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# Asymptotic entropy and Green speed for random walks on groups

## Entropie asymptotique et vitesse Green pour des marches alatoires sur les groupes

Sébastien Blachère      Peter Haïssinsky      Pierre Mathieu

### Abstract

We study asymptotic properties of the Green metric associated to random walks on discrete transient groups. We prove that the rate of escape of the random walk computed in the Green metric equals its asymptotic entropy. Two proofs are given. One relies on integral representations of both quantities with the extended Martin kernel. The other proof (valid only when the volume growth of the group is superpolynomial) relies on a version of the so called fundamental inequality (relating the rate of escape, the entropy and the logarithmic volume growth) extended to random walk with unbounded support.

Nous tudions certaines propri  s asymptotiques de la m  trique Green associe aux marches alatoires sur les groupes discrets transients. Nous dmonstrons que le taux de fuite de la marche alatoire, calcul pour la m  trique Green, est gal    son entropie asymptotique. Deux dmonstrations sont pr  sentes. L'une repose sur la representation integrale des deux quantits avec le noyau de Martin tendu. L'autre (valable seulement lorsque la croissance du volume est superpolynomiale) repose sur l'ingalit dite fondamentale (reliant le taux de fuite, l'entropie et la croissance logarithmique du volume) tendue aux marches alatoires support non born.

## 1 Introduction

Let  $\Gamma$  be a discrete transient group. To each finite generating set, can be associated a Cayley graph of  $\Gamma$  on which the natural graph metric is called the word metric.

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<sup>0</sup>**AMS 2000 subject classification:** 34B27, 60B15

When we study asymptotic properties of a random walk  $(Z_n)$  on  $\Gamma$ , the word metric may not be adapted. A more natural metric is the Green (or hitting) metric, defined by

$$d_G(x, y) = -\ln \mathbb{P}^x[\tau_y < \infty],$$

where  $\tau_y$  is the hitting time of the element  $y$  by the random walk started at  $x$ .

When we assume the random walk to have finite entropy, our main result (Theorem 3.2) states that the asymptotic entropy  $h$  coincides with the rate of escape computed in the Green metric (called the Green speed):

$$\ell_G = \lim_{n \rightarrow \infty} \frac{d_G(e, Z_n)}{n}.$$

We prove this result using an integral representation of  $h$  on the Martin boundary of  $\Gamma$  (Lemma 3.5) and interpreting the Green speed of the random walk as a limit of a Martin kernel (Proposition 3.3). We also give an alternative proof, for groups with superpolynomial growth, for which we extend the inequality  $h \leq l \cdot v$  to any random walk with unbounded support (Proposition 3.7) and any left invariant metric on the group with a finite first moment. Here  $l$  and  $v$  denote the rate of escape and the logarithmic volume growth respectively, both computed in the chosen metric.

The article is organized as follows. In Section 2, we give some properties of the Green metric on transient groups. Section 3 establishes the link between the entropy and the finiteness of the first moment for the Green metric and presents the two proofs of Theorem 3.2. We conclude in Section 4 by some remarks.

In the forthcoming paper [2] we use the Green metric to study fine geometric properties of the harmonic measure on boundaries of hyperbolic groups.

## 2 The Green metric

We will give the definition of the Green metric on transient groups and recall some of its properties from Blachère & Brofferio [1] (some proofs are given here for the sake of completeness).

Let  $\mu$  be a probability measure on a discrete group  $\Gamma$  whose support generates the whole group  $\Gamma$ . (We will always make that generating hypothesis). We do not assume that  $\mu$  is neither symmetric nor finitely supported. Let  $(X_k)$  be a sequence of i.i.d. random variables whose common law is  $\mu$ . The process

$$Z_k \stackrel{\text{def.}}{=} xX_1X_2 \cdots X_k,$$

with  $Z_0 = x \in \Gamma$ , is an irreducible random walk on  $\Gamma$  starting at  $x$  with law  $\mu$ . We denote  $\mathbb{P}^x$  and  $\mathbb{E}^x$ , respectively, the probability and expectation related to a random walk starting at  $x$ . When  $x = e$  (the identity of the group), the exponent will be omitted.

From now, we will always assume the random walk to be transient i.e. it eventually never returns to its starting point. This assumption is always satisfied if  $\Gamma$  is not a finite extension of  $\mathbb{Z}$  or  $\mathbb{Z}^2$  (see Woess [24, Sect. I.3.B]). On finite extension of  $\mathbb{Z}$  or  $\mathbb{Z}^2$ , results depend on the finiteness of the first moment (for the Euclidean norm) of the canonical projection  $\varphi$  of the random walk onto  $\mathbb{Z}$  or  $\mathbb{Z}^2$ :

$$M_1(\mu) \stackrel{\text{def.}}{=} \sum_{x \in \Gamma} \|\varphi(x)\| \mu(x).$$

When  $M_1(\mu) < \infty$ , the random walk is transient if and only if it has a non zero drift ( $\sum_{x \in \Gamma} \varphi(x) \mu(x) \neq 0$ ). There are examples of recurrent and transient random walks with  $M_1(\mu) = \infty$ . There are even examples of transient symmetric random walks on  $\mathbb{Z}$ . For these results and examples, see [21].

The **Green function**  $G(x, y)$  is defined as the expected number of visits at  $y$  for a random walk starting at  $x$ :

$$G(x, y) \stackrel{\text{def.}}{=} \mathbb{E}^x \left[ \sum_{n=0}^{\infty} \mathbb{1}_{\{Z_n=y\}} \right] = \sum_{n=0}^{\infty} \mathbb{P}^x[Z_n = y].$$

Since the random walk is chosen to be transient, the Green function is finite for every  $x$  and  $y$ .

Let  $\tau_y$  be the first hitting time of  $y$  by the random walk:

$$\tau_y \stackrel{\text{def.}}{=} \inf\{k \geq 0 : Z_k = y\}.$$

When  $y$  is never attained, let  $\tau_y = \infty$ . The **hitting probability** of  $y$  starting at  $x$  is

$$F(x, y) \stackrel{\text{def.}}{=} \mathbb{P}^x[\tau_y < \infty].$$

Note that that  $F(x, y) > 0$  since the support of  $\mu$  generates  $\Gamma$ , and  $F$  and  $G$  are invariant by left diagonal multiplication. In particular,  $G(y, y) = G(e, e)$ . A straightforward computation (using the strong Markov property) shows that the functions  $F$  and  $G$  are proportional:

$$G(x, y) = G(y, y)F(x, y) = G(e, e)F(x, y). \quad (1)$$

The metric we will use is the **Green metric** (or Hitting metric, defined in [1]):

$$d_G(x, y) \stackrel{\text{def.}}{=} -\ln F(x, y) = \ln G(e, e) - \ln G(x, y).$$

