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FROM LEFT MODULES TO ALGEBRAS OVER AN OPERAD: APPLICATION TO COMBINATORIAL HOPF ALGEBRAS

MURIEL LIVERNET

ABSTRACT. The purpose of this paper is two fold: we study the behaviour of the forgetful functor from \mathbb{S} -modules to graded vector spaces in the context of algebras over an operad and derive from this theory the construction of combinatorial Hopf algebras. As a byproduct we obtain freeness and cofreeness results for these Hopf algebras.

Let \mathcal{O} denote the forgetful functor from \mathbb{S} -modules to graded vector spaces. Left modules over an operad \mathcal{P} are treated as \mathcal{P} -algebras in the category of \mathbb{S} -modules. We generalize the results obtained by Patras and Reutenauer in the associative case to any operad \mathcal{P} : the functor \mathcal{O} sends \mathcal{P} -algebras to \mathcal{P} -algebras. If \mathcal{P} is a Hopf operad the functor \mathcal{O} sends Hopf \mathcal{P} -algebras to Hopf \mathcal{P} -algebras. If the operad \mathcal{P} is regular one gets two different structures of Hopf \mathcal{P} -algebras in the category of graded vector spaces. We develop the notion of unital infinitesimal \mathcal{P} -bialgebra and prove freeness and cofreeness results for Hopf algebras built from Hopf operads. Finally, we prove that many combinatorial Hopf algebras arise from our theory, as Hopf algebras on the faces of the permutohedra and associahedra.

INTRODUCTION

A \mathbb{S} -module, also named symmetric sequence, is a graded vector space $(V_n)_{n \geq 0}$ together with a right action of the symmetric group S_n on V_n for each n . The present paper is concerned with the study, in an operadic point of view, of the forgetful functor \mathcal{O} from \mathbb{S} -modules to graded vector spaces and its applications.

The category **S-mod** of \mathbb{S} -modules is a tensor category. Motivated by the study of homotopy invariants, Barratt introduced the notion of *twisted Lie algebras* in [3], which are Lie algebras in the category of \mathbb{S} -modules or – in the operad context – *left modules* over the operad $\mathcal{L}ie$. More precisely, a twisted Lie algebra is a \mathbb{S} -module $(L_n)_{n \geq 0}$ together with a bilinear operation

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$[\cdot, \cdot]$ satisfying, for any $a \in L_p, b \in L_q$ and $c \in L_r$ the relations

$$\begin{aligned} [b, a] \cdot \zeta_{p,q} &= -[a, b], \\ [a, [b, c]] + [c, [a, b]] \cdot \zeta_{p+q,r} + [b, [c, a]] \cdot \zeta_{p,q+r} &= 0, \end{aligned}$$

where $\zeta_{p,q}$ is the permutation of S_{p+q} given by $\zeta_{p,q}(i) = q+i$ if $1 \leq i \leq p$ and $\zeta_{p,q}(i) = i-p$ if $p+1 \leq i \leq p+q$. For instance the \mathbb{S} -module $(\mathcal{L}\text{ie}(n))_{n \geq 0}$ is a twisted Lie algebra for the bracket induced by the operadic composition

$$\mathcal{L}\text{ie}(2) \otimes \mathcal{L}\text{ie}(n) \otimes \mathcal{L}\text{ie}(m) \rightarrow \mathcal{L}\text{ie}(n+m).$$

There exists also a notion of *twisted associated algebras*– associative algebras in the tensor category $\mathbf{S}\text{-mod}$ – and a notion of *twisted associated bialgebras* – associative bialgebras in the category $\mathbf{S}\text{-mod}$. Stover proved in [24] that a Cartier-Milnor-Moore theorem relates the categories of twisted associated bialgebras and twisted Lie algebras. Following an idea lying in [24], Patras and Reutenauer proved in [21] that two bialgebras arise naturally from a twisted bialgebra (A, m, Δ) : the *symmetrized bialgebra* $\bar{A} = (A, \hat{m}, \bar{\Delta})$ and the *cosymmetrized bialgebra* $\hat{A} = (A, \bar{m}, \hat{\Delta})$. In [22] Patras and Schocker derived from this construction some known combinatorial Hopf algebras. The first part of this paper is the generalization of these constructions to any operad \mathcal{P} . This generalization is performed in two steps: the first step focus on the algebra constructions and the second step on the coalgebra constructions.

