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► **To cite this version:**

Marc Jungers, Emmanuel Trélat, Hisham Abou-Kandil. Weak Stackelberg strategy with closed-loop information structure. 2010. hal-00086780v2

HAL Id: hal-00086780

<https://hal.science/hal-00086780v2>

Preprint submitted on 24 Aug 2010 (v2), last revised 22 Nov 2010 (v3)

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WEAK STACKELBERG STRATEGY WITH CLOSED-LOOP INFORMATION STRUCTURE

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Key words. Stackelberg strategy, game theory, multi-criteria optimization, closed-loop information structure, bilevel optimization problem.

AMS subject classifications. 91A65, 49N70, 49N90

Abstract. This paper deals with the weak Stackelberg strategy in the case of a closed-loop information structure. Two-player differential games are considered with one leader and one follower. We first derive necessary conditions for the Stackelberg equilibrium in the general case of nonlinear criteria for finite time horizon games which lead to an expression of the optimal controls along the associated trajectory. Then, using focal point theory, the necessary conditions are also shown to be sufficient and lead to cheap control. The set of initial states allowing the existence of an optimal trajectory is emphasized. The Linear Quadratic case is detailed to illustrate these results.

1. Introduction. A Stackelberg game, named after Heinrich von Stackelberg in recognition of his pioneering work on static games [42], designates a two-player noncooperative decision making problem formalized as a hierarchical combination of two optimization problems. The *lower level* decision maker, called the *follower*, selects a strategy optimizing his/her own objective function, depending on the strategy of the *upper level* decision maker, called the *leader*. The leader may decide his/her strategy, optimizing his/her objective function, relative to the decisions of both players by knowing the rational reaction of the follower. Such a problem may be viewed as a particular *bilevel optimization problem* [16, 41, 44]. When the rational reaction set is not reduced to a singleton, the situation is more complex and several formulations exist and have been introduced by Leitmann [25] (see also [4]) and called weak and strong Stackelberg strategies by Breton and *al.* [13] or pessimistic and optimistic ones in [16, 17]. Note that a mixed formulation has been studied in [3] as well as their relations [27].

Game theory being a generic multiobjective optimization framework, the field of applications of Stackelberg strategies is large and includes, for example, economy [5], social behaviors, marketing [20], network communications [10, 23], military intelligence [33]. The Stackelberg strategy for dynamic games was introduced in [15, 35, 36]. We consider here two-player nonzero sum differential games with one leader and one follower.

The information structure in the game is the set of all available information for the players to make their decisions. The methods used to tackle such a Stackelberg optimization problem depend on the specific information structure.

When *open-loop* information structure is considered, no measurement of the state of the system is available and the players are committed to follow a predetermined strategy based on their knowledge of the initial state, the system's model and the cost functional to be minimized. Necessary conditions for obtaining a Stackelberg

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equilibrium with an open-loop information structure are well known [1, 19, 22, 30, 35–38, 43] and are derived from the standard Pontryagin Minimum Principle [24]. The obtained controls in this case are only functions of time.

The Stackelberg strategy is known to be inconsistent in time [18], and dynamic programming cannot help to derive the optimal controls. Note however that the concept of *feedback* Stackelberg control, not considered in the present paper, is defined as the limit of controls issued from dynamic programming on infinitesimally small time subintervals (see [6, 8, 21, 28, 31, 32]). This concept differs from the concept of closed-loop control under consideration in this paper.

For the *closed-loop* information structure case (or more precisely the memoryless closed-loop information structure), each player has access to current state measurements and thus can adapt his strategy in function of the system's evolution. For the closed-loop information structure case, determining the Stackelberg strategy for dynamic games is much harder and has been for a long time an open problem. The main difficulty comes from the presence, in the expression of the rational reaction set of the follower, of the partial derivative of the leader's control with respect to the measurement of the state. Several attempts have been proposed in the literature to overcome this difficulty [8]. Among such techniques, two main approaches could be distinguished.

The first method is dedicated to strong Stackelberg games with a team-optimal approach introduced in [7, 9]. At the first step, the leader and the follower are aiming at optimizing the leader criterion as a team. With some weak assumptions, the optimal value of the leader criterion is attained for a parametrized family of controls for the leader and the follower. At the second step, the parameters of both controls are chosen such that the control of the follower lies in the rational reaction set in response to the control of the leader [39]. This could be interpreted as a threat formulated by the leader towards the follower.

The second approach consists in defining the whole rational reaction set of the follower to a leader control. The resulting optimal control problem turns out to be nonclassical, not solvable *a priori* with the usual Pontryagin Minimum Principle. To solve this kind of nonclassical problem, a variational method is proposed in [34], assuming that this is a normal optimization problem (the possible occurrence of an abnormal case is not mentioned). Moreover, in [29] it is emphasized that this technique does not lead to a solution for all initial states, and the difficulty is bypassed by assuming that the initial state of the system is uniformly distributed over the unit sphere and replacing the optimization criterion with its mean value over the initial state.

In this paper, we derive rigorously necessary conditions for a Stackelberg equilibrium with closed-loop information structure, *along the associated trajectory*, in the same spirit as in [34] by considering all cases. In addition, sufficient conditions of the optimization problem for Linear Quadratic differential games are established using focal times theory. Based on these necessary and/or sufficient conditions, we then characterize all initial states from which there emanates an optimal trajectory. Also, an extension is proposed to associate with every initial state an optimal trajectory by introducing the Jacobian of the leader's control in his own criterion. Note that in [34], although the final result (for Linear Quadratic games only) is correct, some of arguments thereof used to derive the necessary conditions are missing.

The outline of the paper is as follows: in Section 2, the weak Stackelberg strategy is mathematically formulated. Section 3 gathers the necessary conditions for an equi-

librium for the follower (Section 3.1) and for the leader (Section 3.2). A degeneration property of the Stackelberg strategy is emphasized in Section 3.3. These necessary conditions are detailed in the case of Linear Quadratic two-player differential games in Section 3.4. The sufficient conditions are provided for the Linear Quadratic case in Section 4. All these results lead to the two main results of this paper Theorem 3.21 and Theorem 4.5, which ensure the existence of optimal trajectories. Concluding remarks make up Section 5. The main proofs are gathered in Appendix.

2. Preliminaries: Stackelberg strategy. Let $t_f > 0$. Consider a two-player differential game defined on the time horizon $[0, t_f]$ with

$$\dot{x}(t) = f(t, x(t), u_{|t}, v_{|t}), \quad x(0) = x_0, \quad (2.1)$$

where the state vector $x(t)$ is in \mathbb{R}^n , f is of class \mathcal{C}^1 and where $u_{|t}$ and $v_{|t}$ are the values at time t of the controls of both players u and v , m_1 - and m_2 -vector functions lying in the sets of admissible controls \mathcal{U} and \mathcal{V} , which will be defined below.

The two criteria, defined as real-valued functions on $\mathcal{U} \times \mathcal{V}$, are

$$J_i(u, v) = g_i(x(t_f)) + \int_0^{t_f} L_i(t, x(t), u_{|t}, v_{|t}) dt, \quad i \in \{1; 2\}. \quad (2.2)$$

The functions L_1 and L_2 are \mathcal{C}^1 with respect to x , u and v are continuous with respect to time t . The first player, who chooses the control u , aims at minimizing the criterion J_1 and the second player, who chooses the control v , aims at minimizing the criterion J_2 . It is assumed in this paper that there exists an information bias in the game which induces a hierarchy between the two players. We therefore have a Stackelberg differential game. Player 1, called the leader, associated with u announces his strategy first to Player 2. Player 2, associated with v is called the follower. The leader knows how the follower will rationally react to a fixed control u . This bias of information will be used by the leader to minimize his/her own criterion.

DEFINITION 2.1. *The rational reaction set or best response set of the follower is defined by*

$$T : u \in \mathcal{U} \longmapsto Tu \subset \mathcal{V} \quad (2.3)$$

where $Tu = \{v \mid v \text{ minimizes } J_2(u, \bar{v}), \bar{v} \in \mathcal{V}\}$.

DEFINITION 2.2. *A (weak) Stackelberg equilibrium strategy (u^*, v^*) is defined by the minimization problem*

$$\begin{cases} v^* \in Tu^*, \\ u^* \text{ minimizes } \max_{v \in Tu} J_1(u, v). \end{cases} \quad (2.4)$$

REMARK 2.3. *We consider here the usual definition of a Stackelberg equilibrium, called a weak one (see [7, 13]). It can be interpreted as the minimization of the leader's criterion whenever the worst choice of the follower's control among the rational reaction set occurs. It can be viewed as a robust property with respect to the choice of the follower, which could be crucial in automatic control for robust control such as $\mathcal{H}_2/\mathcal{H}_\infty$ -control [26, 45]. An alternative definition is used in [34], called strong Stackelberg strategy, for which the choice of the follower's control among the rational reaction set is made to minimize, jointly with the leader, the leader's criterion (that is a team optimization problem described below). These two definitions coincide*

whenever T is a single-valued map. In the sequel, we do not assume that T is a single-valued map, and we make a weaker assumption (Assumption 3.4) that is related to the criterion J_1 and the set T .

The structure of the controls has to be formalized to make precise the induced optimization problems:

- Whenever the controls are only functions of time t , that is $u|_t = u(t)$ and $v|_t = v(t)$, the game has an *open-loop* information structure. Necessary conditions for obtaining an open-loop Stackelberg solution, derived from the usual Pontryagin Minimum Principle, are well known [1, 35–38].
- The case where the controls are functions of time t and of the current value of the state $x(t)$, $u|_t = u(t, x(t))$ and $v|_t = v(t, x(t))$, is called *closed-loop* Stackelberg strategy. This is the case that we consider in the present paper. The controls are thus designed along the trajectory $x(t)$ associated with the Stackelberg solution.
- Considering $u|_t = u(t, x)$ and $v|_t = v(t, x)$, defined for every $x \in \mathbb{R}^n$, and *not only along the trajectory* $x(t)$ corresponds to the concept of *feedback* Stackelberg solution, in the spirit of the dynamic programming approach [6, 35, 36], even though dynamic programming does not apply rigorously to such a Stackelberg strategy, due to time inconsistency.

As said above, in this paper we consider *closed-loop Stackelberg strategies*. Note that, in the linear quadratic case (see Section 3.4), the values of the closed-loop Stackelberg controls and of the feedback Stackelberg controls coincide *along the associated trajectory*.

Within the framework of a closed-loop information structure, the differential game given by (2.1) is written as

$$\dot{x}(t) = f(t, x(t), u(t, x(t)), v(t, x(t))), \quad x(0) = x_0. \quad (2.5)$$

Throughout the paper, for the sake of clarity, we use the notation $u_x = \frac{\partial u}{\partial x}$ to denote the Jacobian of $u(t, x)$ with respect to the second variable x . Here, \mathcal{U} (resp. \mathcal{V}) denotes an open subset of $\mathcal{L}^\infty([0, t_f] \times \mathbb{R}^n, \mathbb{R}^{m_1})$ (resp. of $\mathcal{L}^\infty([0, t_f] \times \mathbb{R}^n, \mathbb{R}^{m_2})$) such that, for every couple $(u, v) \in \mathcal{U} \times \mathcal{V}$, the associated trajectory $x(\cdot)$, solution of (2.5), is well defined on $[0, t_f]$. More precisely

$$\mathcal{U} = \left\{ u(\cdot, \cdot) \in \mathcal{L}^\infty([0, t_f] \times \mathbb{R}^n, \mathbb{R}^{m_1}), \text{ such that } \frac{\partial u}{\partial x}(t, x(t)) = u_x(t, x(t)) \text{ exists and } u(t, x(t)) \text{ as } u_x(t, x(t)) \text{ are continuous in } x(t) \text{ and piecewise continuous in } t \right\} \quad (2.6)$$

$$\mathcal{V} = \left\{ v(\cdot, \cdot) \in \mathcal{L}^\infty([0, t_f] \times \mathbb{R}^n, \mathbb{R}^{m_2}), \text{ such } v(t, x(t)) \text{ is continuous in } x(t) \text{ and piecewise continuous in } t \right\} \quad (2.7)$$

The main difficulty in a closed-loop Stackelberg strategy is the presence of the partial derivative $\frac{\partial u^*}{\partial x}$ in the necessary conditions for the follower. Different alternatives, surveyed *e.g.* in [8], have been proposed in the literature to overcome the difficulty raised by the presence of the partial derivative $\frac{\partial u^*}{\partial x}$ in the necessary conditions for the follower. The first approach consists in finding an equivalent team problem leading to a global minimization of the leader's cost and obtaining a particular representation of the leader's control [7]. The second approach consists in determining the

follower's rational reaction set and the necessary conditions for the leader optimizing a dynamical problem over an infinite dimensional strategy space subject to dynamical constraints (evolution of the state vector and follower's rational reaction set). In [34], this problem is handled using a variational method, which however does not lead to all solutions. In this paper, based on the Pontryagin Minimum Principle, we derive necessary conditions for a Stackelberg equilibrium, in the sense discussed formerly. Our study permits to compute the values of the controls $u^*(t, x(t))$ and $v^*(t, x(t))$ *along the optimal trajectories*. We do not provide an expression of the Stackelberg controls $u^*(t, x)$ and $v^*(t, x)$ for every x , except in the linear quadratic case (see Section 3.4) where our main result can be made more precise and more explicit. Finally, using the theory of focal points, we provide sufficient conditions for local optimality (which are global in the linear quadratic case).