*Throughout the article, we will call (with some abuse of notation) **metric** any non-negative real function  $d(\cdot, \cdot)$  on  $\Gamma \times \Gamma$  which satisfies the triangular inequality and such that*

$$d(x, y) = 0 = d(y, x) \implies x = y. \quad (2)$$

**Lemma 2.1** ([1] **Lemma 2.1**) *The function  $d_G(\cdot, \cdot)$  is a left invariant metric on  $\Gamma$ .*

**Proof**

As  $F(x, y) \leq 1$ ,  $d_G(\cdot, \cdot)$  is always non-negative. The invariance of  $F(\cdot, \cdot)$  by left diagonal multiplication implies the same property for  $d_G(\cdot, \cdot)$ . Also note that since the random walk is transient we have:

$$\forall x \neq y, 1 > \mathbb{P}^x[\tau'_x < \infty] \geq \mathbb{P}^x[\tau_y < \infty] \mathbb{P}^y[\tau_x < \infty] = F(x, y)F(y, x),$$

where  $\tau'_x \stackrel{\text{def.}}{=} \inf\{k \geq 1 : Z_k = x\}$ . Thus

$$d_G(x, y) = d_G(y, x) = 0 \iff F(x, y) = F(y, x) = 1 \iff x = y.$$

Finally,

$$\mathbb{P}^x[\tau_z < \infty] \geq \mathbb{P}^x[\tau_y < \infty] \mathbb{P}^y[\tau_z < \infty]$$

leads to the triangular inequality:  $d_G(x, z) \leq d_G(x, y) + d_G(y, z)$ .  $\square$

**Exercice 2.2** *Prove that if there exist  $x \neq y \in \Gamma$  such that  $d_G(x, y) = 0$  then  $\Gamma$  is isomorphic to  $\mathbb{Z}$ . (Hint: first show that  $\#\{z \in \text{Supp}(\mu) \setminus \{e\} \text{ s.t. } d_G(e, z) = 0\} \leq 1$ ).*

Observe that, if  $\mu$  is symmetric ( $\mu(x) = \mu(x^{-1})$  for all  $x \in \Gamma$ ), then the Green function  $G$  and  $d_G$  are also symmetric and therefore  $d_G$  becomes a genuine distance on  $\Gamma$ .

For a given finite generating set, we can consider  $\Gamma$  as the vertex set of its associated Cayley graph. The corresponding graph distance is called a **word distance** and denoted with  $d_w(\cdot, \cdot)$ . Different choices of the generating set lead to different word distances in the same quasi-isometry class. When  $\mu$  is symmetric and finitely supported, the two metrics  $d_G(\cdot, \cdot)$  and  $d_w(\cdot, \cdot)$  can be compared:

**Lemma 2.3** ([1] **Lemma 2.2**) *When the measure  $\mu$  is symmetric and finitely supported, there exists a constant  $C$  such that for every  $(x, y)$ ,*

$$C^{-1}d_w(x, y)^{1/2} \leq d_G(x, y) \leq Cd_w(x, y).$$

These bounds are sharp in the sense that examples exist where the Green function decreases as  $\exp(-C|x|)$  or  $\exp(-C|x|^{1/2})$  depending on the direction toward infinity (for instance  $\mathbb{Z} \wr \mathbb{Z}$ , Revelle [20]). The two distances  $d_w$  and  $d_G$  are equivalent as soon as the Green function decreases exponentially with respect to the word distance. This is the case when  $\Gamma$  is non-amenable, as a consequence of Kesten's estimates ([18]):

$$\exists C_{na} > 1 \text{ s.t. } \forall x, y \in \Gamma \text{ and } k \in \mathbb{N} \quad \mathbb{P}^x[Z_k = y] \leq C_{na} \exp(-C_{na}k).$$

On some amenable groups, the two distances  $d_w$  and  $d_G$  are equivalent. This is for instance the case for the simple random walk on the Lamplighter group  $\mathbb{Z} \wr \mathbb{Z}_2$  (Brofferio & Woess [4]).

Throughout the article, the notion of growth of the group  $\Gamma$  always refers to the function  $V_w(n) \stackrel{\text{def.}}{=} \#\{x \in \Gamma \text{ s.t. } d_w(e, x) \leq n\}$  for some (equivalently any) symmetric finite generating set. The group will be said to have

- polynomial growth when  $V_w(n) = O(n^D)$  for some constant  $D$ ;
- superpolynomial growth when  $V_w(n)/n^D$  tends to infinity for every  $D$ ;
- subexponential growth when  $V_w(n) = o(e^{Cn})$  for every constant  $C > 0$ ;
- exponential growth when  $V_w(n)/e^{Cn}$  tends to infinity for some  $C > 0$ .

When the group  $\Gamma$  has exponential growth, an interesting property of the metric  $d_G$  is a precise estimate of the volume growth of its associated balls. Let us define  $B_G(e, n) \stackrel{\text{def.}}{=} \{x \in \Gamma \text{ s.t. } d_G(e, x) \leq n\}$  and  $V_G(n) \stackrel{\text{def.}}{=} \#B_G(e, n)$ .

**Proposition 2.4 ([1] Proposition 2.3)** *When the group  $\Gamma$  has exponential growth and the measure  $\mu$  has a finite support, there exist two constants  $C_a$  and  $C_b$  such that for every integer  $n$ ,*

$$C_a \exp(n) \leq V_G(n) \leq C_b n^3 \exp(n).$$

*The upper bound is also valid when the support is infinite. The  $n^3$  term in the upper bound becomes  $n$  when  $\Gamma$  is non-amenable.*

**Remark 2.5** *Proposition 2.3 in [1] deals with symmetric random walks with finite support. Nevertheless, its proof also works in the non-symmetric case. The upper bound is also valid for random walks with infinite support since its proof is based on the following upper estimate for the transition probability: for any (symmetric or non-symmetric) random walk on a group of exponential growth, assuming the support of  $\mu$  generates the whole group (but without any other restriction on its support),*

$$\exists C_e > 1 \text{ s.t. } \forall x, y \in \Gamma \text{ and } k \in \mathbb{N} \quad \mathbb{P}^x[Z_k = y] \leq C_e \exp(-C_e k^{1/3}), \quad (3)$$

*see Varopoulos [22] and Carlen, Kusuoka & Stroock [5].*

**Remark 2.6** *Assume that  $\Gamma$  is not a finite extension of  $\mathbb{Z}$  or  $\mathbb{Z}^2$ . Then its growth fonction  $V(n)$  satisfies  $V(n) \geq Cn^D$  for some  $D \geq 3$  and therefore the upper bound on the transition probabilities becomes:*

$$\exists C_e > 1 \text{ s.t. } \forall x, y \in \Gamma \text{ and } k \in \mathbb{N} \quad \mathbb{P}^x[Z_k = y] \leq C_e k^{-D/2}. \quad (4)$$

The same argument as in [1] then implies that

$$V_G(n) \leq C_b \exp\left(\frac{D}{D-2} \cdot n\right),$$

for some constant  $C_b$ . Observe that since  $D \geq 3$  we have

$$V_G(n) \leq C_b \exp(3n).$$

In both Remarks 2.5 and 2.6, the reason why the hypothesis of symmetry and finite support can be removed relies on Coulhon [6, Prop. IV.4] (see also [7]). Indeed [6, Prop. IV.4] allows to extend upper bounds of the  $n^{\text{th}}$  convolution power  $\mu_1^n$  of a finitely supported symmetric measure to another measure  $\mu_2^n$  under the following condition:

$$\exists c > 0 \quad \text{s.t.} \quad \forall x, \quad \mu_1(x) \leq c\mu_2(x).$$

For a general measure  $\mu$  whose support generates  $\Gamma$ , there exists  $K$  such that the support of  $\mu^K$  contains any finite symmetric generating set  $\mathcal{S}$  of  $\Gamma$ .

