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# About $L^p$ estimates for the spatially homogeneous Boltzmann equation

Laurent Desvillettes and Clément Mouhot

## Abstract

For the homogeneous Boltzmann equation with (cutoff or non cutoff) hard potentials, we prove estimates of propagation of  $L^p$  norms with a weight  $(1+|x|^2)^{q/2}$  ( $1 < p < +\infty$ ,  $q \in \mathbb{R}_+$  large enough), as well as appearance of such weights. The proof is based on some new functional inequalities for the collision operator, proven by elementary means.

## A propos des estimations $L^p$ pour l'équation de Boltzmann homogène

## Abstract

On prouve la propagation de normes  $L^p$  avec poids  $(1+|x|^2)^{q/2}$  et l'apparition de tels poids pour l'équation de Boltzmann homogène dans le cas des potentiels durs (avec ou sans troncature angulaire). La démonstration est basée sur de nouvelles inégalités fonctionnelles pour l'opérateur de collision, que l'on prouve par des moyens élémentaires

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# 1 Introduction

The spatially homogeneous Boltzmann equation (cf. [5]) writes

$$\frac{\partial f}{\partial t}(t, v) = Q(f, f)(t, v), \quad (1.1)$$

where  $f(t, \cdot) : \mathbb{R}^N \rightarrow \mathbb{R}_+$  is the nonnegative density of particles which at time  $t$  move with velocity  $v$ . The bilinear operator in the right-hand side is defined by

$$Q(g, f)(v) = \int_{\mathbb{R}^N} \int_{S^{N-1}} \left\{ f(v') g(v'_*) - f(v) g(v_*) \right\} B\left(|v - v_*|, \frac{v - v_*}{|v - v_*|} \cdot \sigma\right) d\sigma dv_*. \quad (1.2)$$

In this formula,  $v', v'_*$  and  $v, v_*$  are the velocities of a pair of particles before and after a collision. They are defined by

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,$$

where  $\sigma \in S^{N-1}$ .

We concentrate in this work on hard potentials or hard spheres collision kernels, with or without angular cutoff. More precisely, we suppose that the collision kernel satisfies the following

**Assumptions:** The collision kernel  $B$  is of the form

$$B(x, y) = |x|^\gamma b(|y|), \quad (1.3)$$

where

$$\gamma \in [0, 1] \quad (1.4)$$

and

$$b \in L_{loc}^\infty([-1, 1]), \quad b(y) = O_{y \rightarrow 1^-}((1 - y)^{\frac{-(N-2)+\nu}{2}}), \quad \nu > -3. \quad (1.5)$$

Note that assumption (1.5) is an alternative (and a slightly less general) formulation to the minimal condition necessary for a mathematical treatment of the Boltzmann equation identified in [21, 2], namely the requirement

$$\int_{S^{N-1}} b(\cos \theta)(1 - \cos \theta) d\sigma < +\infty. \quad (1.6)$$

Then, we wish to consider initial data  $f_0 \geq 0$  with finite mass and energy, such that  $f_0(1 + |v|^2)^{q/2} \in L^p(\mathbb{R}^N)$  for some  $1 < p < +\infty$  and  $q \geq 0$  (notice that entropy is thus automatically finite). Existence results under the assumptions of finite mass, energy and

entropy were obtained in [3] for the case of hard potentials with cutoff, in [4] for (non cutoff) soft potentials in dimension 3 under the restriction  $\gamma \geq -1$ , then in [10] and [21] for general kernels (our assumptions on the kernel fall in the setting of [21] for instance). Uniqueness however is proved only in the cutoff case (for optimal result see [17]) and remains an open question in the noncutoff case (except for maxwellian molecules  $\gamma = 0$ , see [20]).

Propagation of moments in  $L^1$  was proven in [13] for Maxwellian molecules with cutoff. Then, for the case of strictly hard potentials with cutoff, it was shown in [6] that all polynomial moments were created immediately when one of them of order strictly bigger than 2 initially existed. This last restriction was later relaxed in [25].

Propagation of moments in  $L^p$  was first obtained by Gustafsson (cf. [11, 12]) thanks to interpolation techniques, under the assumption of angular cutoff. It was recovered by a simpler and more explicit method in [18], thanks to the smoothness properties of the gain part of the Boltzmann's collision operator discovered by P.-L. Lions [14]. As far as appearance of moments in  $L^p$  is concerned, the first result is due to Wennberg in [23], still in the framework of angular cutoff. It is precised in [18].

In this work, we wish to improve these results by presenting an  $L^p$  theory

- first, which is elementary (that is, without abstract interpolations and without using the smoothness properties of Boltzmann's kernel),
- secondly, which includes the non cutoff case,
- finally, without assuming too many moments in  $L^p$  for the initial datum.

Our method is reminiscent of recent works by Mischler and Rodriguez Ricard [16] and Escobedo, Laurençot and Mischler [9] on the Smoluchowsky equation.

Let  $1 < p < +\infty$ . We define the weighted  $L^p$  space  $L_q^p(\mathbb{R}^N)$  by

$$L_q^p(\mathbb{R}^N) = \left\{ f : \mathbb{R}^N \rightarrow \mathbb{R}, \quad \|f\|_{L_q^p(\mathbb{R}^N)} < +\infty \right\},$$

with its norm

$$\|f\|_{L_q^p(\mathbb{R}^N)}^p = \int_{\mathbb{R}^N} |f(v)|^p \langle v \rangle^{pq} dv,$$

and the usual notation  $\langle v \rangle = (1 + |v|^2)^{1/2}$ .

We now state our main theorem

**Theorem 1.1.** *Let  $B$  be a collision kernel satisfying Assumptions (1.3), (1.4), (1.5) and  $q$  such that*

- (i)  $q \in \mathbb{R}_+$  if  $\nu > -1$  (integrable angular kernel),
- (ii)  $pq > 2$  if  $\nu \in (-2, -1]$ ,
- (iii)  $pq > 4$  if  $\nu \in (-3, -2]$ ,

*and  $f_0$  be an initial datum in  $L_{\max(p,2)q+2}^1 \cap L_q^p$ .*

*Then*

- *there exists a (weak) solution to the Boltzmann equation (1.1) with collision kernel  $B$  and initial datum  $f_0$  lying in  $L^\infty([0, +\infty); L_q^p(\mathbb{R}^N))$  (with explicit bounds in this space),*
- *if  $\gamma > 0$ , this solution belongs moreover to  $L^\infty((\tau, +\infty); L_r^p(\mathbb{R}^N))$  for all  $\tau > 0$  and  $r > q$  (still with explicit bounds in this space, the blow up near  $\tau \sim 0^+$  being at worse polynomial).*

**Remarks:** We now discuss the assumptions and the conclusion of this theorem.

1. Our result cannot hold when the hard potentials are replaced by soft potentials. In the case of Maxwellian molecules ( $\gamma = 0$ ), we have uniform (in time) bounds but no appearance of moments (either in  $L^p$  nor in  $L^1$ ) occurs. In the case of the so-called “mollified soft potentials” with cutoff, some bounds growing polynomially in time can be found in [19], based on the regularity property of the gain term of the collision operator.