3. Necessary conditions for a Stackelberg equilibrium. Due to the hierarchy between the two players, necessary conditions are first established for the follower, and then for the leader.

3.1. For the follower. The best response set of the follower Tu^* involved in the definition of Stackelberg equilibrium (2.4) implies that, for a fixed control u^* , the control v^* of the follower solves the following optimization problem:

PROBLEM 3.1.

$$\min_{v \in \mathcal{V}} J_2(u^*, v),$$

subject to

$$\dot{x}(t) = f(t, x(t), u^*(t, x(t)), v(t, x(t))), \quad x(0) = x_0. \quad (3.1)$$

Necessary conditions to solve Problem 3.1 are derived in the next proposition, proved in Appendix.

PROPOSITION 3.2. *Consider a closed-loop Stackelberg pair of controls (u^*, v^*) for the system (3.1), associated with the trajectory $x(\cdot)$, then there exists an absolutely continuous mapping $p_2 : [0, t_f] \rightarrow \mathbb{R}^n$, being a non trivial line vector, such that*

$$\begin{aligned} 0 &= \frac{\partial H_2}{\partial v}(t, x(t), u^*(t, x(t)), v(t, x(t))) \\ &= p_2(t) \frac{\partial f}{\partial v}(t, x(t), u^*(t, x(t)), v(t, x(t))) + \frac{\partial L_2}{\partial v}(t, x(t), u^*(t, x(t)), v(t, x(t))) \end{aligned} \quad (3.2)$$

$$\begin{aligned} \dot{p}_2(t) &= -\frac{dH_2}{dx}(t, x(t), u^*(t, x(t)), v(t, x(t)), p_2(t)) \\ &= -p_2(t) \frac{\partial f}{\partial x}(t, x(t), u^*(t, x(t)), v(t, x(t))) - \frac{\partial L_2}{\partial x}(t, x(t), u^*(t, x(t)), v(t, x(t))) \\ &\quad - p_2(t) \frac{\partial f}{\partial u}(t, x(t), u^*(t, x(t)), v(t, x(t))) \frac{\partial u^*}{\partial x}(t, x(t)) \\ &\quad - \frac{\partial L_2}{\partial u}(t, x(t), u^*(t, x(t)), v(t, x(t))) \frac{\partial u^*}{\partial x}(t, x(t)), \end{aligned} \quad (3.3)$$

$$p_2(t_f) = \frac{\partial g_2(x(t_f))}{\partial x}, \quad (3.4)$$

where H_2 denotes the Hamiltonian of the follower, $H_2(t, x, u, v, p_2) = p_2 f(t, x, u, v) + L_2(t, x, u, v)$. All solutions v of Equations (3.2)-(3.4) are gathered in the set valued mapping $T' : \mathcal{U} \rightarrow \mathcal{V}$. \square

REMARK 3.3. Note that condition (3.3) may seem akin to open-loop control since it does not involve terms in $\frac{\partial v}{\partial x}$. This dependency comes from condition (3.2) that implies that the open-loop and closed-loop control for the follower coincide.

The set valued mapping T' , which is defined by equations (3.2)-(3.4) and satisfies $Tu \subseteq T'u$, will be used in the next subsection to derive necessary conditions for the leader, under a weak assumption on T , T' and the criterion J_1 , as explained next.

3.2. For the leader. Throughout the paper, we make the following assumption, needed to derive Proposition 3.6.

ASSUMPTION 3.4.

$$J_1(u, v') \leq J_1(u, v), \quad \forall v' \in T'u, v \in Tu, u \in \mathcal{U}_{nb}^*, \quad (3.5)$$

where \mathcal{U}_{nb}^* denotes a neighborhood of u^* in \mathcal{U} . □

REMARK 3.5. Note that Assumption 3.4 differs from the main assumption of [34], due to our choice to consider a weak Stackelberg problem instead of a strong one.

PROPOSITION 3.6. Consider a pair of controls (u^*, v^*) associated with Stackelberg solution. The control u^* is defined by Equation (2.4), that is

$$u^* = \arg \min_{u \in \mathcal{U}} \max_{v \in Tu} J_1(u, v). \quad (3.6)$$

Under Assumption 3.4, there holds

$$u^* = \arg \min_{u \in \mathcal{U}} \max_{v \in T'u} J_1(u, v). \quad (3.7)$$

□

REMARK 3.7. Note that, under standard assumptions such as the convexity of J_2 [24, Chap. 5], or the fact that the set Tu^* be reduced to a singleton (as in the linear quadratic case, see Section 3.4), or the fact that $Tu = T'u$ (as assumed in [6, 34] for example), the necessary conditions for the follower are also sufficient (see [24, Chap. 5]) and Assumption 3.4 is fulfilled. In the linear quadratic case in particular, Assumption 3.4 is automatically satisfied.

We also make the following assumption.

ASSUMPTION 3.8. Equation (3.2) has a unique solution v belonging to the rational reaction set Tu^*

$$v(t, x) = \mathcal{S}(t, x(t), p_2(t), u^*(t, x(t))) \in Tu^*, \quad (3.8)$$

with \mathcal{S} continuous with respect to t and \mathcal{C}^1 with respect to x and p_2 .

REMARK 3.9. Note that this assumption is automatically satisfied in the case of linear quadratic games (see Section 3.4).

REMARK 3.10. Assumption 3.8 may be relaxed in the following way, without affecting our main results. Assume that Equation (3.3) is solvable, but admits several (local) solutions. Then, assume that f and L_2 are \mathcal{C}^2 with respect to the variable v , that is $\frac{\partial H_2}{\partial v}$ is \mathcal{C}^1 with respect to v . If the strict Legendre condition holds at every (local) solution, i.e. $\frac{\partial^2 H_2}{\partial v^2}$ is positive definite, then it follows from the Implicit Function Theorem that, locally, every solution v can be written as (3.8). Then, our main results apply to every such (local) solution. We stress again that, in the linear quadratic case (see Section 3.4), there exists a unique global solution.

The leader, with his top hierarchical position with respect to the follower, can impose the control of the follower. The leader knows the reaction of the follower, i.e., he knows the function \mathcal{S} . Then the leader seeks to minimize his own criterion where v is replaced by the function \mathcal{S} . Using the notations $\tilde{L}_1(t, x, p_2, u) = L_1(t, x, u, \mathcal{S}(t, x, p_2, u))$ and

$$\tilde{J}_1(u) = \int_0^{t_f} \tilde{L}_1(t, x(t), p_2(t), u(t, x(t))) dt + g_1(x(t_f)), \quad (3.9)$$

the following problem is considered:

$$\min_{u \in \mathcal{U}} \tilde{J}_1(u) \quad (3.10)$$

under two dynamical constraints:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t, x(t)), \mathcal{S}(t, x(t), p_2(t), u(t, x(t)))), \\ &= F_1(t, x(t), p_2(t), u(t, x(t))), \end{aligned} \quad (3.11)$$

$$\begin{aligned} \dot{p}_2(t) &= -p_2(t) \frac{\partial f}{\partial x}(t, x(t), u(t, x(t)), \mathcal{S}(t, x(t), p_2(t), u(t, x(t)))) \\ &\quad - \frac{\partial L_2}{\partial x}(t, x(t), u(t, x(t)), \mathcal{S}(t, x(t), p_2(t), u(t, x(t)))) \\ &\quad - p_2 \frac{\partial f}{\partial u}(t, x(t), u(t, x(t)), \mathcal{S}(t, x(t), p_2(t), u(t, x(t)))) \frac{\partial u}{\partial x}(t, x(t)) \\ &\quad - \frac{\partial L_2}{\partial u}(t, x(t), u(t, x(t)), \mathcal{S}(t, x(t), p_2(t), u(t, x(t)))) \frac{\partial u}{\partial x}(t, x(t)) \\ &= F_{21}(t, x(t), p_2(t), u(t, x(t))) + F_{22}(t, x(t), p_2(t), u(t, x(t))) \frac{\partial u}{\partial x}(t, x(t)), \end{aligned} \quad (3.12)$$

and $x(0) = x_0$, $p_2(t_f) = \frac{\partial g_2}{\partial x}(x(t_f))$. Denote $\tilde{L}_2(t, x, p_2, u) = L_2(t, x, u, \mathcal{S}(t, x, p_2, u))$, $F_{21}(t, x, p_2, u) = -p_2 \frac{\partial F_1}{\partial x}(t, x, p_2, u) - \frac{\partial \tilde{L}_2}{\partial x}(t, x, p_2, u)$ and finally $F_{22}(t, x, p_2, u) = -p_2 \frac{\partial F_1}{\partial u}(t, x, p_2, u) - \frac{\partial \tilde{L}_2}{\partial u}(t, x, p_2, u)$.

Due to the nonclassical term u_x , the usual Pontryagin Minimum Principle (see [24]) cannot be applied. However it is possible to adapt its proof and derive a version of the Pontryagin Minimum Principle adapted to the system (3.11)-(3.12) (see Appendix). The following proposition is proved in the Appendix.

PROPOSITION 3.11. *If the trajectory $x(\cdot)$, associated to the pair (u^*, v^*) of closed-loop Stackelberg controls, is solution of the Stackelberg problem, then there exist absolutely continuous mappings $\lambda_1, \lambda_2 : [0, t_f] \rightarrow \mathbb{R}^n$, called costate vectors (written as*

line vectors by convention), and a scalar $\lambda^0 \geq 0$, such that

$$0 = \lambda_2(t) \left(\frac{\partial F_{22}}{\partial u}(t, x(t), p_2(t), u(t, x(t)))u_x + \frac{\partial F_{21}}{\partial u}(t, x(t), p_2(t), u(t, x(t))) \right)^T \\ + \lambda_1(t) \frac{\partial F_1}{\partial u}(t, x(t), p_2(t), u(t, x(t))) + \lambda^\circ \frac{\partial \tilde{L}_1}{\partial u}(t, x(t), p_2(t), u(t, x(t))), \quad (3.13)$$

$$0 = \lambda_2^T(t) F_{22}(t, x(t), p_2(t), u(t, x(t))), \quad (3.14) \\ = \lambda_2^T(t) \left(p_2(t) \frac{\partial F_1}{\partial u}(t, x(t), p_2(t), u(t, x(t))) + \frac{\partial \tilde{L}_2}{\partial u}(t, x(t), p_2(t), u(t, x(t))) \right),$$

$$\dot{\lambda}_1(t) = -\lambda_2(t) \left(\frac{\partial F_{21}}{\partial x}(t, x(t), p_2(t), u(t, x(t))) + \frac{\partial F_{22}}{\partial x}(t, x(t), p_2(t), u(t, x(t)))u_x \right)^T \\ - \lambda_1(t) \frac{\partial F_1}{\partial x}(t, x(t), p_2(t), u(t, x(t))) - \lambda^\circ \frac{\partial \tilde{L}_1}{\partial x}(t, x(t), p_2(t), u(t, x(t))), \quad (3.15)$$

$$\dot{\lambda}_2(t) = -\lambda_2(t) \left(\frac{\partial F_{21}}{\partial p_2}(t, x(t), p_2(t), u(t, x(t))) + \frac{\partial F_{22}}{\partial p_2}(t, x(t), p_2(t), u(t, x(t)))u_x \right)^T \\ - \lambda_1(t) \frac{\partial F_1}{\partial p_2}(t, x(t), p_2(t), u(t, x(t))) - \lambda^\circ \frac{\partial \tilde{L}_1}{\partial p_2}(t, x(t), p_2(t), u(t, x(t))), \quad (3.16)$$

for almost every $t \in [0, t_f]$. Moreover, the following relations, called transversality conditions, hold:

$$\lambda_2(0) = 0, \quad \lambda_1(t_f) - \lambda^\circ \frac{\partial g_1}{\partial x}(x(t_f)) + \lambda_2(t_f) \frac{\partial^2 g_2}{\partial x^2}(x(t_f)) = 0. \quad (3.17)$$

□

3.3. Degeneration property. Equation (3.14) implies either that $\lambda_2 \equiv 0$ or $F_{22} \equiv 0$ (or both) along the interval $[0, t_f]$, where $F_{22} = -p_2 \frac{\partial F_1}{\partial u} - \frac{\partial \tilde{L}_2}{\partial u}$. In the general case, the relation $F_{22} \equiv 0$ is not obvious to analyze, however we will see that, in the linear quadratic case, this relation cannot hold under a weak additional assumption. In the general case we conjecture that it cannot hold in generic situations. The next proposition investigates the first case of that alternative, which is, in some sense, the generic one.