Hence, taking  $\mu_2 = \mu^K$ ,  $c = (\min_{x \in \mathcal{S}} \mu_2(x))^{-1}$  and  $\mu_1 = (1/\#\mathcal{S}) \times \delta_{\mathcal{S}}(x)$  (the uniform distribution on  $\mathcal{S}$ ), the measures  $\mu_1$  and  $\mu_2$  satisfy the above condition. Finally, up to changes in the constant  $C_e$ , the estimates (3) and (4) stay valid for  $\mu$ .

**Remark 2.7** Let  $v_G$  be the logarithmic volume growth of the balls for the Green metric:

$$v_G \stackrel{\text{def.}}{=} \limsup_{n \rightarrow \infty} \frac{\ln(\#B_G(e, n))}{n}.$$

When  $\Gamma$  has exponential growth, Proposition 2.4 implies that  $v_G = 1$  if  $\mu$  has a finite support. For measures with infinite support, the upper bound in Proposition 2.4 yields the inequality  $v_G \leq 1$ . When  $\Gamma$  has superpolynomial growth, Remark 2.6 gives  $v_G \leq D/(D-2)$  for any  $D \geq 3$  and therefore, letting  $D$  tend to infinity, we also have  $v_G \leq 1$ .

### 3 Entropy and Green speed

The measure  $\mu$  is now supposed to have finite **entropy**:

$$H(\mu) \stackrel{\text{def.}}{=} - \sum_{x \in \Gamma} \mu(x) \ln \mu(x) < \infty.$$

The first moment of  $\mu$  in the Green metric is, by definition, the expected Green distance between  $e$  and  $Z_1$ , which is also the expected Green distance between  $Z_n$  and  $Z_{n+1}$  for any  $n$  and has the following analytic expression:

$$\mathbb{E}[d_G(e, Z_1)] = \sum_{x \in \Gamma} \mu(x) \cdot d_G(e, x).$$

**Lemma 3.1** *The finiteness of the entropy  $H(\mu)$  implies the finiteness of the first moment of  $\mu$  with respect to the Green metric. When  $\Gamma$  is not a finite extension of  $\mathbb{Z}$  or  $\mathbb{Z}^2$ , both quantities are simultaneously finite.*

**Proof**

By construction, under  $\mathbb{P}$ , the law of  $Z_1 = X_1$  is  $\mu$ . Since  $\mathbb{P}[\tau_x < \infty] \geq \mathbb{P}[Z_1 = x] = \mu(x)$ , we have

$$\sum_{x \in \Gamma} \mu(x) \cdot d_G(e, x) = - \sum_{x \in \Gamma} \mu(x) \cdot \ln(\mathbb{P}[\tau_x < \infty]) \leq - \sum_{x \in \Gamma} \mu(x) \cdot \ln(\mu(x)) = H(\mu).$$

So that  $H(\mu) < \infty \implies \mathbb{E}[d_G(e, X_1)] < \infty$ . For the other implication, we write

$$\begin{aligned} H(\mu) &= - \sum_{x \in \Gamma \text{ s.t. } \mu(x) \leq \exp(-8d_G(e, x))} \mu(x) \cdot \ln(\mu(x)) - \sum_{x \in \Gamma \text{ s.t. } \mu(x) > \exp(-8d_G(e, x))} \mu(x) \cdot \ln(\mu(x)) \\ &\leq - \sum_{n=1}^{\infty} \sum_{x \in \partial B_G(n) \text{ s.t. } \mu(x) \leq \exp(-8d_G(e, x))} \mu(x) \cdot \ln(\mu(x)) + 8 \sum_{x \in \Gamma} \mu(x) \cdot d_G(e, x), \end{aligned}$$

where  $\partial B_G(n) \stackrel{\text{def.}}{=} B_G(e, n) \setminus B_G(e, n-1)$ . In this last inequality, the second term is the first moment. To prove that the first term is finite, we use the inequality  $-a \ln a \leq 2e^{-1} \sqrt{a}$  for  $a \in ]0, 1[$  and the upper bound on the volume from Proposition 2.4 and Remark 2.6. We then obtain the upper bound:

$$2e^{-1} \sum_{n=1}^{\infty} \sum_{x \in \partial B_G(n) \text{ s.t. } \mu(x) \leq \exp(-8d_G(e, x))} \sqrt{\mu(x)} \leq \sum_{n=1}^{\infty} V_G(n) e^{-4n} \leq \sum_{n=1}^{\infty} C_b e^{-n} < \infty.$$

We conclude that  $\mathbb{E}[d_G(e, X_1)] < \infty \implies H(\mu) < \infty$ .  $\square$

Let  $\ell_G$  be the rate of escape of the random walk  $Z_n$  in the Green metric  $d_G(e, \cdot)$ :

$$\ell_G = \ell_G(\mu) \stackrel{\text{def.}}{=} \lim_{n \rightarrow \infty} \frac{d_G(e, Z_n)}{n} = \lim_{n \rightarrow \infty} \frac{-\ln F(e, Z_n)}{n} = \lim_{n \rightarrow \infty} \frac{-\ln G(e, Z_n)}{n},$$

since the functions  $F(\cdot, \cdot)$  and  $G(\cdot, \cdot)$  differ only by a multiplicative constant. We call  $\ell_G$  the **Green speed**. Under the hypothesis that  $\mu$  has finite entropy, by the sub-additive ergodic Theorem (Kingman [19], Derriennic [8]), this limit exists almost surely and in  $L^1$ .

Let us also define the **asymptotic entropy**:

$$h = h(\mu) \stackrel{\text{def.}}{=} \lim_n \frac{H(\mu^n)}{n},$$

where  $\mu^n$  is the  $n^{\text{th}}$  convolution power of the measure  $\mu$ .

Our main result is the following.