Given an operad \mathcal{P} , the following notions are equivalent

- Twisted \mathcal{P} -algebras (see e.g. [12]),
- Left modules over \mathcal{P} (see e.g. [11]) and
- \mathcal{P} -algebras in the category of \mathbb{S} -modules.

Our first question is the following one: given a \mathcal{P} -algebra M in $\mathbf{S}\text{-mod}$ can one endow the graded vector space $\mathcal{O}(M)$ with a \mathcal{P} -algebra structure? This question comes from the observation that $\bigoplus_n \mathcal{P}(n)$ is a \mathcal{P} -algebra in $\mathbf{S}\text{-mod}$ but is not *a priori* a \mathcal{P} -algebra in the category of graded vector spaces. To convince the reader, one can look at the Lie case: if $\mathcal{P} = \mathcal{L}\text{ie}$ then $\bigoplus_n \mathcal{L}\text{ie}(n)$ is a twisted Lie algebra, and the bracket is not anti-symmetric since the action of the symmetric group is non trivial. Our first theorem 2.3.1 states that if we apply a symmetrization to the twisted \mathcal{P} -algebra structure on M then $\mathcal{O}(M)$ is a \mathcal{P} -algebra. If $\mathcal{P} = \mathcal{A}s$ we recover the definition of the symmetrized product \hat{m} of Patras and Reutenauer. Our second theorem 2.4.3 states that another product can be defined if the operad \mathcal{P} is *regular* that is, if the action of the symmetric group is free. Then in case $\mathcal{P} = \mathcal{A}s$ we recover the product \bar{m} defined by Patras and Reutenauer.

The second step of the construction deals with Hopf operads. We define the notion of Hopf \mathcal{P} -algebras in the category $\mathbf{S}\text{-mod}$, so that in case $\mathcal{P} = \mathcal{A}s$ we recover the notion of twisted associative bialgebras. We develop the analogues of the constructions of Patras and Reutenauer: in theorem 3.2.2 we prove that given a Hopf \mathcal{P} -algebra (M, μ_M, Δ_M) in $\mathbf{S}\text{-mod}$ one can endow

$\bar{M} = (\mathcal{O}(M), \hat{\mu}_{\mathcal{O}(M)}, \bar{\Delta}_{\mathcal{O}(M)})$ with a Hopf \mathcal{P} -algebra structure analogous to the symmetrized bialgebra construction. This is the *symmetrized* Hopf \mathcal{P} -algebra associated to M . If \mathcal{P} is regular then there is another Hopf \mathcal{P} -algebra structure on $\hat{M} = (\mathcal{O}(M), \bar{\mu}_{\mathcal{O}(M)}, \hat{\Delta}_{\mathcal{O}(M)})$, analogous to the cosymmetrized bialgebra construction (see theorem 3.3.1): this is the *cosymmetrized* \mathcal{P} -Hopf algebra associated to M . In view of our theory, one can say that the theory developed by Patras and Reutenauer works because the operad \mathcal{A} is regular.

The example of Hopf \mathcal{P} -algebra in **S-mod** that we should have in mind throughout all the article is the \mathbb{S} -module \mathcal{P} itself, if \mathcal{P} is a connected Hopf operad (see section 3.2.3).