PROPOSITION 3.12. *Under the additional assumption that the $m_1 \times m_1$ matrix*

$$\frac{\partial}{\partial u} \left(p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u} \right)^T \quad (3.18)$$

is invertible, there holds

$$\lambda_2 \equiv 0. \quad (3.19)$$

REMARK 3.13. *The fact that $\lambda_2 \equiv 0$ means that the leader does not take into account the rational reaction set of the follower. It is actually not in contradiction with the hierarchical position between the leader and the follower; indeed, in this case*

the leader does not take into account the reaction of the follower, because he can impose his desired control to the follower. The leader is omnipotent with respect to the follower. The condition $\frac{\partial F_{22}}{\partial u}$ invertible formalizes this privileged position of the leader.

Proposition 3.12, under a weak assumption, emphasizes the omnipotence of the leader leading to a degeneration of the Stackelberg strategy. The hierarchical roles of the players seem to disappear. An omnipotent leader is able to impose his/her control to the other player without taking into account the rational reaction set of the follower.

These conditions happen to be more explicit in the linear quadratic case. In the next paragraph we focus on that case, and analyze more deeply the former necessary conditions. Our analysis finally leads to a more precise result on the Stackelberg controls in the linear quadratic case.

3.4. Linear Quadratic case. In this section, we focus on the linear quadratic case, due to its widespread presence in the literature [14], and reformulate and make more precise our previous results. Consider a linear dynamic constraint

$$\dot{x} = Ax + B_1u + B_2v \quad (3.20)$$

and the quadratic criteria

$$J_1(u, v) = \int_0^{t_f} \frac{1}{2} (x^T Q_1 x + u^T R_{11} u + v^T R_{12} v) dt + \frac{1}{2} x(t_f)^T K_{1f} x(t_f), \quad (3.21)$$

$$J_2(u, v) = \int_0^{t_f} \frac{1}{2} (x^T Q_2 x + u^T R_{21} u + v^T R_{22} v) dt + \frac{1}{2} x(t_f)^T K_{2f} x(t_f), \quad (3.22)$$

where the matrices Q_i , R_{ij} and K_{if} are symmetric for $i, j \in \{1, 2\}$, and $Q_i \geq 0$, $R_{ii} > 0$, $R_{12} > 0$, and R_{21} invertible. In what follows, denote $S_{ij} = B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T$ and $S_i = B_i R_{ii}^{-1} B_i^T$ for $i, j \in \{1, 2\}$.

3.4.1. Necessary conditions for the follower. The Hamiltonian associated with the follower (dynamic constraint (3.20) and criterion (3.22)) is

$$H_2(t, x, p_2, u, v) = p_2(Ax + B_1u + B_2v) + \frac{1}{2} (x^T Q_2 x + u^T R_{21} u + v^T R_{22} v). \quad (3.23)$$

Applying the relations (3.2), (3.3), we obtain

$$\dot{p}_2(t) = -\frac{dH_2}{dx}(t, x(t), p_2(t), u(t, x(t))), \quad (3.24)$$

$$= -p_2(t)A - x^T(t)Q_2 - p_2(t)B_1 \frac{\partial u^*}{\partial x}(t, x(t)) - u^T(t, x(t))R_{21} \frac{\partial u^*}{\partial x}(t, x(t)),$$

$$p_2(t_f) = x(t_f)^T K_{2f}, \quad (3.25)$$

$$\frac{\partial H_2}{\partial v} = 0 = p_2(t)B_2 + v^T(t, x(t))R_{22}. \quad (3.26)$$

Since R_{22} is invertible by assumption, the optimal control is

$$v(t, x(t)) = -R_{22}^{-1} B_2^T p_2^T(t) = \mathcal{S}(t, x(t), p_2(t), u(t, x(t))). \quad (3.27)$$

3.4.2. Necessary conditions for the leader. In the case of quadratic criteria, there holds

$$F_1(t, x, p_2, u) = Ax + B_1u - S_2p_2^T, \quad (3.28)$$

$$F_{21}(t, x, p_2, u) = -p_2A - x^TQ_2, \quad (3.29)$$

$$F_{22}(t, x, p_2, u) = -p_2B_1 - u^TR_{21}. \quad (3.30)$$

Using the expression of the optimal control of the follower (3.27), the instantaneous leader's criterion can be written as $\tilde{L}_1(t, x, p_2, u) = \frac{1}{2}(x^TQ_1x + u^TR_{11}u + p_2S_{12}p_2)$. The necessary conditions (3.13), (3.14), (3.15), (3.16) lead to

$$\frac{\partial H}{\partial u} = 0 = \lambda_1(t)B_1 - \lambda_2(t) \left(\frac{\partial u}{\partial x}(t, x(t)) \right)^T R_{21} + \lambda^\circ u^T(t, x(t))R_{11}, \quad (3.31)$$

$$\frac{\partial H}{\partial u_y} = 0 = -\lambda_2^T(t) (p_2(t)B_1 + u^T(t, x(t))R_{21}), \quad (3.32)$$

$$\dot{\lambda}_1(t) = -\lambda_1(t)A + \lambda_2(t)Q_2 - \lambda^\circ x^T(t)Q_1, \quad (3.33)$$

$$\dot{\lambda}_2(t) = \lambda_1(t)S_2 + \lambda_2(t) \left(A + B_1 \left(\frac{\partial u}{\partial x}(t, x(t)) \right) \right)^T - \lambda^\circ p_2(t)S_{12}. \quad (3.34)$$

with the transversality conditions

$$\lambda_1(t_f) = \lambda^\circ x(t_f)^TK_{1f} - \lambda_2(t_f)K_{2f}, \quad \lambda_2(0) = 0. \quad (3.35)$$

From (3.32), as discussed in Section 3.3, either $\lambda_2 \equiv 0$ or $p_2(t)B_1 + u^T(t, x(t))R_{21} \equiv 0$ or both along the interval $[0, t_f]$. Without *a priori* consideration about $p_2(t)B_1 + u^T(t, x(t))R_{21}$, by assuming that $\frac{\partial}{\partial u} \left(p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u} \right) = R_{21}$ is invertible and by Proposition 3.12, we can deduce that $\lambda_2 \equiv 0$.

We next prove by contradiction that $\lambda^\circ \neq 0$. If λ° were equal to 0, then we would infer from (3.33)–(3.35) that λ_1 , like λ_2 is identically equal to zero by Cauchy uniqueness; thus, $(\lambda_1, \lambda_2, \lambda^\circ)$ is trivial, and this is a contradiction with the Pontryagin Minimum Principle. From now on, we normalize the adjoint vector so that $\lambda^\circ = 1$.

From (3.31), we deduce with the invertibility of R_{11} , that

$$u(t, x(t)) = -R_{11}^{-1}B_1^T\lambda_1^T(t). \quad (3.36)$$

Moreover, Equation (3.34) becomes, with $\lambda_2 \equiv 0$ along the interval $[0, t_f]$, $\lambda_1(t)S_2 - p_2(t)S_{12} \equiv 0$. Assuming that the rank of B_2 is maximal, that is, $\text{rank } B_2 = m_2$ (the number of the components of the control v), this relation yields

$$\lambda_1(t)B_2 = p_2(t)B_2R_{22}^{-1}R_{12}. \quad (3.37)$$

Substitution of v from (3.27) into (3.37) gives $R_{12}v(t, x(t)) = -B_2^T\lambda_1^T(t)$. If R_{12} is invertible, then the control v admits two expressions:

$$v(t, x(t)) = -R_{12}^{-1}B_2^T\lambda_1^T(t) = -R_{22}^{-1}B_2^T p_2^T(t). \quad (3.38)$$

We gather the previous necessary conditions for optimality in the following proposition.

PROPOSITION 3.14. *For $x_0 \neq 0$, if the matrices Q_i , R_{ij} and K_{if} are symmetric, if $R_{11} > 0$, $R_{22} > 0$, $R_{12} > 0$, and R_{21} invertible and if $\text{rank } B_2 = m_2$ (B_2 is*

of full rank), then the controls issued from a Stackelberg strategy with a closed-loop information structure are

$$u(t, x(t)) = -R_{11}^{-1} B_1^T \lambda_1^T(t), \quad (3.39)$$

$$v(t, x(t)) = -R_{22}^{-1} B_2^T p_2^T(t) = -R_{12}^{-1} B_2^T \lambda_1^T(t), \quad (3.40)$$

with

$$\dot{x}(t) = Ax(t) + B_1 u(t, x(t)) + B_2 v(t, x(t)), \quad x(0) = x_0, \quad (3.41)$$

$$\dot{p}_2(t) = -p_2(t)A - x^T(t)Q_2 - (p_2(t)B_1 + u^T(t, x(t))R_{21}) \frac{\partial u}{\partial x}(t, x(t)), \quad (3.42)$$

$$\dot{\lambda}_1(t) = -\lambda_1(t)A - x^T(t)Q_1, \quad \lambda_1(t_f) = x_f^T K_{1f}, \quad p_2(t_f) = x_f^T K_{2f}, \quad (3.43)$$

$$\lambda_1(t)B_2 = p_2(t)B_2 R_{22}^{-1} R_{12}. \quad (3.44)$$

REMARK 3.15. As it will be justified below by Proposition 3.16, the case $x_0 = 0$ leads only to the trivial solution, which has only few interest. Thus in the sequel of the paper, x_0 is always considered non trivial to avoid the trivial optimal trajectory.

From (3.32), even if $\lambda_2 \equiv 0$, two cases must be yet considered to precise necessary conditions: either $p_2 B_1 + u^T R_{21} \equiv 0$ or $p_2 B_1 + u^T R_{21} \neq 0$. We next prove that the first case is irrelevant under some additional weak assumptions.

3.4.3. Case $p_2 B_1 + u^T R_{21} \equiv 0$.

PROPOSITION 3.16. If the pair (A^T, Q_1) and one of the pairs (A, B_1) or (A, B_2) satisfy the Kalman condition, then

$$x(t) \equiv \lambda_1^T(t) \equiv p_2^T(t) \equiv 0, \quad \forall t \in [0, t_f]. \quad (3.45)$$

This means that there exists a unique optimal trajectory, which is trivial. \square

The proof of Proposition 3.16 requires the following technical lemma.

LEMMA 3.17. Assuming that the pair (A^T, Q_1) and one of the pairs (A, B_1) or (A, B_2) satisfy the Kalman condition, then the pair $(\mathcal{A}, \mathcal{B})$ satisfies also the Kalman condition, where

$$\mathcal{A} = \begin{bmatrix} A^T & -Q_1 & -Q_2 \\ -S_1 - B_2 R_{12}^{-1} B_2^T & -A & 0 \\ 0 & 0 & -A \end{bmatrix}; \quad \mathcal{B} = \begin{bmatrix} 0 & 0 \\ B_2 & B_1 R_{11}^{-1} R_{21} \\ -B_2 R_{22}^{-1} R_{12} & -B_1 \end{bmatrix}. \quad (3.46)$$

\square

REMARK 3.18. This means that the particular case $p_2 B_1 + u^T R_{21} \equiv 0$ can be discarded under weak assumptions on the system. The leader should be able to observe the system (pair (Q_1, A) observable) and at least one player should be able to control the system ((A, B_1) or (A, B_2) controllable). Once again, it is emphasized that the roles of the players are not symmetric.