2. When the collision kernel  $B$  is not a product of a function of  $x$  by a function of  $y$  (as in Assumption (1.3)), it is likely that Theorem 1.1 still holds provided that the behavior of  $B$  with respect to  $x$  (when  $x \rightarrow +\infty$ ) is that of a nonnegative power and  $B$  satisfies estimate (1.5) uniformly according to  $x$ .

3. The restriction on the weight  $q$  is not a technical one which is likely to be discarded (at least in our method). Indeed as suggested in [1] the noncutoff collision operator behaves roughly like some fractional Laplacian of order  $-\nu/2$  and these derivatives will in fact be supported by the weight, as we shall see. Notice however that there is no condition on  $q$  when  $\nu > -1$ , i.e. in the cutoff case, which is coherent with existing results. Note also that the condition  $f_0 \in L_{2q+2}^1$  is used only to get the uniformity when  $t \rightarrow +\infty$  of the estimates. The local (in time) estimates hold as soon as  $f_0 \in L_{pq+2}^1$ .

4. Finally, Theorem 1.1 can certainly be improved when the collision kernel is non cutoff. In such a case (and under rather not stringent assumption (cf. [1])), it is possible to show that some smoothness is gained, and some  $L^p$  regularity will appear even if it does not initially exist. As a consequence, the assumptions of Theorem 1.1 can certainly be somehow relaxed. One can for example compare Theorem 1.1 to the results of [7] for the Landau equation. We also refer to [8] for “regularized hard potentials” without angular cutoff.

The proof of Theorem 1.1 runs as follows. In Section 2, we give various bounds for quantities like

$$\int_{\mathbb{R}^N} Q(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv.$$

These bounds are applied to the flow of the spatially homogeneous Boltzmann equation in Section 3, and are sufficient to prove Theorem 1.1, except that the bounds may blow up when  $t \rightarrow +\infty$ . Finally in Section 4, we explain why such a blow up cannot take place, and so we conclude the proof of Theorem 1.1. This last part is the only one which is not self-contained. It uses an estimate from [18].

## 2 Functional estimates on the collision operator

In the sequel we shall use the parametrization described in figure 1, where

$$\sigma = \frac{v - v_*}{|v - v_*|}, \quad k = \frac{v' - v'_*}{|v' - v'_*|},$$

and  $\cos \theta = \sigma \cdot k$ . The range of  $\theta$  is  $[0, \pi]$  and  $\sigma$  writes

$$\sigma = \cos \theta k + \sin \theta u,$$

where  $u$  belongs to the sphere of  $\mathbb{S}^{N-1}$  orthogonal to  $k$  (which is isomorphic to  $\mathbb{S}^{N-2}$ ).

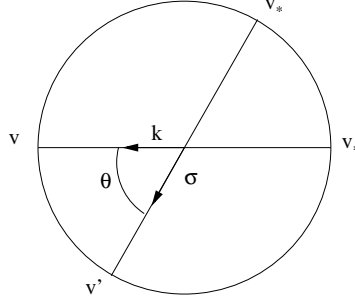


Figure 1: Geometry of binary collisions

Thanks to the change of variable  $\theta \mapsto \pi - \theta$  which exchanges  $v'$  and  $v'_*$ , the *quadratic* collision operator can be written

$$Q(f, f)(v) = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \left\{ f(v') f(v'_*) - f(v) f(v_*) \right\} B_{sym}(|v - v_*|, \cos \theta) d\theta dv_*,$$

where

$$B_{sym}(|v - v_*|, \cos \theta) = [B(|v - v_*|, \cos \theta) + B(|v - v_*|, \cos(\pi - \theta))] 1_{\cos \theta \geq 0}.$$

As a consequence, it is enough to consider the case when  $B(|v - v_*|, \cdot)$  has its support included in  $[0, \pi/2]$ . This is what we shall systematically do in the sequel (Beware that certain propositions are written for the bilinear kernel  $Q(g, f)$  and not for  $Q(f, f)$ : they hold only in fact for the symmetrized collision kernel  $B_{sym}$  defined above).

Recalling that

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma,$$

we use (for all  $F$ ) the formula (cf. [1, Section 3, proof of Lemma 1])

$$\begin{aligned} & \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} B(|v - v_*|, \cos \theta) F(v') dv d\sigma \\ &= \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \frac{1}{\cos^N(\theta/2)} B\left(\frac{|v - v_*|}{\cos(\theta/2)}, \cos \theta\right) F(v) dv d\sigma. \end{aligned} \quad (2.7)$$

Let us prove a first functional estimate independant on the integrability of the angular part of the collision kernel

**Proposition 2.1.** *Let  $B$  be a collision kernel satisfying Assumptions (1.3), (1.4), (1.5). Then, for all  $p > 1$ ,  $q \in \mathbb{R}$  and  $f$  and  $g$  nonnegative, we have*

$$\begin{aligned} \int_{\mathbb{R}^N} Q(g, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv &\leq \\ \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} |v - v_*|^\gamma b(\cos \theta) \left[ (\cos(\theta/2))^{-\frac{N+\gamma}{p'}} - 1 \right] \langle v \rangle^{pq} f^p(v) g(v_*) d\sigma dv_* dv \\ &\quad + \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} \frac{1}{p} (\cos(\theta/2))^{-\frac{N+\gamma}{p'}} |v - v_*|^\gamma b(\cos \theta) \\ &\quad \times [\langle v' \rangle^{pq} - \langle v \rangle^{pq}] f^p(v) g(v_*) d\sigma dv_* dv. \end{aligned} \quad (2.8)$$