**Theorem 3.2** *For a transient random walk on a discrete infinite group  $\Gamma$ , with finite entropy, the asymptotic entropy  $h$  and the Green speed  $\ell_G$  are equal.*



### 3.1 Proof using the Martin boundary

The **Martin kernel** is defined (using (1)) for all  $(x, y) \in \Gamma \times \Gamma$  by

$$K(x, y) \stackrel{\text{def.}}{=} \frac{G(x, y)}{G(e, y)} = \frac{F(x, y)}{F(e, y)}.$$

The Martin kernel continuously extends in a compactification of  $\Gamma$  called the Martin compactification  $\Gamma \cup \partial_M \Gamma$  where  $\partial_M \Gamma$  is the **Martin boundary**. Let us briefly recall the construction of  $\partial_M \Gamma$ : let  $\Psi : \Gamma \rightarrow C(\Gamma)$  be defined by  $y \mapsto K(\cdot, y)$ . Here  $C(\Gamma)$  is the space of real valued functions defined on  $\Gamma$  endowed with the topology of pointwise convergence. It turns out that  $\Psi$  is injective and thus we may identify  $\Gamma$  with its image. The closure of  $\Psi(\Gamma)$  is compact in  $C(\Gamma)$  and, by definition,  $\partial_M \Gamma = \overline{\Psi(\Gamma)} \setminus \Psi(\Gamma)$  is the Martin boundary. In the compact space  $\Gamma \cup \partial_M \Gamma$ , for any initial point  $x$ , the random walk  $Z_n$  almost surely converges to some random variable  $Z_\infty \in \partial_M \Gamma$  (see for instance Dynkin [10] or Woess [24]).

We note that, by means of the Green metric, one can also consider the Martin compactification as a special example of a **Busemann compactification**. We recall that the Busemann compactification of a proper metric space  $(X, d)$  is obtained through the embedding  $\Phi : X \rightarrow C(X)$  defined by  $y \mapsto d(\cdot, y) - d(e, y)$ . ( $e$  is some base point.) In general,  $C(X)$  should be endowed with the topology of uniform convergence on compact sets. The Busemann compactification of  $X$  is the closure of the image  $\Phi(X)$  in  $C(X)$ . We refer to [17] and the references therein for further details.

If one now chooses for  $X$  the group  $\Gamma$  itself and for the distance  $d$  the Green metric, both constructions of the Martin and Busemann compactifications coincide as it is straightforward from the relation:

$$d_G(\cdot, y) - d_G(e, y) = -\ln K(\cdot, y).$$

We first prove that the Green speed can be expressed in terms of the extended Martin kernel. Theorem 3.2 will then be a direct consequence of the formulas in Proposition 3.3 and Lemma 3.5. For that purpose we need to define the reversed law  $\tilde{\mu}$ :

$$\forall x \in \Gamma, \quad \tilde{\mu}(x) \stackrel{\text{def.}}{=} \mu(x^{-1}).$$

Note that  $H(\tilde{\mu}) = H(\mu)$ .

**Proposition 3.3** *Let  $\mu$  be a probability measure on  $\Gamma$  with finite entropy  $H(\mu)$  and whose support generates  $\Gamma$ . Let  $(Z_n)$  be a random walk on  $\Gamma$  of law  $\mu$  (starting at  $e$ ) and let  $\tilde{X}_1$  be an independent random variable of law  $\tilde{\mu}$ . Then*

$$\ell_G = \mathbb{E} \tilde{\mathbb{E}}[-\ln K(\tilde{X}_1, Z_\infty)],$$

where  $\tilde{\mathbb{E}}$  refers to integration with respect to the random variable  $\tilde{X}_1$  and  $\mathbb{E}$  refers to integration with respect to the random walk  $(Z_n)$ .

**Proof**

As  $\mu$  is supposed to have finite entropy,  $\ell_G$  is well defined as an almost sure and  $L^1$  limit. We will prove that the sequence

$$\mathbb{E}[d_G(e, Z_{n+1}) - d_G(e, Z_n)] = \mathbb{E}[-\ln G(e, Z_{n+1}) + \ln G(e, Z_n)],$$

converges to  $\mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1, Z_\infty)]$ . Since its limit in the Cesaro sense is  $\ell_G$ , it implies the formula in Proposition 3.3.

By definition of the reversed law  $\tilde{\mu}$ ,  $\tilde{X}_1^{-1}$  has the same law as  $X_1$  the first increment of the random walk  $(Z_n)$ . Note also that  $X_2 \cdots X_{n+1}$  has the same law as  $Z_n = X_1 \cdots X_n$ . Since we have assumed that  $\tilde{X}_1$  is independent of the sequence  $(Z_n)$ ,  $Z_{n+1} = X_1 \cdot X_2 \cdots X_{n+1}$  has the same law as  $\tilde{X}_1^{-1} \cdot Z_n$  and therefore, using the translation invariance,  $G(e, Z_{n+1})$  has the same law as  $G(\tilde{X}_1, Z_n)$ . Thus,

$$\begin{aligned} \mathbb{E}[-\ln G(e, Z_{n+1}) + \ln G(e, Z_n)] &= \mathbb{E}\tilde{\mathbb{E}}[-\ln G(\tilde{X}_1, Z_n) + \ln G(e, Z_n)] \\ &= \mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1, Z_n)]. \end{aligned}$$

By continuity of the Martin kernel up to the Martin boundary, for every  $x \in \Gamma$ , the sequence  $K(x, Z_n)$  almost surely converges to  $K(x, Z_\infty)$ . We need an integrable bound for  $-\ln K(\tilde{X}_1, Z_n)$  (uniformly in  $n$ ) to justify the convergence of the expectation.

To prove that  $-\ln K(\tilde{X}_1, Z_n)$  cannot go too far in the negatives, we first prove a maximal inequality for the sequence  $(K(\tilde{X}_1, Z_n))$  following Dynkin [10].

**Lemma 3.4** *For any  $a > 0$ ,*

$$\mathbb{P}\tilde{\mathbb{P}}[\sup_n K(\tilde{X}_1, Z_n) \geq a] \leq \frac{1}{a}.$$

where  $\tilde{\mathbb{P}}$  refers to the measure associated to the random variable  $\tilde{X}_1$  and  $\mathbb{P}$  refers to the measure associated to the random walk  $(Z_n)$ .

**Proof**

We fix an integer  $R$ . Let  $\sigma_R$  be the time of the last visit to the ball  $B_G(e, R)$  for the random walk  $(Z_n)$ . (We will only consider this random time for starting points within  $B_G(e, R)$ . Since the random walk is transient,  $\sigma_R$  is well defined and almost surely finite.) Let us define the sequence  $(Z_{\sigma_R - k})$  ( $k \in \mathbb{N}$ ). As this sequence exists (in  $\Gamma$ ) only for  $k \leq \sigma_R$ , we take the following convention for negative indices:

$$\{k > \sigma_R\} \implies \{Z_{\sigma_R - k} \stackrel{\text{def.}}{=} \star\},$$

so that the sequence  $(Z_{\sigma_R - k})_{k \in \mathbb{N}}$  is well defined and takes its values in  $\Gamma \cup \{\star\}$ . Note that  $Z_{\sigma_R}$  takes its value in  $B_G(e, R)$ .