The case \mathcal{P} regular has another advantage: we can define the notion of *unital infinitesimal \mathcal{P} -bialgebra* following the terminology of Loday and Ronco in [16] when $\mathcal{P} = \mathcal{A}$ s. We prove in theorem 4.1.2 that $(\mathcal{O}(M), \bar{\mu}_{\mathcal{O}(M)}, \bar{\Delta}_{\mathcal{O}(M)})$ is a unital infinitesimal \mathcal{P} -bialgebra if M is a Hopf \mathcal{P} -algebra. We prove in theorem 4.1.3 that the graded vector space $\oplus_n \mathcal{P}(n)/S_n$ has also a structure of unital infinitesimal \mathcal{P} -bialgebra. These results combined with the theorem of Loday and Ronco (see 4.2.1) yield the main theorems of our paper, which has some importance in the study of combinatorial Hopf algebras. First of all if $\mathcal{P} = \mathcal{A}$ s and (A, m, Δ) is a twisted associative bialgebra, then the algebra $(A, \bar{m}, \bar{\Delta})$ is a unital infinitesimal bialgebra and consequently is free and cofree (see theorem 4.2.2). Moreover, theorem 4.3.2 asserts that in case \mathcal{P} is a Hopf regular multiplicative operad that is, endowed with a Hopf operad morphism $\mathcal{A} \rightarrow \mathcal{P}$, then any unital infinitesimal \mathcal{P} -bialgebra is isomorphic as a unital infinitesimal \mathcal{P} -bialgebra to the vector space $T(V)$ where V is the space of primitive elements.

Before going through the applications to combinatorial Hopf algebras, let us summarize the results in a table.

Operad \mathcal{P}	\mathbb{S} -module M	graded vector space $\mathcal{O}(M)$	Thm
any	\mathcal{P} -alg (M, μ)	\mathcal{P} -alg $(\mathcal{O}(M), \hat{\mu})$	2.3.1
regular	\mathcal{P} -alg (M, μ)	\mathcal{P} -alg $(\mathcal{O}(M), \hat{\mu})$ \mathcal{P} -alg $(\mathcal{O}(M), \bar{\mu})$	2.4.3
Hopf	Hopf \mathcal{P} -alg (M, μ, Δ)	Hopf \mathcal{P} -alg $(\mathcal{O}(M), \hat{\mu}, \bar{\Delta})$	3.2.2
Hopf regular	Hopf \mathcal{P} -alg (M, μ, Δ)	Hopf \mathcal{P} -alg $(\mathcal{O}(M), \hat{\mu}, \bar{\Delta})$ Hopf \mathcal{P} -alg $(\mathcal{O}(M), \bar{\mu}, \hat{\Delta})$ u.i. \mathcal{P} -bialg $(\mathcal{O}(M), \bar{\mu}, \bar{\Delta})$	3.3.1 4.1.2
mult Hopf regular	Hopf \mathcal{P} -alg (M, μ, Δ)	$(\mathcal{O}(M), \bar{\mu}, \bar{\Delta}) \simeq$ $T(\text{Prim}_{\bar{\Delta}}(\mathcal{O}(M)))$	4.3.2

Given a graded vector space H , the question is: how does a Hopf algebra structure arise on H ? In the last section, we illustrate with examples that many combinatorial Hopf algebras arise from our theory. We distinguish two cases.

If \mathcal{P} is a Hopf multiplicative operad then $H = \bigoplus_n \mathcal{P}(n)$ has two structures of Hopf algebras: the symmetrized Hopf algebra \bar{H} is cofree and the cosymmetrized algebra \hat{H} is free. For instance, we prove that the Malvenuto Reutenauer Hopf algebra arises as the symmetrized Hopf algebra of the operad $\mathcal{A}s$ and the cosymmetrized algebra of the operad $\mathcal{Z}in$ and then is free and cofree. We prove that Hopf algebra structures on the faces of the permutohedra given e.g. by Chapoton in [5], Bergeron and Zabrocky in [4] and Patras and Schocker in [22], arise from the operad $\mathcal{C}TD$ of commutative tridendriform algebras defined by Loday in [14]. We deduce also some freeness results from our theory. This case apply when H_n has an action of the symmetric group.