*Proof of Proposition 2.1.* We first observe that thanks to the pre-post collisional change of variables (that is, the identity  $\int \int \int F(v, v_*, \sigma) d\sigma dv_* dv = \int \int \int F(v', v'_*, \sigma) d\sigma dv'_* dv'$ ):

$$\begin{aligned} &\int_{\mathbb{R}^N} Q(g, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv \\ &= \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} \left\{ g(v'_*) f(v') - g(v_*) f(v) \right\} f^{p-1}(v) \langle v \rangle^{pq} |v - v_*|^\gamma b(\cos \theta) d\sigma dv_* dv \\ &= \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} \left[ \langle v' \rangle^{pq} f^{p-1}(v') f(v) g(v_*) \right. \\ &\quad \left. - \langle v \rangle^{pq} f^p(v) g(v_*) \right] |v - v_*|^\gamma b(\cos \theta) d\sigma dv_* dv. \end{aligned}$$

According to Young's inequality, for all  $\mu \equiv \mu(\theta) > 0$ ,

$$f^{p-1}(v') f(v) = \left( \frac{f(v')}{\mu^{1/p}} \right)^{p-1} (\mu^{1-1/p} f(v)) \leq \left( 1 - \frac{1}{p} \right) \mu^{-1} f^p(v') + \frac{1}{p} \mu^{p-1} f^p(v),$$

so that

$$\begin{aligned} \int_{\mathbb{R}^N} Q(g, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv &\leq \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} \left[ \left( 1 - \frac{1}{p} \right) \mu^{-1} \langle v' \rangle^{pq} f^p(v') \right. \\ &\quad \left. + \frac{1}{p} \mu^{p-1} \langle v' \rangle^{pq} f^p(v) - \langle v \rangle^{pq} f^p(v) \right] g(v_*) |v - v_*|^\gamma b(\cos \theta) d\sigma dv_* dv. \end{aligned}$$

We now use (for a given  $v_*, \theta$ ) formula (2.7) for the first term in this integral. We get

$$\begin{aligned}
\int_{\mathbb{R}^N} Q(g, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv &\leq \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} \left[ \left(1 - \frac{1}{p}\right) \mu^{-1} \langle v \rangle^{pq} (\cos(\theta/2))^{-N-\gamma} f^p(v) \right. \\
&\quad \left. + \frac{1}{p} \mu^{p-1} \langle v' \rangle^{pq} f^p(v) - \langle v \rangle^{pq} f^p(v) \right] g(v_*) |v - v_*|^\gamma b(\cos \theta) d\sigma dv_* dv \\
&= \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} \langle v \rangle^{pq} |v - v_*|^\gamma b(\cos \theta) f^p(v) g(v_*) \\
&\quad \times \left[ \left(1 - \frac{1}{p}\right) \mu^{-1} (\cos(\theta/2))^{-N-\gamma} + \frac{1}{p} \mu^{p-1} - 1 \right] d\sigma dv_* dv \\
&\quad + \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} \frac{1}{p} \mu^{p-1} |v - v_*|^\gamma b(\cos \theta) f^p(v) g(v_*) \times [\langle v' \rangle^{pq} - \langle v \rangle^{pq}] d\sigma dv_* dv.
\end{aligned}$$

We now take the optimal  $\mu = \mu(\theta) > 0$ . This amounts to consider

$$\mu(\theta) = (\cos(\theta/2))^{-\frac{N+\gamma}{p}}.$$

In this way, we get estimate (2.8). □

**Remark:** With the same idea, one could easily obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} Q(g, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv &= \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} \langle v \rangle^{pq} |v - v_*|^\gamma b(\cos \theta) f^p(v) g(v_*) \\
&\quad \times \left[ \left(1 - \frac{1}{p}\right) \mu^{-1} (\cos(\theta/2))^{-N-\gamma} + \frac{1}{p} \mu^{p-1} (\cos(\theta/2))^{pq} - 1 \right] d\sigma dv_* dv \\
&+ \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} \frac{1}{p} \mu^{p-1} |v - v_*|^\gamma b(\cos \theta) f^p(v) g(v_*) [\langle v' \rangle^{pq} - (\cos(\theta/2))^{pq} \langle v \rangle^{pq}] d\sigma dv_* dv,
\end{aligned}$$

so that taking the optimal  $\mu$  given by

$$\mu(\theta) = (\cos(\theta/2))^{-\frac{N+\gamma}{p}-q},$$

the following inequality holds:

$$\begin{aligned}
\int_{\mathbb{R}^N} Q(g, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv &\leq \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} \langle v \rangle^{pq} |v - v_*|^\gamma b(\cos \theta) \left[ (\cos(\theta/2))^{q-\frac{N+\gamma}{p'}} - 1 \right] f^p(v) g(v_*) d\sigma dv_* dv \\
&\quad + \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} \frac{1}{p} (\cos(\theta/2))^{-q(p-1)-\frac{N+\gamma}{p'}} |v - v_*|^\gamma b(\cos \theta) \\
&\quad \left[ \langle v' \rangle^{pq} - (\cos(\theta/2))^{pq} \langle v \rangle^{pq} \right] f^p(v) g(v_*) d\sigma dv_* dv.
\end{aligned}$$



If  $q$  is big enough, i.e. such that

$$q - \frac{N + \gamma}{p'} > 0, \quad (2.9)$$

the first term is strictly negative, and some estimates (in the same spirit as in Lemma 2.3 below) on the term  $[\langle v' \rangle^{pq} - (\cos(\theta/2))^{pq} \langle v \rangle^{pq}]$  for small and large angles  $\theta$  would yield *directly* to

$$\begin{aligned} \int_{\mathbb{R}^N} Q(g, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv &\leq - C \int_{\mathbb{R}^N} g(v_*) dv_* \int_{\mathbb{R}^N} f^p(v) \langle v \rangle^{pq+\gamma} dv \\ &+ D \int_{\mathbb{R}^N} g(v_*) \langle v \rangle^{pq+\gamma} dv_* \int_{\mathbb{R}^N} f^p(v) dv \\ &+ D \int_{\mathbb{R}^N} g(v_*) \langle v \rangle^2 dv_* \int_{\mathbb{R}^N} f^p(v) \langle v \rangle^{pq} dv. \end{aligned}$$

We do not follow in the sequel this line of ideas because we don't want to assume (2.9). We rather choose to make a global splitting between the small and large angles  $\theta$ .

We now deduce from Proposition 2.1 a corollary enabling to bound

$$\int_{\mathbb{R}^N} Q(g, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv$$

in terms of weighted  $L^1$  and  $L^p$  norms of  $f$  and  $g$ . Note that this corollary is almost obvious to prove when the collision kernel is integrable (cutoff case).

