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STACKELBERG STRATEGY WITH CLOSED-LOOP INFORMATION STRUCTURE FOR LINEAR QUADRATIC GAMES

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Abstract. This paper deals with the Stackelberg strategy in the case of a closed-loop information structure. Two players differential games are considered with one leader and one follower. The Stackelberg controls in this case are hard to obtain since the necessary conditions to be satisfied by both players cannot be easily defined. The main difficulty is due to the presence of the partial derivative of the leader's control with respect to state in the necessary condition for the follower. We first derive necessary conditions for the Stackelberg equilibrium in the general case of nonlinear criteria for finite time horizon games. Then, using focal point theory, the necessary conditions are also shown to be sufficient and lead to cheap control. The set of initial states allowing the existence of an optimal trajectory is emphasized. An extension to infinite time horizon games is proposed. The Linear Quadratic case is detailed to illustrate these results.

1. Introduction. The Stackelberg strategy for dynamic games was introduced in [14, 15, 7]. Such a strategy, named after Heinrich von Stackelberg in recognition of his pioneering work on static games [20], exhibits an information bias between players leading to establishing a hierarchy between them. The player who has the ability to enforce his strategy on the other player(s) and knows the rational reaction set of his/her opponent is called the leader. The other player(s) is (are) called the follower(s). We consider here two-players nonzero sum differential games with one leader and one follower.

The information structure in the game is the set of all available information for the players to make their decisions. When open-loop information structure is considered, no measurement of the state of the system is available and the players are committed to follow a predetermined strategy based on their knowledge of the initial state, the system's model and the cost functional to be minimized. For the closed-loop information structure case (or more precisely the memoryless closed-loop information structure), each player has access to state measurements and thus can adapt his strategy in function of the system's evolution.

The necessary conditions for obtaining a Stackelberg equilibrium with an open-loop information structure are well known [1, 14, 15, 16, 17]. The obtained controls in this case are only functions of time. For the closed-loop information structure case, determining the Stackelberg strategy for dynamic games is much harder and has been for a long time an open problem. The main difficulty comes from the presence, in the expression of the rational reaction set of the follower, of the partial derivative of the leader's control with respect to the measurement of the state. Moreover, dynamic programming cannot be used due to the inconsistency of Stackelberg strategy [9]. Several attempts have been proposed in the literature to avoid this difficulty [3]. Among such techniques, two main approaches could be distinguished.

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The first method is based on a team-optimal approach, introduced in [4] for discrete time games. Differential games are treated in [2]. At the first step, the leader and the follower are looking for optimizing the leader criterion as a team. With some weak assumptions, the optimal value of the leader criterion is attained for a parametrized family of controls for the leader and the follower. At the second step, the parameters of both controls are chosen such that the control of the follower lies in the rational reaction set in response to the control of the leader. When the leader is able to influence the criterion of the follower, with some additional assumptions, the leader can achieve the infimum of his/her own criterion [18]. This could be interpreted as a threat formulated by the leader toward the follower.

The second approach consists of defining the whole rational reaction set of the follower to a leader control. The resulting optimal control problem turns out to be nonclassical, not solvable *a priori* with the standard Pontryagin Minimum Principle. To solve this kind of nonclassical problem, a variational method is proposed in [12], assuming that this is a normal optimization problem (the abnormal case is not mentioned). Moreover, in [11] it is emphasized that this technique does not lead to a solution for all initial states, and the difficulty is circumvented by assuming that the initial state of the system is uniformly distributed over the unit sphere and replacing the optimization criterion with its mean value over the initial state.

Other techniques are also used to solve the problem of determining the closed-loop Stackelberg strategy and in [8] a sliding mode approach is proposed.

In this paper, we prove rigorously the necessary conditions for a Stackelberg equilibrium with closed-loop information structure in the same spirit as in [12] by considering all cases. In addition, sufficient conditions of the optimization problem for Linear Quadratic differential games are established using focal times theory. These necessary and sufficient conditions permit to describe the set of initial states associated with optimal trajectories. Also, an extension is proposed to associate with every initial state an optimal trajectory by introducing the Jacobian of the leader's control in his own criterion. Note that in [12], although the final result (for Linear Quadratic games only) is correct, some of arguments thereof used to derive the necessary conditions are erroneous or not precise.

The outline of the paper is as follows: in Section 2, the Stackelberg strategy is mathematically formulated. Section 3 gathers the necessary conditions for an equilibrium for the follower (Section 3.1) and for the leader (Section 3.2). A degenerating property of the Stackelberg strategy is emphasized in Section 3.5. These necessary conditions are detailed in the case of Linear Quadratic two players differential games in Section 3.6. The sufficient conditions are provided for the Linear Quadratic case in Section 4. All these results lead to the two main results of this paper Theorem 3.19 and Theorem 4.4, which ensure the existence of optimal trajectories. Concluding remarks make up Section 5.

2. Preliminaries: Stackelberg strategy. Let $t_f > 0$. Consider a two players differential game defined on the time horizon $[0, t_f]$ with

$$\dot{x}(t) = f(t, x(t), u(t, x(t)), v(t, x(t))), \quad x(0) = x_0, \quad (2.1)$$

where the state vector $x(t)$ is in \mathbb{R}^n , f is of class \mathcal{C}^1 , the controls $u(t, x(t))$ and $v(t, x(t))$ of the two players are respectively m_1 - and m_2 -vector functions lying in the sets of admissible controls \mathcal{U} and \mathcal{V} . Here, \mathcal{U} (resp. \mathcal{V}) denotes an open subset of $\mathcal{L}^\infty([0, t_f], \mathbb{R}^{m_1})$ (resp. of $\mathcal{L}^\infty([0, t_f], \mathbb{R}^{m_2})$) such that, for every couple $(u, v) \in \mathcal{U} \times \mathcal{V}$,

the associated trajectory $x(\cdot)$, solution of (2.1), is well defined on $[0, t_f]$. The two criteria, defined as real-valued functions on $\mathcal{U} \times \mathcal{V}$, are

$$J_1(u, v) = g_1(x(t_f)) + \int_0^{t_f} L_1(t, x(t), u(t, x(t)), v(t, x(t))) dt, \quad (2.2)$$

$$J_2(u, v) = g_2(x(t_f)) + \int_0^{t_f} L_2(t, x(t), u(t, x(t)), v(t, x(t))) dt. \quad (2.3)$$

The first player, who chooses the control $u(t, x(t))$, aims to minimize the criterion J_1 and the second player, who chooses the control $v(t, x(t))$, aims to minimize the criterion J_2 . It is assumed in this paper that there exists an information bias in the game which induces a hierarchy between the two players. Stackelberg strategy is thus well suited to study such differential game.

Player 1, associated with u announces his strategy first and is called the leader while Player 2, associated with v is called the follower. The leader knows how the follower will rationally react to a control u , but the follower does not know the leader's rational reaction.

Define the rational reaction set of the follower by

$$T : \mathcal{U} \rightarrow \mathcal{V} \\ u \mapsto Tu = \{v \mid v \text{ minimizes } J_2(u, \bar{v}), \bar{v} \in \mathcal{V}\}. \quad (2.4)$$

DEFINITION 2.1. *A Stackelberg equilibrium strategy (u^*, v^*) is defined by the minimization problem*

$$\begin{cases} v^* \in Tu^*, \\ u^* \text{ minimizes } \max_{v \in Tu} J_1(u, v). \end{cases} \quad (2.5)$$

Within the framework of a closed-loop information structure, the controls issued from Stackelberg strategy depend on time t and state vector $x(t)$. The main difficulty is the presence of the partial derivative $\frac{\partial u^*}{\partial x}$ in the necessary conditions for the follower. Without this dependence, that is, when u^* is only a function of time t , (as for a Stackelberg strategy with an open-loop information structure) the necessary conditions are well known (see [1, 14, 15, 16, 17]). To avoid the difficulty of the presence of the partial derivative $\frac{\partial u^*}{\partial x}$ in the necessary conditions for the follower, different alternatives have been proposed in the literature and summarized in [3]. The first approach is to find an equivalent team problem leading to a global minimization of the leader's cost and obtaining a particular representation of the leader's control [2]. The second approach consists of determining the follower's rational reaction set and the necessary conditions for the leader optimizing a dynamical problem over an infinite dimensional strategy space subject to dynamical constraints (evolution of the state vector and follower's rational reaction set). In [12], this problem is dealt using a variational method. This method does not lead to all solutions. In this paper, an approach based on the Pontryagin Minimum Principle is provided for deriving rigorously the necessary conditions. Then, using the theory of focal points, sufficient conditions for optimality are obtained.

3. Necessary conditions for a Stackelberg equilibrium. Due to the hierarchy between the two players, necessary conditions must be established first for the follower, and then for the leader.