3.4.4. Case $p_2 B_1 + u^T R_{21} \neq 0$. The relation (3.44) is equivalent to the following two relations:

$$\lambda_1(t_f)B_2 = x^T(t_f)K_{1f}B_2 = p_2(t_f)B_2 R_{22}^{-1} R_{12} = x^T(t_f)K_{2f}B_2 R_{22}^{-1} R_{12}, \quad (3.47)$$

and

$$\dot{\lambda}_1(t)B_2 = \dot{p}_2(t)B_2R_{22}^{-1}R_{12} \quad (3.48)$$

$$= -(\lambda_1(t)A + x^T(t)Q_1)B_2 \quad (3.49)$$

$$= -(p_2(t)A + x^T(t)Q_2)B_2R_{22}^{-1}R_{12} \\ - (p_2(t)B_1 + u^T(t, x(t))R_{21}) \left(\frac{\partial u}{\partial x}(t, x(t)) \right) B_2R_{22}^{-1}R_{12}. \quad (3.50)$$

Hence along the interval $[0, t_f]$

$$(p_2(t)B_1 + u^T(t, x(t))R_{21}) \left(\frac{\partial u}{\partial x}(t, x(t)) \right) B_2 = (\lambda_1(t)A + x^T(t)Q_1)B_2R_{12}^{-1}R_{22} \\ - (p_2(t)A + x^T(t)Q_2)B_2. \quad (3.51)$$

Therefore, (3.44) is equivalent to

$$\left\{ \begin{array}{l} (B_2^TK_{1f} - R_{12}R_{22}^{-1}B_2^TK_{2f})x(t_f) = 0, \\ (p_2(t)B_1 + u^T(t, x(t))R_{21}) \left(\frac{\partial u}{\partial x}(t, x(t)) \right) B_2 \\ \equiv (\lambda_1(t)A + x^T(t)Q_1)B_2R_{12}^{-1}R_{22} - (p_2(t)A + x^T(t)Q_2)B_2. \end{array} \right. \quad (3.52)$$

Equation (3.51) permits to derive an expression of $\frac{\partial u}{\partial x}$, since $p_2B_1 + u^TR_{21} \neq 0$

$$(p_2B_1 + u^TR_{21}) \left(\frac{\partial u}{\partial x} \right) \equiv w_2 + w'_2, \quad (3.53)$$

with

$$w_2 \equiv ((\lambda_1(t)A + x^T(t)Q_1)B_2R_{12}^{-1}R_{22} - (p_2(t)A + x^T(t)Q_2)B_2)(B_2^TB_2)^{-1}B_2^T, \quad (3.54)$$

and $(w'_2)^T \in \text{Ker}(B_2^T)$ (arbitrary).

The constraint (3.47) translates into a constraint on the set of initial points $x_0 \in \mathbb{R}^n$ from which a solution starts.

LEMMA 3.19. *The optimal solutions must emanate from initial conditions x_0 lying in a subspace of \mathbb{R}^n of codimension m_2 (at most).* \square

It should be noted that, given a starting point x_0 lying in the subspace of Lemma 3.19, there exists a unique trajectory starting from x_0 , but it is achieved by all controls which satisfy the relation (3.51). An optimal trajectory induces several possible $\frac{\partial u}{\partial x}$.

This fact appears in [29] where it is assumed that the initial state is uniformly distributed over the unit sphere and replacing the optimization criterion with its mean value over the initial state.

REMARK 3.20. *In the case of an optimization problem without terminal criteria, the relation (3.47) does not reduce the set of initial states x_0 associated with optimal trajectories.*

We gather all previous results in the following theorem.

THEOREM 3.21. *Under the following assumptions:*

- $x_0 \neq 0$,

- Q_i, R_{ij} and K_{if} are symmetric, with $Q_i \geq 0$,
- $R_{11} > 0, R_{22} > 0, R_{12} > 0$ and R_{21} is invertible,
- the pair (A^T, Q_1) and one of the pairs (A, B_1) or (A, B_2) satisfy the Kalman condition,
- $\text{rank } B_2 = m_2$ (B_2 of full rank),

the optimal trajectory satisfies the necessary conditions

$$u(t, x(t)) = -R_{11}^{-1} B_1^T K_1(t) x(t), \quad v(t, x(t)) = -R_{12}^{-1} B_2^T K_1(t) x(t), \quad (3.55)$$

with

$$\dot{x}(t) = (A - (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T) K_1(t)) x(t), \quad x(0) = x_0, \quad (3.56)$$

where K_1 is the unique solution of the matrix differential equation

$$\dot{K}_1(t) = -K_1(t) A - A^T K_1(t) - Q_1 + K_1(t) (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T) K_1(t), \quad K_1(t_f) = K_{1f}. \quad (3.57)$$

Furthermore $\frac{\partial u(t, x(t))}{\partial x}$ satisfies along the interval $[0, t_f]$

$$\begin{aligned} (p_2(t) B_1 + u^T(t, x(t)) R_{21}) \frac{\partial u}{\partial x}(t, x(t)) B_2 \\ = (\lambda_1(t) A + x^T(t) Q_1) B_2 R_{12}^{-1} R_{22} - (p_2(t) A + x^T(t) Q_2) B_2, \end{aligned} \quad (3.58)$$

where

$$\dot{p}_2(t) = -p_2(t) A - x^T(t) Q_2 - (p_2(t) B_1 + u^T(t, x(t)) R_{21}) \frac{\partial u}{\partial x}(t, x(t)), \quad (3.59)$$

$p_2(t_f) = x^T(t_f) K_{2f}$ and $(B_2^T K_{1f} - R_{12} R_{22}^{-1} B_2^T K_{2f}) x(t_f) = 0$.

Theorem 3.21 provides rigorous necessary conditions for closed-loop Stackelberg solutions of generic linear quadratic games. Up to now this problem has remained open and was only partially solved in particular cases in [34]. It should be stressed again that the trajectory associated with closed-loop Stackelberg solution is unique, nevertheless it induces several possible u_x , which satisfy Equations (3.58)-(3.59) and are completely characterized by (3.53). This degree of freedom in the choice of u_x , leading to the same trajectory, requires an additional objective, e.g. arguments related to the robustness or the sensitivity of the Stackelberg solution.

4. Sufficient conditions. In this section, using elements of focal point theory, we derive sufficient optimality conditions, first for the leader, and then for the follower in the case of Linear Quadratic games.

REMARK 4.1. *Note that we use focal points and not conjugate points, since the final state $x(t_f)$ is not fixed.*

4.1. Preliminary comments, focal times. The optimization problem of the leader is $\min_u \hat{J}_1(u)$ where

$$\left\{ \begin{aligned} \hat{J}_1(u) &= \frac{1}{2} \int_0^{t_f} (x^T(t) Q_1 x(t) + u^T(t, x(t)) R_{11} u(t, x(t)) + p_2(t) S_{12} p_2^T(t)) dt \\ &\quad + \frac{1}{2} x^T(t_f) K_{1f} x(t_f), \\ \dot{x}(t) &= A x(t) - S_2 p_2^T(t) + B_1 u(t, x(t)), \\ \dot{p}_2^T(t) &= -A^T p_2^T(t) - Q_2 x(t) - w^T (p_2(t) B_1 + u^T(t, x(t)) R_{21})^T, \end{aligned} \right. \quad (4.1)$$

with $x(0) = x_0$ and $p_2(t_f) = x^T(t_f)K_{1f}$. When $p_2(t)B_1 + u^T(t, x(t))R_{21} \neq 0$, the control w is cheap (see [11] for the concept of cheap control), since it only appears in the dynamics of p_2 , and nowhere else (it does also not appear in the cost); then, we rather consider p_2 as a control. Note that this is a particular case of the so-called Goh transformation (see [11]). Actually, in what follows we consider $\xi = B_2^T p_2^T$ as a control. Then the problem (4.1) can be rewritten as $\min_{(u, \xi)} \hat{J}_1(u, \xi)$ where

$$\begin{cases} \dot{x}(t) = Ax(t) - B_2 R_{22}^{-1} \xi(t) + B_1 u(t, x(t)), \\ \hat{J}_1(u, \xi) = \frac{1}{2} x^T(t_f) K_{1f} x(t_f) \\ \quad + \frac{1}{2} \int_0^{t_f} (x^T(t) Q_1 x(t) + u^T(t, x(t)) R_{11} u(t, x(t)) + \xi^T(t) R_{22}^{-1} R_{12} R_{22}^{-1} \xi(t)) dt. \end{cases} \quad (4.2)$$

REMARK 4.2. *Note that this Linear Quadratic problem with controls (u, ξ) is related to the Team optimal approach in [7]. In this reference, the first step in the research of Stackelberg equilibrium is to obtain the minimum of the criterion of the leader, by a team cooperation between the leader and the follower. Then the follower control is modified to achieve the minimum of the criterion of the follower.*

A necessary condition for the existence of an optimal control of the problem (4.2) is $R_{22}^{-1} R_{12} R_{22}^{-1} \geq 0$. It is equivalent to $R_{12} \geq 0$, since R_{22} is positive definite. When t_f is small, $R_{12} > 0$ is a sufficient condition for the existence of an optimal control (see [24, 40]). In the following, it is assumed that $R_{12} > 0$.

Under this assumption, the optimal controls u and ξ are given by

$$u(t, x(t)) = -R_{11}^{-1} B_1^T \lambda_1^T(t), \quad \xi(t) = R_{22} R_{12}^{-1} B_2^T \lambda_1^T(t). \quad (4.3)$$

Recall that, in order to characterize focal points (see e.g. [11]), we consider the variational system

$$\begin{cases} \delta \dot{x}(t) = A \delta x(t) - (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T) \delta \lambda_1^T(t), \\ \delta \dot{\lambda}_1(t) = -\delta \lambda_1(t) A - \delta x^T(t) Q_1. \end{cases} \quad (4.4)$$

By definition, the first focal time $t_c > 0$ along the trajectory $x(t)$ associated with the controls (u, ξ) is the first positive time at which there exists a solution $(\delta x, \delta \lambda_1)$ satisfying (recall that $x(0) = x_0$ is fixed)

$$\begin{cases} \delta x(0) = 0, \\ \delta \lambda_1(t_c) = \delta x^T(t_c) K_{1f}. \end{cases} \quad (4.5)$$

It is well known that this condition is equivalent to

$$\|K(t)\| \xrightarrow[t \rightarrow t_c; t < t_c]{} +\infty, \quad (4.6)$$

where $K(t)$ is the solution of the Riccati differential equation

$$\begin{cases} \dot{K}(t) = K(t)A + A^T K(t) + Q_1 - K(t) (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T) K(t), \\ K(0) = K_{1f}. \end{cases} \quad (4.7)$$

The first focal time t_c is a finite escape time for the Riccati differential equation (4.7). Note that $K(t) = K_1(t_f - t)$, where $K_1(t)$ is defined by (3.57).

LEMMA 4.3. *Rigorously, since the first focal time is defined by an infimum, a remark is due to comment on its existence. It is well known (see [11, 12]) that, if*

$R_{11} > 0$ and $R_{12} > 0$ then the first focal time t_c is well defined, and t_c is either a positive real number, or is equal to $+\infty$.

REMARK 4.4. If $Q_1 \geq 0$, then Equation (4.7) admits a solution on $[0, +\infty[$. There is no finite escape time for this equation. Thus the first focal time is infinite ($t_c = +\infty$) [2, Corollary 3.6.7 and Example 3.6.8].