**Corollary 2.2.** *Let  $B$  be a collision kernel satisfying Assumptions (1.3), (1.4), (1.5). We consider  $f$  and  $g$  nonnegative and  $q \in \mathbb{R}$ . We suppose moreover that  $pq \geq 2$  if  $\nu \in (-2, -1]$  and  $pq \geq 4$  if  $\nu \in (-3, -2]$ . Then,*

$$\int_{\mathbb{R}^N} Q(g, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv \leq C_{p,N,\gamma}(b) \|g\|_{L^1_{pq+\gamma}} \|f\|_{L^p_{q+\gamma/p}}^p \quad (2.10)$$

where

$$C_{p,N,\gamma}(b) = cst(p, N, \gamma) \left( \int_{\mathbb{S}^{N-1}} b(\cos \theta) (1 - \cos \theta) d\sigma \right),$$

and  $cst(p, N, \gamma)$  is a computable constant depending on  $p$ ,  $N$  and  $\gamma$ .

**Remark:** Since the non cutoff collision operator behaves roughly like some fractional Laplacian of order  $-\nu/2$ , one could wonder how a functional inequality which does not contain derivatives of the function  $f$  can hold. The answer is that the pre-post collisional change of variable and formula (2.7) (which play here the role played by integration by part for differential operators) allow to transfer the derivatives on the weight function  $\langle v \rangle^{pq}$ . This also explains why the restriction on the weight exponent  $q$  depends on the order  $\nu$  of the angular singularity.

*Proof of Corollary 2.2.* Estimate (2.8) can be written

$$\int_{\mathbb{R}^N} Q(g, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv \leq I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} |v - v_*|^\gamma b(\cos \theta) \left[ (\cos(\theta/2))^{-\frac{N+\gamma}{p'}} - 1 \right] \langle v \rangle^{pq} f^p(v) g(v_*) d\sigma dv_* dv, \\ I_2 &= \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} \frac{1}{p} \left[ (\cos(\theta/2))^{-\frac{N+\gamma}{p'}} - 1 \right] |v - v_*|^\gamma b(\cos \theta) [\langle v' \rangle^{pq} - \langle v \rangle^{pq}] f^p(v) g(v_*) d\sigma dv_* dv, \\ I_3 &= \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} \frac{1}{p} |v - v_*|^\gamma b(\cos \theta) [\langle v' \rangle^{pq} - \langle v \rangle^{pq}] f^p(v) g(v_*) d\sigma dv_* dv. \end{aligned}$$

Then the two first terms are easily estimated thanks to the formula

$$\left[ (\cos(\theta/2))^{-\frac{N+\gamma}{p'}} - 1 \right] \sim_{\theta \rightarrow 0} \frac{N + \gamma}{4p'} (1 - \cos \theta).$$

For the last one, we shall need the following lemma, which takes advantage of the symmetry properties of the collision operator:

**Lemma 2.3.** *For all  $\alpha \geq 1$ ,*

$$\left| \int_{u \in \mathbb{S}^{N-2}} [\langle v' \rangle^{2\alpha} - \langle v \rangle^{2\alpha}] du \right| \leq C_\alpha (\sin \theta/2) \langle v \rangle^{2\alpha} \langle v_* \rangle^{2\alpha}, \quad (2.11)$$

and for all  $\alpha \geq 2$ ,

$$\left| \int_{u \in \mathbb{S}^{N-2}} [\langle v' \rangle^{2\alpha} - \langle v \rangle^{2\alpha}] du \right| \leq C_\alpha (\sin \theta/2)^2 \langle v \rangle^{2\alpha} \langle v_* \rangle^{2\alpha}. \quad (2.12)$$

**Remark:** This lemma is reminiscent of the symmetry properties used in the “cancellation lemma” in [2] and [1] in order to give sense to the Boltzmann collision operator for strong angular singularity (i.e.  $\nu \in (-3, -2]$ ).

*Proof of Lemma 2.3.* We note that since

$$|v'|^2 = |v|^2 \cos^2 \theta/2 + |v_*|^2 \sin^2 \theta/2 + 2 \cos \theta/2 \sin \theta/2 |v - v_*| u \cdot v_*,$$

if one introduces (for  $x \in [0, \sqrt{2}/2]$ ) the function

$$R_\alpha(x) = \int_{u \in \mathbb{S}^{N-2}} \left[ \left( 1 + |v|^2 (1 - x^2) + |v_*|^2 x^2 + 2x \sqrt{1 - x^2} |v - v_*| u \cdot v_* \right)^\alpha - (1 + |v|^2)^\alpha \right] du,$$

we get

$$\int_{u \in \mathbb{S}^{N-2}} [(1 + |v'|^2)^\alpha - (1 + |v|^2)^\alpha] du = R_\alpha(\sin \theta/2).$$

But thanks to the change of variables  $u \rightarrow -u$ , we see that  $R_\alpha$  is even. Noticing also that  $R_\alpha(0) = 0$ , we use the identities

$$\begin{aligned} R_\alpha(x) &= x \int_0^1 R'_\alpha(sx) ds, \\ R_\alpha(x) &= x^2 \int_0^1 (1-s) R''_\alpha(sx) ds. \end{aligned}$$

We compute

$$\begin{aligned} R'_\alpha(x) &= \alpha \int_{u \in \mathbb{S}^{N-2}} \left( -2x|v|^2 + 2x|v_*|^2 + 2(1-x^2)^{1/2}|v-v_*|u \cdot v_* \right. \\ &\quad \left. - 2x^2(1-x^2)^{-1/2}|v-v_*|u \cdot v_* \right) \\ &\quad \times \left( 1 + |v|^2(1-x^2) + |v_*|^2x^2 + 2x\sqrt{1-x^2}|v-v_*|u \cdot v_* \right)^{\alpha-1} du \end{aligned}$$

and

$$\begin{aligned} R''_\alpha(x) &= \alpha(\alpha-1) \int_{u \in \mathbb{S}^{N-2}} \left( -2x|v|^2 + 2x|v_*|^2 + 2(1-x^2)^{1/2}|v-v_*|u \cdot v_* \right. \\ &\quad \left. - 2x^2(1-x^2)^{-1/2}|v-v_*|u \cdot v_* \right)^2 \\ &\quad \times \left( 1 + |v|^2(1-x^2) + |v_*|^2x^2 + 2x\sqrt{1-x^2}|v-v_*|u \cdot v_* \right)^{\alpha-2} du \\ &\quad + \alpha \int_{u \in \mathbb{S}^{N-2}} \left( -2|v|^2 + 2|v_*|^2 - 2x(1-x^2)^{-1/2} \right. \\ &\quad \left. |v-v_*|u \cdot v_* - 2|v-v_*|u \cdot v_* (2x(1-x^2)^{-1/2} + x^3(1-x^2)^{-3/2}) \right) \\ &\quad \times \left( 1 + |v|^2(1-x^2) + |v_*|^2x^2 + 2x\sqrt{1-x^2}|v-v_*|u \cdot v_* \right)^{\alpha-1} du. \end{aligned}$$

