x) = 0$ . In addition, the fluid limit  $\mathbb{Q}_x^\beta$  is deterministic on  $\mathbf{D}([0, T_x], \mathbf{X})$ , with support function  $\mu(\cdot; x)$ .

We may apply Theorem 1.8. From Theorem 2.1 and Proposition 2.14 we have the following.

THEOREM 2.15. *Consider the SRWM Markov chain with target distribution  $\pi$  on  $\mathcal{P}_\delta$  and increment distribution  $q$  having a moment of order  $p \geq 2$  and satisfying (17). Then, for any  $1 \leq u \leq p/(2 - m\delta)$ , the SRWM Markov chain is  $(f_u, r_u)$ -ergodic with*

$$f_u(x) = 1 + |x|^{p-u(2-m\delta)}, \quad r_u(t) \sim t^{u-1}.$$

EXAMPLE 3. Consider the subexponential Weibullian family derived from Example 1,

$$(26) \quad \pi(x_1, x_2) \propto (1 + x_1^2 + x_2^2 + x_1^8 x_2^2)^\delta \exp(-(x_1^2 + x_2^2)^\delta).$$

The contour curves are displayed in Figure 2. Since  $\pi \in \mathcal{P}_\delta$ , Proposition 2.14 shows that  $\pi$  is subexponential and regular in the tails with  $\beta = 1 - 2\delta$  and  $\ell_\infty(x) = -2\delta n(x)$ . Taking  $q \sim \mathcal{N}(0, \sigma^2 \text{Id})$ ,  $\Delta_\infty(x) = -\sigma^2 \delta n(x)$  and the (Carathéodory) solutions of the initial value problem  $\dot{\mu} = |\mu|^{-(1-2\delta)} \Delta_\infty(\mu)$ ,  $\mu(0) = x$  are given by  $\mu(t; x) = [|x|^{2(1-\delta)} - 2\sigma^2 \delta (1-\delta)t]^{0.5(1-\delta)^{-1}} n(x) \times \mathbb{1}_{|x|^{2(1-\delta)} - 2\sigma^2 \delta (1-\delta)t \geq 0}$ . Here, again, the gradient  $\ell(x)$  (even properly normalized) does not have a limit as  $|x| \rightarrow \infty$  along the  $x$ -axis, but the fluid limit model is simple to determine. Hence, the ergodicity of the SRWM sampler with target distribution (26) may be established [note that for this example, the theory developed in Fort and Moulines (2003) and Douc et al. (2004) does not apply]. The functions  $\Delta$  and  $\Delta_\infty$  are displayed in Figure 3. The flow of the initial value problem  $\dot{\mu} = h(\mu)$  for a set of initial conditions on the unit sphere in  $(0, \pi/2)$  is displayed in Figure 4, together with trajectories of the interpolated process.

EXAMPLE 4. Consider the mixture of bivariate Weibull distributions [see Patra and Dey (1999) for applications],

$$(27) \quad \begin{aligned} \pi(x) \propto & \alpha (x' \Gamma_1^{-1} x)^{\delta-1} \exp(-(1/2)(x' \Gamma_1^{-1} x)^\delta) \\ & + (1-\alpha) (x' \Gamma_2^{-1} x)^{\delta-1} \exp(-(1/2)(x' \Gamma_2^{-1} x)^\delta), \end{aligned}$$

where  $\Gamma_i$ ,  $i = \pm 1, 2$ , are defined in Example 2 and  $0 < \alpha < 1$ . Similarly to Example 2, the curvature of the contour curve on the diagonal tends to infinity; nevertheless, the target density is regular in the tails over  $\mathbf{O} = \mathbb{X} \setminus \{x = (x_1, x_2) \in \mathbb{R}^2, |x_1| = |x_2|\}$ . More precisely, we have the following.

LEMMA 2.16. *For any  $\varepsilon > 0$ , there exist  $M$  and  $K$  such that*

$$(28) \quad \sup_{|x| \geq K, ||x_1| - |x_2|| \geq M} ||x|^\beta \Delta(x) - \Delta_\infty(x)| \leq \varepsilon,$$

where  $\beta \stackrel{\text{def}}{=} 1 - 2\delta$  and  $\Delta_\infty(x) \stackrel{\text{def}}{=} -m_2(q_0) |x|^\beta \delta (x' \Gamma_2^{-1} x)^{\delta-1} \Sigma \Gamma_2^{-1} x$  if  $|x_1| > |x_2|$  and  $\Delta_\infty(x) \stackrel{\text{def}}{=} -m_2(q_0) |x|^\beta \delta (x' \Gamma_1^{-1} x)^{\delta-1} \Sigma \Gamma_1^{-1} x$  otherwise.

We can then establish the analog of Proposition 2.10 for the target distribution (27), again assuming that the proposal distribution  $q$  has compact support. The details are omitted for brevity. From the discussions above, the SRWM Markov chain with target distribution  $\pi$  given by (27) is  $(f_u, r_s)$ -ergodic with  $f_u(x) = 1 + |x|^u$  and  $r_s(t) \sim t^s$  for all  $u \geq 0$ ,  $s \geq 0$ .

**3. Conclusions.** ODE techniques provide a general and powerful approach to establishing stability and ergodic theorems for a Markov chain. In typical applications, the assumptions of this paper hold for *any*  $p > 0$  and,

consequently, the ergodic Theorem 1.4 asserts that the mean of *any* function with polynomial growth converges to its steady-state mean faster than *any* polynomial rate. The counterexample presented in Gamarnik and Meyn (2005) shows that, in general, it is impossible to obtain a geometric rate of convergence, even when  $\Delta$ ,  $\{\epsilon_k\}$  and the function  $f$  are bounded.

The ODE method developed within the queueing networks research community has undergone many refinements and has been applied in many very different contexts. Some of these extensions might serve well in other applications, such as MCMC. In particular, we should note the following points.

(i) Control variates have been proposed previously in MCMC to speed convergence and construct stopping rules [Robert (1998)]. The fluid model is a convenient tool for constructing control variates for application in the simulation of networks. The resulting simulators show dramatic performance improvements in numerical experiments: a hundredfold variance reduction is obtained in experiments presented in Henderson and Meyn (1997) and Henderson et al. (2003) based on marginal additional computational effort. Moreover, analytical results demonstrate that the asymptotic behavior of the controlled estimators are greatly improved [Meyn (2005, 2006, 2007)]. It is likely that both the theory and methodology can be extended to other applications.

(ii) A current focus of interest in the networks community is the reflected diffusion model obtained under a “heavy traffic scaling.” An analog of “heavy-traffic” in MCMC is the case  $\beta > 0$  considered in this paper; the larger scaling is necessary to obtain a nonstatic fluid limit (see Theorem 1.2). We have maintained  $\beta < 1$  in order to obtain a deterministic limit. With  $\beta = 1$ , we expect that a diffusion limit will be obtained for the scaled MH algorithm under general conditions. This will be an important tool in the subexponential case. In the fluid setting of this paper, when  $\beta > 0$ , it is necessary to assume a great deal of regularity on the densities  $\pi$  and  $q$  appearing in the MH algorithm to obtain a meaningful fluid limit model. We expect that very different regularity assumptions will be required to obtain a diffusion limit and that new insights will be obtained from properties of the resulting diffusion model.

#### 4. Proofs of the main results.

4.1. *State-dependent drift conditions.* In this section, we improve the state-dependent drift conditions proposed by Filonov (1989) for discrete state space and later extended by Meyn and Tweedie (1994) for general state space Markov chains [see also Meyn and Tweedie (1993) and Robert (2000) for additional references and comments].

Following Nummelin and Tuominen (1983), we denote by  $\Lambda$  the set of nondecreasing sequences  $r = \{r(n)\}_{n \in \mathbb{N}}$  satisfying  $\lim_{n \rightarrow \infty} \downarrow \log r(n)/n = 0$ ,

that is,  $\log r(n)/n$  converges to zero monotonically from above. A sequence  $r \in \Lambda$  is said to be *subgeometric*. Examples include polynomial sequences  $r(n) = (n+1)^\delta$  with  $\delta > 0$  and *truly* subexponential sequences,  $r(n) = (n+1)^\delta e^{cn^\gamma}$  [ $c > 0$  and  $\gamma \in (0, 1)$ ]. Denote by  $\mathcal{C}$  the set of functions

$$(29) \quad \mathcal{C} \stackrel{\text{def}}{=} \left\{ \phi: [1, \infty) \rightarrow \mathbb{R}^+, \phi \text{ is concave, monotone nondecreasing,} \right. \\ \left. \text{differentiable and } \inf_{\{v \in [1, \infty)\}} \phi(v) > 0, \lim_{v \rightarrow \infty} \phi'(v) = 0 \right\}.$$

For  $\phi \in \mathcal{C}$ , define  $H_\phi(v) \stackrel{\text{def}}{=} \int_1^v (1/\phi(x)) dx$ . The function  $H_\phi: [1, \infty) \rightarrow [0, \infty)$  is increasing and  $\lim_{v \rightarrow \infty} H_\phi(v) = \infty$ ; see [Douc et al. (2004), Section 2].

Define, for  $u \geq 0$ ,  $r_\phi(u) \stackrel{\text{def}}{=} \phi \circ H_\phi^{-1}(u) / \phi \circ H_\phi^{-1}(0)$ , where  $H_\phi^{-1}$  is the inverse of  $H_\phi$ . The function  $u \mapsto r_\phi(u)$  is log-concave and thus the sequence  $\{r_\phi(k)\}_{k \geq 0}$  is subgeometric. Polynomial functions  $\phi(v) = v^\alpha$ ,  $\alpha \in (0, 1)$  are associated with polynomial sequences  $r_\phi(k) = (1 + (1 - \alpha)k)^{\alpha/(1-\alpha)}$ .

**PROPOSITION 4.1.** *Let  $f: \mathsf{X} \rightarrow [1, \infty)$  and  $V: \mathsf{X} \rightarrow [1, \infty)$  be measurable functions,  $\varepsilon \in (0, 1)$  be a constant and  $C \in \mathcal{X}$  be a set. Assume that  $\sup_C f/V < \infty$  and that there exists a stopping time  $\tau \geq 1$  such that, for any  $x \notin C$ ,*

$$(30) \quad \mathbb{E}_x \left[ \sum_{k=0}^{\tau-1} f(\Phi_k) \right] \leq V(x) \quad \text{and} \quad \mathbb{E}_x[V(\Phi_\tau)] \leq (1 - \varepsilon)V(x).$$

*Then, for all  $x \notin C$ ,  $\mathbb{E}_x[\sum_{k=0}^{\tau_C} f(\Phi_k)] \leq (\varepsilon^{-1} \vee \sup_C f/V)V(x)$ . If, in addition, we assume that  $\sup_{x \in C} \{f(x) + \mathbb{E}_x[V(\Phi_1)]\} < \infty$ , then  $\sup_{x \in C} \mathbb{E}_x[\sum_{k=0}^{\tau_C} f(\Phi_k)] < \infty$ .*

**PROOF.** Set  $\tilde{\tau} \stackrel{\text{def}}{=} \tau \mathbb{1}_{C^c}(\Phi_0) + \mathbb{1}_C(\Phi_0)$  and define recursively the sequence  $\{\tau^n\}$  by  $\tau^0 \stackrel{\text{def}}{=} 0$ ,  $\tau^1 \stackrel{\text{def}}{=} \tilde{\tau}$  and  $\tau^n \stackrel{\text{def}}{=} \tau^{n-1} + \tilde{\tau} \circ \theta^{\tau^{n-1}}$ , where  $\theta$  is the shift operator. For any  $n \in \mathbb{N}$ , define by  $\bar{\Phi}_n = \Phi_{\tau^n}$  the chain sampled at the instants  $\{\tau^n\}_{n \geq 0}$ .  $\{\bar{\Phi}_n\}_{n \geq 0}$  is a Markov chain with transition kernel  $\bar{P}(x, A) \stackrel{\text{def}}{=} \mathbb{P}_x(\Phi_{\tilde{\tau}} \in A)$ ,  $x \in \mathsf{X}$ ,  $A \in \mathcal{X}$ . Equation (30) implies that

$$(31) \quad \bar{P}V(x) = \mathbb{E}_x[V(\Phi_{\tilde{\tau}})] \leq V(x) - F(x) \quad \text{for all } x \notin C,$$

where  $F(x) \stackrel{\text{def}}{=} \varepsilon \mathbb{E}_x[\sum_{k=0}^{\tilde{\tau}-1} f(\Phi_k)]$ . Let  $\bar{\tau}_C \stackrel{\text{def}}{=} \inf\{n \geq 1, \bar{\Phi}_n \in C\}$ . Applying the Markov property and the bound  $\tau_C \leq \tau^{\bar{\tau}_C}$ , we obtain, for all  $x \notin C$ ,

$$\begin{aligned} \mathbb{E}_x \left[ \sum_{k=0}^{\tau_C} f(\Phi_k) \right] &\leq \mathbb{E}_x \left[ \sum_{k=0}^{\bar{\tau}_C-1} \sum_{j=0}^{\tilde{\tau} \circ \theta^{\tau^k} - 1} f(\Phi_{j+\tau^k}) \right] + \mathbb{E}_x[f(\Phi_{\tau^{\bar{\tau}_C}}) \mathbb{1}_{\{\tau^{\bar{\tau}_C} < \infty\}}] \\ &\leq \varepsilon^{-1} \mathbb{E}_x \left[ \sum_{k=0}^{\bar{\tau}_C-1} F(\bar{\Phi}_k) \right] + \left( \sup_C \frac{f}{V} \right) \mathbb{E}_x[V(\Phi_{\tau^{\bar{\tau}_C}}) \mathbb{1}_{\{\tau^{\bar{\tau}_C} < \infty\}}]. \end{aligned}$$