Let us call  $\mathcal{F}_k$  the  $\sigma$ -algebra generated by  $(Z_{\sigma_R}, \dots, Z_{\sigma_R-k})$  and observe that

$$\mathbb{1}_{\{k \leq \sigma_R\}} \in \mathcal{F}_k,$$

since  $\{k \leq \sigma_R\}$  means that none of  $Z_{\sigma_R}, \dots, Z_{\sigma_R-k}$  equals  $\star$ . With the convention that, for any  $x \in \Gamma$ ,  $K(x, \star) = 0$ , we can define, for any  $x$  in  $\Gamma$ , the non-negative sequence  $(K(x, Z_{\sigma_R-k}))$  ( $k \in \mathbb{N}$ ). This sequence is adapted to the filtration  $(\mathcal{F}_k)$  and we will prove, following Dynkin [10, §6,7], that it is a supermartingale with respect to  $(\mathcal{F}_k)$ .

Namely, for any positive integer  $k$  and any sequence  $z_0, z_1, \dots, z_{k-1}$  in  $\Gamma \cup \{\star\}$  (with  $z_0 \in B_G(e, R)$ ), let us check that

$$\begin{aligned} \mathbb{E} \left[ K(x, Z_{\sigma_R-k}) \prod_{j=0}^{k-1} \mathbb{1}_{\{Z_{\sigma_R-j}=z_j\}} \right] \\ = (K(x, z_{k-1}) - \delta_x(z_{k-1})G(e, x)^{-1}) \cdot \mathbb{E} \left[ \prod_{j=0}^{k-1} \mathbb{1}_{\{Z_{\sigma_R-j}=z_j\}} \right]. \quad (5) \end{aligned}$$

We first compute the left-hand side of (5) in the case where none of  $z_0, z_1, \dots, z_{k-1}$  equals  $\star$ . Using first that  $K(x, \star) = 0$ ,

$$\begin{aligned} \sum_{z_k \in \Gamma \cup \{\star\}} \mathbb{P}[Z_{\sigma_R} = z_0, \dots, Z_{\sigma_R-(k-1)} = z_{k-1}, Z_{\sigma_R-k} = z_k] \cdot K(x, z_k) \\ = \sum_{z_k \in \Gamma} \mathbb{P}[Z_{\sigma_R} = z_0, \dots, Z_{\sigma_R-k} = z_k] \cdot K(x, z_k) \\ = \sum_{z_k \in \Gamma} \mathbb{P}[k \leq \sigma_R, Z_{\sigma_R} = z_0, \dots, Z_{\sigma_R-k} = z_k] \cdot K(x, z_k), \end{aligned}$$

since none of  $z_0, \dots, z_k$  equals  $\star$ , it means in particular

$$\bigcap_{j=0}^k \{Z_{\sigma_R-j} = z_j\} \subset \{k \leq \sigma_R\}.$$

Then,

$$\begin{aligned}
& \sum_{z_k \in \Gamma \cup \{\star\}} \mathbb{P}[Z_{\sigma_R} = z_0, \dots, Z_{\sigma_R-(k-1)} = z_{k-1}, Z_{\sigma_R-k} = z_k] \cdot K(x, z_k) \\
&= \sum_{z_k \in \Gamma} \sum_{m=k}^{\infty} \mathbb{P}[\sigma_R = m, Z_m = z_0, \dots, Z_{m-k} = z_k] \cdot K(x, z_k) \\
&= \sum_{z_k \in \Gamma} \sum_{m=k}^{\infty} \mathbb{P}[Z_{m-k} = z_k] \mu(z_k^{-1} z_{k-1}) \cdots \mu(z_1^{-1} z_0) \mathbb{P}^{z_0}[\sigma_R = 0] \cdot K(x, z_k) \\
&= \mu(z_{k-1}^{-1} z_{k-2}) \cdots \mu(z_1^{-1} z_0) \mathbb{P}^{z_0}[\sigma_R = 0] \sum_{z_k \in \Gamma} G(e, z_k) \mu(z_k^{-1} z_{k-1}) \cdot K(x, z_k) \\
&= \mu(z_{k-1}^{-1} z_{k-2}) \cdots \mu(z_1^{-1} z_0) \mathbb{P}^{z_0}[\sigma_R = 0] \sum_{z_k \in \Gamma} G(x, z_k) \mu(z_k^{-1} z_{k-1}) \\
&= \mu(z_{k-1}^{-1} z_{k-2}) \cdots \mu(z_1^{-1} z_0) \mathbb{P}^{z_0}[\sigma_R = 0] (G(x, z_{k-1}) - \delta_x(z_{k-1})) .
\end{aligned}$$

Using the same kind of computation we get that the right-hand side of (5) equals

$$\begin{aligned}
& \sum_{m=k-1}^{\infty} \mathbb{P}[\sigma_R = m, Z_m = z_0, \dots, Z_{m-(k-1)} = z_{k-1}] (K(x, z_{k-1}) - \delta_x(z_{k-1}) G(e, x)^{-1}) \\
&= \sum_{m=k-1}^{\infty} \mathbb{P}[Z_{m-(k-1)} = z_{k-1}] \mu(z_{k-1}^{-1} z_{k-2}) \cdots \mu(z_1^{-1} z_0) \mathbb{P}^{z_0}[\sigma_R = 0] \\
&\quad \times (K(x, z_{k-1}) - \delta_x(z_{k-1}) G(e, x)^{-1}) \\
&= \mu(z_{k-1}^{-1} z_{k-2}) \cdots \mu(z_1^{-1} z_0) \mathbb{P}^{z_0}[\sigma_R = 0] (G(x, z_{k-1}) - \delta_x(z_{k-1})) .
\end{aligned}$$

So (5) is true as soon as  $z_0, \dots, z_{k-1}$  take values in  $\Gamma$ . Now suppose that  $z_j = \star$  for some  $j \leq k-1$ , then

$$\{Z_{\sigma_R-j} = z_j\} \implies \{Z_{\sigma_R-(k-1)} = \star\} \implies \{Z_{\sigma_R-k} = \star\} .$$

Since  $K(x, \star) = 0$ , the left-hand side of (5) is null. To check that the right-hand side is also null, observe that

$$z_{k-1} \neq \star \implies \mathbb{E} \left[ \prod_{j=0}^{k-1} \mathbb{1}_{Z_{\sigma_R-j} = z_j} \right] = 0 ,$$

and

$$z_{k-1} = \star \implies K(x, z_{k-1}) = 0 \quad \text{and} \quad \delta_x(z_{k-1}) = 0 ,$$

since  $x \in \Gamma$ . The proof of (5) is now complete. Since the Green function is non-negative, we deduce from (5)

$$\mathbb{E} \left[ K(x, Z_{\sigma_R-k}) \prod_{j=0}^{k-1} \mathbb{1}_{\{Z_{\sigma_R-j} = z_j\}} \right] \leq K(x, z_{k-1}) \cdot \mathbb{E} \left[ \prod_{j=0}^{k-1} \mathbb{1}_{\{Z_{\sigma_R-j} = z_j\}} \right] ,$$

thus proving the supermartingale property of the sequence  $(K(x, Z_{\sigma_R-k}))$  ( $k \in \mathbb{N}$ ).