3.1. For the follower. Equation (2.5) implies that, for a fixed control u^* , the control v^* minimizes $J_2(u^*, v)$. It can be reformulated in terms of a classical optimal control problem as:

$$\begin{aligned} \dot{x}(t) &= f(t, x, u^*(t, x(t)), v(t, x(t))), \quad x(0) = x_0, \\ \min_{v \in \mathcal{V}} J_2(u^*, v). \end{aligned} \quad (3.1)$$

The Hamiltonian H_2 associated with problem (3.1) is given by

$$H_2 = p_2 f + p_2^\circ L_2, \quad (3.2)$$

where p_2 is a line vector with the same number of components as the state vector x , and $p_2^\circ \geq 0$ is a constant scalar. Necessary conditions for the follower are issued from the Pontryagin Minimum Principle [13] are:

$$\frac{\partial H_2}{\partial v} = 0 = p_2 \frac{\partial f}{\partial v} + p_2^\circ \frac{\partial L_2}{\partial v}, \quad (3.3)$$

$$\dot{p}_2 = -\frac{dH_2}{dx} = -p_2 \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \frac{\partial u^*}{\partial x} \right) - p_2^\circ \left(\frac{\partial L_2}{\partial x} + \frac{\partial L_2}{\partial u} \frac{\partial u^*}{\partial x} \right). \quad (3.4)$$

Note that there is no term $\frac{\partial v}{\partial x}$ in (3.4) due to (3.3). Since the final state $x(t_f)$ is free, the above differential equations (3.3) and (3.4) are associated with the transversality condition

$$p_2(t_f) = p_2^\circ \frac{\partial g_2(x(t_f))}{\partial x}. \quad (3.5)$$

Moreover $(p_2(t_f), p_2^\circ)$ is necessarily nontrivial. Then from (3.4) and (3.5), it is clear that $p_2^\circ \neq 0$. It is possible to normalize the costate vector with $p_2^\circ = 1$, since $(p_2(t_f), p_2^\circ)$ is defined up to a multiplicative scalar.

3.2. For the leader. Assuming that Equation (3.3) is solvable, note S a possible solution

$$v(t, x) = S(t, x, p_2, u^*). \quad (3.6)$$

This assumption is a weak and a natural one. The leader, with his top hierarchical position with respect to the follower, can impose the control of the follower. The leader knows the reaction of the follower, i.e., he knows the function S . Then the leader seeks to minimize his own criterion where v is replaced by the function S .

REMARK 3.1. *If Equation (3.3) has several solutions, then our study can be applied to each of them (at least according to the Implicit Function Theorem). For the Linear Quadratic case, there exists a unique (global) solution (see Section 3.6).*

Using the notations

$$\tilde{L}_1(t, x, p_2, u) = L_1(t, x, u, S(t, x, p_2, u)) \quad (3.7)$$

and

$$\tilde{J}_1(u) = \int_0^{t_f} \tilde{L}_1(t, x(t), p_2(t), u(t, x(t))) dt + g_1(x(t_f)), \quad (3.8)$$

the following problem is considered:

$$\min_{u \in \mathcal{U}} \tilde{J}_1(u) \quad (3.9)$$

under two dynamical constraints:

$$\begin{aligned} \dot{x} &= f(t, x(t), u(t, x(t)), S(t, x(t), p_2(t), u(t, x(t)))) \\ &= \tilde{f}(t, x(t), p_2(t), u(t, x(t))) \\ &= F_1(t, x(t), p_2(t), u(t, x(t))), \end{aligned} \quad (3.10)$$

$$\begin{aligned} \dot{p}_2 &= -p_2(t) \frac{\partial f}{\partial x}(t, x(t), u(t, x(t)), S(t, x(t), p_2(t), u(t, x(t)))) \\ &\quad - \frac{\partial L_2}{\partial x}(t, x(t), u(t, x(t)), S(t, x(t), p_2(t), u(t, x(t)))) \\ &\quad - p_2 \frac{\partial f}{\partial u}(t, x(t), u(t, x(t)), S(t, x(t), p_2(t), u(t, x(t)))) \frac{\partial u}{\partial x}(t, x(t)) \\ &\quad - \frac{\partial L_2}{\partial u}(t, x(t), u(t, x(t)), S(t, x(t), p_2(t), u(t, x(t)))) \frac{\partial u}{\partial x}(t, x(t)) \\ &= F_{21}(t, x(t), p_2(t), u(t, x(t))) + F_{22}(t, x(t), p_2(t), u(t, x(t))) \frac{\partial u}{\partial x}(t, x(t)), \end{aligned} \quad (3.11)$$

and $x(0) = x_0$, $p_2(t_f) = \frac{\partial g_2}{\partial x}(x(t_f))$. Denote

$$\tilde{L}_2(t, x(t), p_2(t), u(t, x(t))) = L_2(t, x(t), u(t, x(t)), S(t, x(t), p_2(t), u(t, x(t)))),$$

$$F_1 = \tilde{f}, \quad F_{21} = -p_2 \frac{\partial \tilde{f}}{\partial x} - \frac{\partial \tilde{L}_2}{\partial x}, \quad \text{and} \quad F_{22} = -p_2 \frac{\partial \tilde{f}}{\partial u} - \frac{\partial \tilde{L}_2}{\partial u}.$$

PROPOSITION 3.2. *The necessary conditions for the leader for a closed loop Stackelberg equilibrium are*

$$0 = \lambda_1 \frac{\partial F_1}{\partial u} + \lambda_2 \left(\frac{\partial F_{21}}{\partial u} + \frac{\partial F_{22}}{\partial u} u_y \right)^T + \lambda^\circ \frac{\partial \tilde{L}_1}{\partial u}, \quad (3.12)$$

$$0 = \lambda_2^T F_{22} = \lambda_2^T \left(p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u} \right), \quad (3.13)$$

$$\dot{\lambda}_1 = -\lambda_1 \frac{\partial F_1}{\partial x} - \lambda_2 \left(\frac{\partial F_{21}}{\partial x} + \frac{\partial F_{22}}{\partial x} u_y \right)^T - \lambda^\circ \frac{\partial \tilde{L}_1}{\partial x}, \quad (3.14)$$

$$\dot{\lambda}_2 = -\lambda_1 \frac{\partial F_1}{\partial p_2} - \lambda_2 \left(\frac{\partial F_{21}}{\partial p_2} + \frac{\partial F_{22}}{\partial p_2} u_y \right)^T - \lambda^\circ \frac{\partial \tilde{L}_1}{\partial p_2}. \quad (3.15)$$

where $\lambda_1 \in \mathbb{R}^n$ and $\lambda_2 \in \mathbb{R}^n$ are the two line costate vectors respectively associated with the dynamical constraints (3.10) and (3.11), and $\lambda^\circ \geq 0$ a constant scalar.

The occurrence of the Jacobian $u_y = \frac{\partial u}{\partial y}$ of $u(t, y)$ with respect to the second variable y in the dynamic (3.11) prevents the use of the classical Pontryagin Minimum Principle. Two different approaches are proposed here to solve this nonclassical

problem. The first approach is based on an affine formulation of the leader control and allows to go back to the classical Pontryagin Minimum Principle. The second one is based on a general variational study which leads to prove a modified version of the Pontryagin Minimum Principle adapted to the system (3.11).

3.2.1. First proof of Proposition 3.2. This optimization problem associated with the dynamics (3.10) and (3.11) and with the criterion (3.8) can be reformulated as the auxiliary problem by noting $z = \begin{pmatrix} x \\ p_2^T \end{pmatrix} \in \mathbb{R}^{2n}$

$$\begin{cases} \dot{z}(t) = F(t, z(t), u(t, h(z(t))), u_y(t, h(z(t)))), \\ \min J(u) = \int_0^{t_f} L(t, z(t), u(t, h(z(t))), u_y(t, h(z(t))))dt + g(h(z(t_f))), \end{cases} \quad (3.16)$$

where $u_y = \frac{\partial u}{\partial y}$ is the Jacobian of u with respect to the second variable of u , and $h(z) = h((x^T, p_2^T)^T) = x$ is a projection. It is important to note that u is not a function of the whole extended state $(x^T, p_2^T)^T$, but only of a projection of the state. The next lemma gives an equivalent and more standard formulation for this auxiliary optimization problem.

LEMMA 3.3. *The optimal control problem*

$$(P_1) \begin{cases} \dot{z}(t) = F(t, z(t), u(t, h(z(t))), u_y(t, h(z(t)))), \\ \min J_1(u) = \int_0^{t_f} L(t, z(t), u(t, h(z(t))), u_y(t, h(z(t))))dt + g(h(z(t_f))), \end{cases} \quad (3.17)$$

is equivalent to the classical optimal control problem

$$(P_2) \begin{cases} \dot{z}(t) = F(t, z(t), w_1(t), w_2(t)), \\ \min \check{J}_1(u) = \int_0^{t_f} L(t, z(t), w_1(t), w_2(t))dt + g(h(z(t_f))). \end{cases} \quad (3.18)$$

Furthermore, the respective optimal controls are related by

$$u(t, y) = \langle w_2(t), (y - h(z(t))) \rangle + w_1(t). \quad (3.19)$$

Proof. Let $u(t, y)$ be an optimal control for (P_1) , associated with the trajectory $z(\cdot)$. Then using $w_1(t) = u(t, h(z(t)))$ and $w_2(t) = u_y(t, h(z(t)))$, the controls (w_1, w_2) generate for (P_2) the same trajectory $z(\cdot)$. Therefore, $\inf \check{J}_1 \leq \inf J_1$.

On the other hand, if (w_1, w_2) is an optimal control for (P_2) , associated with the trajectory $z(\cdot)$, then the control $u(t, y) = w_2(t)(y - h(z(t))) + w_1(t)$ generates the same trajectory $z(\cdot)$, and thus $\inf J_1 \leq \inf \check{J}_1$.

Hence, $\inf J_1 = \inf \check{J}_1$.