Then, for  $x \in [0, \sqrt{2}/2]$ , if  $\alpha \geq 1$ , we get

$$|R'_\alpha(x)| \leq C_\alpha \langle v \rangle^{2\alpha} \langle v_* \rangle^{2\alpha},$$

and if  $\alpha \geq 2$ ,

$$|R''_\alpha(x)| \leq C_\alpha \langle v \rangle^{2\alpha} \langle v_* \rangle^{2\alpha}.$$

This concludes the proof of Lemma 2.3. □

Let us come back to the proof of Corollary 2.2. We have

$$I_3 = \int_{\mathbb{R}^{2N}} \int_0^\pi \frac{1}{p} |v-v_*|^\gamma b(\cos \theta) R_\alpha(\sin \theta/2) (\sin \theta)^{N-2} f^p(v) g(v_*) d\theta dv_* dv$$

for  $\alpha = (pq)/2$ . Lemma 2.3 and the equality

$$(\sin \theta/2)^2 = \frac{(1 - \cos \theta)}{2}$$

conclude the proof. □

We now turn to an estimate which holds when the (angular part of the) collision kernel has its support in  $[\theta_0, \pi/2]$  for some  $\theta_0 > 0$ . As we shall see later on, this term is the “dominant part” of the same quantity when the (angular part of the) collision kernel has its support in  $[0, \pi/2]$ .

**Proposition 2.4.** *Let  $B$  satisfy Assumptions (1.3), (1.4), (1.5). We suppose moreover that  $b$  has its support in  $[\theta_0, \pi/2]$ . Then, for all  $p > 1$ ,  $q \geq 0$  and  $f$  nonnegative with bounded  $L^1_{pq+2}$  norm, we have*

$$\int_{\mathbb{R}^N} Q(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv \leq C^+(b) \|f\|_{L^p_q}^p - K^-(b) \|f\|_{L^p_{q+\gamma/p}}^p \quad (2.13)$$

with

$$C^+(b) = C^+ \left( \int_{\mathbb{S}^{N-1}} b d\sigma \right), \quad K^-(b) = K^- \left( \int_{\mathbb{S}^{N-1}} b d\sigma \right),$$

where  $C^+$ ,  $K^-$  are strictly positive constants. Both depend on an upper bound on  $\|f\|_{L^1_{pq+2}}$  and on a lower bound on  $\|f\|_{L^1}$ ;  $C^+$  also depends on  $\theta_0$ .

**Remark:** This estimate could be deduced from the results of [18], but we shall give here an elementary self-contained proof, in the same spirit as that of the proof of Proposition 2.1.

*Proof of Proposition 2.4.* Let us write the quantity to be estimated

$$\begin{aligned} \int_{\mathbb{R}^N} Q(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv &\leq \int_{\mathbb{R}^N} Q^+(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv \\ &\quad - \int_{\mathbb{R}^N} Q^-(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv, \end{aligned}$$

splitting as usual the operator between its gain and loss parts (remember that the small angles have been cutoff). On one hand, using  $|v - v_*|^\gamma \geq [\langle v \rangle^\gamma - \text{cst} \langle v_* \rangle^\gamma]$  we get

$$- \int_{\mathbb{R}^N} Q^-(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv \leq -K_0 \|b\|_{L^1(\mathbb{S}^{N-1})} \|f\|_{L^p_{q+\gamma/p}}^p + C_0 \|b\|_{L^1(\mathbb{S}^{N-1})} \|f\|_{L^p_q}^p$$

for some constant  $K_0 > 0$  depending on a lower bound on  $\|f\|_{L^1}$  and  $C_0 > 0$  depending on an upper bound on the  $\|f\|_{L^1_\gamma}$ . On the other hand,

$$\int_{\mathbb{R}^N} Q^+(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv = \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} f'_* f' f^{p-1} \langle v \rangle^{pq} B dv dv_* d\sigma$$

can be split into

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} f'_* (f j_r)' f^{p-1} \langle v \rangle^{pq} B dv dv_* d\sigma, \\ I_2 &= \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} f'_* (f j_{rc})' f^{p-1} \langle v \rangle^{pq} B dv dv_* d\sigma, \end{aligned}$$

with  $j_r(v) = 1_{|v| \leq r}$  and  $j_{r^c} = 1 - j_r$ . This means that we treat separately large and small velocities. Then

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} f_* (f j_r) (f')^{p-1} \langle v' \rangle^{pq} B \, dv \, dv_* \, d\sigma \\
&\leq \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} f_* \left[ \left(1 - \frac{1}{p}\right) \mu_1^{-1} f^p(v') + \frac{1}{p} \mu_1^{p-1} (f j_r)^p(v) \right] \langle v' \rangle^{pq} B \, dv \, dv_* \, d\sigma \\
&\leq \|b\|_{L^1(\mathbb{S}^{N-1})} \left[ \left(1 - \frac{1}{p}\right) \mu_1^{-1} (\cos \pi/4)^{-N-\gamma} \|f\|_{L_\gamma^1} \|f\|_{L_{q+\gamma/p}^p}^p \right. \\
&\quad \left. + \frac{1}{p} \mu_1^{p-1} \|f\|_{L_{pq+\gamma}^1} \|f j_r\|_{L_{q+\gamma/p}^p}^p \right],
\end{aligned}$$

and thus

$$\begin{aligned}
I_1 \leq \|b\|_{L^1(\mathbb{S}^{N-1})} &\left[ \left(1 - \frac{1}{p}\right) \mu_1^{-1} (\cos \pi/4)^{-N-\gamma} \|f\|_{L_\gamma^1} \|f\|_{L_{q+\gamma/p}^p}^p \right. \\
&\quad \left. + \frac{1}{p} \mu_1^{p-1} r^\gamma \|f\|_{L_{pq+\gamma}^1} \|f\|_{L_q^p}^p \right]. \quad (2.14)
\end{aligned}$$

As for  $I_2$ , we get

$$I_2 = \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} f' (f j_{r^c})_* f^{p-1} \langle v \rangle^{pq} \tilde{B} \, dv \, dv_* \, d\sigma$$

thanks to the change of variable  $\sigma \rightarrow -\sigma$ . Now  $\tilde{B}$  has compact support in  $[\pi/2, \pi - \theta_0]$ . Then we compute