The optimization problem for the follower is $\min \hat{J}_2$ where

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 u(t, x(t)) + B_2 v(t, x(t)), & x(0) = x_0, \\ \hat{J}_2 = \frac{1}{2} x^T(t_f) K_{2f} x(t_f) \\ \quad + \frac{1}{2} \int_0^{t_f} (x^T(t) Q_2 x(t) + u^T(t, x(t)) R_{21} u(t, x(t)) + v^T(t, x(t)) R_{22} v(t, x(t))) dt \end{cases} \quad (4.8)$$

with $v(t, x(t)) = -R_{22}^{-1} B_2^T p_2^T(t)$ where $p_2(t_f) = x^T(t_f) K_{2f}$ and

$$\dot{p}_2(t) = -p_2(t)A - x^T(t)Q_2 - (p_2(t)B_1 + u^T(t, x(t))R_{21}) \frac{\partial u}{\partial x}(t, x(t)). \quad (4.9)$$

The variational system along the trajectory $x(\cdot)$ is

$$\delta \dot{x}(t) = A \delta x(t) + B_1 \frac{\partial u}{\partial x} \delta x - S_2 p_2^T(t), \quad (4.10)$$

$$\begin{aligned} \delta \dot{p}_2(t) = & -\delta p_2(t)A - \delta x^T(t)Q_2 - (p_2(t)B_1 + u^T(t, x(t))R_{21}) \frac{\partial^2 u}{\partial x^2}(t, x(t)) \delta x \\ & - \left(\delta p_2(t)B_1 + \left(\frac{\partial u}{\partial x}(t, x(t)) \delta x(t) \right)^T R_{21} \right) \frac{\partial u}{\partial x}(t, x(t)). \end{aligned} \quad (4.11)$$

Here, due to the freedom in the choice of w'_2 to obtain u_x by relation (3.53), we choose $u(t, x(t))$ affine with respect to $x(t)$, thus $\frac{\partial^2 u}{\partial x^2} = 0$. Equation (4.11) then rewrites

$$\delta \dot{p}_2(t) = -\delta p_2(t)A - \delta x^T(t)Q_2 - \left(\delta p_2(t)B_1 + \left(\frac{\partial u}{\partial x}(t, x(t)) \delta x(t) \right)^T R_{21} \right) \frac{\partial u}{\partial x}(t, x(t)). \quad (4.12)$$

By definition, the first focal time t'_c along the trajectory $x(t)$ associated with the control v is the first time at which there exists a solution $(\delta x, \delta p_2)$ of (4.10)-(4.11) such that $\delta x(0) = 0$ and $\delta p_2(t'_c) = \delta x^T(t'_c) K_{2f}$. For each choice of admissible term u_x (that is choice of w'_2) satisfying relation (3.58), there exists a first focal time t'_c .

4.2. Sufficient conditions for Stackelberg Strategy. We gather the previous remarks in the following result.

THEOREM 4.5. Under the assumptions of Theorem 3.21, let w'_2 be a function of time t such that $w'_2 \in (B_2^T)^{\perp}$. This choice of w'_2 leads to design the Jacobian u_x satisfying (3.53): $(p_2 B_1 + u^T R_{21}) u_x = w_2 + w'_2$. Let

$$T^* = \min(t_c, t'_c) > 0, \quad (4.13)$$

where t_c is the first focal time of the Riccati differential equation (3.57) and t'_c is the first focal time of the system (4.10)-(4.12), induced by u_x , that is by w'_2 .

For every $t_f < T^*$, there exists a unique solution of the Riccati differential equation (3.57). Denoting $x(t, x_0)$ the obtained trajectory, let

$$\mathcal{H} = \{x_0 \in \mathbb{R}^n \mid (B_2^T K_{1f} - R_{12} R_{22}^{-1} B_2^T K_{2f}) x(t_f, x_0) = 0\}. \quad (4.14)$$

Then, for every $x_0 \in \mathcal{H}$, there exists a unique optimal solution of the optimization problem (4.8) on $[0, t_f]$ associated with w'_2 . The optimal controls (u^*, v^*) associated with this unique optimal trajectory satisfy $(p_2 B_1 + u^T R_{21})u_x = w_2 + w'_2$ and furthermore

$$u(t, x(t)) = -R_{11}^{-1} B_1^T K_1(t)x(t), \quad v(t, x(t)) = -R_{12}^{-1} B_2^T K_1(t)x(t). \quad (4.15)$$

In addition, for every $x_0 \notin \mathcal{H}$, there exists no optimal trajectory starting from x_0 . \square

REMARK 4.6. Theorem 4.5 is a result of existence of closed-loop Stackelberg strategies for linear-quadratic differential games, which is new, to the best of our knowledge.

REMARK 4.7. The sufficient conditions for optimality are developed in the Linear Quadratic case, and are global in that case. It is also, by the same argument, possible to express similar sufficient conditions in the general case of nonlinear criteria. However they are not developed here, because their expressions are more technical and because they lead only to local optimality results [12, chap. 9].

REMARK 4.8. The assumption $R_{12} > 0$ is required to derive Theorem 4.5. This assumption is used in a crucial way in order to derive Lemma 4.3 (more precisely, to derive the inequality (A.59)). It is natural to make such an assumption when inspecting the minimization criterion $\hat{J}_1(u, \xi)$ defined by (4.2): indeed, as explained few lines above (4.2), the problem degenerates into a cheap control problem. In this sense, $\xi = B_2^T p_2^T$ may then be considered as a control, and therefore it is clear that one has to assume that $R_{12} > 0$ in order to ensure nice coercivity properties for the quadratic criterion $\hat{J}_1(u, \xi)$.

REMARK 4.9. From Remark 4.4, the assumption $Q_1 \geq 0$ ensures that $t_c = +\infty$. A lower bound of $T^* = \min(t_c, t'_c)$ corresponds to a lower bound of t'_c the first focal time of the non-linear variational system (4.10)-(4.12). It depends implicitly on the choice of u_x , that is the choice of w'_2 . The problem of determining a lower bound of t'_c is open.

4.3. Extension: weighting of u_x in criteria. The problem is degenerated, because for each $x_0 \in \mathcal{H}$, there may exist an infinite choice of terms $\frac{\partial u^*}{\partial x}$. A way to yield a unique $\frac{\partial u^*}{\partial x}$ is to include a weight on the term $\frac{\partial u^*}{\partial x}$ in the criterion J_1 of the leader, as in [34]. Then the leader takes into account a restriction on the Jacobian of its control. The leader is no more omnipotent.

The new criterion of the leader is then

$$J_1(u, v) = \frac{1}{2} \int_0^{t_f} \left[x^T(t) Q_1 x(t) + u^T(t, x(t)) R_{11} u(t, x(t)) + v^T(t, x(t)) R_{12} v(t, x(t)) \right. \\ \left. + \sum_{j=1}^{m_1} \left(\frac{\partial u_j}{\partial x}(t, x(t)) \right) R_j \left(\frac{\partial u_j}{\partial x}(t, x(t)) \right)^T \right] dt + \frac{1}{2} x(t_f)^T K_{1f} x(t_f), \quad (4.16)$$

where u_j are the m_1 components of the control u , and $R_j \in \mathbb{R}^{n \times n}$, ($\forall j = 1, \dots, m_1$) are symmetric positive definite matrices.

Nothing changes for the follower. However the necessary conditions for the leader

are modified as follows

$$\frac{\partial H}{\partial u} = 0 = \lambda_1(t)B_1 - \lambda_2(t) \left(\frac{\partial u}{\partial x}(t, x(t)) \right)^T R_{21} + \lambda^\circ u^T(t, x(t))R_{11}, \quad (4.17)$$

$$\frac{\partial H}{\partial u_y} = 0 = \left(\lambda_2(t) (B_1^T p_2^T(t) + R_{21}u(t, x(t)))_j + \frac{\partial u_j}{\partial x}(t, x(t))R_j \right)_{j=1, \dots, m_1}. \quad (4.18)$$

The other necessary conditions (3.15) and (3.16) are the same. Equations (4.17) and (4.18) are easily solvable, without considering different cases. In this framework, λ_2 cannot be trivial. As in [34], we infer from (4.18) that

$$\frac{\partial u_j^*}{\partial x}(t, x(t)) = \left((B_1^T p_2^T(t) + R_{21}u(t, x(t)))_j \lambda_2(t) R_j^{-1} \right)_{j=1, \dots, m_1}. \quad (4.19)$$

For the sake of simplicity, we next assume, as in [34], that $R_j = R > 0$, for every $j = 1, \dots, m_1$. Then

$$\frac{\partial u^*}{\partial x}(t, x(t)) = (B_1^T p_2^T(t) + R_{21}u(t, x(t))) \lambda_2(t) R^{-1}. \quad (4.20)$$

Plugging this expression into (4.17), we get

$$R_{11}u(t, x(t)) = -B_1^T \lambda_1^T(t) + R_{21}B_1^T p_2^T(t) \lambda_2(t) R^{-1} \lambda_2^T(t) + R_{21}^2 u \lambda_2(t) R^{-1} \lambda_2^T(t), \quad (4.21)$$

or

$$(R_{11} - \lambda_2(t)R^{-1}\lambda_2^T(t)R_{21}^2) u(t, x(t)) = -B_1^T \lambda_1^T(t) + R_{21}B_1^T p_2^T(t) \lambda_2(t) R^{-1} \lambda_2^T(t). \quad (4.22)$$

REMARK 4.10. For $t = 0$, $\lambda_2(0) = 0$, then $R_{11} - \lambda_2(0)R^{-1}\lambda_2^T(0)R_{21}^2 = R_{11} > 0$ is invertible. For $t \geq 0$ small enough, the matrix $R_{11} - \lambda_2(t)R^{-1}\lambda_2^T(t)R_{21}^2$ is invertible.

As long as $R_{11} - \lambda_2(t)R^{-1}\lambda_2^T(t)R_{21}^2$ is invertible, the optimal control is

$$u(t, x(t)) = (R_{11} - \lambda_2(t)R^{-1}\lambda_2^T(t)R_{21}^2)^{-1} (-B_1^T \lambda_1^T(t) + R_{21}B_1^T p_2^T(t) \lambda_2(t) R^{-1} \lambda_2^T(t)) \quad (4.23)$$

The nonlinear optimization problem becomes

$$\begin{aligned} \dot{x}(t) &= Ax(t) - S_2 p_2^T(t) + B_1 (R_{11} - \lambda_2(t)R^{-1}\lambda_2^T(t)R_{21}^2)^{-1} \\ &\quad \times (-B_1^T \lambda_1^T(t) + R_{21}B_1^T p_2^T(t) \lambda_2(t) R^{-1} \lambda_2^T(t)), \end{aligned} \quad (4.24)$$

$$\dot{p}_2(t) = -p_2(t)A - x^T(t)Q_2 - \|p_2(t)B_1 + u^T(t, x(t))R_{21}\|^2 \lambda_2(t)R^{-1}, \quad (4.25)$$

$$\dot{\lambda}_1(t) = -\lambda_1(t)A + \lambda_2(t)Q_2 - x^T(t)Q_1, \quad (4.26)$$

$$\begin{aligned} \dot{\lambda}_2(t) &= \lambda_1(t)S_2 + \lambda_2(t) (A^T + R^{-1}\lambda_2^T(t) (p_2(t)B_1 + u^T(t, x(t))R_{21}) B_1^T) \\ &\quad - p_2(t)S_{12}. \end{aligned} \quad (4.27)$$

with boundary conditions

$$x(0) = x_0, \quad p_2(t_f) = x^T(t_f)K_{2f}, \quad (4.28)$$

$$\lambda_2(0) = 0, \quad \lambda_1(t_f) = x^T(t_f)K_{1f} - \lambda_2(t_f)K_{2f}. \quad (4.29)$$

REMARK 4.11. For $R = \gamma Id$, if we let γ tend to $+\infty$, then we recover the necessary conditions for the strategy of Stackelberg with an open-loop information

structure. Note that this coincidence is obtained only by taking the limit $\gamma \rightarrow +\infty$ without modifying the criterion of the leader.

REMARK 4.12. *These conditions are necessary conditions. As previously, the theory of focal points leads to sufficient conditions associated with the Stackelberg strategy with closed-loop information structure including a weight for u_x in the criterion of the leader, namely, given $x_0 \in \mathbb{R}^n$. For t_f less than the global focal time of the system, there exists only one trajectory starting from x_0 solution of (4.24)-(4.29) associated with the optimal control (u, u_x) (4.23), (4.20).*

5. Conclusion. In this paper the weak Stackelberg strategy with a closed-loop information structure is studied. The framework is restricted to two-player differential games. Necessary conditions for a closed-loop Stackelberg equilibrium are derived considering all cases. It is shown that the Stackelberg strategy may degenerate whenever the leader is *omnipotent* and can impose his control to the follower. The focal times theory provides sufficient conditions for the optimization problems of the two players. The Linear Quadratic case is used to illustrate the obtained necessary and sufficient conditions. Moreover in this linear quadratic case, the control $u(t, x)$ is obtained for each state x . An extension is proposed to allow an optimal trajectory starting from any initial state by including, in the criterion, the Jacobian of his/her control in the criterion of the leader.

Appendix A. Proof of Theorems.