We use similar arguments to compute the expectation of the value of the supermartingale at time  $k = 0$ :  $\mathbb{E}[K(x, Z_{\sigma_R})]$  which turns out not to depend on  $R$ .

$$\begin{aligned}
\mathbb{E}[K(x, Z_{\sigma_R})] &= \sum_{m=0}^{\infty} \sum_{z \in B_G(e, R)} \mathbb{P}[\sigma_R = m, Z_m = z] \cdot K(x, z) \\
&= \sum_{m=0}^{\infty} \sum_{z \in B_G(e, R)} \mathbb{P}^z[\sigma_R = 0] \mathbb{P}[Z_m = z] \cdot K(x, z) \\
&= \sum_{z \in B_G(e, R)} \mathbb{P}^z[\sigma_R = 0] \cdot G(x, z) \\
&= \sum_{z \in B_G(e, R)} \mathbb{P}^z[\sigma_R = 0] \sum_{m=0}^{\infty} \mathbb{P}^x[Z_m = z] \\
&= \sum_{z \in B_G(e, R)} \sum_{m=0}^{\infty} \mathbb{P}^x[\sigma_R = m, Z_{\sigma_R} = z] \\
&= 1.
\end{aligned}$$

We can now use Doob's maximal inequality for non-negative supermartingales, see for instance [3, Prop. 5.13], to get that:

$$\forall x \in \Gamma, \quad \mathbb{P}[\sup_k K(x, Z_{\sigma_R-k}) \geq a] \leq \frac{1}{a}.$$

So,

$$\mathbb{P}\tilde{\mathbb{P}}[\sup_k K(\tilde{X}_1, Z_{\sigma_R-k}) \geq a] \leq \frac{1}{a},$$

and, letting  $R$  tend to infinity,

$$\mathbb{P}\tilde{\mathbb{P}}[\sup_n K(\tilde{X}_1, Z_n) \geq a] \leq \frac{1}{a}.$$

□

Let us go back to the proof of Proposition 3.3: Lemma 3.4 implies that, for any  $b > 0$ ,

$$\mathbb{P}\tilde{\mathbb{P}}[\sup_n \ln K(\tilde{X}_1, Z_n) \geq b] \leq e^{-b},$$

and therefore  $\mathbb{E}\tilde{\mathbb{E}}[\sup_n \ln K(\tilde{X}_1, Z_n) \mathbb{1}_{K(\tilde{X}_1, Z_n) \geq 1}] < \infty$ .

On the other hand, we have

$$K(x, Z_n) = \frac{\mathbb{P}^x[\tau_{Z_n} < \infty]}{\mathbb{P}^e[\tau_{Z_n} < \infty]} \geq \frac{\mathbb{P}^x[\tau_e < \infty] \cdot \mathbb{P}^e[\tau_{Z_n} < \infty]}{\mathbb{P}^e[\tau_{Z_n} < \infty]} = \mathbb{P}^e[\tau_{x^{-1}} < \infty] \geq \tilde{\mu}(x),$$

and

$$\tilde{\mathbb{E}}[-\ln \tilde{\mu}(\tilde{X}_1)] = H(\tilde{\mu}) = H(\mu) < \infty.$$

Writing that

$$\begin{aligned} |\ln K(\tilde{X}_1, Z_n)| &= \ln K(\tilde{X}_1, Z_n) \mathbb{1}_{K(\tilde{X}_1, Z_n) \geq 1} - \ln K(\tilde{X}_1, Z_n) \mathbb{1}_{K(\tilde{X}_1, Z_n) \leq 1} \\ &\leq \ln K(\tilde{X}_1, Z_n) \mathbb{1}_{K(\tilde{X}_1, Z_n) \geq 1} - \ln \tilde{\mu}(\tilde{X}_1), \end{aligned}$$

we conclude that the random variable  $\sup_n |\ln K(\tilde{X}_1, Z_n)|$  is integrable. We can therefore apply the dominated convergence theorem to deduce that the sequence  $\mathbb{E}[-\ln G(e, Z_{n+1}) + \ln G(e, Z_n)]$  converges to

$$\mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1, Z_\infty)].$$

□

**Lemma 3.5** *Let  $\Gamma$  be discrete group and  $\mu$  be a probability measure on  $\Gamma$  whose support generates  $\Gamma$ , with finite entropy  $H(\mu)$ . Then*

$$h = \mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1, Z_\infty)].$$

**Proof**

Recall that  $\tilde{\mu}$  is the law of  $\tilde{X}_1$ . We have

$$\mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1, Z_\infty)] = \int_{\Gamma} \int_{\partial_M \Gamma} -\ln(K(x, \xi)) d\nu(\xi) d\tilde{\mu}(x),$$

where  $\nu_y(\cdot)$  is the harmonic measure on the Martin boundary  $\partial_M \Gamma$  for a random walk (of law  $\mu$ ) starting at  $y$  and  $\nu(\cdot) = \nu_e(\cdot)$ . By the Martin boundary convergence Theorem, see Hunt [13] or Woess [24, Th. 24.10], the Martin kernel  $K(x, \xi)$  is the Radon-Nikodym derivative of  $\nu_x$  by  $\nu$  at  $\xi$ . Therefore

$$\mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1, Z_\infty)] = \int_{\Gamma} \int_{\partial_M \Gamma} -\ln \left( \frac{d\nu_x(\xi)}{d\nu(\xi)} \right) d\nu(\xi) d\mu(x^{-1}).$$

We will make the following change of variables as in Kaimanovich [14, Sect. 1.7.3.]. As  $\partial_M \Gamma$  is stable by left multiplication, the change of variables  $\xi \mapsto x^{-1}\xi$  gives  $\nu_x(\xi) \mapsto \nu(\xi)$  and  $\nu(\xi) \mapsto \nu_{x^{-1}}(\xi)$ . Hence, changing also  $x$  into  $x^{-1}$ , gives

$$\begin{aligned} \mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1, Z_\infty)] &= \int_{\Gamma} \int_{\partial_M \Gamma} -\ln \left( \frac{d\nu(\xi)}{d\nu_x(\xi)} \right) d\nu_x(\xi) d\mu(x) \\ &= \int_{\Gamma} \int_{\partial_M \Gamma} \ln \left( \frac{d\nu_x(\xi)}{d\nu(\xi)} \right) d\nu_x(\xi) d\mu(x). \quad (6) \end{aligned}$$

Observe that  $d\nu_x(\xi)/d\nu(\xi)$  is the Radon-Nikodym derivative of the joint law of  $(\tilde{X}_1^{-1}, Z_\infty)$  with respect to the product measure  $\mu(\cdot) \otimes \nu(\cdot)$ . Therefore (6) means that  $\mathbb{E}\mathbb{E}[-\ln K(\tilde{X}_1, Z_\infty)]$  is the relative entropy of the joint law of  $(\tilde{X}_1^{-1}, Z_\infty)$  with respect to  $\mu(\cdot) \otimes \nu(\cdot)$ , which equals the asymptotic entropy  $h$  (see Derriennic [9] who actually takes the latter as the definition of the asymptotic entropy and proves that both definitions coincide.)  $\square$

## 3.2 Proof using the "fundamental" inequality

We now present a different proof of Theorem 3.2 in the case of groups with superpolynomial growth.