Both optimization problems are equivalent, in the sense that if a trajectory is optimal for one problem, then it is also optimal for the other, and the optimal values of both criteria coincide. The associated optimal controls are related by (3.19).

REMARK 3.4. *We choose here an affine representation of $u(t, y)$, but we can choose another one, like in [12]. This choice allows to reduce the optimal control research only to the term w_1 in (3.19), along the optimal trajectory since w_2 does not*

appear anymore in the control. That is, along the optimal trajectory the control is only a function of time t .

This lemma allows a direct application of the Pontryagin Minimum Principle with the following Hamiltonian (3.20) to obtain necessary conditions for the leader ($\lambda \in \mathbb{R}^{2n}$ denotes the line costate vector associated with the state $z \in \mathbb{R}^{2n}$ and $\lambda^\circ \geq 0$ a constant scalar):

$$H = \lambda F(t, z, w_1, w_2) + \lambda^\circ L(t, z, w_1, w_2). \quad (3.20)$$

The necessary conditions are given by

$$\frac{\partial H}{\partial w_1} = 0 = \lambda F_u + \lambda^\circ L_u, \quad (3.21)$$

$$\frac{\partial H}{\partial w_2} = 0 = \lambda F_{u_y} + \lambda^\circ L_{u_y}, \quad (3.22)$$

$$\dot{\lambda} = -\frac{dH}{dz} = -\lambda \frac{\partial F}{\partial z} - \lambda^\circ \frac{\partial L}{\partial z}. \quad (3.23)$$

Introducing the additional notations $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^{2n}$. Then

$$F(t, z, u, u_z) = \begin{pmatrix} F_1(t, x, p_2, u) \\ (F_{21}(t, x, p_2, u) + F_{22}(t, x, p_2, u)u_y)^T \end{pmatrix}. \quad (3.24)$$

The Hamiltonian (3.20) takes the form

$$H = \lambda_1 F_1 + \lambda_2 (F_{21} + F_{22}u_y)^T + \lambda^\circ \tilde{L}_1(t, x, p_2, u). \quad (3.25)$$

The necessary conditions (3.21), (3.22), (3.23) can be rewritten as

$$\frac{\partial H}{\partial u} = 0 = \lambda_1 \frac{\partial F_1}{\partial u} + \lambda_2 \left(\frac{\partial F_{21}}{\partial u} + \frac{\partial F_{22}}{\partial u} u_y \right)^T + \lambda^\circ \frac{\partial \tilde{L}_1}{\partial u}, \quad (3.26)$$

$$\frac{\partial H}{\partial u_y} = 0 = \lambda_2^T F_{22} = \lambda_2^T \left(p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u} \right), \quad (3.27)$$

$$\dot{\lambda}_1 = -\lambda_1 \frac{\partial F_1}{\partial x} - \lambda_2 \left(\frac{\partial F_{21}}{\partial x} + \frac{\partial F_{22}}{\partial x} u_y \right)^T - \lambda^\circ \frac{\partial \tilde{L}_1}{\partial x}, \quad (3.28)$$

$$\dot{\lambda}_2 = -\lambda_1 \frac{\partial F_1}{\partial p_2} - \lambda_2 \left(\frac{\partial F_{21}}{\partial p_2} + \frac{\partial F_{22}}{\partial p_2} u_y \right)^T - \lambda^\circ \frac{\partial \tilde{L}_1}{\partial p_2}. \quad (3.29)$$

This proves Proposition 3.2.

3.3. Second proof of Proposition 3.2. In the necessary condition (3.11) the term $\frac{\partial u}{\partial x}$ comes out. The optimization problem for the leader is then nonclassical.

Define the extended state $Z = \begin{pmatrix} x \\ p_2^T \\ x^\circ \end{pmatrix} \in \mathbb{R}^{2n+1}$, where x° is the instantaneous cost associated with the criterion of the leader satisfying

$$\dot{x}^\circ = \tilde{L}_1(t, x, p_2, u), \quad x^\circ(0) = 0. \quad (3.30)$$

The extended system admits the dynamics

$$\dot{Z} = \tilde{F}(t, Z, u, u_y^T) = \begin{pmatrix} F_1(t, x, p_2, u) \\ (F_{21}(t, x, p_2, u) + F_{22}(t, x, p_2, u)u_y)^T \\ \tilde{L}_1(t, x, p_2, u) \end{pmatrix}, \quad (3.31)$$

where $u = u(t, h(Z))$ is a function of time t and of the projection $h(Z) = x$.

Solving this kind of problem requires to define the end-point mapping and singular controls.

DEFINITION 3.5. *The end-point mapping at time t_f of system (3.31) with initial state Z_0 is the mapping*

$$\begin{aligned} E_{Z_0, t_f} : \mathcal{U} \subset L^\infty([0, t_f] \times \mathbb{R}^n, \mathbb{R}^{m_1}) &\longrightarrow \mathbb{R}^{2n+1} \\ u &\longmapsto Z_u(t_f) \end{aligned} \quad (3.32)$$

where Z_u is the solution of (3.31), associated to u , starting from Z_0 . Here \mathcal{U} denotes the open set of controls $u \in \mathcal{L}^\infty([0, t_f] \times \mathbb{R}^n, \mathbb{R}^{m_1})$ such that the solution $Z_u(\cdot)$ of (3.31), associated with u and starting from Z_0 , is well defined on $[0, t_f]$.

Note that, if \tilde{F} is of class \mathcal{C}^p , $p \geq 1$, then E_{Z_0, t_f} is also of class \mathcal{C}^p .

To compute the Fréchet first derivative, consider a fixed control δu on \mathcal{U} and note $Z + \delta Z$ the trajectory associated with the control $u + \delta u$ [19, 10]. A Taylor series development of \tilde{F} leads to

$$\begin{aligned} \frac{d(Z + \delta Z)}{dt} &= \tilde{F}(t, Z + \delta Z, u(t, h(Z + \delta Z)) + \delta u(t, h(Z + \delta Z)), \\ &\quad u_y(t, h(Z + \delta Z))^T + \delta u_y(t, h(Z + \delta Z))^T) \end{aligned} \quad (3.33)$$

Furthermore a Taylor series development of the control u gives

$$\begin{aligned} u(t, h(Z + \delta Z)) &= u(t, h(Z)) + h_Z(Z)\delta Z + o(\delta Z) \\ &= u(t, h(Z)) + u_y(t, h(Z))h_Z(Z)\delta Z + o(\delta Z) \end{aligned}$$

By identification in these two Taylor series developments, we have, at the first order,

$$\frac{d(\delta Z)}{dt} = \tilde{F}_Z \delta Z + \tilde{F}_u u_y h_Z \delta Z + \tilde{F}_u \delta u + \tilde{F}_{u_y} u_{yy} h_Z \delta Z + \tilde{F}_{u_y} \delta u_y^T \quad (3.34)$$

$$= \underbrace{\left(\tilde{F}_Z + \tilde{F}_u u_y h_Z + \tilde{F}_{u_y} u_{yy} h_Z \right)}_A \delta Z + \underbrace{\tilde{F}_u}_{B} \delta u + \underbrace{\tilde{F}_{u_y}}_C \delta u_y^T \quad (3.35)$$

Using the transition matrix M verifying $\dot{M} = AM$ and $M(0) = Id$ (Id denoting the identity matrix), it follows that

$$dE_{Z_0, t_f}(u) \cdot \delta u = \delta Z(t_f) = M(t_f) \int_0^{t_f} M^{-1}(s) \left(B(s)\delta u(s) + C(s)\delta u_y^T(s) \right) ds. \quad (3.36)$$

DEFINITION 3.6. *A control $u \in \mathcal{U}$ is said to be singular on $[0, t_f]$ if the Fréchet first derivative of the end-point mapping is not surjective.*

We next provide a Hamiltonian characterization of singular controls. If u is singular, then there exists a vector $\psi \in \mathbb{R}^{2n+1}$ $\psi \neq 0$ such that

$$\psi \cdot dE_{Z_0, t_f}(u)\delta u = 0, \quad \forall \delta u. \quad (3.37)$$

DEFINITION 3.7. A singular control is said to be of corank 1 if

$$\text{codim Im } dE_{X_0, t_f}(u) = 1.$$

In other words, ψ is the unique vector (up to a multiplying scalar) verifying (3.37).

Set $\psi(t) = \psi M(t_f)M^{-1}(t)$, then

$$\dot{\psi} = -\psi A = -\psi \left(\tilde{F}_Z + \tilde{F}_u u_y h_Z + \tilde{F}_{u_y} u_{yy} h_Z \right). \quad (3.38)$$

Furthermore the relation (3.37) is verified for every control δu , and thus

$$\int_0^{t_f} \psi(t) \left(B(t) \delta u(t, x) + C(t) \delta u_y^T(t, x) \right) dt = 0. \quad (3.39)$$

This relation is verified for all controls u functions of t and x . In particular it is also verified for controls u functions of t only. For this kind of controls, the relation (3.39) becomes

$$\int_0^{t_f} \psi(t) \left(B(t) \delta u(t) \right) dt = 0. \quad (3.40)$$

It implies that almost everywhere on $[0, t_f]$

$$\psi(t) B(t) = 0. \quad (3.41)$$

The relation (3.39) can be simplified as

$$\int_0^{t_f} \psi(t) \left(C(t) \delta u_y^T(t, x) \right) dt = 0. \quad (3.42)$$

Hence, almost everywhere on $[0, t_f]$, there holds

$$\psi(t) C(t) = 0. \quad (3.43)$$

REMARK 3.8. This argument holds because the time is an argument of the controls u . If the admissible controls are pure feedback controls $u(x)$ and independent of time, the relations (3.41) and (3.43) do not hold anymore. The condition (3.40) in this case gives a constraint linking $B(t)$ and $C(t)$.