Furthermore, (31) and the comparison theorem [Meyn and Tweedie (1993), Theorem 11.3.2] applied to the sampled chain  $\{\bar{\Phi}_n\}_{n \geq 0}$  yields

$$\mathbb{E}_x \left[ \sum_{k=0}^{\bar{\tau}_C-1} F(\bar{\Phi}_k) \right] + \mathbb{E}_x[V(\bar{\Phi}_{\bar{\tau}_C})\mathbb{1}_{\{\bar{\tau}_C < \infty\}}] \leq V(x), \quad x \notin C,$$

which concludes the proof of the first claim. The second claim follows by writing, for  $x \in C$ ,

$$\begin{aligned} \mathbb{E}_x \left[ \sum_{k=0}^{\tau_C} f(\Phi_k) \right] &\leq 2 \sup_C f + \mathbb{E}_x \left[ \mathbb{1}_{\{X_1 \notin C\}} \sum_{k=1}^{\tau_C} f(\Phi_k) \right] \\ &\leq 2 \sup_C f + \mathbb{E}_x \left[ \mathbb{1}_{\{X_1 \notin C\}} \mathbb{E}_{X_1} \left[ \sum_{k=0}^{\tau_C-1} f(\Phi_k) \right] \right] \\ &\leq 2 \sup_C f + \left( \varepsilon^{-1} \vee \sup_C f/V \right) \mathbb{E}_x[\mathbb{1}_{\{X_1 \notin C\}} V(X_1)]. \quad \square \end{aligned}$$

PROPOSITION 4.2. *Assume that the conditions of Proposition 4.1 are satisfied with  $f(x) = \phi \circ V(x)$  for  $x \notin C$  with  $\phi \in \mathcal{C}$ . Then, for  $x \notin C$ ,  $\mathbb{E}_x[\sum_{k=0}^{\tau_C-1} r_{\tilde{\phi}}(k)] \leq M^{-1}V(x)$  and  $\sup_{x \in C} \mathbb{E}_x[\sum_{k=0}^{\tau_C-1} r_{\tilde{\phi}}(k)] < \infty$ , where,*

$$(32) \quad \text{for all } t \quad \tilde{\phi}(t) \stackrel{\text{def}}{=} \phi(Mt) \quad \text{and} \quad M \stackrel{\text{def}}{=} \left[ \varepsilon^{-1} \vee \sup_C \phi \circ V/V \right]^{-1}.$$

PROOF. It is known that  $U(x) \stackrel{\text{def}}{=} \mathbb{E}_x[\sum_{k=0}^{\sigma_C} \phi \circ V(\Phi_k)]$ , where  $\sigma_C \stackrel{\text{def}}{=} \inf\{k \geq 0, \Phi_k \in C\}$ , solves the equations  $PU(x) = U(x) - \phi \circ V(x)$ ,  $x \notin C$  and  $U(x) = \phi \circ V(x)$ ,  $x \in C$  [see Meyn and Tweedie (1993), Theorem 14.2.3]. By Proposition 4.1,  $U(x) \leq M^{-1}V(x)$  for all  $x \notin C$ . Hence,

$$(33) \quad PU(x) \leq U(x) - \tilde{\phi} \circ U(x), \quad x \notin C.$$

From (33) and Douc et al. (2004), Proposition 2.2,  $\mathbb{E}_x[\sum_{k=0}^{\tau_C-1} r_{\tilde{\phi}}(k)] \leq U(x) \leq M^{-1}V(x)$ , for  $x \notin C$ . The proof is concluded by noting that for  $x \in C$ ,

$$\begin{aligned} \mathbb{E}_x \left[ \sum_{k=0}^{\tau_C-1} r_{\tilde{\phi}}(k) \right] &\leq r_{\tilde{\phi}}(0) + \mathbb{E}_x \left[ \mathbb{1}_{\{\Phi_1 \notin C\}} \sum_{k=1}^{\tau_C-1} r_{\tilde{\phi}}(k) \right] \\ &\leq r_{\tilde{\phi}}(0) + M^{-1} \sup_{x \in C} PV(x) < \infty. \quad \square \end{aligned}$$

THEOREM 4.3. *Suppose that  $\{\Phi_n\}_{n \geq 0}$  is a phi-irreducible and aperiodic Markov chain. Assume that there exist a function  $\phi \in \mathcal{C}$ , a measurable function  $V : \mathcal{X} \rightarrow [1, \infty)$ , a stopping time  $\tau \geq 1$ , a constant  $\varepsilon \in (0, 1)$  and a petite*



set  $C \subset \mathcal{X}$  such that

$$(34) \quad \mathbb{E}_x \left[ \sum_{k=0}^{\tau-1} \phi \circ V(\Phi_k) \right] \leq V(x), \quad x \notin C,$$

$$(35) \quad \mathbb{E}_x[V(\Phi_\tau)] \leq (1 - \varepsilon)V(x), \quad x \notin C,$$

$$(36) \quad \sup_C \{V + PV\} < \infty.$$

$P$  is then positive Harris recurrent with invariant probability  $\pi$  and:

- (1) for all  $x \in \mathbf{X}$ ,  $\lim_{n \rightarrow \infty} r_{\tilde{\phi}}(n) \|P^n(x, \cdot) - \pi\|_{\text{TV}} = 0$ , where  $\tilde{\phi}$  is defined in (32);
- (2) for all  $x \in \mathbf{X}$ ,  $\lim_{n \rightarrow \infty} \|P^n(x, \cdot) - \pi\|_{\phi \circ V} = 0$ ;
- (3) the fundamental kernel  $Z$  is a bounded linear transformation from  $L_\infty^{\phi \circ V}$  to  $L_\infty^V$ .

PROOF. (1–2) By Tuominen and Tweedie [(1994), Theorem 2.1], it is sufficient to prove that

$$\sup_{x \in C} \mathbb{E}_x \left[ \sum_{k=0}^{\tau_C-1} r_{\tilde{\phi}}(k) \right] < \infty, \quad \sup_{x \in C} \mathbb{E}_x \left[ \sum_{k=0}^{\tau_C-1} \phi \circ V(\Phi_k) \right] < \infty$$

and, for all  $x \in \mathbf{X}$ ,

$$\mathbb{E}_x \left[ \sum_{k=0}^{\tau_C-1} r_{\tilde{\phi}}(k) \right] < \infty, \quad \mathbb{E}_x \left[ \sum_{k=0}^{\tau_C-1} \phi \circ V(\Phi_k) \right] < \infty.$$

In Proposition 4.2 we show that the stated assumptions imply such bounds.

(3) By Glynn and Meyn (1996), Theorem 2.3, it is sufficient to prove that there exist constants  $b, c < \infty$  such that for all  $x \in \mathbf{X}$ ,  $PW(x) \leq W(x) - \phi \circ V(x) + b1_C(x)$ , with  $W(x) \leq cV(x)$ . This follows from Proposition 4.1, which shows that  $\sup_{x \in C} \mathbb{E}_x[\sum_{k=0}^{\tau_C} \phi \circ V(\Phi_k)] < \infty$  and  $\mathbb{E}_x[\sum_{k=0}^{\tau_C} \phi \circ V(\Phi_k)] \leq cV(x)$  for all  $x \notin C$ .  $\square$

Using an interpolation technique, we derive a rate of convergence associated with some  $g$ -norm,  $0 \leq g \leq \phi \circ V$ .

COROLLARY 4.4 (Theorem 4.3). *For any pair of functions  $(\alpha, \beta)$  satisfying  $\alpha(u)\beta(v) \leq u + v$ , for all  $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$  and all  $x \in \mathbf{X}$ ,*

$$\lim_n \alpha(r_{\tilde{\phi}}(n)) \|P^n(x, \cdot) - \pi\|_{\beta(\phi \circ V) \vee 1} = 0.$$

A pair of functions  $(\alpha, \beta)$  satisfying this condition can be constructed by using Young's inequality [Krasnosel'skij and Rutitskij (1961)].

4.2. *Proof of Theorem 1.2.* We preface the proof with a preparatory lemma. For any process  $\{\epsilon_k\}_{k \geq 1}$ , define

$$(37) \quad M_\infty(\epsilon, n) \stackrel{\text{def}}{=} \sup_{1 \leq l \leq n} \left| \sum_{k=1}^l \epsilon_k \right|.$$

LEMMA 4.5. *Assume B1 and B2.*

(i) *For all  $\kappa > 0$ ,  $J$  and  $K$  integers with  $J < K$ ,*

$$\begin{aligned} & \sup_{0 \leq k \leq k+j \leq K, 0 \leq j \leq J} |\Phi_{k+j} - \Phi_k| \\ & \leq 8M_\infty(\epsilon, K) + 2N(\beta, \Delta)\kappa^{-\beta}J + N(\beta, \Delta) + 2\kappa, \end{aligned}$$

where  $N(\beta, \Delta)$  is given in B2.

(ii) *For all  $0 \leq \alpha \leq \beta$  and all  $T > 0$ , there exists  $M$  such that*

$$\lim_{r \rightarrow \infty} \sup_{x \in X} \mathbb{P}_x \left( \sup_{0 \leq k \leq k+j \leq \lfloor Tr^{1+\alpha} \rfloor} |\Phi_{k+j} - \Phi_k| \geq Mr \right) = 0.$$

(iii) *For all  $T > 0$  and  $\varepsilon > 0$ , there exists  $\delta > 0$  such that*

$$\lim_{r \rightarrow \infty} \sup_{x \in X} \mathbb{P}_x \left( \sup_{0 \leq k \leq k+j \leq \lfloor Tr^{1+\beta} \rfloor, 0 \leq j \leq \lfloor \delta r^{1+\beta} \rfloor} |\Phi_{k+j} - \Phi_k| \geq \varepsilon r \right) = 0.$$

PROOF. (i) Let  $0 \leq j \leq J$  and  $0 \leq k \leq K - j$ . On the set  $\bigcap_{l=0}^{j-1} \{|\Phi_{k+l}| > \kappa\}$ ,

$$\begin{aligned} (38) \quad |\Phi_{k+j} - \Phi_k| &= \left| \sum_{l=k}^{k+j-1} \{\Phi_{l+1} - \Phi_l\} \right| \\ &\leq \left| \sum_{l=k+1}^{k+j} \epsilon_l \right| + \sum_{l=k}^{k+j-1} |\Delta(\Phi_l)| \\ &\leq \left| \sum_{l=k+1}^{k+j} \epsilon_l \right| + \sum_{l=k}^{k+j-1} |\Phi_l|^{-\beta} N(\beta, \Delta) \\ &\leq 2M_\infty(\epsilon, K) + J\kappa^{-\beta} N(\beta, \Delta). \end{aligned}$$

Consider now the case when  $|\Phi_{k+l}| \leq \kappa$  for some  $0 \leq l \leq j - 1$ . Define

$$\tau_j \stackrel{\text{def}}{=} \inf\{0 \leq l \leq j - 1, |\Phi_{k+l}| \leq \kappa\}$$

and

$$\sigma_j \stackrel{\text{def}}{=} \sup\{0 \leq l \leq j - 1, |\Phi_{k+l}| \leq \kappa\} + 1,$$

which are, respectively, the first hitting time and the last exit time before  $j$  of the ball of radius  $\kappa$ . Write  $\Phi_{k+j} - \Phi_k = (\Phi_{k+j} - \Phi_{k+\sigma_j}) + (\Phi_{k+\sigma_j} - \Phi_{k+\tau_j}) + (\Phi_{k+\tau_j} - \Phi_k)$  and consider the three terms separately. The first term is nonnull if  $\sigma_j < j$ ; hence,

$$\begin{aligned} |\Phi_{k+j} - \Phi_{k+\sigma_j}| &\leq \left| \sum_{l=k+\sigma_j+1}^{k+j} \epsilon_l \right| + \sum_{l=k+\sigma_j}^{k+j-1} |\Delta(\Phi_l)| \\ &\leq 2M_\infty(\epsilon, K) + J\kappa^{-\beta}N(\beta, \Delta) \end{aligned}$$

since, by the definition of  $\sigma_j$ ,  $|\Phi_{k+l}| > \kappa$  for all  $\sigma_j \leq l \leq j-1$ . Similarly, for the third term,

$$\begin{aligned} (39) \quad |\Phi_{k+\tau_j} - \Phi_k| &\leq \left| \sum_{l=k+1}^{k+\tau_j} \epsilon_l \right| + \sum_{l=k}^{k+\tau_j-1} |\Delta(\Phi_l)| \\ &\leq 2M_\infty(\epsilon, K) + J\kappa^{-\beta}N(\beta, \Delta) \end{aligned}$$

since, by the definition of  $\tau_j$ ,  $|\Phi_l| > \kappa$  for all  $0 \leq l < \tau_j$ . Finally, the second term is bounded by

$$\begin{aligned} |\Phi_{k+\sigma_j} - \Phi_{k+\tau_j}| &\leq |\Phi_{k+\sigma_j} - \Phi_{k+\sigma_j-1}| + |\Phi_{k+\sigma_j-1}| + |\Phi_{k+\tau_j}| \\ &\leq N(\beta, \Delta) + 2M_\infty(\epsilon, K) + 2\kappa. \end{aligned}$$

Combining the inequalities above yields the desired result.