There is a general obvious link between the Green speed and the asymptotic entropy:

**Lemma 3.6** *For any random walk with finite entropy  $H(\mu)$ , we have  $\ell_G \leq h$ .*

**Proof**

The sequence  $\frac{1}{n}d_G(e, Z_n)$  converges to  $\ell_G$  in  $L^1$ . Therefore

$$\ell_G = \lim_{n \rightarrow \infty} \frac{-\sum_{x \in \Gamma} \mu^n(x) \ln \left( \sum_{k=0}^{\infty} \mu^k(x) \right)}{n} \leq \lim_{n \rightarrow \infty} \frac{-\sum_{x \in \Gamma} \mu^n(x) \ln \mu^n(x)}{n} = h.$$

$\square$

Our aim is to prove the other inequality and deduce that  $h = \ell_G$ .

We rely on the so-called fundamental inequality:

$$h \leq \ell_G \cdot v_G, \tag{7}$$

which holds as soon as  $\mu$  has finite entropy. On groups with superpolynomial growth, Remark 2.7 gives  $v_G \leq 1$  and therefore inequality (7) implies that  $h \leq \ell_G$  and we conclude that  $h = \ell_G$ . Thus, all that remains to be done in order to complete the proof of Theorem 3.2 in the case of groups with superpolynomial growth is justify (7). This is the content of the next Proposition.

A version of inequality (7), when the speed and volume growth are computed in a word metric, is proved by Guivarc'h [12] and is discussed in great details by Vershik [23]. The same proofs as in [12] or [23] would apply to any invariant metric on  $\Gamma$ , for instance the Green metric, provided  $\mu$  has finite support. The fundamental inequality is also known to hold for measures with unbounded support and a finite first moment in a word metric. See for instance [11, Lem. 6] or [16] but note that their argument seems to apply only to word metrics and observe that the Green metric is not a word metric in general: as a matter of fact it need not even be a geodesic metric. We shall derive the fundamental inequality in the Green metric, under the mere assumption that the entropy of  $\mu$  is finite.

We present our result in a general setting (for any invariant metric and group) since it has its own interest.

**Proposition 3.7** *Let  $\mu$  be the law of the increment of a random walk on a discrete group  $\Gamma$ , starting at a point  $e$ , and let  $d(\cdot, \cdot)$  be a left invariant metric. Under the following hypothesis*

- *The measure  $\mu$  has finite entropy,*
- *The measure  $\mu$  has finite first moment with respect to the metric  $d$ ,*
- *The logarithmic volume growth  $v \stackrel{\text{def.}}{=} \limsup_{n \rightarrow \infty} \frac{\ln(\#B(e, n))}{n}$  is finite,*

*the asymptotic entropy  $h$ , the rate of escape  $\ell \stackrel{\text{def.}}{=} \lim_n \frac{d(e, Z_n)}{n}$  (limit both in  $L^1$  and almost surely) and the logarithmic volume growth  $v$  satisfy the following inequality:*

$$h \leq \ell \cdot v.$$

### Proof

The proof relies on the idea of Guivarc'h [12, Prop. C.2]. Fix  $\varepsilon > 0$  and, for all integer  $n$ , let  $B_\varepsilon^n \stackrel{\text{def.}}{=} B(e, (\ell + \varepsilon)n)$  (here the balls are defined for the metric  $d(e, \cdot)$ ). We split  $\Gamma \setminus B_\varepsilon^n$  into a sequence of annuli: choose  $K > \ell + \varepsilon$  and define

$$\begin{aligned} \mathcal{C}_\varepsilon^{n, K} &\stackrel{\text{def.}}{=} B(e, Kn) \setminus B_\varepsilon^n \\ \forall i \geq 1, \mathcal{C}_i^{n, K} &\stackrel{\text{def.}}{=} B(e, 2^i Kn) \setminus B(e, 2^{i-1} Kn). \end{aligned}$$

Define the conditional entropy

$$H(\mu | A) \stackrel{\text{def.}}{=} - \sum_{x \in A} \frac{\mu(x)}{\mu(A)} \ln \frac{\mu(x)}{\mu(A)}.$$

The entropy of  $\mu^n$  can then be written as

$$\begin{aligned} H(\mu^n) &= \mu^n(B_\varepsilon^n) \cdot H(\mu^n | B_\varepsilon^n) + \mu^n(\mathcal{C}_\varepsilon^{n, K}) \cdot H(\mu^n | \mathcal{C}_\varepsilon^{n, K}) \\ &\quad + \sum_{i=1}^{\infty} \mu^n(\mathcal{C}_i^{n, K}) \cdot H(\mu^n | \mathcal{C}_i^{n, K}) + H'_n, \end{aligned} \quad (8)$$

where

$$H'_n \stackrel{\text{def.}}{=} -\mu^n(B_\varepsilon^n) \cdot \ln(\mu^n(B_\varepsilon^n)) - \mu^n(\mathcal{C}_\varepsilon^{n, K}) \cdot \ln(\mu^n(\mathcal{C}_\varepsilon^{n, K})) - \sum_{i=1}^{\infty} \mu^n(\mathcal{C}_i^{n, K}) \cdot \ln(\mu^n(\mathcal{C}_i^{n, K})). \quad (9)$$

We will repeatedly use the fact that the entropy of any probability measure supported by a finite set is maximal for the uniform measure and then equals the logarithm of the volume. First observe that

$$H(\mu^n | B_\varepsilon^n) \leq \ln(\#B_\varepsilon^n) \leq (\ell + \varepsilon) \cdot v \cdot n + o(n),$$



and thus the first term in (8) satisfies

$$\lim_n \frac{\mu^n(B_\varepsilon^n) \cdot H(\mu^n | B_\varepsilon^n)}{n} \leq (\ell + \varepsilon) \cdot v.$$

For the second term in (8), we get that

$$H(\mu^n | \mathcal{C}_\varepsilon^{n,K}) \leq \ln(\#\mathcal{C}_\varepsilon^{n,K}) \leq K \cdot v \cdot n + o(n).$$