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} (f j_{r^c})_* f (f')^{p-1} \langle v' \rangle^{pq} \tilde{B} \, dv \, dv_* \, d\sigma \\
&\leq \int_{\mathbb{R}^{2N} \times \mathbb{S}^{N-1}} (f j_{r^c})_* \left[ \left(1 - \frac{1}{p}\right) \mu_2^{-1} f^p(v') + \frac{1}{p} \mu_2^{p-1} f^p(v) \right] \langle v' \rangle^{pq} \tilde{B} \, dv \, dv_* \, d\sigma \\
&\leq \|b\|_{L^1(\mathbb{S}^{N-1})} \left[ \left(1 - \frac{1}{p}\right) \mu_2^{-1} (\sin \theta_0/2)^{-N-\gamma} \|f j_{r^c}\|_{L_\gamma^1} \|f\|_{L_{q+\gamma/p}^p}^p \right. \\
&\quad \left. + \frac{1}{p} \mu_2^{p-1} \|f j_{r^c}\|_{L_{pq+\gamma}^1} \|f\|_{L_{q+\gamma/p}^p}^p \right]
\end{aligned}$$

by using again formula (2.7) and thus

$$\begin{aligned}
I_2 \leq \|b\|_{L^1(\mathbb{S}^{N-1})} &\left[ \left(1 - \frac{1}{p}\right) \mu_2^{-1} (\sin \theta_0/2)^{-N-\gamma} (1 + r^2)^{(\gamma-2)/2} \|f\|_{L_2^1} \|f\|_{L_{q+\gamma/p}^p}^p \right. \\
&\quad \left. + \frac{1}{p} \mu_2^{p-1} \|f\|_{L_{pq+\gamma}^1} \|f\|_{L_{q+\gamma/p}^p}^p \right]. \quad (2.15)
\end{aligned}$$

Gathering (2.14) and (2.15), we obtain for the gain part

$$\begin{aligned} \int_{\mathbb{R}^N} Q^+(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv \leq \\ \|b\|_{L^1(\mathbb{S}^{N-1})} \left[ \frac{1}{p} \mu_1^{p-1} (1+r^2)^{\gamma/2} \|f\|_{L_{pq+\gamma}^1} \right] \|f\|_{L_q^p}^p \\ + \|b\|_{L^1(\mathbb{S}^{N-1})} \left[ \left(1 - \frac{1}{p}\right) \mu_1^{-1} (\cos \pi/4)^{-N-\gamma} \right. \\ \left. + \left(1 - \frac{1}{p}\right) \mu_2^{-1} (\sin \theta_0/2)^{-N-\gamma} (1+r^2)^{(\gamma-2)/2} + \frac{1}{p} \mu_2^{p-1} \right] \|f\|_{L_{pq+\gamma}^1} \|f\|_{L_{q+\gamma/p}^p}^p. \end{aligned}$$

For some  $\theta_0 > 0$  fixed, one can first choose  $\mu_2$  small enough, then  $r$  big enough (remember that  $\gamma - 2 < 0$ ), then  $\mu_1$  big enough, in such a way that

$$\left[ \left(1 - \frac{1}{p}\right) \mu_1^{-1} (\cos \pi/4)^{-N-\gamma} + \left(1 - \frac{1}{p}\right) \mu_2^{-1} (\sin \theta_0/2)^{-N-\gamma} r^{\gamma-2} + \frac{1}{p} \mu_2^{p-1} \right] \|f\|_{L_{pq+\gamma}^1} \leq \frac{K_0}{2}.$$

We thus get the wanted estimate by combining the estimates for the gain part and the loss part.  $\square$

We now can gather Corollary 2.2 with Proposition 2.4 in order to get the

**Proposition 2.5.** *Let  $B$  satisfy Assumptions (1.3), (1.4), (1.5),  $p$  belong to  $(1, +\infty)$ , and  $q \geq 0$ . We suppose moreover that  $pq \geq 2$  if  $\nu \in (-2, -1]$  and  $pq \geq 4$  if  $\nu \in (-3, -2]$ . Then, for  $f$  nonnegative with bounded  $L_{pq+2}^1$  norm, we have*

$$\int_{\mathbb{R}^N} Q(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv \leq C^+ \|f\|_{L_q^p}^p - K^- \|f\|_{L_{q+\gamma/p}^p}^p \quad (2.16)$$

for some positive constants  $C^+$  and  $K^-$ , depending on an upper bound on  $\|f\|_{L_{pq+2}^1}$  and on a lower bound on  $\|f\|_{L^1}$ .

*Proof of Proposition 2.5.* The proof is straightforward and based on a splitting of  $b$  of the form

$$b = b_c^{\theta_0} + b_r^{\theta_0}, \quad (2.17)$$

where  $b_c^{\theta_0} = b 1_{\theta \in [\theta_0, \pi/2]}$  stands for the “cutoff” part,  $b_r^{\theta_0} = 1 - b_c^{\theta_0}$  for the remaining part, and  $\theta_0 \in (0, \pi/2]$  is some fixed positive angle. We split the corresponding collision operator as  $Q = Q_c + Q_r$ . It remains then to apply Corollary 2.2 to

$$\int_{\mathbb{R}^N} Q_r(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv$$

and Proposition 2.4 to

$$\int_{\mathbb{R}^N} Q_c(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv.$$

Observing that

$$\int_{\mathbb{S}^{N-1}} b_r^{\theta_0}(\cos \theta) (1 - \cos \theta) d\sigma \rightarrow_{\theta_0 \rightarrow 0} 0,$$

we see that the term corresponding to  $Q_r$  can be absorbed by the damping (nonpositive) part of  $Q_c$ , for  $\theta_0$  small enough.  $\square$

### 3 Application to the flow of the equation

In this section, we denote by  $K$  any strictly positive constant which can be replaced by a smaller strictly positive constant, and by  $C$  any constant which can be replaced by a larger constant. We precise the dependance with respect to time when this is useful.

We now prove Theorem 1.1 without trying to get bounds which are uniform when  $t \rightarrow +\infty$ . We notice that a solution  $f(t, \cdot)$  at time  $t \geq 0$  of the Boltzmann equation (given by the results of [3], [4] and [21]) satisfies:

$$\frac{d}{dt} \int_{\mathbb{R}^N} f^p(v) \langle v \rangle^{pq} dv = p \int_{\mathbb{R}^N} Q(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv.$$

We also recall that (under our assumptions on the initial datum), such a solution  $f(t, \cdot)$  has a constant mass  $\|f(t, \cdot)\|_{L^1}$ . The  $L_q^p$  integrability of the initial datum  $f_0$  implies that this initial datum has bounded entropy, then the  $H$ -Theorem ensures that the entropy remains uniformly bounded for all times (by the initial entropy). Also its moment of order  $2 + pq$  in  $L^1$  is propagated and remains uniformly bounded for all times with explicit constant (see for instance [25]).