(ii) From the previous inequality applied with  $\kappa = \ell r > 0$  and  $K = J = \lfloor Tr^{1+\alpha} \rfloor$ , it holds that

$$\begin{aligned} &\mathbb{P}_x \left( \sup_{0 \leq k \leq k+j \leq \lfloor Tr^{1+\alpha} \rfloor} |\Phi_{k+j} - \Phi_k| \geq 4Mr \right) \\ &\leq 4^p M^{-p} r^{-p} \sup_{x \in \mathbf{X}} \mathbb{E}_x [M_\infty^p(\epsilon, \lfloor Tr^{1+\alpha} \rfloor)] \\ &\quad + \mathbb{1}\{N(\beta, \Delta) \geq Mr\} + \mathbb{1}\{2N(\beta, \Delta)T \geq \ell^\beta Mr^{-\alpha+\beta}\} + \mathbb{1}\{2\ell \geq M\}. \end{aligned}$$

By Lemma A.1, the expectation tends to zero uniformly for  $x \in \mathbf{X}$ . The second term tends to zero when  $r \rightarrow \infty$ . The remaining two terms are zero with  $\ell$  and  $M$  chosen so that  $\ell^{1+\beta} > N(\beta, \Delta)T$  and  $M > 2\ell$ .

(iii) The proof follows similarly upon setting  $K = \lfloor Tr^{1+\beta} \rfloor$ ,  $J = \lfloor \delta r^{1+\beta} \rfloor$  and  $\kappa = \ell r$ .  $\square$

**PROOF OF THEOREM 1.2.** Let  $\alpha \leq \beta$ . A sequence of probability measures on  $\mathbf{D}(\mathbb{R}^+, \mathbf{X})$  is said to be  $\mathbf{D}(\mathbb{R}^+, \mathbf{X})$ -tight if it is tight in  $\mathbf{D}(\mathbb{R}^+, \mathbf{X})$  and if every weak limit of a subsequence is continuous. By [Billingsley (1999), Theorem 13.2, (13.7), page 140 and Corollary, page 142], the sequence of probability measures  $\{\mathbb{Q}_{r_n; x_n}^\alpha\}_{n \geq 0}$  is  $\mathbf{C}(\mathbb{R}_+, \mathbf{X})$ -tight if (a)  $\lim_{a \rightarrow \infty} \limsup_n \mathbb{Q}_{r_n; x_n}^\alpha \{\eta : |\eta(0)| \geq$

$a\} = 0$ , (b)  $\limsup_{n \rightarrow \infty} \mathbb{Q}_{r_n; x_n}^\alpha \{\eta : \sup_{0 \leq t \leq T} |\eta(t) - \eta(t-)| \geq a\} = 0$  and (c) for all  $\kappa > 0$  and  $\varepsilon > 0$ , there exist  $\delta \in (0, 1)$  such that  $\limsup_n \mathbb{Q}_{r_n; x_n}^\alpha \{\eta : w(\eta, \delta) \geq \varepsilon\} \leq \kappa$ , where  $w(\eta, \delta) \stackrel{\text{def}}{=} \sup_{0 \leq s \leq t \leq T, |t-s| \leq \delta} |\eta(t) - \eta(s)|$ . Properties (a)–(c) follow immediately from Lemma 4.5. Choose  $\alpha < \beta$ . Let  $\{r_n\}$  and  $\{x_n\}$  be sequences such that  $\lim_n r_n = \infty$  and  $\lim_n x_n = x$ . Let  $\varepsilon > 0$ . We have, for all  $n$  sufficiently large that  $|x_n - x| \leq \varepsilon/2$ ,

$$\mathbb{P}_{r_n x_n} \left( \sup_{0 \leq t \leq T} |\eta_{r_n}^\alpha(t; x_n) - x| \geq \varepsilon \right) \leq \mathbb{P}_{r_n x_n} \left( \sup_{0 \leq k \leq \lfloor T r_n^{1+\alpha} \rfloor} |\Phi_k - r_n x_n| \geq (\varepsilon/2) r_n \right)$$

and we have (b), again by Lemma 4.5(ii).  $\square$

4.3. *Proof of Theorem 1.4.* We preface the proof by establishing a uniform integrability condition for the martingale increment sequence  $\{\epsilon_k\}_{k \geq 1}$  and then for the Markov chain  $\{\Phi_k\}_{k \geq 0}$ .

LEMMA 4.6. *Assume B1. Then, for all  $T > 0$ ,*

$$(40) \quad \lim_{b \rightarrow \infty} \sup_{|x| \geq 1} |x|^{-p} \mathbb{E}_x [M_\infty^p(\epsilon, \lfloor T |\Phi_0|^{1+\beta} \rfloor) \mathbb{1} \{M_\infty(\epsilon, \lfloor T |\Phi_0|^{1+\beta} \rfloor) \geq b |\Phi_0|\}] = 0.$$

PROOF. Set  $T_{\Phi_0} \stackrel{\text{def}}{=} \lfloor T |\Phi_0|^{1+\beta} \rfloor$ . For  $K \geq 0$ , set  $\bar{\epsilon}_k \stackrel{\text{def}}{=} \epsilon_k \mathbb{1} \{|\epsilon_k| \leq K\}$  and  $\tilde{\epsilon}_k \stackrel{\text{def}}{=} \epsilon_k \mathbb{1} \{|\epsilon_k| \geq K\}$ . By Lemma A.2, there exists a constant  $C$  (depending only on  $p$ ) such that

$$\begin{aligned} & \mathbb{E}_x [M_\infty^p(\epsilon, T_{\Phi_0}) \mathbb{1} \{M_\infty(\epsilon, T_{\Phi_0}) \geq b |\Phi_0|\}] \\ & \leq C \mathbb{E}_x [M_\infty^p(\bar{\epsilon}, T_{\Phi_0}) \mathbb{1} \{M_\infty(\bar{\epsilon}, T_{\Phi_0}) \geq (b/2) |\Phi_0|\}] + C \mathbb{E}_x [M_\infty^p(\tilde{\epsilon}K, T_{\Phi_0})]. \end{aligned}$$

Consider the first term on the right-hand side of the previous inequality. Using Lemma A.3 with  $a > 1 \vee 2/p$  and Lemma A.1 yields

$$\begin{aligned} & |x|^{-p} \mathbb{E}_x [M_\infty^p(\bar{\epsilon}, \lfloor T |\Phi_0|^{1+\beta} \rfloor) \mathbb{1} \{M_\infty(\bar{\epsilon}, T_{\Phi_0}) \geq (b/2) |\Phi_0|\}] \\ & \leq (b/2)^{-(a-1)p} |x|^{-ap} \mathbb{E}_x [M_\infty^{ap}(\bar{\epsilon}, T_{\Phi_0})] \leq CA(\bar{\epsilon}, ap) b^{-(a-1)p} |x|^{-a(1-\beta)p/2}, \end{aligned}$$

where  $A(\bar{\epsilon}, ap) \stackrel{\text{def}}{=} \sup_{x \in X} \mathbb{E}_x [|\bar{\epsilon}_1|^{ap}]$ . Note that, by construction,  $A(\bar{\epsilon}, ap) \leq K^{ap}$ . Similarly, Lemma A.1 implies that  $\mathbb{E}_x [M_\infty^p(\tilde{\epsilon}K, T_{\Phi_0})] \leq CA(\tilde{\epsilon}K, p) \times T^{p/2} |x|^{\{p(1+\beta)/2\} \vee (1+\beta)}$ , where  $A(\tilde{\epsilon}K, p) \stackrel{\text{def}}{=} \sup_{x \in X} \mathbb{E}_x [|\tilde{\epsilon}_1|^p]$ . Therefore, since  $p \geq 1 + \beta$ ,  $\sup_{|x| \geq 1} |x|^{-p} \mathbb{E}_x [M_\infty^p(\tilde{\epsilon}K, T_{\Phi_0})] \leq CT^{p/2} A(\tilde{\epsilon}K, p)$ . Combining the two last inequalities, we have

$$\begin{aligned} & \sup_{|x| \geq 1} |x|^{-p} \mathbb{E}_x [M_\infty^p(\epsilon, T_{\Phi_0}) \mathbb{1} \{M_\infty(\epsilon, T_{\Phi_0}) \geq b |\Phi_0|\}] \\ & \leq C \{K^{ap} b^{-p(a-1)} + A(\tilde{\epsilon}K, p)\}, \end{aligned}$$

which goes to 0 by setting  $K \stackrel{\text{def}}{=}} K(b) = \log(b)$ .  $\square$

PROPOSITION 4.7. *Assume B1 and B2. Then, for all  $T > 0$ ,*

$$(41) \quad \sup_{x \in X} (1 + |x|)^{-p} \mathbb{E}_x \left[ \sup_{0 \leq k \leq \lfloor T|\Phi_0|^{1+\beta} \rfloor} |\Phi_k|^p \right] < \infty,$$

$$(42) \quad \lim_{K \rightarrow \infty} \sup_{|x| \geq 1} |x|^{-p} \mathbb{E}_x \left[ \sup_{0 \leq k \leq \lfloor T|\Phi_0|^{1+\beta} \rfloor} |\Phi_k|^p \right. \\ \left. \times \mathbb{1} \left\{ \sup_{0 \leq k \leq \lfloor T|\Phi_0|^{1+\beta} \rfloor} |\Phi_k| \geq K|\Phi_0| \right\} \right] = 0.$$

PROOF. Set  $T_{\Phi_0} = \lfloor T|\Phi_0|^{1+\beta} \rfloor$ . For all  $r \geq 1$ , applying Lemma 4.5(i) with  $K = J = \lfloor T|\Phi_0|^{1+\beta} \rfloor$  and  $\kappa = |\Phi_0|$  yields

$$(43) \quad \sup_{0 \leq k \leq T_{\Phi_0}} |\Phi_k|^r \leq C \{1 + |\Phi_0|^r + M_\infty^r(\epsilon, T_{\Phi_0})\}$$

for some constant  $C$  depending upon  $r, \beta, N(\beta, \Delta)$  and  $T$ . The first assertion is then a consequence of Lemma A.1. Inequality (43) applied with  $r = 1$  implies that there exist constants  $a, b > 0$  such that for all  $|x| \geq 1$  and all large enough  $K$ ,

$$\left\{ \sup_{0 \leq k \leq T_{\Phi_0}} |\Phi_k| \geq K|\Phi_0| \right\} \subset \{M_\infty(\epsilon, T_{\Phi_0}) \geq (aK - b)|\Phi_0|\} \quad \mathbb{P}_x\text{-a.s.}$$

Hence, for large enough  $K$  and an appropriately chosen constant  $C$ ,

$$\sup_{|x| \geq 1} |x|^{-p} \mathbb{E}_x \left[ \sup_{0 \leq k \leq T_{\Phi_0}} |\Phi_k|^p \mathbb{1} \left\{ \sup_{0 \leq k \leq T_{\Phi_0}} |\Phi_k| \geq K|\Phi_0| \right\} \right] \\ \leq C \sup_{|x| \geq 1} \mathbb{P}_x [M_\infty(\epsilon, T_{\Phi_0}) \geq (aK - b)|\Phi_0|] \\ + C \sup_{|x| \geq 1} |x|^{-p} \mathbb{E}_x [M_\infty^p(\epsilon, T_{\Phi_0}) \mathbb{1} \{M_\infty(\epsilon, T_{\Phi_0}) \geq (aK - b)|\Phi_0|\}].$$