In the second case, we assume that there exists a Hopf multiplicative regular operad \mathcal{P} such that $H = \bigoplus_n \mathcal{P}(n)/S_n$. Our theory implies that H is a Hopf \mathcal{P} -algebra hence a Hopf algebra and is also a unital infinitesimal \mathcal{P} -bialgebra. We prove that the Hopf algebra of planar trees described by Chapoton in [5], and the one of planar binary trees described by Loday and Ronco in [15] arise this way. As a byproduct we obtain freeness results for these Hopf algebras.

The organization of the present paper is as follows. After some preliminaries in section 1 we define in section 2 the notion of algebras over an operad \mathcal{P} in the category of \mathbb{S} -modules and in the category of graded vector spaces. We explore the structures of \mathcal{P} -algebra on the underlying graded vector space of a \mathcal{P} -algebra in $\mathbf{S}\text{-mod}$. We prove that such a structure always exists and when the operad is regular one has an additional structure. We compare this result to the one obtained by Patras and Reutenauer [21] in the case of twisted associative algebras. In section 3 we study Hopf operads and the consequences on the underlying graded vector space of a Hopf \mathcal{P} -algebra in $\mathbf{S}\text{-mod}$. The section 4 is devoted to the unital infinitesimal \mathcal{P} -bialgebra notion and to the freeness theorems. We develop in section 5 the application to combinatorial Hopf algebras by way of examples.

Throughout the paper, the ground field is denoted by \mathbf{k} and all vector spaces are \mathbf{k} -vector spaces.

CONTENTS

Introduction	1
1. \mathbb{S} -modules and related functors	5
1.1. The symmetric group	5
1.2. Graded vector spaces and \mathbb{S} -modules.	6
1.3. Endofunctors induced by a \mathbb{S} -module	8

1.4. Proposition	9
2. Algebras over an operad	9
2.1. Operads	10
2.2. Algebras over an operad	10
2.3. Relating \mathcal{P} -algebras in S-mod and in grVect	11
2.4. Regular Operads	12
2.5. Multiplicative operads	14
3. Hopf algebras over a Hopf operad	14
3.1. Hopf operads	14
3.2. Hopf algebras over a Hopf operad	16
3.3. Regular Hopf operads	19
3.4. Application to multiplicative Hopf operads	21
4. Unital infinitesimal \mathcal{P} -bialgebras	22
4.1. Unital infinitesimal \mathcal{P} -bialgebras	22
4.2. Rigidity for twisted bialgebras	25
4.3. Rigidity for unital infinitesimal \mathcal{P} -bialgebras	26
5. Application to combinatorial Hopf algebras	28
5.1. The Hopf algebra $T(V)$	29
5.2. The Malvenuto-Reutenauer Hopf algebra	30
5.3. Hopf algebra structures on the faces of the permutohedron.	31
5.4. Hopf algebra structures on the faces of the associahedron	35
References	40

1. \mathbb{S} -MODULES AND RELATED FUNCTORS

1.1. The symmetric group. In this section we develop some material on the symmetric group needed in the paper. The set $\{1, \dots, n\}$ is written $[n]$. For any set of integers S , the set $\{p + s, s \in S\}$ is denoted $p + S$. For any sets $S \subset [n]$ and $T \subset [m]$ the set $S \times T$ is the subset $S \cup (T + n)$ of $[n + m]$.

Any permutation $\sigma \in S_n$ is written $(\sigma_1, \dots, \sigma_n)$ which $\sigma_i = \sigma(i)$. There is a natural injection

$$\begin{aligned} S_n \times S_m &\rightarrow S_{n+m} \\ (\sigma, \tau) &\mapsto \sigma \times \tau = (\sigma_1, \dots, \sigma_n, \tau_1 + n, \dots, \tau_m + n). \end{aligned}$$

The *standardisation* of a sequence of distinct integers (a_1, \dots, a_p) is the unique permutation $\sigma \in S_p$ following the conditions

$$\sigma(i) < \sigma(j) \Leftrightarrow a_i < a_j, \quad \forall i, j.$$

For instance

$$\text{st}(2, 13, 9, 4) = (1, 4, 3, 2).$$

Any subset $A = \{a_1 < \dots < a_p\} \subset [n]$ induces a map

$$\begin{aligned} S_n &\rightarrow S_p \\ \sigma &\mapsto \sigma|_A = \text{st}(\sigma(a_1), \dots, \sigma(a_p)). \end{aligned}$$

For instance

$$(2, 6, 1, 3, 5, 4)|_{\{1,2,4\}} = \text{st}(2, 6, 3) = (1, 3, 2).$$

If A is the empty set then $\sigma|_{\emptyset} = 1_0 \in S_0$.