Preliminaries

We start with preliminaries essentially borrowed from [24,40]. First of all, consider a usual optimal control problem:

$$\begin{cases} \min C(u) \\ \text{under } \dot{x}(t) = f(t, x(t), u(t)), \text{ with } C(u) = \int_0^{t_f} f^0(t, x(t), u(t)) dt, \end{cases} \quad (\text{A.1})$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. A usual way to derive the Pontryagin Minimum Principle for such an optimal control problem is to extend the control system with a new state variable representing the cost, in the following way. Define the extended state $z = \begin{pmatrix} x \\ x^0 \end{pmatrix} \in \mathbb{R}^{n+1}$, with $x^0(0) = 0$ and $\tilde{f} = \begin{pmatrix} f \\ f^0 \end{pmatrix}$. Consider the extended control system

$$\dot{z}(t) = \tilde{f}(t, z(t), u(t)). \quad (\text{A.2})$$

The associated end-point mapping e_{z_0, t_f} at time t_f is defined by

$$\begin{aligned} e_{z_0, t_f} : \mathcal{U} &\longrightarrow \mathbb{R}^{n+1} \\ v &\longmapsto z_u(t_f), \end{aligned} \quad (\text{A.3})$$

with z_u the trajectory solution of (A.2) associated to the control u , and \mathcal{U} is the set of admissible controls. The crucial remark which is at the basis of the proof of the Minimum Principle is the following: if a trajectory $x(\cdot)$, associated with a control u on $[0, t_f]$ is optimal, then the end-point mapping e_{z_0, t_f} is not locally surjective at u , then it follows from an implicit function argument that the first differential of the end-point mapping at u is not surjective (at least in the case where there is no constraint on the controls). This fact leads to a Lagrange multipliers type equation which finally leads to the well known Pontryagin Minimum Principle (see [40] for details).

In the present paper we are not dealing with such a classic optimal control problem, however the previous reasoning may be adapted, even though our controls now depend also on $x(t)$, and we first derive a proof of Proposition 3.2.

Proof of Proposition 3.2.

Define the extended state $z = \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \in \mathbb{R}^{n+1}$ (and the projector $q(z) = x$), where \hat{x} is the instantaneous cost associated with the criterion of the follower such that $\dot{\hat{x}}(t) = L_2(t, x(t), u^*(t, x(t)), v(t, x(t)))$, $\hat{x}(0) = 0$, which leads to the dynamic of the extended state z :

$$\dot{z}(t) = \hat{f}(t, z(t), v(t, x(t))) = \begin{pmatrix} f \\ L_2 \end{pmatrix}. \quad (\text{A.4})$$

It is pointed out that the function $t \mapsto u^*(t, x(t))$ is fixed and the control of the follower $v(t, x(t))$ is the optimization variable.

DEFINITION A.1. *The end-point mapping at time t_f of system (A.4) with the initial state $z_0 = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ is the mapping*

$$e_{z_0, t_f} : \begin{array}{l} \mathcal{V} \longrightarrow \mathbb{R}^{n+1} \\ v \longmapsto z_v(t_f) \end{array} \quad (\text{A.5})$$

where z_v is the solution of (A.4), associated to v , starting from z_0 , and \mathcal{V} is the set of admissible controls.

To compute the Fréchet first derivative, consider a fixed control δv such that v and $v + \delta v$ belong to \mathcal{V} and denote $z + \delta z$ the trajectory associated with the latter control [24, 40]. An expansion to the first order of \hat{f} leads to

$$\begin{aligned} \frac{d(z + \delta z)}{dt} &= \hat{f}(t, z + \delta z, u^*(t, q(z + \delta z)), v(t, q(z + \delta z)) + \delta v(t, q(z + \delta z))) + o(\delta z), \\ &= \hat{f}(t, z, u^*(t, q(z)), v(t, q(z))) + \hat{f}_z(t, z, u^*(t, q(z)), v(t, q(z)))\delta z \\ &\quad + \hat{f}_u(t, z, u^*(t, q(z)), v(t, q(z)))u_y^* + o(\delta z). \end{aligned} \quad (\text{A.6})$$

Furthermore, to the first order,

$$\begin{aligned} v(t, q(z + \delta z)) &= v(t, q(z) + q_z(z)\delta z + o(\delta z)) = v(t, q(z)) + v_y(t, q(z))q_z(z)\delta z + o(\delta z), \\ u^*(t, q(z + \delta z)) &= u^*(t, q(z)) + u_y^*(t, q(z))q_z(z)\delta z + o(\delta z), \\ \delta v(t, q(z + \delta z)) &= \delta v(t, q(z)) + o(\delta z). \end{aligned}$$

Plugging these Taylor series expansions into relation (A.6), we have, at the first order,

$$\frac{d(\delta z)}{dt} = \hat{f}_z\delta z + \hat{f}_u u_y^* q_z \delta z + \hat{f}_v \delta v + \hat{f}_v v_y q_z(z)\delta z \quad (\text{A.7})$$

$$= \underbrace{\left(\hat{f}_z + \hat{f}_u u_y^* q_z + \hat{f}_v v_y q_z(z) \right)}_{a(t)} \delta z + \underbrace{\hat{f}_v}_{b(t)} \delta v. \quad (\text{A.8})$$

Using the transition matrix $\Phi(t)$ satisfying $\dot{\Phi}(t) = a\Phi(t)$ and $\Phi(0) = Id$ (Id denoting the identity matrix), it follows that

$$de_{z_0, t_f}(v) \cdot \delta v = \delta z(t_f) = \Phi(t_f) \int_0^{t_f} \Phi^{-1}(s)b(s)\delta v ds. \quad (\text{A.9})$$

If a closed-loop Stackelberg control $v^* \in \mathcal{V}$ of the follower is optimal, then the first derivative of the end-point mapping, $de_{z_0, t_f}(v^*)$, is not surjective, and hence there exists a line vector $\tilde{\phi} \in \mathbb{R}^{n+1}$, $\tilde{\phi} \neq 0$ such that

$$\tilde{\phi} \cdot de_{z_0, t_f}(v^*)\delta v = 0, \quad \forall \delta v \in \mathcal{V}. \quad (\text{A.10})$$

Set $\phi(t) = \tilde{\phi}\Phi(t_f)\Phi^{-1}(t)$, then the relation (A.10) is satisfied for every control δv , and thus $\int_0^{t_f} \phi(t)b(t)\delta v dt = 0$. It implies that almost everywhere on $[0, t_f]$

$$\phi(t)b(t) = 0. \quad (\text{A.11})$$

Furthermore, derivating $\phi(t) = \tilde{\phi}\Phi(t_f)\Phi^{-1}(t)$ with respect to t , we obtain

$$\dot{\phi}(t) = -\phi(t)a = -\phi(t) \left(\hat{f}_z + \hat{f}_u u_y^* q_z + \hat{f}_v v_y q_z(z) \right) = -\phi(t) \left(\hat{f}_z + \hat{f}_u u_y^* q_z \right), \quad (\text{A.12})$$

the last equality holds due to relation (A.11). Denoting $\phi(t) = (p_2(t) \ p_2^\circ(t))$, we obtain that p_2° is a constant scalar. Finally the initial condition z_0 being fixed and the final condition $z(t_f)$ being free, the standard transversality condition associated with the system (A.4) implies that $p_2^\circ \neq 0$, because $\tilde{\phi} = (p_2(t_f) \ p_2^\circ) \neq 0$. We normalize the costate vector so that $p_2^\circ = 1$, since $(p_2(t_f), p_2^\circ)$ is defined up to a multiplicative scalar. The transversality condition leads to Equation (3.4) and p_2 satisfies Equation (3.3). In addition relation (A.11) could be reformulated into Equation (3.2).

Proof of Proposition 3.6.

For each $u \in \mathcal{U}$, the inclusion $Tu \subseteq T'u$ implies that

$$\max_{v \in Tu} J_1(u, v) \leq \max_{v' \in T'u} J_1(u, v'). \quad (\text{A.13})$$

With Assumption 3.4, in a neighborhood \mathcal{U}_{nb}^* of $u^* \in \mathcal{U}$, we have

$$\max_{v \in Tu} J_1(u, v) = \max_{v' \in T'u} J_1(u, v'), \quad \forall u \in \mathcal{U}_{nb}^*. \quad (\text{A.14})$$

Thus u^* , defined by (3.6) is also given by (3.7).

Proof of Proposition 3.11.

As previously, we define the extended state

$$Z = (x^T \ p_2 \ x^\circ)^T \in \mathbb{R}^{2n+1}, \quad (\text{A.15})$$

where x° is the instantaneous cost associated with the criterion of the leader satisfying $\dot{x}^\circ(t) = \tilde{L}_1(t, x, p_2, u)$, $x^\circ(0) = 0$.

The extended system is subject to the dynamics

$$\dot{Z}(t) = \tilde{F}(t, Z, u, u_y^T) = \begin{pmatrix} F_1(t, x, p_2, u) \\ (F_{21}(t, x, p_2, u) + F_{22}(t, x, p_2, u)u_y)^T \\ \tilde{L}_1(t, x, p_2, u) \end{pmatrix}, \quad (\text{A.16})$$

where $u = u(t, h(Z))$ is a function of time t and of the projection $h(Z) = x$.

The *end-point mapping* at time t_f of system (A.16) with initial state Z_0 is the mapping

$$E_{Z_0, t_f} : \begin{array}{ll} \mathcal{U} & \longrightarrow \mathbb{R}^{2n+1} \\ u & \longmapsto Z_u(t_f) \end{array} \quad (\text{A.17})$$

where Z_u is the solution of (A.16), associated to u , starting from Z_0 . Here \mathcal{U} denotes the open set of controls $u \in \mathcal{L}^\infty([0, t_f] \times \mathbb{R}^n, \mathbb{R}^{m_1})$ such that the solution $Z_u(\cdot)$ of (A.16), associated with u and starting from Z_0 , is well defined on $[0, t_f]$.

Note that, if \tilde{F} is of class \mathcal{C}^p , $p \geq 1$, then E_{Z_0, t_f} is also of class \mathcal{C}^p .

To compute the Fréchet first derivative, we proceed as in [24, 40], consider a fixed control δu on \mathcal{U} and note $Z + \delta Z$ the trajectory associated with the control $u + \delta u$. An expansion to the first order of \tilde{F} leads to

$$\begin{aligned} \frac{d(Z + \delta Z)}{dt} &= \tilde{F}(t, Z + \delta Z, u(t, h(Z + \delta Z))) + \delta u(t, h(Z + \delta Z)), \\ &u_x(t, h(Z + \delta Z))^T + \delta u_x(t, h(Z + \delta Z))^T + o(\delta Z). \end{aligned} \quad (\text{A.18})$$

Furthermore an expansion to the first order of the control u gives

$$\begin{aligned} u(t, h(Z + \delta Z)) &= u(t, h(Z) + h_Z(Z)\delta Z + o(\delta Z)) \\ &= u(t, h(Z)) + u_x(t, h(Z))h_Z(Z)\delta Z + o(\delta Z) \end{aligned}$$

Therefore, at the first order,

$$\frac{d(\delta Z)}{dt} = \tilde{F}_Z \delta Z + \tilde{F}_u u_x h_Z \delta Z + \tilde{F}_u \delta u + \tilde{F}_{u_x} u_{xx} h_Z \delta Z + \tilde{F}_{u_x} \delta u_x^T \quad (\text{A.19})$$

$$= \underbrace{\left(\tilde{F}_Z + \tilde{F}_u u_x h_Z + \tilde{F}_{u_x} u_{xx} h_Z \right)}_A \delta Z + \underbrace{\tilde{F}_u}_B \delta u + \underbrace{\tilde{F}_{u_x}}_C \delta u_x^T \quad (\text{A.20})$$

Using the transition matrix M defined by $\dot{M}(t) = A(t)M(t)$ and $M(0) = Id$ (Id denoting the identity matrix), it follows that

$$dE_{Z_0, t_f}(u) \cdot \delta u = \delta Z(t_f) = M(t_f) \int_0^{t_f} M^{-1}(s) \left(B(s)\delta u(s) + C(s)\delta u_y^T(s) \right) ds. \quad (\text{A.21})$$

If u is the control of the leader in a closed-loop Stackelberg solution, then there exists a vector $\tilde{\psi} \in \mathbb{R}^{2n+1}$, $\tilde{\psi} \neq 0$ such that

$$\tilde{\psi} \cdot dE_{Z_0, t_f}(u)\delta u = 0, \quad \forall \delta u \in \mathcal{U}. \quad (\text{A.22})$$

Set $\psi(t) = \tilde{\psi}M(t_f)M^{-1}(t)$, then

$$\dot{\psi}(t) = -\psi(t)A(t) = -\psi(t) \left(\tilde{F}_Z + \tilde{F}_u u_y h_Z + \tilde{F}_{u_y} u_{yy} h_Z \right). \quad (\text{A.23})$$

Furthermore the relation (A.22) holds for every control δu , and thus

$$\int_0^{t_f} \psi(t) \left(B(t)\delta u(t, x) + C(t)\delta u_y^T(t, x) \right) dt = 0. \quad (\text{A.24})$$

This relation is satisfied for all controls u functions of t and x . In particular it is also satisfied for controls u functions of t only. For such a control, the relation (A.24) becomes $\int_0^{t_f} \psi(t) \left(B(t)\delta u(t) \right) dt = 0$. It implies that almost everywhere on $[0, t_f]$, $\psi(t)B(t) = 0$. Then, (A.24) leads to $\int_0^{t_f} \psi(t) \left(C(t)\delta u_y^T(t, x) \right) dt = 0$. Hence, almost everywhere on $[0, t_f]$, there holds $\psi(t)C(t) = 0$.