The proof of (42) follows from Lemma 4.6.  $\square$

PROPOSITION 4.8. *Assume B1 and B2 and that there exist  $T < \infty$  and  $\rho \in (0, 1)$  such that*

$$(44) \quad \limsup_{|x| \rightarrow \infty} \mathbb{P}_x(\sigma > \tau) = 0, \quad \text{with } \sigma \stackrel{\text{def}}{=} \inf \{k \geq 0, |\Phi_k| < \rho|\Phi_0|\},$$

where  $\tau \stackrel{\text{def}}{=} \sigma \wedge \lceil T|\Phi_0|^{1+\beta} \rceil$ . It then follows that (a) there exists  $M$  such that  $\sup_{|x| \geq M} |x|^{-p} \mathbb{E}_x [|\Phi_\tau|^p] < 1$  and

$$(b) \quad \mathbb{E}_x \left[ \sum_{k=0}^{\tau-1} |\Phi_k|^p \right] \leq C|x|^{p+1+\beta}.$$

PROOF. Set  $T_{\Phi_0} = \lceil T|\Phi_0|^{1+\beta} \rceil$ . For any  $K \geq 0$ ,

$$\begin{aligned}
& |x|^{-p} \mathbb{E}_x[|\Phi_\tau|^p] \\
(45) \quad &= |x|^{-p} \mathbb{E}_x[\mathbb{1}\{\tau = \sigma\} |\Phi_\tau|^p] + |x|^{-p} \mathbb{E}_x[\mathbb{1}\{\sigma > T_{\Phi_0}\} |\Phi_{T_{\Phi_0}}|^p] \\
&\leq \rho^p + |x|^{-p} \mathbb{E}_x[|\Phi_{T_{\Phi_0}}|^p \mathbb{1}\{|\Phi_{T_{\Phi_0}}| \geq K|\Phi_0|\}] + K^p \mathbb{P}_x[\sigma > T_{\Phi_0}].
\end{aligned}$$

By Proposition 4.7, one may choose  $K$  sufficiently large so that

$$(46) \quad \sup_{|x| \geq 1} |x|^{-p} \mathbb{E}_x[|\Phi_{T_{\Phi_0}}|^p \mathbb{1}\{|\Phi_{T_{\Phi_0}}| \geq K|\Phi_0|\}] < 1 - \rho^p.$$

Since  $\limsup_{|x| \rightarrow \infty} \mathbb{P}_x[\sigma > T_{\Phi_0}] = 0$ , the proof of (a) follows. Since  $\tau \leq T_{\Phi_0}$ , (b) follows from (41) and the bound  $\mathbb{E}_x[\sum_{k=0}^{\tau-1} |\Phi_k|^p] \leq CT|x|^{1+\beta} \times \mathbb{E}_x[\sup_{1 \leq k \leq T_{\Phi_0}} |\Phi_k|^p]$ .  $\square$

The following elementary proposition relates the stability of the fluid limit model to the condition (44) on the stopping time  $\sigma$ . We introduce the polygonal process that agrees with  $\Phi_k/r$  at the points  $t = kr^{-(1+\alpha)}$  and is defined by linear interpolation

$$\begin{aligned}
(47) \quad \tilde{\eta}_r^\alpha(t; x) &= r^{-1} \sum_{k \geq 0} \{(k+1 - tr^{1+\alpha})\Phi_k + (tr^{1+\alpha} - k)\Phi_{k+1}\} \\
&\quad \times \mathbb{1}\{k \leq tr^{1+\alpha} < (k+1)\}.
\end{aligned}$$

Denote by  $\tilde{\mathbb{Q}}_{r;x}^\alpha$  the image probability on  $\mathcal{C}(\mathbb{R}^+, \mathbf{X})$  of  $\mathbb{P}_{rx}$  by  $\tilde{\eta}_r^\alpha(t; x)$ . The introduction of this process allows for an easier characterization of the open and closed sets of  $\mathcal{C}([0, T], \mathbf{X})$  equipped with the uniform topology, than the open and closed sets of  $\mathcal{D}([0, T], \mathbf{X})$  equipped with the Skorokhod topology. For any sequences  $\{r_n\}_n \subset \mathbb{R}^+$  such that  $r_n \rightarrow +\infty$  and  $\{x_n\} \subset \mathbf{X}$  such that  $x_n \rightarrow x$ , the family of probability measures  $\{\tilde{\mathbb{Q}}_{r_n;x_n}^\alpha\}$  is tight and converges weakly to  $\mathbb{Q}_x^\alpha$ , the weak limit of the sequence  $\{\mathbb{Q}_{r_n;x_n}^\alpha\}_{n \in \mathbb{N}}$ . This can be proved following the same lines as in the proof of Theorem 1.2 [see, e.g., Billingsley (1999), Theorem 7.3]. Details are omitted.

PROPOSITION 4.9. *Assume B1, B2 and that the  $\beta$ -fluid limit model  $\{\mathbb{Q}_x^\beta, x \in \mathbf{X}\}$  is stable. Then, (44) is satisfied.*

PROOF. Let  $\{y_n\} \subset \mathbf{X}$  be any sequence of initial states with  $|y_n| \rightarrow \infty$  as  $n \rightarrow \infty$ . Set  $r_n \stackrel{\text{def}}{=} |y_n|$  and  $x_n \stackrel{\text{def}}{=} y_n/|y_n|$ . One may extract a subsequence  $\{x_{n_j}\} \subseteq \{x_n\}$  such that  $\lim_{j \rightarrow \infty} x_{n_j} = x$  for some  $x$ ,  $|x| = 1$ . By Theorem 1.2, there exist subsequences  $\{r_{m_j}\} \subseteq \{r_{n_j}\}$  and  $\{x_{m_j}\} \subseteq \{x_{n_j}\}$  and a  $\beta$ -fluid

limit  $\mathbb{Q}_x^\beta$  such that  $\tilde{\mathbb{Q}}_{r_{m_j}; x_{m_j}}^\beta \Rightarrow \mathbb{Q}_x^\beta$ . By construction,

$$\begin{aligned} \mathbb{P}_{r_{m_j}; x_{m_j}}(\sigma > \tau) &\leq \mathbb{P}_{r_{m_j}; x_{m_j}}\left(\inf_{0 \leq t \leq T} |\tilde{\eta}_{r_{m_j}}^\beta(t; x_{m_j})| \geq \rho\right) \\ &= \tilde{\mathbb{Q}}_{r_{m_j}; x_{m_j}}^\beta\left(\eta \in \mathcal{C}(\mathbb{R}^+, \mathbf{X}) : \inf_{0 \leq t \leq T} |\eta(t)| \geq \rho\right). \end{aligned}$$

By the Portmanteau theorem, since the set  $\{\eta \in \mathcal{C}(\mathbb{R}^+, \mathbf{X}), \inf_{[0, T]} |\eta| \geq \rho\}$  is closed, we have

$$\limsup_{j \rightarrow \infty} \tilde{\mathbb{Q}}_{r_{m_j}; x_{m_j}}^\beta\left(\inf_{0 \leq t \leq T} |\eta(t)| \geq \rho\right) \leq \mathbb{Q}_x^\beta\left(\inf_{0 \leq t \leq T} |\eta(t)| \geq \rho\right) = 0.$$

Because  $\{y_n\}$  is an arbitrary sequence, this relation implies (44).  $\square$

PROOF OF THEOREM 1.4. This follows immediately from Theorem 4.3, using Propositions 4.8 and 4.9.  $\square$

4.4. *Proof of Proposition 1.5.* In this proof, we see the  $\beta$ -fluid limit  $\mathbb{Q}_x^\beta$  as the weak limit of  $\tilde{\mathbb{Q}}_{r_n; x_n}^\beta$  for some sequences  $\{r_n\} \subset \mathbb{R}_+$  and  $\{x_n\} \subset \mathbf{X}$  satisfying  $\lim_{n \rightarrow \infty} r_n = \infty$  and  $\lim_{n \rightarrow \infty} x_n = x$ . Fix  $s, t$  such that  $s < t$ . We prove that

$$(48) \quad \mathbb{Q}_x^\beta\left(\mathbf{A}(s, t) \cap \left\{\eta \in \mathcal{C}([s, t], \mathbf{X}) : \sup_{s \leq u \leq t} \left|\eta(u) - \eta(s) - \int_s^u h \circ \eta(y) dy\right| > 0\right\}\right) = 0.$$

Let  $\mathbf{U}$  be an open set such that  $\bar{\mathbf{U}} \subseteq \mathbf{O}$ , where  $\bar{\mathbf{U}}$  denotes the closure of the set  $\mathbf{U}$ . For any  $\delta > 0$ ,  $M > 0$  and  $m > 0$ ,  $s \leq u < w \leq t$ , define

$$(49) \quad \mathbf{A}_{\delta, m, M}^{\mathbf{U}}(u, w) \stackrel{\text{def}}{=} \left\{\eta \in \mathcal{C}([s, t], \mathbf{X}), \eta([u, w]) \subset \mathbf{U} \cap \mathbf{C}_{m, M}, \sup_{u \leq v \leq w} \left|\eta(v) - \eta(u) - \int_u^v h \circ \eta(x) dx\right| > \delta\right\},$$

where  $\mathbf{C}_{m, M} \stackrel{\text{def}}{=} \{x \in \mathbf{X}, m \leq |x| \leq M\}$ . Since  $\delta$ ,  $m$ ,  $M$ ,  $\mathbf{U}$ ,  $u$  and  $w$  are arbitrary, (48) holds whenever  $\mathbb{Q}_x^\beta[\mathbf{A}_{\delta, m, M}^{\mathbf{U}}(u, w)] = 0$ . By the Portmanteau theorem, since the set  $\mathbf{A}_{\delta, m, M}^{\mathbf{U}}(u, w)$  is open in the uniform topology,

$$\mathbb{Q}_x^\beta[\mathbf{A}_{\delta, m, M}^{\mathbf{U}}(u, w)] \leq \liminf_{n \rightarrow \infty} \tilde{\mathbb{Q}}_{r_n; x_n}^\beta[\mathbf{A}_{\delta, m, M}^{\mathbf{U}}(u, w)]$$

and the property will follow if we can prove that the right-hand side of the

previous inequality is null. To that end, we write

$$\begin{aligned}
& \tilde{\eta}_{r_n}^\beta(v; x_n) - \tilde{\eta}_{r_n}^\beta(u; x_n) - \int_u^v h \circ \tilde{\eta}_{r_n}^\beta(y; x_n) dy \\
&= \tilde{\eta}_{r_n}^\beta(v; x_n) - \tilde{\eta}_{r_n}^\beta(\lfloor vr_n^{1+\beta} \rfloor r_n^{-(1+\beta)}; x_n) \\
&\quad + \tilde{\eta}_{r_n}^\beta(\lfloor ur_n^{1+\beta} \rfloor r_n^{-(1+\beta)}; x_n) - \tilde{\eta}_{r_n}^\beta(u; x_n) \\
&\quad + r_n^{-1} \sum_{k=\lfloor ur_n^{1+\beta} \rfloor}^{\lfloor vr_n^{1+\beta} \rfloor - 1} \{\Phi_{k+1} - \Phi_k\} - \int_u^v h \circ \tilde{\eta}_{r_n}^\beta(t; x_n) dt \\
&\leq 2\chi_1 + \chi_2 + \chi_3 + 2r_n^{-1} M_\infty(\epsilon, \lfloor tr_n^{1+\beta} \rfloor),
\end{aligned}$$

where we have defined

$$\begin{aligned}
\chi_1 &\stackrel{\text{def}}{=} \sup_{u \leq v \leq w} \left\{ \left| \tilde{\eta}_{r_n}^\beta(v; x_n) - \tilde{\eta}_{r_n}^\beta(\lfloor vr_n^{1+\beta} \rfloor r_n^{-(1+\beta)}; x_n) \right. \right. \\
&\quad \left. \left. + \left| \int_{\lfloor vr_n^{1+\beta} \rfloor r_n^{-(1+\beta)}}^v h \circ \tilde{\eta}_{r_n}^\beta(t; x_n) dt \right| \right\}, \\
\chi_2 &= \sum_{j=\lfloor ur_n^{1+\beta} \rfloor}^{\lfloor wr_n^{1+\beta} \rfloor - 1} |r_n^{-1} \Delta(r_n \tilde{\eta}_{r_n}^\beta(jr_n^{-(1+\beta)}); x_n) - r_n^{-(1+\beta)} h(\tilde{\eta}_{r_n}^\beta(jr_n^{-(1+\beta)})x_n)|, \\
\chi_3 &= \sum_{j=\lfloor ur_n^{1+\beta} \rfloor}^{\lfloor wr_n^{1+\beta} \rfloor - 1} \left| r_n^{-(1+\beta)} h(\tilde{\eta}_{r_n}^\beta(jr_n^{-(1+\beta)}); x_n) - \int_{jr_n^{-(1+\beta)}}^{(j+1)r_n^{-(1+\beta)}} h \circ \tilde{\eta}_{r_n}^\beta(t; x_n) dt \right|.
\end{aligned}$$

Denote by  $\omega_{m,M,U}$  the modulus of continuity of  $h$  on  $U \cap \mathcal{C}_{m,M}$ . Since  $h$  is continuous on  $U$ ,  $\lim_{\lambda \rightarrow 0} \omega_{m,M,U}(\lambda) = 0$ . On the event  $\{\tilde{\eta}_{r_n}^\beta(t; x_n) \in U \cap \mathcal{C}_{m,M}\}$ ,

$$\begin{aligned}
\chi_1 &\leq r_n^{-1} \left( 1 + \sup_{|x| \geq m} |h(x)| \right) \sup_{1 \leq j \leq \lfloor tr_n^{1+\beta} \rfloor} |\Phi_{j+1} - \Phi_j|, \\
\chi_2 &\leq (t-s+1)m^{-\beta} \sup_{\{x \in U, |x| \geq m\}} |r_n^\beta |x|^\beta \Delta(r_n x) - \Delta_\infty(x)|
\end{aligned}$$

and, for any  $\lambda > 0$ ,

$$\chi_3 \leq (t-s+1) \left( \omega_{m,M,U}(\lambda) + \sup_{|x| \geq m} |h(x)| \mathbb{1} \left\{ \sup_{1 \leq j \leq \lfloor tr_n^{1+\beta} \rfloor} |\Phi_{j+1} - \Phi_j| \geq \lambda r_n \right\} \right).$$

By Lemma 4.5, for any  $\delta > 0$ ,  $\lim_{n \rightarrow \infty} \mathbb{P}_{r_n x_n}(\sup_{1 \leq j \leq \lfloor tr_n^{1+\beta} \rfloor} |\Phi_{j+1} - \Phi_j| \geq \delta r_n) = 0$ . On the other hand,  $\lim_{n \rightarrow \infty} \sup_{\{x \in U, |x| \geq mr_n\}} |x|^\beta \Delta(x) - \Delta_\infty(x) = 0$ . Therefore, for any  $\delta > 0$ , one may choose  $\lambda$  small enough so that

$$\lim_{n \rightarrow \infty} \mathbb{P}_{r_n x_n}(\tilde{\eta}_{r_n}^\beta(t; x_n) \in U \cap \mathcal{C}_{m,M}, (2\chi_1 + \chi_2 + \chi_3) \geq \delta) = 0.$$



The proof follows from Lemma A.1.