Then Propostion 2.5 gives the following a priori differential inequality,

$$\frac{d}{dt} \|f\|_{L_q^p}^p \leq C \|f\|_{L_q^p}^p - K \|f\|_{L_{q+\gamma/p}^p}^p. \quad (3.18)$$

In particular,

$$\frac{d}{dt} \|f\|_{L_q^p}^p \leq C \|f\|_{L_q^p}^p. \quad (3.19)$$

According to Gronwall's lemma, the norm  $\|f\|_{L_q^p}$  remains bounded (on all intervals  $[0, T]$  for  $T > 0$ ) if it is initially finite.

Let us now turn to the question of appearance of higher moments in  $L^p$  (when  $\gamma > 0$ ). Let  $r > 0$ . Using Hölder's inequality, we see that

$$\|f\|_{L_r^p} \leq \|f\|_{L_{q_1}^p}^\theta \|f\|_{L_{q_2}^p}^{1-\theta}$$

with  $r = \theta q_1 + (1 - \theta) q_2$ . Thus with  $q_2 = 0$  and  $q_1 = r + \gamma/p$ , we get

$$\|f\|_{L_r^p} \leq \|f\|_{L_{r+\gamma/p}^p}^{\frac{r}{r+\gamma/p}} \|f\|_{L^p}^{\frac{\gamma/p}{r+\gamma/p}}$$

Therefore,

$$\|f\|_{L_{r+\gamma/p}^p} \geq K_T \|f\|_{L_r^p}^{1+\frac{\gamma}{pr}},$$

where  $K_T = \left(\sup_{t \in [0, T]} \|f\|_{L^p}(t)\right)^{-\frac{\gamma}{r\gamma}}$ . But this last quantity is finite (thanks to estimate (3.19)). We thus obtain the following a priori differential inequality on  $\|f\|_{L_r^p}^p$ :

$$\frac{d}{dt} \|f\|_{L_r^p}^p \leq -K_T \left(\|f\|_{L_r^p}^p\right)^{1+\frac{\gamma}{pr}} + C \|f\|_{L_r^p}^p$$

Using a standard argument (first used by Nash for parabolic equations) of comparison with the Bernoulli differential equation

$$y' = -K_T y^{1+\frac{\gamma}{pr}} + C y,$$

whose solutions can be computed explicitly, we see that for all  $0 < t \leq T$ ,

$$\|f\|_{L_r^p}(t) < +\infty,$$

more precisely

$$\|f\|_{L_r^p}(t) \leq \left[ \frac{C}{K_T \left(1 - e^{-\frac{C\gamma}{pr}t}\right)} \right]^{r/\gamma}. \quad (3.20)$$

This concludes the proof of Theorem 1.1 for local in times bounds. It remains to study more accurately the behavior of these bounds when  $t$  goes to infinity.

### Remarks:

1. Notice that the upper bound (3.20) cannot be optimal since for example if  $\|f_0\|_{L_q^p} < +\infty$  then  $\|f\|_{L_q^p} < +\infty$  uniformly on  $[0, T]$  by the argument above, and the a priori differential inequality (3.18) implies that the quantity  $\|f\|_{L_{q+\gamma/p}^p}$  is integrable at  $t \sim 0^+$ , which is not necessarily the case of the right-hand side term in (3.20).

2. Note that in the previous computation, one should use approximate solutions of the Boltzmann equation in order to give a completely rigorous proof. For example, solutions of the equation

$$\begin{cases} \partial_t f_\varepsilon &= Q(f_\varepsilon, f_\varepsilon) + \varepsilon \Delta_v f_\varepsilon, \\ f_\varepsilon(0, \cdot) &= f_{in} * \phi_\varepsilon, \end{cases}$$

where  $\phi_\varepsilon$  is a sequence of mollifiers, can be used. This point does not lead to any difficulties.

3. It is also possible to get a slightly less stringent condition on the  $L^1$  moments of the initial data  $f_0$  by using the appearance of the  $L^1$  moments of  $f$  (in the case  $\gamma > 0$ ).



## 4 Behavior for large times

The goal of this section is to conclude the proof of Theorem 1.1 by showing that the bounds on the  $L^p$  moments are uniform when  $t \rightarrow +\infty$ .

Our starting point is a stronger result than Proposition 2.4, which is a particular case of a result proven in [18] (where the result holds for every collision kernel which satisfies angular integrability), and is based on the regularity property of the gain term of the cutoff collision kernel. This result writes:

**Proposition 4.1** (cf. [18], Theorem 4.1). *Let  $B$  satisfy Assumptions (1.3), (1.4), (1.5). We suppose moreover that  $b$  has its support in  $[\theta_0, \pi/2]$ . Then, for all  $p > 1$ ,  $q \geq 0$  and  $f$  nonnegative with bounded entropy and  $L^1_{2q+2}$  norm, we have*

$$\int_{\mathbb{R}^N} Q(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv \leq C^+(b) \|f\|_{L^p_q}^{p(1-\varepsilon)} - K^-(b) \|f\|_{L^p_{q+\gamma/p}}^p \quad (4.21)$$

with

$$C^+(b) = C^+ \left( \int_{\mathbb{S}^{N-1}} b d\sigma \right), \quad K^-(b) = K^- \left( \int_{\mathbb{S}^{N-1}} b d\sigma \right),$$

and  $C^+$ ,  $K^-$  are positive constants. Both depend on an upper bound on the entropy and the  $L^1_{2q+2}$  norm of  $f$  and a lower bound on  $\|f\|_{L^1}$ ;  $C^+$  also depends on  $\theta_0$ . Finally  $\varepsilon \in (0, 1)$  is a constant depending only on the dimension  $N$  and  $p$ .