Let $H = \psi \tilde{F}(t, Z, u, u_y)$ be the Hamiltonian associated with this optimization problem. The last equations can be rewritten as a Hamiltonian characterization of a singular control almost everywhere on $[0, t_f]$:

$$\dot{Z} = \tilde{F}(t, Z, u, u_y) = \frac{\partial H}{\partial \psi}, \quad (3.44)$$

$$\dot{\psi} = -\psi \left(\tilde{F}_Z + \tilde{F}_u u_y h_Z + \tilde{F}_{u_y} u_{yy} h_Z \right) = -\frac{dH}{dZ}, \quad (3.45)$$

$$\frac{\partial H}{\partial u} = \psi(t) B(t) = 0, \quad (3.46)$$

$$\frac{\partial H}{\partial u_y} = \psi(t) C(t) = 0. \quad (3.47)$$

LEMMA 3.9. If a control u is optimal for the optimization problem composed of the dynamical constraints (3.10) and (3.11) and the criterion (3.8), then it is singular for the extended dynamic (3.31).

A proof can be found in [5].

Denoting $\psi = (\lambda_1 \ \lambda_2 \ \lambda^\circ)$, one obtains

$$0 = \lambda_1 \frac{\partial F_1}{\partial u} + \lambda_2 \left(\frac{\partial F_{21}}{\partial u} + \frac{\partial F_{22}}{\partial u} u_y \right)^T + \lambda^\circ \frac{\partial \tilde{L}_1}{\partial u}, \quad (3.48)$$

$$0 = \lambda_2^T F_{22} = \lambda_2^T \left(p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u} \right), \quad (3.49)$$

$$\dot{\lambda}_1 = -\lambda_1 \frac{\partial F_1}{\partial x} - \lambda_2 \left(\frac{\partial F_{21}}{\partial x} + \frac{\partial F_{22}}{\partial x} u_y \right)^T - \lambda^\circ \frac{\partial \tilde{L}_1}{\partial x}, \quad (3.50)$$

$$\dot{\lambda}_2 = -\lambda_1 \frac{\partial F_1}{\partial p_2} - \lambda_2 \left(\frac{\partial F_{21}}{\partial p_2} + \frac{\partial F_{22}}{\partial p_2} u_y \right)^T - \lambda^\circ \frac{\partial \tilde{L}_1}{\partial p_2}, \quad (3.51)$$

which are the necessary conditions given by Proposition 3.2 for a closed-loop Stackelberg equilibrium.

3.4. Transversality conditions. Some part of the initial and final values of the extended state z are imposed by the transversality condition for the follower optimization problem (3.5) and by the initial state $x(0) = x_0$. We can formalize these conditions by defining two sets M_0 and M_1

$$\begin{pmatrix} x(0) \\ p_2^T(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ p_2^T(0) \end{pmatrix} \in M_0, \quad \begin{pmatrix} x(t_f) \\ p_2^T(t_f) \end{pmatrix} = \begin{pmatrix} x(t_f) \\ \frac{\partial g_2}{\partial x}(h(z(t_f))) \end{pmatrix} \in M_1, \quad (3.52)$$

where

$$M_0 = \{x_0\} \times \mathbb{R}^n = \left\{ \begin{pmatrix} x \\ p_2^T \end{pmatrix} \mid F_0 \begin{pmatrix} x \\ p_2^T \end{pmatrix} = x - x_0 = 0 \right\}, \quad (3.53)$$

and

$$M_1 = \left\{ \begin{pmatrix} x \\ p_2^T \end{pmatrix} \mid F_1 \begin{pmatrix} x \\ p_2^T \end{pmatrix} = \frac{\partial g_2}{\partial x}(h(z(t_f))) - p_2 = 0 \right\}. \quad (3.54)$$

The tangent manifolds $T_{z(0)}M_0$ and $T_{z(t_f)}M_1$ are defined by

$$T_{z(0)}M_0 = \{(0, \alpha) \in \mathbb{R}^{2n} \mid \alpha \in \mathbb{R}^n\}, \quad (3.55)$$

and

$$T_{z(t_f)}M_1 = \left\{ \left(\beta, \beta \frac{\partial^2 g_2}{\partial x^2} \right) \mid \beta \in \mathbb{R}^n \right\}. \quad (3.56)$$

The transversality conditions can be written as (see [19, p. 104])

$$\lambda(0) \perp T_{z_0}M_0, \quad (3.57)$$

$$\lambda(t_f) - \lambda^\circ \frac{\partial g_1}{\partial z}(h(z(t_f))) \perp T_{z(t_f)}M_1, \quad (3.58)$$

and lead to

$$\lambda_2(0) = 0, \quad (3.59)$$

$$\lambda_1(t_f) - \lambda^\circ \frac{\partial g_1}{\partial x}(x(t_f)) + \lambda_2(t_f) \frac{\partial^2 g_2}{\partial x^2}(x(t_f)) = 0. \quad (3.60)$$

3.5. Degenerating property. In this section, under a weak assumption, the omnipotence of the leader is emphasized leading to a degeneracy of the Stackelberg strategy. The hierarchical roles of the players seem to disappear. An omnipotent leader is able to impose his/her control to the other player without taking into account the rational reaction set of the follower.

PROPOSITION 3.10. *Equation (3.13) implies that $\lambda_2 \equiv 0$ or $F_{22} \equiv 0$ (or both). Under the additional assumption that the $m_1 \times m_1$ matrix*

$$\frac{\partial}{\partial u} \left(p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u} \right)^T \quad (3.61)$$

is invertible, there holds

$$\lambda_2 \equiv 0. \quad (3.62)$$

Proof. The proof is obtained by contradiction. The term $\lambda_2^T F_{22}$ in (3.13) is the product of a column vector (λ_2^T) and a line vector $F_{22} = p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u}$, since λ_2 is a line costate vector. The triviality of this term induces that all components of λ_2 or all components of F_{22} are trivial (or both).

Assume that $\lambda_2 \neq 0$, then $F_{22} = p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u} \equiv 0$. If furthermore

$$\frac{\partial F_{22}}{\partial u} = \frac{\partial}{\partial u} \left(p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u} \right) \quad (3.63)$$

is invertible, then the Implicit Function Theorem applied to the function F_{22} with respect to the variable u allows to write locally along the trajectory the control

$$u = u(t, x, p_2). \quad (3.64)$$

The system in (x, p_2) is rewritten as

$$\begin{cases} \dot{x} &= F_1(t, x, p_2, u(t, x, p_2)) \\ \dot{p}_2 &= F_{21}(t, x, p_2, u(t, x, p_2)), \quad \text{because } F_{22} = 0. \end{cases} \quad (3.65)$$

Since the dynamics and the criterion are independent of u_y , we can deduce that any control u_y is extremal for the optimization problem. But the relation (3.12) is a constraint on u_y . Hence

$$\lambda_2 \equiv 0. \quad (3.66)$$

□

REMARK 3.11. *The fact that $\lambda_2 \equiv 0$ means that the leader does not take into account the rational reaction set of the follower. It seems to be in contradiction with the hierarchical position between the leader and the follower. In fact, the leader does not take into account the reaction of the follower, because he can impose his desired control to the follower. The leader is omnipotent with respect to the follower. The condition $\frac{\partial F_{22}}{\partial u}$ invertible formalizes this privileged position of the leader.*

3.6. Linear Quadratic case. In this section, the obtained results are applied to the linear quadratic case. Consider a linear dynamic constraint

$$\dot{x} = Ax + B_1u + B_2v \quad (3.67)$$

and the quadratic criteria

$$J_1(u, v) = \frac{1}{2} \int_0^{t_f} (x^T Q_1 x + u^T R_{11} u + v^T R_{12} v) dt + \frac{1}{2} x(t_f)^T K_{1f} x(t_f), \quad (3.68)$$

$$J_2(u, v) = \frac{1}{2} \int_0^{t_f} (x^T Q_2 x + u^T R_{21} u + v^T R_{22} v) dt + \frac{1}{2} x(t_f)^T K_{2f} x(t_f), \quad (3.69)$$

$$(3.70)$$

where the matrices Q_i , R_{ij} and K_{if} are symmetric for $i, j \in \{1, 2\}$, and R_{22} and R_{11} are invertible.