4.5. *Proof of Theorem 1.6.* We preface the proof by a lemma showing that the fluid limits are uniformly bounded.

LEMMA 4.10. *Assume B1 and B2.*

(i) *For any  $T > 0$  and  $\rho > 0$ , there exists  $\delta > 0$  such that, for any  $\beta$ -fluid limit  $\mathbb{Q}_x^\beta$ ,*

$$(50) \quad \mathbb{Q}_x^\beta \left( \eta \in \mathcal{C}(\mathbb{R}^+, \mathbf{X}), \sup_{0 \leq t \leq u \leq t + \delta \leq T} |\eta(u) - \eta(t)| \leq \rho \right) = 1.$$

(ii) *For any  $T > 0$ , there exists  $K > 0$  such that, for any  $\beta$ -fluid limit  $\mathbb{Q}_x^\beta$ ,*

$$(51) \quad \mathbb{Q}_x^\beta \left( \eta \in \mathcal{C}(\mathbb{R}^+, \mathbf{X}), \sup_{0 \leq t \leq T} |\eta(t) - \eta(0)| \geq K \right) = 0.$$

PROOF. (i) Let  $\{r_n\} \subset \mathbb{R}_+$  and  $\{x_n\} \subset \mathbf{X}$  be two sequences such that  $\lim_{n \rightarrow \infty} r_n = +\infty$ ,  $\lim_{n \rightarrow \infty} x_n = x$  and  $\mathbb{Q}_{r_n; x_n}^\beta \Rightarrow \mathbb{Q}_x^\beta$ . By the Portmanteau theorem, since the set  $\{\eta \in \mathcal{C}(\mathbb{R}^+, \mathbf{X}), \sup_{0 \leq t \leq u \leq t + \delta \leq T} |\eta(u) - \eta(t)| \leq \rho\}$  is closed, it follows that

$$\begin{aligned} & \mathbb{Q}_x^\beta \left( \eta \in \mathcal{C}(\mathbb{R}^+, \mathbf{X}), \sup_{0 \leq t \leq u \leq t + \delta \leq T} |\eta(u) - \eta(t)| \leq \rho \right) \\ & \geq \limsup_n \tilde{\mathbb{Q}}_{r_n; x_n}^\beta \left( \eta \in \mathcal{C}(\mathbb{R}^+, \mathbf{X}), \sup_{0 \leq t \leq u \leq t + \delta \leq T} |\eta(u) - \eta(t)| \leq \rho \right). \end{aligned}$$

By definition of the process  $\tilde{\eta}_{r_n}^\beta(\cdot; x_n)$ ,

$$\begin{aligned} & \tilde{\mathbb{Q}}_{r_n; x_n}^\beta \left( \eta \in \mathcal{C}(\mathbb{R}^+, \mathbf{X}), \sup_{0 \leq t \leq u \leq t + \delta \leq T} |\eta(u) - \eta(t)| > \rho \right) \\ & \leq \mathbb{P}_{r_n x_n} \left( \sup_{0 \leq k < k + j \leq T r_n^{1+\beta}, 0 \leq j \leq \delta r_n^{1+\beta}} |\Phi_{k+j} - \Phi_k| > \rho r_n \right) \end{aligned}$$

and the proof follows from Lemma 4.5(iii).

(ii) The proof follows from (i) by considering the decomposition

$$\sup_{0 \leq t \leq T} |\eta(t) - \eta(0)| \leq \sum_{q=0}^{\lfloor T/\delta \rfloor} \sup_{q\delta \leq u \leq (q+1)\delta} |\eta(u) - \eta(q\delta)|. \quad \square$$

PROOF OF THEOREM 1.6. Under the stated assumptions,  $\mu([0, T_x]; \cdot)$  is a compact subset of  $\mathcal{O}$ . Since  $\mathcal{O}$  is open, there exists  $\rho > 0$  such that

$$\{y \in \mathbf{X}, d(y, \mu([0, T_x]; x)) \leq 2\rho\} \subset \mathcal{O},$$

where, for  $x \in \mathbf{X}$  and  $A \subset \mathbf{X}$ ,  $d(x, A)$  is the distance from  $x$  to the set  $A$ . By Lemma 4.10(i), there exists  $\delta > 0$  such that

$$\mathbb{Q}_x^\beta \left( \eta \in \mathbf{C}(\mathbb{R}_+, \mathbf{X}), \sup_{0 \leq t \leq u \leq t + \delta \leq T_x} |\eta(u) - \eta(t)| \leq \rho \right) = 1.$$

Since  $\mathbb{Q}_x^\beta(\eta \in \mathbf{C}(\mathbb{R}_+, \mathbf{X}), \eta(0) = x = \mu(0; x)) = 1$ , we have

$$\mathbb{Q}_x^\beta(\eta \in \mathbf{C}(\mathbb{R}_+, \mathbf{X}), \eta([0, \delta]) \subset \mathbf{O}) = 1.$$

By Proposition 1.5, this yields  $\mathbb{Q}_x^\beta = \delta_{\mu(\cdot; x)}$  on  $\mathbf{C}([0, \delta], \mathbf{X})$ . By repeated application of Lemma 4.10(i), it is readily proved by induction that  $\mathbb{Q}_x^\beta = \delta_{\mu(\cdot; x)}$  on  $\mathbf{C}([(q-1)\delta, q\delta] \cap [0, T_x], \mathbf{X})$  for any integer  $q \geq 1$ .  $\square$

4.6. *Proof of Theorem 1.9.* Let  $x$  be such that  $|x| = 1$ . By Lemma 4.10, there exists  $K$  depending on  $T_0$  such that  $\mathbb{Q}_x^\beta(\eta: \sup_{[0, T_0]} |\eta(\cdot)| \leq K) = 1$  for any  $\beta$ -fluid limit  $\mathbb{Q}_x^\beta$ . Set  $T = T_0 + T_K$ , where  $T_0$  and  $T_K$  are defined by (11) and (12), respectively.

By definition, for any set  $\mathbf{H}$ ,  $\mathbf{H} \subset \Omega_{\mathbf{H}}$ ; therefore, there exists an increasing sequence  $\{\mathbf{H}_n\}$  of compact subsets of  $\mathbf{O}$  such that  $\mathbf{H}_n \subsetneq \mathbf{H}_{n+1}$  and  $\mathbf{O} = \bigcup_n \Omega_{\mathbf{H}_n}$  (note that  $\Omega_{\mathbf{H}_n} \subseteq \Omega_{\mathbf{H}_{n+1}}$ ). This implies that

$$\begin{aligned} & \mathbb{Q}_x^\beta \left( \eta: \inf_{[0, T]} |\eta(\cdot)| > \rho_K \right) \\ &= \mathbb{Q}_x^\beta \left( \eta: \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \eta([0, T_0]) \cap \mathbf{O} \neq \emptyset \right) \\ &= \lim \uparrow_n \mathbb{Q}_x^\beta \left( \eta: \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \eta([0, T_0]) \cap \Omega_{\mathbf{H}_n} \neq \emptyset \right). \end{aligned}$$

$\lim \uparrow_n$  stands for a limit that converges monotonically from below. We prove that for any  $n$ , the term in the right-hand side is zero. To that end we start by proving that for any compact set  $\mathbf{H} \subset \mathbf{O}$  and any real numbers  $0 \leq q \leq T_0$ ,

$$\begin{aligned} & \mathbb{Q}_x^\beta \left( \eta: \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \eta(q) \in \Omega_{\mathbf{H}} \right) \\ (52) \quad &= \mathbb{Q}_x^\beta \left( \eta: \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \eta(q + \cdot) = \mu(\cdot; \eta(q)) \right. \\ & \quad \left. \text{on } [0, T_{\eta(q)}], \eta(q) \in \Omega_{\mathbf{H}} \right). \end{aligned}$$

We will then establish that

$$\begin{aligned} & \mathbb{Q}_x^\beta \left( \eta: \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \eta(q + \cdot) = \mu(\cdot; \eta(q)) \right. \\ (53) \quad & \quad \left. \text{on } [0, T_{\eta(q)}], \eta(q) \in \Omega_{\mathbf{H}} \right) = 0. \end{aligned}$$

Since  $\mathbb{Q}_x^\beta(\mathbf{C}(\mathbb{R}^+, \mathbf{X})) = 1$ , (52) and (53) imply that

$$\begin{aligned} & \mathbb{Q}_x^\beta \left( \eta : \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \eta([0, T_0]) \cap \Omega_{\mathbf{H}_n} \neq \emptyset \right) \\ & \leq \sum_{q \in \mathcal{Q}} \mathbb{Q}_x^\beta \left( \eta : \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \eta(q + \cdot) = \mu(\cdot; \eta(q)) \right. \\ & \quad \left. \text{on } [0, T_{\eta(q)}], \eta(q) \in \Omega_{\mathbf{H}'_n} \right) = 0, \end{aligned}$$

where  $\mathbf{H}'_n \supset \mathbf{H}_n$  is a compact set of  $\mathbf{O}$  and  $\mathcal{Q} \subset [0, T_0]$  is a denumerable dense set. This concludes the proof.

We now turn to the proof of (52) and (53). Since  $\Omega_{\mathbf{H}}$  is a compact set of  $\mathbf{O}$ , there exists  $\varepsilon > 0$  (depending on  $\mathbf{H}$ ) such that  $\{y \in \mathbf{X}, d(y, \Omega_{\mathbf{H}}) \leq 2\varepsilon\} \subsetneq \mathbf{O}$ . By Lemma 4.10, one may choose  $\delta > 0$  small enough (depending on  $T$  and  $\varepsilon$ ) so that

$$\mathbb{Q}_x^\beta \left( \eta \in \mathbf{C}(\mathbb{R}^+, \mathbf{X}) : \sup_{0 \leq t \leq u \leq t + \delta \leq T} |\eta(u) - \eta(t)| \leq \varepsilon \right) = 1.$$

Therefore, for any compact set  $\mathbf{H} \subset \mathbf{O}$  and  $q \in \mathcal{Q}$ ,

$$\begin{aligned} & \mathbb{Q}_x^\beta \left( \eta : \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \eta(q) \in \Omega_{\mathbf{H}} \right) \\ & = \mathbb{Q}_x^\beta \left( \eta : \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \eta(q) \in \Omega_{\mathbf{H}}, \sup_{0 \leq t \leq u \leq t + \delta \leq T} |\eta(u) - \eta(t)| \leq \varepsilon \right) \\ & = \mathbb{Q}_x^\beta \left( \eta : \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \eta(q) \in \Omega_{\mathbf{H}}, \eta([q, (q + \delta) \wedge T]) \subset \mathbf{O} \right). \end{aligned}$$

By Proposition 1.5, on the set  $\mathbf{A}(q, q + \delta)$ ,  $\eta(q + \cdot) = \mu(\cdot; \eta(q))$  on  $[0, \delta \wedge T_{\eta(q)}]$ ,  $\mathbb{Q}_x^\beta$ -a.s. Hence,

$$\begin{aligned} & \mathbb{Q}_x^\beta \left( \eta : \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \eta(q) \in \Omega_{\mathbf{H}} \right) \\ & = \mathbb{Q}_x^\beta \left( \eta : \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \eta(q) \in \Omega_{\mathbf{H}}, \eta(q + \cdot) = \mu(\cdot; \eta(q)), \right. \\ & \quad \left. \text{on } [0, \delta \wedge T_{\eta(q)}] \right). \end{aligned}$$

By repeated application of Proposition 1.5, for any integer  $l > 0$ ,

$$\begin{aligned} & \mathbb{Q}_x^\beta \left( \eta : \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \eta(q) \in \Omega_{\mathbf{H}} \right) \\ & = \mathbb{Q}_x^\beta \left( \eta : \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \eta(q) \in \Omega_{\mathbf{H}}, \eta(q + \cdot) = \mu(\cdot; \eta(q)), \right. \end{aligned}$$

$$\text{on } [0, l\delta \wedge T_{\eta(q)}]),$$

which concludes the proof of (52).

$$\begin{aligned} & \mathbb{Q}_x^\beta \left( \eta : \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \eta(q + \cdot) = \mu(\cdot; \eta(q)) \text{ on } [0, T_{\eta(q)}], \eta(q) \in \Omega_{\mathbf{H}} \right) \\ & \leq \mathbb{Q}_x^\beta \left( \eta : \inf_{[0, T]} |\eta(\cdot)| > \rho_K, \inf_{[0, T_0 + T_K]} |\eta| \leq \rho_K \right) = 0 \end{aligned}$$

since  $T = T_0 + T_K$ , which concludes the proof of (53).