On the other hand,  $\ell$  is also the limit in probability of  $d(e, Z_n)/n$ , hence  $\forall \varepsilon > 0$ ,  $\lim_n \mu^n(B_\varepsilon^n) = 1$ . Therefore  $\lim_n \mu^n(\mathcal{C}_\varepsilon^{n,K}) = 0$  and the second term in (8) satisfies

$$\lim_n \frac{\mu^n(\mathcal{C}_\varepsilon^{n,K}) \cdot H(\mu^n | \mathcal{C}_\varepsilon^{n,K})}{n} = 0.$$

For the third term in (8), as before, we have

$$H(\mu^n | \mathcal{C}_i^{n,K}) \leq \ln(\#\mathcal{C}_i^{n,K}) \leq 2^i K \cdot v \cdot n + o(n),$$

and, by the definition of  $\mathcal{C}_i^{n,K}$ ,

$$\mu^n(\mathcal{C}_i^{n,K}) = \mathbb{E} \left[ \mathbb{I}_{\{Z_n \in \mathcal{C}_i^{n,K}\}} \right] \leq \mathbb{E} \left[ \frac{d(e, Z_n)}{2^{i-1} K n} \cdot \mathbb{I}_{\{Z_n \in \mathcal{C}_i^{n,K}\}} \right]. \quad (10)$$

So,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{\infty} \mu^n(\mathcal{C}_i^{n,K}) \cdot H(\mu^n | \mathcal{C}_i^{n,K}) &\leq \left( \frac{2v}{n} + o\left(\frac{1}{n}\right) \right) \mathbb{E} \left[ d(e, Z_n) \sum_{i=1}^{\infty} \mathbb{I}_{\{Z_n \in \mathcal{C}_i^{n,K}\}} \right] \\ &= \left( \frac{2v}{n} + o\left(\frac{1}{n}\right) \right) \mathbb{E} \left[ d(e, Z_n) \cdot \mathbb{I}_{\{d(e, Z_n) > K n\}} \right]. \end{aligned}$$

As  $d(e, Z_n) \leq \sum_{k=1}^n d(e, X_k)$ ,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{\infty} \mu^n(\mathcal{C}_i^{n,K}) \cdot H(\mu^n | \mathcal{C}_i^{n,K}) &\leq \left( \frac{2v}{n} + o\left(\frac{1}{n}\right) \right) \\ &\quad \times \sum_{j=1}^n \mathbb{E} \left[ d(e, X_j) \cdot \mathbb{I}_{\{\sum_{k=1}^n d(e, X_k) > K n\}} \right] \\ &= (2v + o(1)) \mathbb{E} \left[ d(e, X_1) \cdot \mathbb{I}_{\{\sum_{k=1}^n d(e, X_k) > K n\}} \right], \end{aligned}$$

since  $X_1, \dots, X_n$  are i.i.d., so that the random variables

$$Y_j \stackrel{\text{def.}}{=} d(e, X_j) \cdot \mathbb{I}_{\{\sum_{k=1}^n d(e, X_k) > K n\}},$$

have the same distribution.

By the strong law of large numbers, the sequence  $\frac{1}{n} \sum_{k=1}^n d(e, X_k)$  almost surely converges to  $\mathbb{E}[d(e, X_1)] = m < \infty$ . As a consequence, for any  $K > m$ , we have

$$d(e, X_1) \cdot \mathbb{1}_{\{\sum_{k=1}^n d(e, X_k) > Kn\}} \xrightarrow{\text{a.s.}} 0. \quad (11)$$

Moreover, as

$$d(e, X_1) \cdot \mathbb{1}_{\{\sum_{k=1}^n d(e, X_k) > Kn\}} \leq d(e, X_1),$$

which is integrable, the limit in (11) occurs also in  $L^1$ . Then,

$$\lim_n \frac{1}{n} \sum_{i=1}^{\infty} \mu^n(\mathcal{C}_i^{n,K}) \cdot H(\mu^n | \mathcal{C}_i^{n,K}) = 0.$$

We are left with  $H'_n$ . As  $\lim_n \mu^n(B_\varepsilon^n) = 1$  and  $\lim_n \mu^n(\mathcal{C}_\varepsilon^{n,K}) = 0$ ,

$$\lim_n -\mu^n(B_\varepsilon^n) \cdot \ln(\mu^n(B_\varepsilon^n)) - \mu^n(\mathcal{C}_\varepsilon^{n,K}) \cdot \ln(\mu^n(\mathcal{C}_\varepsilon^{n,K})) = 0.$$

For the last term in (9), note that (10) gives

$$\mu^n(\mathcal{C}_i^{n,K}) \leq \frac{1}{2^{i-1}Kn} \sum_{k=1}^n \mathbb{E}[d(e, X_k)] \leq \frac{m}{2^{i-1}K}.$$

Together with the inequality  $-\ln(a) \leq 2e^{-1}\sqrt{a}$  seen in Lemma 3.1, we get

$$-\sum_{i=1}^{\infty} \mu^n(\mathcal{C}_i^{n,K}) \cdot \ln(\mu^n(\mathcal{C}_i^{n,K})) \leq 2e^{-1} \sum_{i=1}^{\infty} \sqrt{\mu^n(\mathcal{C}_i^{n,K})} < \infty.$$

So  $\lim_n H'_n/n = 0$ .

Finally, taking the limit  $n \rightarrow \infty$ , we deduce from (8) that  $h \leq (\ell + \varepsilon) \cdot v$  for any  $\varepsilon$ , so  $h \leq \ell \cdot v$ .  $\square$

## 4 Remarks

**Remark 4.1** *The proof of Theorem 3.2 using the Martin boundary relies on the translation invariance of  $\Gamma$  but the hypothesis that the graph is a Cayley graph of a discrete group seems too strong. It would be interesting to extend this proof to the case of space homogeneous Markov chains (see [15]).*

**Remark 4.2** *The difference between the two proofs of Theorem 3.2 is the case of groups with polynomial growth. In that case the logarithmic volume growth for the word metric is null. Then, Lemma 3.6 and Proposition 3.7 imply that  $h = 0 = \ell_G$  if the random walk has a finite first moment for the word metric. Is there an example of a random walk on a discrete group with polynomial growth, with finite entropy but infinite first moment for the word metric, and non-zero asymptotic entropy?*

**Remark 4.3** *Theorem 3.2 shows that the obvious inequality in the proof of Lemma 3.6 is indeed an equality. Therefore, the correct order for*

$$\frac{-\sum_{x \in \Gamma} \mu^n(x) \ln \left( \sum_{k=0}^{\infty} \mu^k(x) \right)}{n}$$

*is given by the single term for  $k = n$ .*

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Sébastien Blachère, Peter Haïssinsky, Pierre Mathieu  
 blachere@cmi.univ-mrs.fr, phaissin@cmi.univ-mrs.fr, pmathieu@cmi.univ-mrs.fr  
 Université Aix-Marseille 1  
 CMI, 39 rue Joliot-Curie  
 13453 Marseille Cedex, France