3.6.1. Necessary conditions for the follower. The Hamiltonian associated with the follower (dynamic constraint (3.67) and criterion (3.69)) is

$$H_2 = p_2(Ax + B_1u + B_2v) + \frac{1}{2}(x^T Q_2 x + u^T R_{21} u + v^T R_{22} v). \quad (3.71)$$

Applying the relations (3.3), (3.4), we obtain

$$\dot{p}_2 = -\frac{dH_2}{dx} = -p_2 A - x^T Q_2 - p_2 B_1 \frac{\partial u^*}{\partial x} - u^T R_{21} \frac{\partial u^*}{\partial x}, \quad (3.72)$$

$$p_2(t_f) = x(t_f)^T K_{2f}, \quad (3.73)$$

$$\frac{\partial H_2}{\partial v} = 0 = p_2 B_2 + v^T R_{22}. \quad (3.74)$$

Since R_{22} is invertible by assumption, the optimal control is

$$v = -R_{22}^{-1} B_2^T p_2^T = S(t, x, p_2, u). \quad (3.75)$$

3.6.2. Necessary conditions for the leader. In the case of quadratic criteria, there holds

$$F_1(t, x, p_2, u) = Ax + B_1u - B_2 R_{22}^{-1} B_2^T p_2^T, \quad (3.76)$$

$$F_{21}(t, x, p_2, u) = -p_2 A - x^T Q_2, \quad (3.77)$$

$$F_{22}(t, x, p_2, u) = -p_2 B_1 - u^T R_{21}. \quad (3.78)$$

By injecting the expression of the optimal control of the follower (3.75), the instantaneous leader's criterion can be written as

$$\tilde{L}_1(t, x, p_2, u) = \frac{1}{2} (x^T Q_1 x + u^T R_{11} u + p_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T p_2). \quad (3.79)$$

The necessary conditions (3.12), (3.13), (3.14), (3.15) lead to

$$\frac{\partial H}{\partial u} = 0 = \lambda_1 B_1 - \lambda_2 \left(\frac{\partial u}{\partial x} \right)^T R_{21} + \lambda^\circ u^T R_{11}, \quad (3.80)$$

$$\frac{\partial H}{\partial u_y} = 0 = -\lambda_2^T (p_2 B_1 + u^T R_{21}), \quad (3.81)$$

$$\dot{\lambda}_1 = -\lambda_1 A + \lambda_2 Q_2 - \lambda^\circ x^T Q_1, \quad (3.82)$$

$$\dot{\lambda}_2 = \lambda_1 B_2 R_{22}^{-1} B_2^T + \lambda_2 \left(A + B_1 \left(\frac{\partial u}{\partial x} \right)^T \right)^T - \lambda^\circ p_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T, \quad (3.83)$$

$$\dot{\lambda}^\circ = 0, \quad (3.84)$$

with the transversality conditions

$$\lambda_1(t_f) = \lambda^\circ x(t_f)^T K_{1f} - \lambda_2(t_f) K_{2f}, \quad (3.85)$$

$$\lambda_2(0) = 0. \quad (3.86)$$

From Proposition 3.10, and assuming that $\frac{\partial}{\partial u} \left(p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u} \right) = R_{21}$ is invertible,

we can deduce that $\lambda_2 \equiv 0$.

Since the vector $(\lambda_1, \lambda_2, \lambda^\circ)$ at $t = t_f$ cannot be zero, and according to the transversality condition (3.85), we can take $\lambda^\circ = 1$.

From (3.80), we deduce with the invertibility of R_{11} , that

$$u = -R_{11}^{-1} B_1^T \lambda_1^T. \quad (3.87)$$

Moreover, Equation (3.83) becomes, with $\lambda_2 \equiv 0$,

$$\lambda_1 B_2 R_{22}^{-1} B_2^T - p_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \equiv 0. \quad (3.88)$$

Assuming that the rank of B_2 is maximal, that is, $\text{rank } B_2 = m_2$ (the number of the components of the control v), this relation yields

$$\lambda_1 B_2 = p_2 B_2 R_{22}^{-1} R_{12}. \quad (3.89)$$

In this expression the optimal control v , given by (3.75) can be recognized. The optimal control v verifies also

$$R_{12} v = -B_2^T \lambda_1^T. \quad (3.90)$$

If R_{12} is invertible, then the control v admits two expressions:

$$v = -R_{12}^{-1} B_2^T \lambda_1^T = -R_{22}^{-1} B_2^T p_2^T. \quad (3.91)$$

We gather the necessary conditions for optimality obtained in the following theorem.

PROPOSITION 3.12. *For $x_0 \neq 0$, if the matrices Q_i , R_{ij} and K_{if} are symmetric, if $R_{11} > 0$, $R_{22} > 0$, $R_{12} > 0$, and R_{21} invertible and if $\text{rank } B_2 = m_2$ (B_2 is of full rank), then the controls issued from a Stackelberg strategy with a closed-loop information structure are*

$$u = -R_{11}^{-1} B_1^T \lambda_1^T, \quad (3.92)$$

$$v = -R_{22}^{-1} B_2^T p_2^T = -R_{12}^{-1} B_2^T \lambda_1^T, \quad (3.93)$$

with

$$\dot{x} = Ax + B_1 u + B_2 v, \quad x(0) = x_0, \quad (3.94)$$

$$\dot{p}_2 = -p_2 A - x^T Q_2 - \left(p_2 B_1 + u^T R_{21} \right) \frac{\partial u}{\partial x}, \quad p_2(t_f) = x_f^T K_{2f}, \quad (3.95)$$

$$\dot{\lambda}_1 = -\lambda_1 A - x^T Q_1, \quad \lambda_1(t_f) = x_f^T K_{1f}, \quad (3.96)$$

$$\lambda_1 B_2 = p_2 B_2 R_{22}^{-1} R_{12}. \quad (3.97)$$

At this step, two cases are considered: $p_2 B_1 + u^T R_{21} = 0$ and $p_2 B_1 + u^T R_{21} \neq 0$ to derive the necessary conditions. It is shown that the first case is irrelevant under some additional weak assumptions. The investigation of the second case shows that it is relevant.

3.6.3. Case $p_2 B_1 + u^T R_{21} = 0$.

LEMMA 3.13. *If the pair (Q_1, A) is observable, and at least one of the pairs (A, B_1) and (A, B_2) is controllable, then*

$$x(t) = \lambda_1^T(t) = p_2^T(t) = 0, \quad \forall t \in [0, t_f]. \quad (3.98)$$

This means that the only optimal trajectory is the trivial one.

REMARK 3.14. *Equations (3.98) imply in particular, that $x(0) = 0$. If $x(0) = x_0 \neq 0$, there does not exist any optimal trajectory starting from x_0 .*

Proof. With the condition

$$p_2 B_1 + u^T R_{21} = 0, \quad (3.99)$$

the term $\frac{\partial u}{\partial x}$ does not appear anymore in the necessary conditions (3.94)-(3.97). Derivating with respect to time the relation (3.99) does not induce necessary conditions for $\frac{\partial u}{\partial x}$.

However assuming that R_{21} is invertible, the control u admits two representations

$$u = -R_{11}^{-1} B_1^T \lambda_1^T = -R_{21}^{-1} B_1^T p_2^T. \quad (3.100)$$

From this relation and from (3.97), necessary conditions about $x(t_f)$ are developed by successive derivation with respect to time.

$$\begin{cases} \lambda_1 B_2 - p_2 B_2 R_{22}^{-1} R_{12} = 0, \\ \lambda_1 B_1 R_{11}^{-1} R_{21} - p_2 B_1 = 0. \end{cases} \quad (3.101)$$

These two relations can be rewritten for every $t \in [0, t_f]$ as

$$\begin{pmatrix} x & \lambda_1 & p_2 \end{pmatrix} \underbrace{\begin{bmatrix} 0 & 0 \\ B_2 & B_1 R_{11}^{-1} R_{21} \\ -B_2 R_{22}^{-1} R_{12} & -B_1 \end{bmatrix}}_{\mathcal{B}} = 0. \quad (3.102)$$

Injecting (3.99) into the dynamics of x , λ_1 and p_2 , we obtain the autonomous differential system

$$\frac{d}{dt} \begin{pmatrix} x \\ \lambda_1^T \\ p_2^T \end{pmatrix} = \begin{pmatrix} x \\ \lambda_1^T \\ p_2^T \end{pmatrix} \underbrace{\begin{bmatrix} A^T & -Q_1 & -Q_2 \\ -B_1 R_{11}^{-1} B_1^T - B_2 R_{12}^{-1} B_2^T & -A & 0 \\ 0 & 0 & -A \end{bmatrix}}_{\mathcal{A}} \quad (3.103)$$

The k -order derivation of (3.102) with respect to time, at time $t = t_f$, gives

$$\begin{bmatrix} x^T(t_f) & x^T(t_f) K_{1f} & x^T(t_f) K_{2f} \end{bmatrix} \mathcal{A}^k \mathcal{B} = 0, \quad \forall k \in \mathbb{N}. \quad (3.104)$$

Assuming that the pair $(\mathcal{A}, \mathcal{B})$ is controllable, these conditions imply that $x(t_f) = 0$. However the autonomous linear system in x , λ_1 and p_2 with end value conditions $x(t_f) = \lambda_1^T(t_f) = p_2^T(t_f) = 0$ imposes, by a backward integration of (3.103)

$$x(t) = \lambda_1^T(t) = p_2^T(t) = 0, \quad \forall t \in [0, t_f]. \quad (3.105)$$

The only optimal trajectory in this case is the trivial one. \square

Sufficient conditions are derived below to ensure that the pair $(\mathcal{A}, \mathcal{B})$ is controllable.

PROPOSITION 3.15. *Assuming*

- *the pair (Q_1, A) is observable,*
- *at least one of the pairs (A, B_1) or (A, B_2) is controllable,*

then the pair $(\mathcal{A}, \mathcal{B})$ is controllable.