## 5. Proofs for Section 2.

### 5.1. Proofs of Section 2.1.

PROOF OF PROPOSITION 2.4. Define

$$(54) \quad \bar{\Delta}(x) \stackrel{\text{def}}{=} - \int_{\mathbb{R}_x} yq(y) \lambda^{\text{Leb}}(dy).$$

Introduce, for any  $\delta > 0$ , the  $\delta$ -zone  $C_x(\delta)$  around  $C_x$ ,

$$(55) \quad C_x(\delta) \stackrel{\text{def}}{=} \{y + sn(y), y \in C_x, -\delta \leq s \leq \delta\}.$$

By [Jarnier and Hansen \[\(2000\), Theorem 4.1\]](#), we may bound the measure of the  $\delta$ -zone's intersection with the ball  $\mathbf{B}(0, K)$ , for any  $K > 0$  and all  $|x|$  large enough,

$$\lambda^{\text{Leb}}(C_x(\delta) \cap \mathbf{B}(0, K)) \leq \delta \left( \frac{|x| + K}{|x| - K} \right)^{d-1} \frac{\lambda^{\text{Leb}}\{\mathbf{B}(0, 3K)\}}{K},$$

where the  $x$ -dependent term tends to 1 as  $|x|$  tends to infinity. From this, it follows, using the fact that  $\int |y|q(y) \lambda^{\text{Leb}}(dy) < \infty$ , that for any  $K > 0$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$(56) \quad \limsup_{|x| \rightarrow \infty} \int_{E_x(\delta, K)} |y|q(y) \lambda^{\text{Leb}}(dy) < \epsilon,$$

where  $E_x(\delta, K) \stackrel{\text{def}}{=} C_x(\delta) \cap \mathbf{B}(0, K)$ . For arbitrary, but fixed,  $\epsilon > 0$ , choose  $K > 0$  such that  $\int_{\mathbf{B}^c(0, K)} |y|q(y) \lambda^{\text{Leb}}(dy) \leq \epsilon$ . Then choose  $\delta > 0$  such that (56) holds. By construction, for  $y \in \mathbb{R}_x$ ,  $\pi(x + y)/\pi(x) \leq 1$  and (56) implies that

$$(57) \quad \limsup_{|x| \rightarrow \infty} \int_{\mathbb{R}_x \cap E_x(\delta, K)} |y| \frac{\pi(x + y)}{\pi(x)} q(y) \lambda^{\text{Leb}}(dy) \leq \epsilon,$$

$$(58) \quad \limsup_{|x| \rightarrow \infty} \int_{\mathbb{R}_x \cap \mathbf{B}^c(0, K)} |y| \frac{\pi(x + y)}{\pi(x)} q(y) \lambda^{\text{Leb}}(dy) \leq \epsilon.$$

From (15), for  $y \in \mathbb{R}_x$  such that  $y$  has radial distance at least  $\delta$  to  $\mathbb{C}_x$ , the acceptance probability satisfies  $\pi(x+y)/\pi(x) \leq \epsilon/K$  for all  $|x|$  sufficiently large [see Jarner and Hansen (2000), page 351] and (56) shows that

$$(59) \quad \limsup_{|x| \rightarrow \infty} \int_{\mathbb{R}_x \cap \mathbb{E}_x^\epsilon(\delta, K) \cap \mathbb{B}(0, K)} |y| \frac{\pi(x+y)}{\pi(x)} q(y) \lambda^{\text{Leb}}(dy) \leq \epsilon.$$

By combining (14), (54), (57), (58) and (59),  $\limsup_{|x| \rightarrow \infty} |\Delta(x) - \bar{\Delta}(x)| \leq 3\epsilon$  and since  $\epsilon$  is arbitrary,  $\lim_{|x| \rightarrow \infty} |\Delta(x) - \bar{\Delta}(x)| = 0$ .  $\square$

PROOF OF PROPOSITION 2.6. Set  $z = (z_1, \dots, z_d) \stackrel{\text{def}}{=} \Sigma^{-1/2}y$  and  $v = n(\Sigma^{1/2}u)$ . Then,

$$\begin{aligned} \int_{\{y, y'u \geq 0\}} yq(y) \lambda^{\text{Leb}}(dy) &= \Sigma^{1/2} \int_{\{z, v'z \geq 0\}} zq_0(z) \lambda^{\text{Leb}}(dz) \\ &= \Sigma^{1/2}v \int_{\mathbb{X}} z_1 \mathbb{1}_{\{z_1 \geq 0\}} q_0(z) dz. \end{aligned}$$

The proof follows.  $\square$

5.2. *Proof of Lemma 2.9.* Let  $\delta$  and  $M$  be constants to be specified later. Write  $\Delta(x) - \Delta_\infty(x) \stackrel{\text{def}}{=} \sum_{i=1}^4 A_i(\delta, M, x)$ , where

$$\begin{aligned} A_1(\delta, M, x) &\stackrel{\text{def}}{=} \int_{\{y, |y| \leq M, |y'\Gamma_2^{-1}x| \geq \delta|x|\}} \frac{\pi(x+y)}{\pi(x)} \mathbb{1}_{R_{\infty, x}}(y) yq(y) \lambda^{\text{Leb}}(dy), \\ A_2(\delta, M, x) &\stackrel{\text{def}}{=} \int_{\{y, |y| \leq M, |y'\Gamma_2^{-1}x| \geq \delta|x|\}} \left( \frac{\pi(x+y)}{\pi(x)} - 1 \right) (\mathbb{1}_{R_x}(y) - \mathbb{1}_{R_{\infty, x}}(y)) \\ &\quad \times yq(y) \lambda^{\text{Leb}}(dy), \\ A_3(\delta, M, x) &\stackrel{\text{def}}{=} \int_{\{y, |y| \leq M, |y'\Gamma_2^{-1}x| \leq \delta|x|\}} \left\{ \left( \frac{\pi(x+y)}{\pi(x)} - 1 \right) \mathbb{1}_{R_x}(y) + \mathbb{1}_{R_{\infty, x}}(y) \right\} \\ &\quad \times yq(y) \lambda^{\text{Leb}}(dy), \\ A_4(\delta, M, x) &\stackrel{\text{def}}{=} \int_{\{y, |y| \geq M\}} \left\{ \left( \frac{\pi(x+y)}{\pi(x)} - 1 \right) \mathbb{1}_{R_x}(y) + \mathbb{1}_{R_{\infty, x}}(y) \right\} yq(y) \lambda^{\text{Leb}}(dy). \end{aligned}$$

For  $x = (x_1, x_2)$  such that  $|x_1| - |x_2| \geq 2M$  and  $|y| \leq M$ , and  $|x_1 + y_1| \geq |x_1| - M \geq |x_2| + M \geq |x_2 + y_2|$ , it is easily shown that

$$(60) \quad \begin{aligned} &(1 - \alpha) \exp(-0.5y'\Gamma_2^{-1}y - x'\Gamma_2^{-1}y) \\ &\leq \frac{\pi(x+y)}{\pi(x)} \\ &\leq (1 - \alpha)^{-1} \exp(-0.5y'\Gamma_2^{-1}y - x'\Gamma_2^{-1}y). \end{aligned}$$

If  $y \in R_{\infty,x} \cap \{z: |x'\Gamma_2^{-1}z| \geq \delta|x|\}$ , then, by (60),  $\pi(x+y)/\pi(x) \leq (1-\alpha)^{-1}e^{-\delta|x|}$ , which implies that  $|A_1(\delta, M, x)| \leq (1-\alpha)^{-1}e^{-\delta|x|} \int |y|q(y)\lambda^{\text{Leb}}(dy)$ . Furthermore, for any  $K$  such that  $(1-\alpha)^{-1}e^{-\delta K} \leq 1$  and  $x$  such that  $\|x_1\| - \|x_2\| \geq 2M$  and  $|x| \geq K$ ,  $R_{\infty,x} \cap \{y: |y| \leq M, |x'\Gamma_2^{-1}y| \geq \delta|x|\} \subseteq R_x$ . This property yields to the bound

$$(61) \quad \begin{aligned} & \left| \frac{\pi(x+y)}{\pi(x)} - 1 \right| \mathbb{1}_{R_x}(y) - \mathbb{1}_{R_{\infty,x}}(y) \mathbb{1}_{\{|y| \leq M, |x'\Gamma_2^{-1}y| \geq \delta|x|\}} \\ & \leq \mathbb{1}_{R_x \setminus R_{\infty,x}}(y) \mathbb{1}_{\{|y| \leq M, |x'\Gamma_2^{-1}y| \geq \delta|x|\}}. \end{aligned}$$

Again using (60) for  $y \in R_x \cap \{|y| \leq M\}$ ,  $(1-\alpha)e^{-0.5a^2M^2}e^{-x'\Gamma_2^{-1}y} \leq \pi(x+y)/\pi(x) \leq 1$ . On the other hand, for  $y \notin R_{\infty,x}$  satisfying  $|x'\Gamma_2^{-1}y| \geq \delta|x|$ , we have  $x'\Gamma_2^{-1}y \leq -\delta|x|$ , showing that

$$\begin{aligned} & y \in R_x \setminus R_{\infty,x} \cap \{z, |z| \leq M, |x'\Gamma_2^{-1}z| \geq \delta|x|\} \\ & \implies (1-\alpha)e^{-0.5a^2M^2}e^{\delta K} \leq \pi(x+y)/\pi(x) \leq 1. \end{aligned}$$

For fixed  $M$ , we choose  $K$  such that  $(1-\alpha)e^{-0.5a^2M^2}e^{\delta K} > 1$ , which implies that the right-hand side in (61) is zero and thus  $A_2(\delta, M, x) = 0$ . Finally, consider  $A_i(\delta, M, x)$ ,  $i = 3, 4$ . Noting that

$$\left| \left( \frac{\pi(x+y)}{\pi(x)} - 1 \right) \mathbb{1}_{R_x}(y) + \mathbb{1}_{R_{\infty,x}}(y) \right| \leq 2,$$

the proof follows from the bounds

$$(62) \quad |A_3(\delta, M, x)| \leq 2M \int \mathbb{1}_{\{|y| \leq M, |x'\Gamma_2^{-1}y| \leq \delta|x|\}} |y|q(y)\lambda^{\text{Leb}}(dy),$$

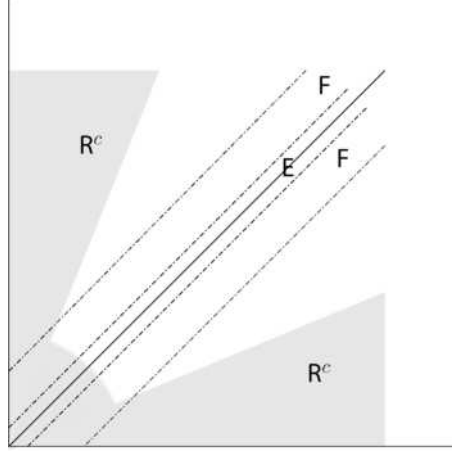
$$(63) \quad |A_4(\delta, M, x)| \leq 2 \int_{|y| \geq M} |y|q(y)\lambda^{\text{Leb}}(dy).$$

These terms are arbitrarily small for convenient constants  $M$  and  $\delta$ .