Gathering now Corollary 2.2 with Proposition 4.1, we get the

**Proposition 4.2.** *Let  $B$  satisfy Assumptions (1.3), (1.4), (1.5),  $p$  belong to  $]1, +\infty[$  and  $q \geq 0$ . We suppose moreover that  $pq \geq 2$  if  $\nu \in (-2, -1]$  and  $pq \geq 4$  if  $\nu \in (-3, -2]$ . Then, for  $f$  nonnegative with bounded entropy and  $L^1_{\max\{pq, 2q\}+2}$  norm, we have*

$$\int_{\mathbb{R}^N} Q(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv \leq C^+ \|f\|_{L^p_q}^{p(1-\varepsilon)} - K^- \|f\|_{L^p_{q+\gamma/p}}^p \quad (4.22)$$

for some positive constants  $C^+$  and  $K^-$  depending on an upper bound on  $\|f\|_{L^1_{\max\{pq, 2q\}+2}}$ , an upper bound on the entropy and a lower bound on  $\|f\|_{L^1}$ . Finally  $\varepsilon \in (0, 1)$  is a constant depending only on the dimension  $N$  and  $p$ .

*Proof of Proposition 4.2.* The proof is exactly the same as that of Proposition 2.5. It is based on the splitting

$$b = b_c^{\theta_0} + b_r^{\theta_0}$$

and the use of Corollary 2.2 for

$$\int_{\mathbb{R}^N} Q_r(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv$$

and Proposition 4.1 for

$$\int_{\mathbb{R}^N} Q_c(f, f)(v) f^{p-1}(v) \langle v \rangle^{pq} dv.$$

□

We now can prove that the bound on the  $L^p$  moments is uniform for large times. Indeed, Proposition 4.2 leads to the following a priori differential inequality on  $y(t) = \|f(t, \cdot)\|_{L_q^p}^p$ :

$$y' \leq C y^{1-\varepsilon} - K y.$$

Then, by a maximum principle, we see that  $y(t)$  is bounded on  $[\tau, +\infty[$  as soon as it is finite at time  $\tau$ . The explicit estimate is in fact:

$$\forall t \geq \tau, \quad y(t) \leq \max \left\{ y(\tau); \left( \frac{C}{K} \right)^{1/\varepsilon} \right\}.$$

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## References

- [1] ALEXANDRE R., DESVILLETES L., VILLANI C., WENNBORG B. Entropy dissipation and long range interactions. *Arch. Rat. Mech. Anal.* 152 (2000), 327–355.
- [2] ALEXANDRE R., VILLANI C. On the Boltzmann equation for long-range interactions. *Comm. Pure Appl. Math.* 55 (2002), 30–70.
- [3] ARKERYD, L. On the Boltzmann Equation. *Arch. Rat. Mech. Anal.* 45 (1972), 1–34.
- [4] ARKERYD, L. Intermolecular forces of infinite range and the Boltzmann equation. *Arch. Rat. Mech. Anal.* 77 (1981), 11–21.
- [5] CERCIGNANI, C. *The Boltzmann Equation and its Applications*. Springer, 1988.
- [6] DESVILLETES, L. Some applications of the method of moments for the homogeneous Boltzmann and Kac equations. *Arch. Rat. Mech. Anal.* 123 (1993), 387–404.
- [7] DESVILLETES, L., AND VILLANI, C. On the spatially homogeneous Landau equation for hard potentials. Part I: Existence, uniqueness and smoothness. *Comm. Partial Differential Equations* 25, 1/2 (2000), 179–259.
- [8] DESVILLETES, L., AND WENNBORG, B., Smoothness of the solution of the spatially homogeneous Boltzmann equation without cutoff. To appear in *Comm. Partial Differential Equations*.
- [9] ESCOBEDO, M., LAURENÇOT P., AND MISCHLER, S. On a kinetic equation for coalescing particles. Prépublication Inria (2003).

- [10] GOUDON, T. On Boltzmann equations and Fokker-Planck asymptotics: influence of grazing collisions. *J. Stat. Phys.* 89, 3-4 (1997), 751–776.
- [11] GUSTAFSSON, T.  $L^p$ -estimates for the nonlinear spatially homogeneous Boltzmann equation. *Arch. Rat. Mech. Anal.* 92 (1986), 23–57.
- [12] GUSTAFSSON, T. Global  $L^p$ -properties for the spatially homogeneous Boltzmann equation. *Arch. Rat. Mech. Anal.* 103 (1988), 1–38.
- [13] IKENBERRY, E. AND TRUESDELL, C. On the pressures and the flux of energy in a gas according to Maxwell’s kinetic theory. I *Arch. Rat. Mech. Anal.* 5 (1956), 1–54.
- [14] LIONS, P.L. Compactness in Boltzmann’s equation via Fourier integral operators and applications. I, II, III. *J. Math. Kyoto Univ.* 34 (1994), 539–584.
- [15] LIONS, P.L. Regularity and compactness for Boltzmann collision operators without angular cut-off. *C.R. Acad. Sci. Paris* 326, Série I, 1 (1998), 37–41.
- [16] MISCHLER, S. AND RODRIGUEZ RICARD, M. Existence globale pour l’équation de Smoluchowski continue non homogène et comportement asymptotique des solutions. *C.R. Acad. Sc. Paris* 336, Série I, (2003), 407–412.
- [17] MISCHLER, S. AND WENNBERG, B. On the spatially homogeneous Boltzmann equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 16 (1999), no. 4, 467–501.
- [18] MOUHOT, C., AND VILLANI, C. Regularity theory for the homogeneous Boltzmann with angular cut-off. To appear in *Arch. Rat. Mech. Anal.*
- [19] TOSCANI, G., AND VILLANI, C. On the trend to equilibrium for some dissipative systems with slowly increasing a priori bounds. *J. Statist. Phys.* 98 (2000), 1279–1309.
- [20] TOSCANI, G., AND VILLANI, C. Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas. *J. Statist. Phys.* 94 (1999), 619–637.
- [21] VILLANI, C. On a new class of weak solutions for the spatially homogeneous Boltzmann and Landau equations. *Arch. Rat. Mech. Anal.* 143 (1998), 273–307.
- [22] VILLANI, C. Regularity estimates via the entropy dissipation for the spatially homogeneous Boltzmann equation without cut-off. *Rev. Mat. Iberoam.* 15 (1999), 335–352.
- [23] WENNBERG, B. On moments and uniqueness for solutions to the space homogeneous Boltzmann equation. *Transport Theory Statist. Phys.* 23 (1994), 533–539.
- [24] WENNBERG, B. The Povzner Inequality and Moments in the Boltzmann Equation. *Rendiconti del Circolo Matematico di Palermo*, Ser II (1996), Suppl. 45, 673–681.

- [25] WENNBERG, B. Entropy dissipation and moment production for the Boltzmann equation. *J. Statist. Phys.* 86 (1997), no. 5-6, 1053–1066.

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