Proof. The proof uses the controllability Hautus test. The pair $(\mathcal{A}, \mathcal{B})$ is controllable if and only if the matrix

$$\begin{bmatrix} \mathcal{A} - \alpha I & \vdots & \mathcal{B} \end{bmatrix} \quad (3.106)$$

is of full rank, for every $\alpha \in \mathbb{C}$. The proof consists of showing that all line vectors $\begin{pmatrix} z_1^T & z_2^T & z_3^T \end{pmatrix}$ verify

$$\begin{pmatrix} z_1^T & z_2^T & z_3^T \end{pmatrix} \begin{bmatrix} \mathcal{A} - \alpha I & \vdots & \mathcal{B} \end{bmatrix} = 0, \quad (3.107)$$

are trivial. Developing Equation (3.107), we have

$$-z_1^T Q_1 = z_2^T (A - \alpha I_n), \quad (3.108)$$

$$-z_1^T Q_2 = z_3^T (A - \alpha I_n), \quad (3.109)$$

$$z_1^T (A^T - \alpha I_n) = z_2^T (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T), \quad (3.110)$$

$$z_2^T B_2 = z_3^T B_2 R_{22}^{-1} R_{12}, \quad (3.111)$$

$$z_2^T B_1 R_{11}^{-1} R_{21} = z_3^T B_1. \quad (3.112)$$

Multiplying by z_1 Equation (3.108) and by z_2 Equation (3.110), we obtain

$$-z_1^T Q_1 z_1 = z_2^T (A - \alpha I_n) z_1 = z_2^T (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T) z_2 \quad (3.113)$$

The first term is nonpositive and the last term is nonnegative, hence both are zero. It follows that

$$z_1^T Q_1 = 0, \quad z_2^T B_1 = 0, \quad z_2^T B_2 = 0. \quad (3.114)$$

Plugging these relations in (3.108), (3.112), one gets

$$z_1^T (A^T - \alpha I_n) = 0, \quad z_1^T Q_1 = 0, \quad (3.115)$$

$$z_2^T (A - \alpha I_n) = 0, \quad z_2^T B_2 = 0, \quad z_2^T B_1 = 0, \quad (3.116)$$

$$z_3^T (A - \alpha I_n) = 0, \quad z_3^T B_2 = 0, \quad z_3^T B_1 = 0. \quad (3.117)$$

The relations (3.115) corresponds to the observability Hautus test of the pair (Q_1, A) , the relations (3.116) to the controllability Hautus test of the pair (A, B_1) or (A, B_2) , and the relations (3.117) to the controllability Hautus test of the pair (A, B_1) or (A, B_2) . The assumptions of controllability and observability lead to z_1, z_2 and z_3 trivial. \square

REMARK 3.16. *This means that the particular case $p_2 B_1 + u^T R_{21} = 0$ can be avoided with weak assumptions on the system. The leader should be able to observe the system (pair (Q_1, A) observable) and at least one player should be able to control the system ((A, B_1) or (A, B_2) controllable). Once again, it is emphasized that the roles of the players are not symmetric.*

We can then assume that

$$p_2 B_1 + u^T R_{21} \neq 0. \quad (3.118)$$

3.6.4. Case $p_2 B_1 + u^T R_{21} \neq 0$. The relation (3.97) is equivalent to both relations: (3.97) at time $t = t_f$ and his derivative with respect to time, that is,

$$\lambda_1(t_f) B_2 = x^T(t_f) K_{1f} B_2 = p_2(t_f) B_2 R_{22}^{-1} R_{12} = x^T(t_f) K_{2f} B_2 R_{22}^{-1} R_{12}, \quad (3.119)$$

and

$$\dot{\lambda}_1 B_2 = \dot{p}_2 B_2 R_{22}^{-1} R_{12} \quad (3.120)$$

$$= (\lambda_1 A + x^T Q_1) B_2 \quad (3.121)$$

$$= \left(p_2 A + x^T Q_2 + (p_2 B_1 + u^T R_{21}) \left(\frac{\partial u}{\partial x} \right) \right) B_2 R_{22}^{-1} R_{12}. \quad (3.122)$$

Hence

$$(p_2 B_1 + u^T R_{21}) \left(\frac{\partial u}{\partial x} \right) B_2 = (\lambda_1 A + x^T Q_1) B_2 R_{12}^{-1} R_{22} - (p_2 A + x^T Q_2) B_2. \quad (3.123)$$

Therefore, (3.97) is equivalent to

$$\begin{cases} \left(B_2^T K_{1f} - R_{12} R_{22}^{-1} B_2^T K_{2f} \right) x(t_f) = 0, \\ \left(p_2 B_1 + u^T R_{21} \right) \left(\frac{\partial u}{\partial x} \right) B_2 = (\lambda_1 A + x^T Q_1) B_2 R_{12}^{-1} R_{22} - (p_2 A + x^T Q_2) B_2. \end{cases} \quad (3.124)$$

Equation (3.123) permits to derive an expression of $\frac{\partial u}{\partial x}$, since $p_2 B_1 + u^T R_{21} \neq 0$

$$(p_2 B_1 + u^T R_{21}) \left(\frac{\partial u}{\partial x} \right) = w_2 + w'_2, \quad (3.125)$$

with

$$w_2 = \left((\lambda_1 A + x^T Q_1) B_2 R_{12}^{-1} R_{22} - (p_2 A + x^T Q_2) B_2 \right) \frac{B_2^T}{\|B_2\|^2}, \quad (3.126)$$

and $(w'_2)^T \in \text{Ker}(B_2^T)$ (arbitrary).

The constraint (3.119) translates into a constraint on the set of initial points $x_0 \in \mathbb{R}^n$ from which an solution starts.

LEMMA 3.17. *The existing solutions are associated with a set of x_0 included in a subset of \mathbb{R}^n with a m_2 -codimension (at worst).*

Proof. Here, like in a classical Linear Quadratic problems, it is possible to look for an optimal solution by assuming that the costate vector $\lambda_1(t)$ is linear with respect to the state $x(t)$.

$$\lambda_1^T(t) = K_1(t)x(t). \quad (3.127)$$

The matrix $K_1(t) \in \mathbb{R}^{n \times n}$ verifies

$$\dot{K}_1 x + K_1 (Ax - (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T) K_1 x) = -A^T K_1 x - Q_1 x. \quad (3.128)$$

This should be true $\forall x$, hence $K_1(t)$ is the solution of the following Riccati differential equation

$$\begin{cases} \dot{K}_1 = -K_1 A - A^T K_1 - Q_1 + K_1 (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T) K_1, \\ K_1(t_f) = K_{1f}. \end{cases} \quad (3.129)$$

The existence of a solution of the optimization problem is assured classically "a la Riccati" and by the uniqueness of an optimal trajectory. This is justified *a posteriori* in the following by using the theory of focal times.

By reinjecting $\lambda_1^T = K_1 x$ in (3.91) and (3.100), the state $x(t)$ has the dynamical constraint

$$\begin{cases} \dot{x}(t) = (A - (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T) K_1(t)) x(t) = \tilde{A}x(t), \\ x(0) = x_0. \end{cases} \quad (3.130)$$

Let $\mathcal{M}(t)$ be the transition matrix associated with (3.130). Then $x(t) = \mathcal{M}(t)x_0$. Then the constraint (3.119) becomes

$$(B_2^T K_{1f} - R_{12} R_{22}^{-1} B_2^T K_{2f}) \mathcal{M}(t_f)x_0 = 0. \quad (3.131)$$

This is a m_2 -codimension (at worst) condition on the initial states x_0 . \square

It should be noted that the optimal trajectory is unique, in this case, but it is achieved by all controls which verify the relation (3.123). An optimal trajectory induces several $\frac{\partial u}{\partial x}$.

This explains the idea in [11] to assume that the initial state is uniformly distributed over the unit sphere and replacing the optimization criterion with its mean value over the initial state.

REMARK 3.18. *In the case of optimization problem without terminal criteria, the relation (3.119) does not reduce the set of initial state x_0 associated with optimal trajectories.*

We gather all previous results in the following theorem.

THEOREM 3.19. *If*

- $x_0 \neq 0$,
- Q_i , R_{ij} and K_{if} are symmetric,
- $R_{11} > 0$, $R_{22} > 0$ and R_{12} and R_{21} are invertible,
- the pair (Q_1, A) is observable,
- at least one of the pairs (A, B_1) and (A, B_2) is controllable,
- rank $B_2 = m_2$ (B_2 of full rank),

then the optimal trajectory verifies the necessary conditions

$$u(t, x(t)) = -R_{11}^{-1} B_1^T K_1(t)x(t), \quad v(t, x(t)) = -R_{12}^{-1} B_2^T K_1(t)x(t), \quad (3.132)$$

with

$$\dot{x}(t) = (A - (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T) K_1(t))x(t), \quad x(0) = x_0, \quad (3.133)$$

where

$$\dot{K}_1 = -K_1 A - A^T K_1 - Q_1 - K_1 (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T) K_1, \quad K_1(t_f) = K_{1f}. \quad (3.134)$$

Futhermore $\frac{\partial u(t, x(t))}{\partial x}$ verifies

$$(p_2 B_1 + u^T R_{21}) \frac{\partial u}{\partial x} B_2 = (\lambda_1 A + x^T Q_1) B_2 R_{12}^{-1} R_{22} - (p_2 A + x^T Q_2) B_2, \quad (3.135)$$

where

$$\dot{p}_2 = -p_2 A - x^T Q_2 - (p_2 B_1 + u^T R_{21}) \frac{\partial u}{\partial x}, \quad p_2(t_f) = x^T(t_f) K_{2f}, \quad (3.136)$$

and

$$(B_2^T K_{1f} - R_{12} R_{22}^{-1} B_2^T K_{2f})x(t_f) = 0. \quad (3.137)$$

In the next section, the theory of focal times allows to obtain sufficient conditions for the Stackelberg strategy with a closed-loop information structure.