### 5.3. Proof of Proposition 2.10.

5.3.1. *Proof of condition (i) of Theorem 1.9.* The only difficulty here stems from the irregularity of the ODE for initial conditions on the diagonals. Consider the  $\beta$ -fluid limit  $\mathbb{Q}_{u_\star}^\beta$  with initial condition  $u_\star \stackrel{\text{def}}{=} (1/\sqrt{2}, 1/\sqrt{2})$  (the other cases can be dealt with similarly). Set  $v_\star \stackrel{\text{def}}{=} (1/\sqrt{2}, -1/\sqrt{2})$  and define  $V(x) = |\langle v_\star, x \rangle|$ . Since the increment distribution is assumed to be bounded, there exists a positive constant  $C_q$  such that  $|\Phi_1 - \Phi_0| \leq C_q$ ,  $\mathbb{P}_x$ -a.s. for all  $x \in \mathbf{X}$ . By Lemma 2.9, we may choose constants  $\gamma \in (0, 1)$ ,  $m > 0$ ,  $M_0 > C_q$  and  $R$  such that

$$(64) \quad R \cap E^c \subset \{x \in \mathbf{X}, |\langle v_\star, \Delta(x) \rangle| \geq m, \langle v_\star, x \rangle \langle v_\star, \Delta(x) \rangle > 0\},$$

FIG. 9. The complement of the zone  $R$  and the strips  $E$  and  $F$ .

where (see Figure 9)

$$(65) \quad E \stackrel{\text{def}}{=} \{x, V(x) \leq M_0\} \quad \text{and} \quad R \stackrel{\text{def}}{=} \{x \in \mathsf{X}, |x| \geq R, |\langle v_*, n(x) \rangle| \leq \gamma\}.$$

For  $\delta > 0$ , define the stopping time  $\kappa(\delta)$  as the infimum of the following three stopping times

$$(66) \quad \kappa_1(\delta) \stackrel{\text{def}}{=} \inf\{k \geq 0, |\langle v_*, \Phi_k \rangle| \geq 2\delta|\Phi_0|\},$$

$$(67) \quad \kappa_2 \stackrel{\text{def}}{=} \inf\{k \geq 0, |\Phi_k - \Phi_0| \geq (1/2)|\Phi_0|\},$$

$$(68) \quad \kappa_3 \stackrel{\text{def}}{=} \inf\{k \geq 0, |\Phi_k| < R\}.$$

We will establish the following drift condition: there exist constants  $b > 0$  and  $C$  such that for all  $\delta \in (0, \gamma/4)$ ,

$$(69) \quad \mathbb{E}[V(\Phi_{k+1})|\mathcal{F}_k] \geq V(\Phi_k) + m - b\mathbb{1}_E(\Phi_k) \quad \text{on the set } \{k < \kappa(\delta)\},$$

$$(70) \quad \mathbb{E}_x \left[ \sum_{k=0}^{\kappa(\delta)-1} \mathbb{1}_E(\Phi_k) \right] \leq C,$$

with the convention that  $\sum_a^b = 0$  when  $a > b$ . We postpone the proof of (69) and (70) and show how these drift conditions allow us to obtain condition (i). On the event  $\{k < \kappa(\delta)\}$ ,  $|\Phi_k| \geq R$ ,  $(1/2)|\Phi_0| \leq |\Phi_k| \leq (3/2)|\Phi_0|$  and  $|\langle v_*, n(\Phi_k) \rangle| \leq 4\delta \leq \gamma$ . Therefore, for all  $x \in \mathsf{X}$ ,  $\mathbb{P}_x$ -a.s.,

$$(71) \quad \{k < \kappa(\delta)\} \subset \{\Phi_k \in R\}.$$

Condition (69) yields, for any constant  $N > 0$ ,

$$m\mathbb{E}_x[\kappa(\delta) \wedge N] \leq \mathbb{E}_x[V(\Phi_{\kappa(\delta) \wedge N})\mathbb{1}\{\kappa(\delta) \geq 1\}] + b\mathbb{E}_x \left[ \sum_{k=0}^{\kappa(\delta) \wedge N - 1} \mathbb{1}_E(\Phi_k) \right].$$

The definitions of  $\kappa(\delta)$  and  $C_q$  imply that  $\mathbb{E}_x[V(\Phi_{\kappa(\delta)\wedge N})\mathbb{1}\{\kappa(\delta) \geq 1\}] \leq 2\delta|x| + C_q$  for all  $N$ , which, with (70), yields the bound

$$(72) \quad m\mathbb{E}_x[\kappa(\delta)] \leq 2\delta|x| + bC + C_q.$$

Let  $\{x_n\}$  be a sequence of initial states such that  $\lim_{n \rightarrow \infty} x_n = u_\star$  and  $\{r_n\}$  be a sequence of scaling constants,  $\lim_{n \rightarrow \infty} r_n = +\infty$ . By Lemma 4.10, there exists  $T_0$  such that  $\mathbb{Q}_{u_\star}^\beta\{\sup_{t \in [0, T_0]} |\eta(t) - \eta(0)| < 1/4\} = 1$ . Furthermore, we have  $1/2 \leq |x_n| \leq 3/2$  for all  $n$  large enough. Then, by the Portmanteau theorem,

$$\begin{aligned} & \mathbb{Q}_{u_\star}^\beta\{\eta, \eta([0, T_0]) \cap \mathbf{O} = \emptyset\} \\ &= \lim_{\delta \downarrow 0^+} \mathbb{Q}_{u_\star}^\beta\left\{\eta, \sup_{t \in [0, T_0]} |\eta(t) - \eta(0)| < 1/4, \sup_{t \in [0, T_0]} |\langle v_\star, \eta(t) \rangle| < \delta\right\} \\ &\leq \lim_{\delta \downarrow 0^+} \liminf_{n \rightarrow \infty} \mathbb{P}_{r_n x_n} \left\{ \sup_{0 \leq k \leq 2T_0|\Phi_0|/3} |\Phi_k - \Phi_0| < (1/2)|\Phi_0|, \right. \\ &\quad \left. \sup_{0 \leq k \leq 2T_0|\Phi_0|/3} |\langle v_\star, \Phi_k \rangle| < 2\delta|\Phi_0| \right\} \\ &\leq \lim_{\delta \downarrow 0^+} \liminf_{n \rightarrow \infty} \mathbb{P}_{r_n x_n}(\kappa(\delta) \geq 2T_0|\Phi_0|/3) = 0, \end{aligned}$$

where the last equality stems from (72). This proves Theorem 1.9(i).

We now prove (69). Since  $\mathbb{E}[\Phi_{k+1}|\mathcal{F}_k] = \Phi_k + \Delta(\Phi_k)$ , Jensen's inequality implies that  $\mathbb{E}_x[V(\Phi_{k+1})|\mathcal{F}_k] \geq |\langle v_\star, \Phi_k + \Delta(\Phi_k) \rangle|$ . Furthermore, by (64) and (71),  $\{k < \kappa(\delta), \Phi_k \in \mathbf{E}^c\} \subset \{\Phi_k \in \mathbf{R} \cap \mathbf{E}^c\}$ , which implies that  $|\langle v_\star, \Phi_k + \Delta(\Phi_k) \rangle| - |\langle v_\star, \Phi_k \rangle| = |\langle v_\star, \Delta(\Phi_k) \rangle| \geq m$  since, on  $\mathbf{R} \cap \mathbf{E}^c$ ,  $\langle v_\star, x \rangle$  and  $\langle v_\star, \Delta(x) \rangle$  have the same sign and  $\langle v_\star, \Delta(x) \rangle$  is lower bounded. On the set  $\{k < \kappa(\delta), \Phi_k \in \mathbf{E}\}$ , we write  $V(\Phi_{k+1}) \geq V(\Phi_k) - C_q$  so that  $\mathbb{E}[V(\Phi_{k+1})|\mathcal{F}_k] \geq V(\Phi_k) + m - (C_q + m)$ . This concludes the proof of (69).

Finally, we prove (70). For  $A \in \mathcal{X}$ , we denote by  $\sigma_A \stackrel{\text{def}}{=} \inf\{k \geq 0, \Phi_k \in A\}$  the first hitting time on  $A$ . For notational simplicity, we write  $\kappa$  instead of  $\kappa(\delta)$ . Define recursively  $\sigma^{(1)} \stackrel{\text{def}}{=} \sigma_{\mathbf{E} \cap \mathbf{R}}$  and, for all  $k \geq 2$ ,  $\sigma^{(k)} \stackrel{\text{def}}{=} \sigma^{(k-1)} + \tau \circ \theta^{\sigma^{(k-1)}} + \sigma^{(1)} \circ \theta^{\tau \circ \theta^{\sigma^{(k-1)}} + \sigma^{(k-1)}}$ , where  $\tau \stackrel{\text{def}}{=} \kappa \wedge k_\star$ ,  $k_\star$  being an integer whose value will be specified later. With this notation,

$$(73) \quad \mathbb{E}_x \left[ \sum_{k=0}^{\kappa-1} \mathbb{1}_{\mathbf{E}}(\Phi_k) \right] \leq k_\star \sum_{q \geq 1} \mathbb{P}_x(\sigma^{(q)} < \kappa).$$

Furthermore, for all  $q \geq 2$ , the strong Markov property yields the bound

$$\mathbb{P}_x(\sigma^{(q)} < \kappa) \leq \mathbb{P}_x(\sigma^{(q-1)} < \kappa) \sup_{y \in \mathbf{E} \cap \mathbf{R}} \mathbb{P}_y(\tau + \sigma^{(1)} \circ \theta^\tau < \kappa).$$



Therefore, by (73), (70) holds, provided that  $\sup_{x \in \mathbf{E} \cap \mathbf{R}} \mathbb{P}_x(\tau + \sigma^{(1)} \circ \theta^\tau < \kappa) < 1$ . For all  $x \in \mathbf{E} \cap \mathbf{R}$ , it is easily seen that

$$(74) \quad \begin{aligned} & \mathbb{P}_x(\tau + \sigma^{(1)} \circ \theta^\tau < \kappa) \\ &= \mathbb{P}_x(\tau < \kappa) - \mathbb{E}_x(\mathbb{1}\{\tau < \kappa\} \mathbb{1}\{\Phi_\tau \in \mathbf{E}^c \cap \mathbf{R}\} \mathbb{P}_{\Phi_\tau}[\kappa \leq \sigma^{(1)}]) \end{aligned}$$

$$(75) \quad \leq 1 - \inf_{x \in \mathbf{E}^c \cap \mathbf{R}} \mathbb{P}_x(\kappa \leq \sigma^{(1)}) \{\mathbb{P}_x(\tau = \kappa) + \mathbb{P}_x(\tau = k_\star, \Phi_{k_\star} \in \mathbf{E}^c \cap \mathbf{R})\},$$

showing that the conditions

$$(76) \quad \inf_{x \in \mathbf{E} \cap \mathbf{R}} \mathbb{P}_x(\{\tau < k_\star\} \cup \{\tau = k_\star, \Phi_{k_\star} \in \mathbf{E}^c \cap \mathbf{R}\}) > 0,$$

$$(77) \quad \inf_{x \in \mathbf{E}^c \cap \mathbf{R}} \mathbb{P}_x(\kappa \leq \sigma^{(1)}) > 0$$

imply (70). We first prove (76). Choose  $\tilde{\gamma} \in (\gamma, 1)$  such that the four half-planes  $\{z, \langle z, \Gamma_i^{-1} u_{\star, \tilde{\gamma}}^\pm \rangle < 0\}$  ( $i = 1, 2$ ) have a nonempty intersection, where  $u_{\star, \tilde{\gamma}}^-$  and  $u_{\star, \tilde{\gamma}}^+$  are the unit vectors defining the edges of the cone  $\mathbf{C}_{\tilde{\gamma}} \stackrel{\text{def}}{=} \{z \in \mathbf{X}, |\langle v_\star, n(z) \rangle| \leq \tilde{\gamma}\}$ . Define

$$(78) \quad \mathbf{W} \stackrel{\text{def}}{=} \{z, 0 \leq |z| \leq C_q, \langle z, \Gamma_i^{-1} u_{\star, \tilde{\gamma}}^\pm \rangle \leq 0, i = 1, 2\}.$$

Since any vector  $y$  in the cone  $\mathbf{C}_{\tilde{\gamma}}$  can be written as a linear combination of the vectors  $u_{\star, \tilde{\gamma}}^-$  and  $u_{\star, \tilde{\gamma}}^+$  with positive weights, for any  $y \in \mathbf{C}_{\tilde{\gamma}}$  and  $z \in \mathbf{W}$ ,  $\langle z, \Gamma_i^{-1} y \rangle \leq 0$ ,  $i = 1, 2$ , which implies that

$$\begin{aligned} & \langle z, \nabla \pi(y) \rangle \\ &= -\alpha \langle z, \Gamma_1^{-1} y \rangle \exp(-0.5y' \Gamma_1^{-1} y) - (1 - \alpha) \langle z, \Gamma_2^{-1} y \rangle \exp(-0.5y' \Gamma_2^{-1} y) \\ &\geq 0. \end{aligned}$$

By choosing  $R$  large enough [see (65)], we can assume, without loss of generality, that for all  $x \in \mathbf{R}$  and  $z \in \mathbf{W}$ ,  $x + tz \in \mathbf{C}_{\tilde{\gamma}}$  for all  $t \in (0, 1)$ . Thus,  $\pi(x + z) = \pi(x) + \int_0^1 \langle \nabla \pi(x + tz), z \rangle dt \geq 0$  and we have  $\pi(x + z) \geq \pi(x)$ , showing that  $\mathbf{W} \subset \mathbf{A}_x$ . Finally, we write  $\mathbf{W}$  as the union of two disjoint sets  $\mathbf{W}^-$ ,  $\mathbf{W}^+$ , where  $\mathbf{W}^+ \stackrel{\text{def}}{=} \{z \in \mathbf{W}, \langle v_\star, z \rangle \geq 0\}$ . Since, for  $x \in \mathbf{R}$ ,  $\mathbf{W} \subset \mathbf{A}_x$ , for any  $0 \leq c \leq C_q$ , we have

$$\begin{aligned} & \inf_{x \in \mathbf{R}, \langle v_\star, x \rangle \geq 0} \mathbb{P}_x(|\langle v_\star, \Phi_1 \rangle| \geq |\langle v_\star, \Phi_0 \rangle| + c) \\ &\geq \int_{\mathbf{W}^+} \mathbb{1}\{y, |\langle v_\star, y \rangle| \geq c\} q(y) \lambda^{\text{Leb}}(dy) > 0. \end{aligned}$$

An analogous lower bound holds for all  $x \in \mathbf{R}$  such that  $\langle v_\star, x \rangle \leq 0$ . These inequalities, combined with repeated applications of the Markov property, yield (76), by choosing  $k_\star$  such that  $k_\star c \geq M_0$ .