Let $H(t, Z, u, u_y) = \psi(t) \tilde{F}(t, Z, u, u_y)$ be the Hamiltonian associated with this optimization problem. The last equations can be rewritten almost everywhere on $[0, t_f]$ as:

$$\begin{aligned} \dot{Z}(t) &= \tilde{F}(t, Z(t), u(t, h(Z(t))), u_y(t, h(Z(t)))) \\ &= \frac{\partial H}{\partial \psi}(t, Z(t), u(t, h(Z(t))), u_y(t, h(Z(t)))) \end{aligned} \quad (\text{A.25})$$

$$\begin{aligned} \dot{\psi}(t) &= -\psi(t) \left(\tilde{F}_Z(t, Z(t), u(t, h(Z(t))), u_y(t, h(Z(t)))) \right. \\ &\quad + \tilde{F}_u(t, Z(t), u(t, h(Z(t))), u_y(t, h(Z(t)))) u_y(t, h(Z(t))) h_Z(Z(t)) \\ &\quad \left. + \tilde{F}_{u_y}(t, Z(t), u(t, h(Z(t))), u_y(t, h(Z(t)))) u_{yy}(t, h(Z(t))) h_Z(Z(t)) \right), \end{aligned} \quad (\text{A.26})$$

$$= -\frac{dH}{dZ}(t, Z(t), u(t, h(Z(t))), u_y(t, h(Z(t)))) \quad (\text{A.27})$$

$$\frac{\partial H}{\partial u} = \psi(t) B(t) = 0, \quad \frac{\partial H}{\partial u_y} = \psi(t) C(t) = 0. \quad (\text{A.28})$$

Denoting $\psi = (\lambda_1 \ \lambda_2 \ \lambda^\circ)$, one obtains the necessary conditions (3.13)-(3.16) given by Proposition 3.11 for a closed-loop Stackelberg equilibrium. Finally, some part of the initial and final values of the extended state Z , defined by (A.15) are imposed by the transversality condition for the follower optimization problem (3.4) and by the initial state $x(0) = x_0$. We can formalize these conditions by defining two sets M_0 and M_1

$$\begin{pmatrix} x(0) \\ p_2^T(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ p_2^T(0) \end{pmatrix} \in M_0, \quad \begin{pmatrix} x(t_f) \\ p_2^T(t_f) \end{pmatrix} = \begin{pmatrix} x(t_f) \\ \frac{\partial g_2}{\partial x}(h(Z(t_f))) \end{pmatrix} \in M_1, \quad (\text{A.29})$$

where

$$M_0 = \{x_0\} \times \mathbb{R}^n = \left\{ \begin{pmatrix} x \\ p_2^T \end{pmatrix} \mid \mathcal{F}_0 \begin{pmatrix} x \\ p_2^T \end{pmatrix} \triangleq x - x_0 = 0 \right\}, \quad (\text{A.30})$$

$$M_1 = \left\{ \begin{pmatrix} x \\ p_2^T \end{pmatrix} \mid \mathcal{F}_1 \begin{pmatrix} x \\ p_2^T \end{pmatrix} \triangleq \frac{\partial g_2}{\partial x}(h(Z(t_f))) - p_2 = 0 \right\}. \quad (\text{A.31})$$

The tangent manifolds $T_{Z(0)}M_0$ and $T_{Z(t_f)}M_1$ are defined by

$$T_{Z(0)}M_0 = \{(0, \alpha) \in \mathbb{R}^{2n} \mid \alpha \in \mathbb{R}^n\}, \quad T_{Z(t_f)}M_1 = \left\{ \left(\beta, \beta \frac{\partial^2 g_2}{\partial x^2} \right) \mid \beta \in \mathbb{R}^n \right\}. \quad (\text{A.32})$$

The transversality conditions can be written as (see [40, p. 104])

$$\lambda(0) \perp T_{Z_0}M_0, \quad \lambda(t_f) - \lambda^\circ \frac{\partial g_1}{\partial Z}(h(Z(t_f))) \perp T_{Z(t_f)}M_1, \quad (\text{A.33})$$

and lead to Equations (3.17).

Proof of Proposition 3.12.

The proof goes by contradiction. The term $\lambda_2^T F_{22}$ in (3.14) is the product of a column vector (λ_2^T) and a line vector $F_{22} = p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u}$, since λ_2 is a line costate

vector. The triviality of this term induces that all components of λ_2 or all components of F_{22} are trivial (or both).

Assume that $\lambda_2 \neq 0$, then $F_{22} = p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u} \equiv 0$. If furthermore $\frac{\partial F_{22}}{\partial u} = \frac{\partial}{\partial u} \left(p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u} \right)$ is invertible, then the Implicit Function Theorem applied to the function F_{22} with respect to the variable u permits to write locally along the trajectory the control $u = u(t, x, p_2)$.

The system in (x, p_2) is rewritten as

$$\begin{cases} \dot{x}(t) &= F_1(t, x, p_2, u(t, x, p_2)) \\ \dot{p}_2(t) &= F_{21}(t, x, p_2, u(t, x, p_2)), \end{cases} \quad \text{because } F_{22} = 0. \quad (\text{A.34})$$

Since the dynamics and the criterion do not depend on u_y , we can deduce that any control u_y is extremal for the optimization problem. But the relation (3.13) is a constraint on u_y . The relation (3.19) follows. \square

Proof of Lemma 3.17.

The proof uses the controllability Hautus test. The pair $(\mathcal{A}, \mathcal{B})$ satisfies the Kalman condition if and only if the matrix $\begin{bmatrix} \mathcal{A} - \alpha I \\ \mathcal{B} \end{bmatrix}$ is of full rank, for every $\alpha \in \mathbb{C}$. The proof consists in showing that all line vectors $\begin{pmatrix} z_1^T & z_2^T & z_3^T \end{pmatrix}$ satisfy

$$\begin{pmatrix} z_1^T & z_2^T & z_3^T \end{pmatrix} \begin{bmatrix} \mathcal{A} - \alpha I \\ \mathcal{B} \end{bmatrix} = 0, \quad (\text{A.35})$$

are trivial. Developing Equation (A.35), we have

$$-z_1^T Q_1 = z_2^T (A - \alpha I_n), \quad (\text{A.36})$$

$$-z_1^T Q_2 = z_3^T (A - \alpha I_n), \quad (\text{A.37})$$

$$z_1^T (A^T - \alpha I_n) = z_2^T (S_1 + B_2 R_{12}^{-1} B_2^T), \quad (\text{A.38})$$

$$z_2^T B_2 = z_3^T B_2 R_{22}^{-1} R_{12}, \quad (\text{A.39})$$

$$z_2^T B_1 R_{11}^{-1} R_{21} = z_3^T B_1. \quad (\text{A.40})$$

Multiplying Equation (A.36) by z_1 and Equation (A.38) by z_2 , we obtain

$$-z_1^T Q_1 z_1 = z_2^T (A - \alpha I_n) z_1 = z_2^T (S_1 + B_2 R_{12}^{-1} B_2^T) z_2 \quad (\text{A.41})$$

The first term is nonpositive ($Q_1 \geq 0$) and the last term is nonnegative, hence both are zero. It follows that $z_1^T Q_1 = 0$, $z_2^T B_1 = 0$ and $z_2^T B_2 = 0$. Plugging these relations into (A.36), (A.40), one gets

$$z_1^T (A^T - \alpha I_n) = 0, \quad z_1^T Q_1 = 0, \quad (\text{A.42})$$

$$z_2^T (A - \alpha I_n) = 0, \quad z_2^T B_2 = 0, \quad z_2^T B_1 = 0, \quad (\text{A.43})$$

$$z_3^T (A - \alpha I_n) = 0, \quad z_3^T B_2 = 0, \quad z_3^T B_1 = 0. \quad (\text{A.44})$$

The relations (A.42) correspond to the observability Hautus test of the pair (Q_1, A) , the relations (A.43) and (A.44) to the controllability Hautus test of the pair (A, B_1) or (A, B_2) . The assumptions of controllability and observability lead to z_1, z_2 and z_3 trivial. \square

Proof of Proposition 3.16.

With the condition

$$p_2 B_1 + u^T R_{21} \equiv 0, \quad (\text{A.45})$$

the term $\frac{\partial u}{\partial x}$ does not appear anymore in the necessary conditions (3.41)-(3.44). Derivating with respect to time the relation (A.45) does not induce necessary conditions for $\frac{\partial u}{\partial x}$.

However assuming that R_{21} is invertible, the control u admits two representations

$$u(t, x(t)) = -R_{11}^{-1} B_1^T \lambda_1^T(t) = -R_{21}^{-1} B_1^T p_2^T(t). \quad (\text{A.46})$$

From this relation and from (3.44), necessary conditions about $x(t_f)$ are developed by successive derivations with respect to time,

$$\begin{cases} \lambda_1(t) B_2 - p_2(t) B_2 R_{22}^{-1} R_{12} = 0, \\ \lambda_1(t) B_1 R_{11}^{-1} R_{21} - p_2(t) B_1 = 0. \end{cases} \quad (\text{A.47})$$

These two relations can be rewritten for every $t \in [0, t_f]$ as

$$\begin{pmatrix} x(t) & \lambda_1(t) & p_2(t) \end{pmatrix} \mathcal{B} = 0. \quad (\text{A.48})$$

Plugging (A.45) into the dynamics of x , λ_1 and p_2 , we obtain the autonomous differential system

$$\frac{d}{dt} \begin{pmatrix} x^T(t) & \lambda_1(t) & p_2(t) \end{pmatrix} = \begin{pmatrix} x^T(t) & \lambda_1(t) & p_2(t) \end{pmatrix} \mathcal{A} \quad (\text{A.49})$$

The k -order derivation of (A.48) with respect to time, at time $t = t_f$, gives

$$\begin{bmatrix} x^T(t_f) & x^T(t_f) K_{1f} & x^T(t_f) K_{2f} \end{bmatrix} \mathcal{A}^k \mathcal{B} = 0, \quad \forall k \in \mathbb{N}. \quad (\text{A.50})$$

The assumptions of Lemma 3.17 are checked. It leads to the controllability of the pair $(\mathcal{A}, \mathcal{B})$, which implies that $x(t_f) = 0$. Furthermore the autonomous linear system in x , λ_1 and p_2 with end value conditions $x(t_f) = \lambda_1^T(t_f) = p_2^T(t_f) = 0$ imposes, by a backward integration of (A.49)

$$x(t) \equiv \lambda_1^T(t) \equiv p_2^T(t) \equiv 0, \quad \forall t \in [0, t_f]. \quad (\text{A.51})$$

The unique optimal trajectory in this case is the trivial one. \square

Proof of Lemma 3.19.

Similarly as in the classic linear quadratic problem, we seek a solution in the form $\lambda_1^T(t) = K_1(t)x(t)$. Then, the matrix $K_1(t) \in \mathbb{R}^{n \times n}$ must satisfy

$$\dot{K}_1(t)x(t) + K_1(t) (Ax(t) - (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T) K_1(t)x(t)) = -A^T K_1(t)x(t) - Q_1 x(t). \quad (\text{A.52})$$

This relation should hold for every x , which leads to define $K_1(t)$ as the solution of the following Riccati differential equation

$$\begin{cases} \dot{K}_1(t) = -K_1(t)A - A^T K_1(t) - Q_1 + K_1(t) (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T) K_1(t), \\ K_1(t_f) = K_{1f}. \end{cases} \quad (\text{A.53})$$

The existence of a solution of the optimization problem is ensured in a standard way "a la Riccati" and by the uniqueness of an optimal trajectory. This is justified *a posteriori* in the following by using the theory of focal times.