4. Sufficient conditions. In this section, in order to obtain sufficient conditions, the theory of focal times is required (and no conjugate times, because the final state $x(t_f)$ is free). Sufficient conditions for optimality are first derived for the leader, and then for the follower in the case of Linear Quadratic games.

4.1. Sufficient conditions for the leader. The optimization problem of the leader is

$$\begin{cases} \dot{x} = Ax - B_2 R_{22}^{-1} B_2^T p_2^T + B_1 u, \\ \dot{p}_2^T = -A^T p_2^T - Q_2 x - w^T (p_2 B_1 + u^T R_{21})^T, \\ \hat{J}_1(u) = \frac{1}{2} \int_0^{t_f} (x^T Q_1 x + u^T R_{11} u + p_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T p_2^T) dt + \frac{1}{2} x^T(t_f) K_{1f} x(t_f), \end{cases} \quad (4.1)$$

with $x(0) = x_0$ and $p_2(t_f) = x^T(t_f) K_{1f}$. When $p_2 B_1 + u^T R_{21} \neq 0$, the control w is cheap [5], and p_2 can be also be considered as a control. More precisely, $y = B_2 p_2^T$ is considered as a control. Then the problem (4.1) can be rewritten as

$$\begin{cases} \dot{x} = Ax - B_2 R_{22}^{-1} y + B_1 u, \\ \hat{J}_1(u, y) = \frac{1}{2} \int_0^{t_f} (x^T Q_1 x + u^T R_{11} u + y^T R_{22}^{-1} R_{12} R_{22}^{-1} y) dt + \frac{1}{2} x^T(t_f) K_{1f} x(t_f). \end{cases} \quad (4.2)$$

REMARK 4.1. *Note that this Linear Quadratic problem with controls (u, y) can be linked to the Team optimal approach in [2]. In this reference, the first step in the research of Stackelberg equilibrium is to obtain the minimum of the criterion of the leader, by a team cooperation between the leader and the follower. Then the follower control is modified to achieve the minimum of the criterion of the follower.*

The Linear Quadratic problem corresponds to find the optimal control (u, y) . A necessary condition for the existence of an optimal control is $R_{22}^{-1} R_{12} R_{22}^{-1} \geq 0$. It is equivalent to $R_{12} \geq 0$, since R_{22} is positive definite. A sufficient condition for the existence of an optimal control, when t_f is small is $R_{12} > 0$ [19]. In the following, it is assumed that $R_{12} > 0$.

To characterize the focal points, by taking

$$u = -R_{11}^{-1} B_1^T \lambda_1^T, \quad y = R_{22} R_{12}^{-1} B_2^T \lambda_1^T, \quad (4.3)$$

we consider the variational system

$$\begin{cases} \delta \dot{x} = A \delta x - (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T) \delta \lambda_1^T, \\ \delta \dot{\lambda}_1 = -\lambda_1 A - \delta x^T Q_1. \end{cases} \quad (4.4)$$

We search the first focal time denoted $t_c > 0$ (see [6]), such that there exists a solution $(\delta x, \delta \lambda_1)$ verifying $(x(0) = x_0)$ is fixed)

$$\begin{cases} \delta x(0) = 0, \\ \delta \lambda_1(t_c) = \delta x^T(t_c) K_{1f}. \end{cases} \quad (4.5)$$

This is equivalent to

$$\|K(t)\| \begin{array}{l} \longrightarrow +\infty, \\ t \rightarrow t_c \\ t < t_c \end{array} \quad (4.6)$$

where $K(t)$ is the solution of the Riccati differential equation

$$\begin{cases} \dot{K} = KA + A^TK + Q_1 - K(B_1R_{11}^{-1}B_1^T + B_2R_{12}^{-1}B_2^T)K, \\ K(0) = K_{1f}. \end{cases} \quad (4.7)$$

The first focal time t_c is a finite escape time for the Riccati differential equation (4.7). The matrix $K(t)$ can be viewed as the matrix $K_1(t)$ after a change of time $t \in [0, t_f] \mapsto (t_f - t)$.

LEMMA 4.2. *If $R_{11} > 0$ and $R_{12} > 0$ then $t_c > 0$.*

Proof. From Equation (4.1), one gets

$$x(t) = e^{tA}x_0 - \int_0^t e^{(t-s)A} (B_2R_{22}^{-1}y(s) - B_1u(s)) ds. \quad (4.8)$$

Given $t_f > 0$, for every $t \in [0, t_f]$, there exist scalar constants $C_k \geq 0$ verifying

$$\|x(t)\| \leq C_1\|x_0\| + C_2\sqrt{t_f} \left[\left(\int_0^{t_f} \|y(s)\|^2 ds \right)^{\frac{1}{2}} + \left(\int_0^{t_f} \|u(s)\|^2 ds \right)^{\frac{1}{2}} \right]. \quad (4.9)$$

Hence

$$\begin{aligned} \left\| \int_0^{t_f} x^T(s)Q_1x(s)ds \right\| &\leq C_3\|x_0\|^2 + C_4t_f^2 \left(\int_0^{t_f} \|y(s)\|^2 ds + \int_0^{t_f} \|u(s)\|^2 ds \right) \\ &\quad C_5t_f + C_5t_f^2 \int_0^{t_f} \|y(s)\|^2 ds + C_5t_f^2 \int_0^{t_f} \|u(s)\|^2 ds \end{aligned} \quad (4.10)$$

In addition, by assuming $R_{11} > 0$ and $R_{22}^{-1}R_{12}R_{22}^{-1} > 0$,

$$\left\| \int_0^{t_f} u^T(s)R_{11}u(s)ds \right\| \geq C_6 \int_0^{t_f} \|u(s)\|^2 ds, \quad (4.11)$$

$$\left\| \int_0^{t_f} y^T(s)R_{22}^{-1}R_{12}R_{22}^{-1}y(s)ds \right\| \geq C_6 \int_0^{t_f} \|y(s)\|^2 ds. \quad (4.12)$$

Using these inequalities and (4.2), we can compute a lower bound of the criterion $\hat{J}_1(u, y)$

$$2\hat{J}_1(u, y) \geq (C_6 - (C_4 + C_5)t_f^2) \left[\int_0^{t_f} \|u\|^2 ds + \int_0^{t_f} \|y\|^2 ds \right] \quad (4.13)$$

$$+ x^T(t_f)K_{1f}x(t_f) - C_3\|x_0\|^2 - C_5t_f. \quad (4.14)$$

For t_f enough small

$$t_f \leq \sqrt{\frac{C_6}{C_4 + C_5}}, \quad (4.15)$$

the criterion $\hat{J}_1(u, y)$ is finite bounded, then $0 < t_f < t_c$. \square

REMARK 4.3. *If $Q_1 \geq 0$, then Equation (4.7) admits a solution on $[0, +\infty[$. There is no finite escape time for this equation. There is no first focal time ($t_c \rightarrow +\infty$).*

4.2. Sufficient conditions for the follower. The optimization problem for the follower is

$$\begin{cases} \dot{x} = Ax + B_1 u(t, x) + B_2 v, & x(0) = x_0, \\ \hat{J}_2 = \frac{1}{2} \int_0^{t_f} (x^T Q_2 x + u^T R_{21} u + v^T R_{22} v) dt + \frac{1}{2} x^T(t_f) K_{2f} x(t_f). \end{cases} \quad (4.16)$$

with

$$v = -R_{22}^{-1} B_2^T p_2^T \quad (4.17)$$

where

$$\dot{p}_2 = -p_2 A - x^T Q_2 - (p_2 B_1 + u^T R_{21}) \frac{\partial u}{\partial x}, \quad p_2(t_f) = x^T(t_f) K_{2f}. \quad (4.18)$$

The variational system along the trajectory $x(\cdot)$ is

$$\delta \dot{x} = A \delta x + B_1 \frac{\partial u}{\partial x} \delta x - B_2 R_{22}^{-1} B_2^T p_2^T, \quad (4.19)$$

$$\begin{aligned} \delta \dot{p}_2 = & -\delta p_2 A - \delta x^T Q_2 - (p_2 B_1 + u^T R_{21}) \frac{\partial^2 u}{\partial x^2} \delta x \\ & - \left(\delta p_2 B_1 + \left(\frac{\partial u}{\partial x} \delta x \right)^T R_{21} \right) \frac{\partial u}{\partial x}. \end{aligned} \quad (4.20)$$

with $\delta x(0) = 0$ and $\delta p_2(t_f) = \delta x^T(t_f) K_{2f}$. Here $u(t, x)$ is affine with respect to x , then $\frac{\partial^2 u}{\partial x^2} = 0$. Equation (4.20) then rewrites

$$\delta \dot{p}_2 = -\delta p_2 A - \delta x^T Q_2 - \left(\delta p_2 B_1 + \left(\frac{\partial u}{\partial x} \delta x \right)^T R_{21} \right) \frac{\partial u}{\partial x}. \quad (4.21)$$

A focal time t'_c characterizes the existence of a solution $(\delta x, \delta p_2)$ such that $\delta x(0) = 0$ and $\delta p_2(t'_c) = \delta x^T(t'_c) K_{2f}$. For each choice of the term $\frac{\partial u}{\partial x}$, there exists a first focal time t'_c .