We now prove (77). Let  $M_1 > M_0$  and set  $\mathbf{F} \stackrel{\text{def}}{=} \{x, V(x) \leq M_1\}$ . By Lemma A.1, we may choose  $J \geq 1$  and then  $M_1 > M_0$  large enough so that, for all  $x \in \mathbf{X}$ ,

$$(79) \quad \mathbb{P}_x \left( \sup_{j \geq J} j^{-1} \left| \sum_{l=1}^j \epsilon_l \right| \geq m \right) < 1/2,$$

$$\mathbb{P}_x \left( \sup_{j \leq J} \left| \sum_{l=1}^j \epsilon_l \right| \geq M_1 - M_0 \right) < 1/2.$$

It is easily seen that, using the strong Markov property,

$$\inf_{x \in \mathbf{E}^c \cap \mathbf{R}} \mathbb{P}_x(\kappa \leq \sigma_{\mathbf{E} \cap \mathbf{R}}) \geq \inf_{x \in \mathbf{E}^c \cap \mathbf{R}} \mathbb{P}_x(\sigma_{\mathbf{F}^c \cap \mathbf{R}} < \sigma_{\mathbf{E} \cap \mathbf{R}}) \inf_{x \in \mathbf{F}^c \cap \mathbf{R}} \mathbb{P}_x(\kappa \leq \sigma_{\mathbf{E} \cap \mathbf{R}}).$$

The first term of the right-hand side of the previous relation can be shown to be positive, using arguments which are similar to those used in the proof of (76). We write  $\langle v_*, \Phi_k \rangle = \langle v_*, \Phi_0 \rangle + \sum_{l=1}^k \langle v_*, \Delta(\Phi_{l-1}) \rangle + \sum_{l=1}^k \langle v_*, \epsilon_l \rangle$ . Let  $x \in \mathbf{F}^c \cap \mathbf{R}$ .  $\mathbb{P}_x$ -a.s., since  $|\Phi_l - \Phi_{l-1}| \leq C_q \leq M_0$ , on the event  $\{1 \leq k \leq \sigma_{\mathbf{E} \cap \mathbf{R}} < \kappa\}$ ,  $|\langle v_*, \Phi_k \rangle| \geq M_0$ ,  $\langle v_*, \Phi_0 \rangle \langle v_*, \Phi_j \rangle > 0$  and  $\langle v_*, \Phi_0 \rangle \langle v_*, \Delta(\Phi_j) \rangle > 0$  for all  $0 \leq j < k$ , which implies that

$$\begin{aligned} |\langle v_*, \Phi_k \rangle| &\geq |\langle v_*, \Phi_0 \rangle| + \sum_{l=1}^k |\langle v_*, \Delta(\Phi_{l-1}) \rangle| - \left| \sum_{l=1}^k \langle v_*, \epsilon_l \rangle \right| \\ &\geq M_1 + km - \left| \sum_{l=1}^k \langle v_*, \epsilon_l \rangle \right|. \end{aligned}$$

Thus, for all  $x \in \mathbf{F}^c \cap \mathbf{R}$ , using the definition (79) of  $J$  and  $M_1$ , we have

$$\mathbb{P}_x \{J \leq \sigma_{\mathbf{E} \cap \mathbf{R}} < \kappa\} \leq \sup_{x \in \mathbf{X}} \mathbb{P}_x \left\{ \sup_{j \geq J} j^{-1} \left| \sum_{l=1}^j \langle v_*, \epsilon_l \rangle \right| \geq m \right\} < 1/2,$$

$$\mathbb{P}_x \{ \sigma_{\mathbf{E} \cap \mathbf{R}} < \kappa \wedge J \} \leq \sup_{x \in \mathbf{X}} \mathbb{P}_x \left\{ \sup_{j \leq J} \left| \sum_{l=1}^j \epsilon_l \right| \geq (M_1 - M_0) \right\} < 1/2,$$

which proves  $\inf_{x \in \mathbf{F}^c \cap \mathbf{R}} \mathbb{P}_x(\kappa \leq \sigma_{\mathbf{E} \cap \mathbf{R}}) > 0$  and therefore (77).

5.3.2. *Proof of B4 and the conditions (ii)–(iii) of Theorem 1.9.* Assume that  $x \in \mathbf{C} \stackrel{\text{def}}{=} \{x, 0 < |x_2| < x_1\}$  (the three other cases are similar). By Lemma 2.9,  $h(x) = -c_q n(\Gamma_2^{-1}x)$  for all  $x \in \mathbf{C}$ , which is locally Lipschitz. Hence, there exists a unique maximal solution  $\mu(\cdot; x)$  on  $[0, T_x]$  satisfying  $\mu(0; x) = x$  and  $\mu(t; x) \in \mathbf{C}$  for all  $t \leq T_x$ , showing B4. Since, for  $t \in [0, T_x)$ ,  $d/dt |\mu(t; x)|^2 = 2|\mu(t; x)| \langle n(\mu(t; x)), h \circ \mu(t; x) \rangle < -2c_q |a|^{-1} |\mu(t; x)|$ , the norm of the ODE solution is bounded by  $|\mu(t; x)| \leq (|x| - c_q |a|^{-1} t)_+$  for all  $0 \leq t \leq |x| |a| c_q^{-1}$ , which implies condition (ii), provided that  $T_x \geq T_K$  for

all  $x \in \mathbb{C} \cap \mathbb{B}(0, K)$ . This result follows from the fact that the boundaries of  $\mathbb{C}$  are repulsive: consider the relative neighborhood in  $\mathbb{C}$ ,  $\mathbb{V} \stackrel{\text{def}}{=} \mathbb{V}_1 \cup \mathbb{V}_2$ , of the boundaries where  $\mathbb{V}_1 \stackrel{\text{def}}{=} \{x : x_1 > 0, \langle v_*, x \rangle > 0, \langle x, \Gamma_2^{-1} v_* \rangle < 0\}$  and  $\mathbb{V}_2 \stackrel{\text{def}}{=} \{x : x_1 > 0, \langle u_*, x \rangle > 0, \langle x, \Gamma_2^{-1} u_* \rangle < 0\}$ . Assume that there exists  $s \in [0, T_x]$  such that  $\mu(s; x) \in \mathbb{V}_1$  (the other case can be handled similarly). Since  $t \mapsto \mu(t; x)$  is continuous and  $\mathbb{V}_1$  is a relative open subset of  $\mathbb{C}$ , there exists  $\delta$  such that for all  $0 \leq t \leq \delta$ ,  $\mu(s+t; x) \in \mathbb{V}_1$ . This implies that for all  $0 \leq t \leq \delta$ ,

$$\begin{aligned} & \langle v_*, \mu(s+t; x) \rangle - \langle v_*, \mu(s; x) \rangle \\ &= -c_q \int_0^t |\Gamma_2^{-1} \mu(s+u; x)|^{-1} \langle v_*, \Gamma_2^{-1} \mu(s+u; x) \rangle du > 0, \end{aligned}$$

showing that, in  $\mathbb{V}_1$ , the distance to the boundary always increases. The properties above also imply condition (iii) of Theorem 1.9.

#### APPENDIX: TECHNICAL LEMMAS

LEMMA A.1. *Let  $\{\varepsilon_k\}_{k \geq 1}$  be an  $L^p$ -martingale difference sequence adapted to the filtration  $\{\mathcal{F}_k\}_{k \geq 0}$ . For any  $p > 1$ , there exists a constant  $C$  (depending only on  $p$ ) such that*

$$(80) \quad \mathbb{E} \left[ \sup_{1 \leq l \leq n} \left| \sum_{k=1}^l \varepsilon_k \right|^p \right] \leq C \sup_{k \geq 1} \mathbb{E}[|\varepsilon_k|^p] n^{1 \vee p/2},$$

$$(81) \quad \mathbb{P} \left[ \sup_{n \leq l} l^{-1} \left| \sum_{k=1}^l \varepsilon_k \right| \geq M \right] \leq C \sup_{k \geq 1} \mathbb{E}[|\varepsilon_k|^p] M^{-p} n^{-p+1 \vee p/2}.$$

PROOF. For  $p > 1$ , applying in sequence the Doob maximal inequality, by the Burkholder inequality for  $L^p$ -martingales, there exists a constant  $C_p$  such that

$$\mathbb{E} \left[ \sup_{1 \leq l \leq n} \left| \sum_{k=1}^l \varepsilon_k \right|^p \right] \leq C_p \mathbb{E} \left[ \left| \sum_{k=1}^n |\varepsilon_k|^2 \right|^{p/2} \right].$$

Equation (80) follows from the Minkovski inequality for  $p \geq 2$ ,

$$(82) \quad \mathbb{E} \left[ \sup_{1 \leq l \leq n} \left| \sum_{k=1}^l \varepsilon_k \right|^p \right] \leq C_p \sup_{k \geq 1} \mathbb{E}[|\varepsilon_k|^p] n^{p/2},$$

and the subadditivity inequality for  $1 < p \leq 2$ ,

$$(83) \quad \mathbb{E} \left[ \sup_{1 \leq l \leq n} \left| \sum_{k=1}^l \varepsilon_k \right|^p \right] \leq C_p \sup_{k \geq 1} \mathbb{E}[|\varepsilon_k|^p] n.$$

Equation (81) follows from [Birbaum and Marshall \(1961\)](#), Theorem 1.  $\square$

LEMMA A.2. *Let  $X, Y$  be two nonnegative random variables. Then, for any  $p \geq 1$ , there exists a constant  $C_p$  (depending only on  $p$ ) such that, for any  $M > 0$ ,*

$$\mathbb{E}[(X + Y)^p \mathbb{1}\{X + Y > M\}] \leq C_p(\mathbb{E}[X^p \mathbb{1}\{X \geq M/2\}] + \mathbb{E}[Y^p]).$$

PROOF. Note that  $\mathbb{1}\{X + Y \geq M\} \leq \mathbb{1}\{X \geq M/2\} + \mathbb{1}\{X \leq M/2\} \mathbb{1}\{Y \geq M/2\}$ . Therefore,

$$\begin{aligned} \mathbb{E}(X^p \mathbb{1}\{X + Y \geq M\}) &\leq \mathbb{E}(X^p \mathbb{1}\{X \geq M/2\}) + (M/2)^p \mathbb{P}(Y \geq M/2) \\ &\leq \mathbb{E}(X^p \mathbb{1}\{X \geq M/2\}) + \mathbb{E}(Y^p). \end{aligned}$$

The proof then follows from the fact that  $(X + Y)^p \leq 2^{p-1}(X^p + Y^p)$ .  $\square$

LEMMA A.3. *Let  $X$  be a nonnegative random variable. For any  $p \geq 0$ ,  $a > 1$  and  $M$ , we have*

$$\mathbb{E}[X^p \mathbb{1}\{X \geq M\}] \leq M^{-(a-1)p} \mathbb{E}[X^{ap}].$$

PROOF.

$$\begin{aligned} \mathbb{E}[X^p \mathbb{1}\{X \geq M\}] &\leq (\mathbb{E}[X^{ap}])^{1/a} (\mathbb{P}[X \geq M])^{(a-1)/a} \\ &\leq (\mathbb{E}[X^{ap}])^{1/a} (M^{-ap} \mathbb{E}[X^{ap}])^{(a-1)/a}. \quad \square \end{aligned}$$

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