Plugging $\lambda_1^T(t) = K_1(t)x(t)$ into (3.38) and (A.46), the state $x(t)$ has the dynamical constraint

$$\begin{cases} \dot{x}(t) = (A - (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T) K_1(t)) x(t) = \tilde{A}x(t), \\ x(0) = x_0. \end{cases} \quad (\text{A.54})$$

Let $\mathcal{M}(t)$ be the transition matrix associated with (A.54). Then $x(t) = \mathcal{M}(t)x_0$. Then the constraint (3.47) becomes

$$(B_2^T K_{1f} - R_{12} R_{22}^{-1} B_2^T K_{2f}) \mathcal{M}(t_f)x_0 = 0. \quad (\text{A.55})$$

This is a m_2 -codimension (at most) condition on the initial states x_0 . \square

Proof of Lemma 4.3.

From Equation (4.1), one gets $x(t) = e^{tA}x_0 - \int_0^t e^{(t-s)A} (B_2 R_{22}^{-1} \xi(s) - B_1 u(s)) ds$. There exist scalar constants $C_k \geq 0$ such that for a given $t_f > 0$, for every $t \in [0, t_f]$

$$\|x(t)\| \leq C_1 \|x_0\| + C_2 \sqrt{t_f} \left[\left(\int_0^{t_f} \|\xi(s)\|^2 ds \right)^{\frac{1}{2}} + \left(\int_0^{t_f} \|u(s)\|^2 ds \right)^{\frac{1}{2}} \right]. \quad (\text{A.56})$$

Hence

$$\begin{aligned} \left\| \int_0^{t_f} x^T(s) Q_1 x(s) ds \right\| &\leq C_3 \|x_0\|^2 + C_4 t_f^2 \left(\int_0^{t_f} \|\xi(s)\|^2 ds + \int_0^{t_f} \|u(s)\|^2 ds \right) \\ &\quad + C_5 t_f + C_5 t_f^2 \int_0^{t_f} \|\xi(s)\|^2 ds + C_5 t_f^2 \int_0^{t_f} \|u(s)\|^2 ds \end{aligned} \quad (\text{A.57})$$

In addition, assuming $R_{11} > 0$ and $R_{22}^{-1} R_{12} R_{22}^{-1} > 0$,

$$\left\| \int_0^{t_f} u^T(s) R_{11} u(s) ds \right\| \geq C_6 \int_0^{t_f} \|u(s)\|^2 ds, \quad (\text{A.58})$$

$$\left\| \int_0^{t_f} \xi^T(s) R_{22}^{-1} R_{12} R_{22}^{-1} \xi(s) ds \right\| \geq C_6 \int_0^{t_f} \|\xi(s)\|^2 ds. \quad (\text{A.59})$$

Using these inequalities and (4.2), we can compute a lower bound of the criterion $\hat{J}_1(u, \xi)$

$$\begin{aligned} 2\hat{J}_1(u, \xi) &\geq (C_6 - (C_4 + C_5) t_f^2) \left[\int_0^{t_f} \|u\|^2 ds + \int_0^{t_f} \|\xi\|^2 ds \right] \\ &\quad + x^T(t_f) K_{1f} x(t_f) - C_3 \|x_0\|^2 - C_5 t_f. \end{aligned} \quad (\text{A.60})$$

For $t_f > 0$ small enough, $t_f \leq \sqrt{\frac{C_6}{C_4 + C_5}}$, the criterion $\hat{J}_1(u, \xi)$ is finite, then $0 < t_f < t_c$. \square

REFERENCES

- [1] H. ABOU-KANDIL AND P. BERTRAND, *Analytical Solution for an Open-Loop Stackelberg Game*, IEEE Transactions on Automatic Control, AC-30 (1985), pp. 1222–1224.

- [2] H. ABOU-KANDIL, G. FREILING, V. IONESCU, AND G. JANK, *Matrix Riccati Equations in Control and Systems Theory*, Birkhäuser, 2003.
- [3] A. ABOUSSOROR AND P. LORIDAN, *Strong-weak Stackelberg problems in finite dimensional spaces*, *Serdica Mathematical Journal*, 21 (1995), pp. 151–170.
- [4] ———, *Existence of solutions of two-level optimization problems with nonunique lower-level solutions*, *Journal of Mathematical Analysis and Applications*, 254 (2001), pp. 348–357.
- [5] A. BAGCHI, *Stackelberg Differential Games in Economic Models*, *Lecture Notes in Control and Information Sciences*, Springer Verlag, June 1984.
- [6] T. BAŞAR AND A. HAURIE, *Feedback equilibria in differential games with structural and modal uncertainties*, JAE Press Inc. Connecticut, May 1984, ch. *Advances in Large Scale Systems*, chapter 1.
- [7] T. BAŞAR AND G. J. OLSDER, *Team-optimal closed-loop Stackelberg strategies in hierarchical control problems*, *Automatica*, 16 (1980), pp. 409–414.
- [8] ———, *Dynamic Noncooperative Game Theory*, SIAM, 1995.
- [9] T. BAŞAR AND H. SELBUZ, *Closed-loop Stackelberg strategies with applications in the optimal control of multilevel systems*, *IEEE Transactions on Automatic Control*, AC-24 (1979), pp. 166–179.
- [10] T. BAŞAR AND R. SRIKANT, *A stackelberg network game with a large number of followers*, *Journal of Optimization Theory and Applications*, 115 (2002), pp. 479–490.
- [11] B. BONNARD AND M. CHYBA, *The role of singular trajectories in control theory*, *Math. & Appl.* 40, Springer Verlag, 2003.
- [12] B. BONNARD, L. FAUBOURG, AND E. TRÉLAT, *Mécanique céleste et contrôle de systèmes spatiaux*, *Math. & Appl.* 51, Springer Verlag, 2006.
- [13] M. BRETON, A. ALJ, AND A. HAURIE, *Sequential Stackelberg equilibria in two-person games*, *Journal of Optimization Theory and Applications*, 59 (1988), pp. 71–97.
- [14] B. CHEN AND P.-A. ZADROZNY, *An anticipative feedback solution for the infinite-horizon, linear-quadratic, dynamic, Stackelberg game*, *Journal of Economic Dynamics and Control*, 26 (2002), pp. 1397–1416.
- [15] C. I. CHEN AND J. B. CRUZ, *Stackelberg solution for two-person games with biased information patterns*, *IEEE Transactions on Automatic Control*, AC-17 (1972), pp. 791–797.
- [16] S. DEMPE, *Essays and Surveys in Global Optimization*, Springer Verlag, 2005, ch. *Bilevel programming*, pp. 165–193.
- [17] S. DEMPE AND N. GADHI, *Second order optimality conditions for bilevel set optimization problems*, *Journal of Global Optimization*, 47 (2010), pp. 233–245.
- [18] E. DOCKNER, S. JØRGENSEN, N. VAN LONG, AND G. SORGER, *Differential games in economics and management science*, Cambridge University Press, 2000.
- [19] G. FREILING, G. JANK, AND S. R. LEE, *Existence and uniqueness of open-loop stackelberg equilibria in linear-quadratic differential games*, *Journal of Optimization Theory and Applications*, 110 (2001), pp. 515–544.
- [20] X. HE, A. PRASAD, S. P. SETHI, AND G. J. GUTIERREZ, *A survey of Stackelberg differential game models in supply and marketing channels*, *Journal of Systems Science and Systems Engineering*, 16 (2007), pp. 385–413.
- [21] M. JUNGERS, *Matrix block formulation of closed-loop memoryless Stackelberg strategy for discrete-time games*, in *47th IEEE Conference on Decision and Control (CDC)*, Cancun, Mexico, December 2008.
- [22] M. JUNGERS AND C. OARĂ, *Anti-palindromic pencil formulation for open-loop Stackelberg strategy in discrete-time*, in *19th International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, Budapest, Hungary, soumis, July 2010, pp. 2265–2268.
- [23] S. LASAULCE, Y. HAYEL, R. EL AZOUZI, AND M. DEBBAH, *Introducing hierarchy in energy games*, *IEEE Transactions on Wireless Communications*, 8 (2009), pp. 3833–3843.
- [24] E.B. LEE AND L. MARKUS, *Foundations of Optimal Control Theory*, New York: Wiley, 1967.
- [25] G. LEITMANN, *On generalized stackelberg strategies*, *Journal of Optimization Theory and Applications*, 26 (1978), p. 1978.
- [26] D.J.N. LIMEBEER, B.D.O. ANDERSON, AND H. HENDEL, *A Nash game approach to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control*, *IEEE Transactions on Automatic Control*, 39 (1994), pp. 69–82.
- [27] P. LORIDAN AND J. MORGAN, *Weak via strong Stackelberg problem: New results*, *Journal of Global Optimization*, 8 (1996), pp. 263–287.
- [28] L. MALLOZZI AND J. MORGAN, *Existence of a feedback equilibrium for two-stage Stackelberg games*, *IEEE Transactions on Automatic Control*, 42 (1997), pp. 1612–1614.
- [29] J. MEDANIC, *Closed-loop Stackelberg strategies in linear-quadratic problems*, *IEEE Transactions on Automatic Control*, 23 (1978), pp. 632–637.
- [30] P.-Y. NIE, *Dynamic Stackelberg games under open-loop complete information*, *Journal of The*

- Franklin Institute, 342 (2005), pp. 737–748.
- [31] P.-Y. NIE, L.-H. CHEN, AND M. FUKUSHIMA, *Dynamic programming approach to discrete time dynamic feedback Stackelberg games with independent and dependent followers*, European Journal of Operational Research, 169 (2006), pp. 310–328.
 - [32] P.-Y. NIE, M.-Y. LAI, AND S.-J. ZHU, *Dynamic feedback Stackelberg games with non-unique solutions*, Nonlinear Analysis, 69 (2008), pp. 1904–1913.
 - [33] A.-J. NOVAK, G. FEICHTINGER, AND G. LEITMANN, *A differential game related to terrorism: Nash and Stackelberg strategies*, Journal of Optimization Theory and Applications, 144 (2010), pp. 533–555.
 - [34] G. P. PAPAVALASSILOPOULOS AND J. B. CRUZ, *Nonclassical control problems and Stackelberg games*, IEEE Transactions on Automatic Control, 24 (1979), pp. 155–166.
 - [35] M. SIMAAN AND J. B. CRUZ, *Additional aspects of the Stackelberg strategy in nonzero-sum games*, Journal of Optimization Theory and Applications, 11 (1973), pp. 613–626.
 - [36] ———, *On the Stackelberg strategy in nonzero-sum games*, Journal of Optimization Theory and Applications, 11 (1973), pp. 533–555.
 - [37] A. W. STARR AND Y. C. HO, *Further properties of nonzero-sum differential games*, Journal of Optimization Theory and Applications, 3 (1969), pp. 207–219.
 - [38] ———, *Nonzero-sum differential games*, Journal of Optimization Theory and Applications, 3 (1969), pp. 184–206.
 - [39] B. TOLWINSKI, *Closed-loop Stackelberg solution to a multistage linear-quadratic game*, Journal of Optimization Theory and Applications, 34 (1981), pp. 485–501.
 - [40] E. TRÉLAT, *Contrôle optimal: théorie et applications*, Vuibert, 2005.
 - [41] L. N. VICENTE AND P. H. CALAMAI, *Bilevel and multilevel programming: a bibliography review*, Journal of Global Optimization, 5 (1994), pp. 291–306.
 - [42] H. VON STACKELBERG, *Marktform und Gleichgewicht*, Springer, 1934.
 - [43] X. YASHAN, *Stackelberg equilibirums of open-loop differential games*, in Proceedings of the 26th Chinese Control Conference, Zhangjiajie, Hunan, China, July 2007, pp. 446–450.
 - [44] J. J. YE, *Optimal strategies for bilevel dynamic problems*, SIAM Journal on Control and Optimization, 35 (1997), pp. 512–531.
 - [45] K. ZHOU, J. C. DOYLE, AND K. GLOVER, *Robust and Optimal Control*, Prentice Hall, New Jersey, 1996.