4.3. Sufficient conditions for Stackelberg Strategy. The sufficient conditions ensuring the existence of an optimal trajectory are summed up in the following theorem.

THEOREM 4.4. *Assume that*

- $x_0 \neq 0$,
- Q_i, R_{ij} and K_{if} are symmetric,
- $R_{11} > 0, R_{22} > 0, R_{12} > 0$, and R_{21} invertible,
- the pair (Q_1, A) is observable,
- at least one of the pairs (A, B_1) and (A, B_2) is controllable,
- rank $B_2 = m_2$ (B_2 is of full rank).

Let w'_2 a function of time t such that $w'_2 \in (B_2^T)^\perp$, and let

$$T^* = \min(t_c, t'_c) > 0. \quad (4.22)$$

For every $t_f < T^*$, there exists a unique solution of the Riccati differential equation (3.129). Denoting $x(t, x_0)$ the solution of Equation (3.130), let

$$\mathcal{H} = \{x_0 \in \mathbb{R}^n \mid (B_2^T K_{1f} - R_{12} R_{22}^{-1} B_2^T K_{2f}) x(t_f, x_0) = 0\}. \quad (4.23)$$

Then, for every $x_0 \in \mathcal{H}$, there exists a unique optimal solution of the optimization problem on $[0, t_f]$ associated with w'_2 . The optimal controls (u, v) associated with verify (3.125) and furthermore

$$u(t, x(t)) = -R_{11}^{-1} B_1^T K_1(t) x(t), \quad v(t, x(t)) = -R_{12}^{-1} B_2^T K_1(t) x(t). \quad (4.24)$$

In addition, for every $x_0 \notin \mathcal{H}$, there does not exist any optimal trajectory starting from x_0 .

REMARK 4.5. Theorem 4.4 provides a new result of existence of closed-loop Stackelberg strategies for linear-quadratic differential games.

REMARK 4.6. The sufficient conditions for optimality are developed in the Linear Quadratic case. The optimality results are here global. It is also, by the same argument, possible to express similar sufficient conditions in the general case of nonlinear criteria. However they are not developed here, because their expressions are more technical and because they lead only to local optimality results [6, chap. 9].

4.4. Extension: weighting of $\frac{\partial u}{\partial x}$ in criteria. The problem is degenerated,

because for each $x_0 \in \mathcal{H}$, there can exist an infinite choice of terms $\frac{\partial u^*}{\partial x}$. A way to yield a unique $\frac{\partial u^*}{\partial x}$ is to include a weighting on the term $\frac{\partial u^*}{\partial x}$ in the criterion J_1 of the leader. Then the leader takes into account a restriction on the Jacobian of its control. The leader is no more omnipotent.

The new criteria of the leader is then

$$J_1(u, v) = \frac{1}{2} \int_0^{t_f} \left[x^T Q_1 x + u^T R_{11} u + v^T R_{12} v + \sum_{j=1}^{m_1} \left(\frac{\partial u_j}{\partial x} \right) R_j \left(\frac{\partial u_j}{\partial x} \right)^T \right] dt + \frac{1}{2} x(t_f)^T K_{1f} x(t_f), \quad (4.25)$$

where u_j are the m_1 components of the control u , and $R_j \in \mathbb{R}^{n \times n}$, ($\forall j = 1, \dots, m_1$) are symmetric positive definite matrices.

There is no change for the follower. However the necessary conditions for the leader are modified as follows

$$\frac{\partial H}{\partial u} = 0 = \lambda_1 B_1 - \lambda_2 \left(\frac{\partial u}{\partial x} \right)^T R_{21} + \lambda^\circ u^T R_{11}, \quad (4.26)$$

$$\frac{\partial H}{\partial u_j} = 0 = \left(\lambda_2 (B_1^T p_2^T + R_{21} u)_j + \frac{\partial u_j}{\partial x} R_j \right)_{j=1, \dots, m_1}. \quad (4.27)$$

The other necessary conditions (3.14) and (3.15) are unchanged. Equations (4.26) and (4.26) are easily solvable, without considering different cases. In this framework, λ_2 is not trivial anymore. One gets a nonlinear optimization problem on boundary values. The computation leads to the result of [12]. More precisely from (4.27), we obtain

$$\frac{\partial u_j^*}{\partial x} = \left((B_1^T p_2^T + R_{21} u)_j \lambda_2 R_j^{-1} \right)_{j=1, \dots, m_1}. \quad (4.28)$$

In order to simplify like in [12], it is assumed that $R_j = R > 0$, $\forall j = 1, \dots, m_1$. Then

$$\frac{\partial u}{\partial x} = (B_1^T p_2^T + R_{21} u) \lambda_2 R^{-1}. \quad (4.29)$$

By reinjecting this expression in the equation (4.26),

$$R_{11}u = -B_1^T \lambda_1^T + R_{21} B_1^T p_2^T \lambda_2 R^{-1} \lambda_2^T + R_{21}^2 u \lambda_2 R^{-1} \lambda_2^T, \quad (4.30)$$

or

$$(R_{11} - \lambda_2 R^{-1} \lambda_2^T R_{21}^2) u = -B_1^T \lambda_1^T + R_{21} B_1^T p_2^T \lambda_2 R^{-1} \lambda_2^T. \quad (4.31)$$

REMARK 4.7. For $t = 0$, $\lambda_2(0) = 0$, then $R_{11} - \lambda_2 R^{-1} \lambda_2^T R_{21}^2 = R_{11} > 0$ is invertible. For $t \geq 0$ small enough, the matrix $R_{11} - \lambda_2(t) R^{-1} \lambda_2^T(t) R_{21}^2$ is invertible. As long as $R_{11} - \lambda_2(t) R^{-1} \lambda_2^T(t) R_{21}^2$ is invertible, the optimal control verifies

$$u = (R_{11} - \lambda_2 R^{-1} \lambda_2^T R_{21}^2)^{-1} (-B_1^T \lambda_1^T + R_{21} B_1^T p_2^T \lambda_2 R^{-1} \lambda_2^T) \quad (4.32)$$

The nonlinear optimization problem becomes

$$\begin{aligned} \dot{x} &= Ax - B_2 R_{22}^{-1} B_2 p_2^T \\ &\quad + B_1 (R_{11} - \lambda_2 R^{-1} \lambda_2^T R_{21}^2)^{-1} (-B_1^T \lambda_1^T + R_{21} B_1^T p_2^T \lambda_2 R^{-1} \lambda_2^T), \end{aligned} \quad (4.33)$$

$$\dot{p}_2 = -p_2 A - x^T Q_2 - \|p_2 B_1 + u^T R_{21}\|^2 \lambda_2 R^{-1}, \quad (4.34)$$

$$\dot{\lambda}_1 = -\lambda_1 A + \lambda_2 Q_2 - x^T Q_1, \quad (4.35)$$

$$\begin{aligned} \dot{\lambda}_2 &= \lambda_1 B_2 R_{22}^{-1} B_2^T + \lambda_2 (A^T + R^{-1} \lambda_2^T (p_2 B_1 + u^T R_{21}) B_1^T) \\ &\quad - p_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T. \end{aligned} \quad (4.36)$$

with boundary conditions

$$x(0) = x_0, \quad (4.37)$$

$$p_2(t_f) = x^T(t_f) K_{2f}, \quad (4.38)$$

$$\lambda_1(t_f) = x^T(t_f) K_{1f} - \lambda_2(t_f) K_{2f}, \quad (4.39)$$

$$\lambda_2(0) = 0. \quad (4.40)$$

REMARK 4.8. If $R = \gamma I$ and $\gamma \rightarrow +\infty$, we obtain at the limit $\frac{\partial u}{\partial x} = 0$ and we recognize the necessary conditions for the strategy of Stackelberg with an open-loop information structure. Note that to obtain this result, only an infinite weighting on $\frac{\partial u}{\partial x}$ is needed in the criterion of the leader.

These conditions are necessary conditions. Like previously, the theory of focal points leads to sufficient conditions associated with the Stackelberg strategy with closed-loop information structure including a weighting for $\frac{\partial u}{\partial x}$ in the criterion of the leader, namely, given $x_0 \in \mathbb{R}^n$. For t_f less than the global focal time of the system, there exists only one trajectory starting from x_0 solution of (4.33)-(4.40) associated with the optimal control $\left(u, \frac{\partial u}{\partial x}\right)$ (4.32), (4.29).

5. Conclusion. In this paper the Stackelberg strategy with a closed-loop information structure is studied. The framework is restricted to two players differential games. Necessary conditions for obtaining a closed-loop Stackelberg equilibrium are derived considering all cases. It is shown that the Stackelberg strategy could degenerate if the leader is omnipotent and can impose his control to the follower. The focal

times theory provides sufficient conditions for the optimization problems of the two players. The Linear Quadratic case is used to illustrate the obtained necessary and sufficient conditions. An extension is proposed to allow an optimal trajectory starting from any initial state by including, in the criterion, the Jacobian of his/her control in the criterion of the leader.

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