A (p_1, \dots, p_r) -*shuffle* is a permutation of $S_{p_1+\dots+p_r}$ of type

$$(\tau_1^1, \dots, \tau_{p_1}^1, \dots, \tau_1^r, \dots, \tau_{p_r}^r)^{-1}$$

with $\tau_1^k < \dots < \tau_{p_k}^k$ for all $1 \leq k \leq r$. The set of all (p_1, \dots, p_r) -shuffles is denoted by $\text{Sh}_{p_1, \dots, p_r}$. For simplicity, a (p_1, \dots, p_r) -shuffle is written as a r -uple (A_1, \dots, A_r) where $A_1 \sqcup \dots \sqcup A_r$ is a partition of the set $[p_1 + \dots + p_r]$. Some of the A_i 's may be empty.

For instance $(\{2, 5\}, \{1, 3, 4\})$ denotes the $(2, 3)$ -shuffle $(3, 1, 4, 5, 2)$. Recall that $\text{Sh}_{p_1, \dots, p_r}$ constitutes a set of right coset representatives for $S_{p_1} \times \dots \times S_{p_r} \subset S_{p_1+\dots+p_r}$, i.e. any $\sigma \in S_{p_1+\dots+p_r}$ has a unique factorization

$$\sigma = (\sigma_1 \times \dots \times \sigma_r)\alpha,$$

where $\sigma_i \in S_{p_i}$ and where α is a (p_1, \dots, p_r) -shuffle. More precisely, if $r = 2$

$$\sigma = (\sigma|_{\sigma^{-1}([p])} \times \sigma|_{\sigma^{-1}(p+[q])})(\sigma^{-1}([p]), \sigma^{-1}(p+[q])).$$

1.2. Graded vector spaces and \mathbb{S} -modules.

1.2.1. *Definition.* A *graded vector space* A is a collection $\{A_n\}_{n \geq 0}$ of \mathbf{k} -vector spaces A_n indexed by the non-negative integers. One can define also A as $A = \bigoplus_n A_n$. A map $A \rightarrow B$ of graded vector spaces is a collection of linear morphisms $A_n \rightarrow B_n$. The category of graded vector spaces is denoted **grVect**.

A \mathbb{S} -*module* M is a graded vector space together with a right S_n -action $M_n \otimes \mathbf{k}[S_n] \rightarrow M_n$ for each $n \geq 0$. A map $M \rightarrow N$ of \mathbb{S} -modules is a collection $M_n \rightarrow N_n$ of morphisms of right S_n -modules. The category of \mathbb{S} -modules is denoted **S-mod**.

There is a forgetful functor

$$\mathcal{O} : \mathbf{S}\text{-mod} \rightarrow \mathbf{grVect}$$

which forgets the action of the symmetric group.

1.2.2. Tensor product.

The category **grVect** is a linear symmetric monoidal category with the following *tensor product*:

$$(A \otimes B)_n = \bigoplus_{p+q=n} A_p \otimes B_q.$$

The symmetry isomorphism $\tau : A \otimes B \rightarrow B \otimes A$ is given by

$$\begin{array}{ccc} \tau : A_p \otimes B_q & \rightarrow & B_q \otimes A_p \\ a \otimes b & \mapsto & b \otimes a \end{array}$$

The symmetry isomorphism τ induces a left action of the symmetric group S_k on $A^{\otimes k}$, for $A \in \mathbf{grVect}$.

The category $\mathbf{S-mod}$ is a linear symmetric monoidal category with the following *tensor product*:

$$\begin{aligned} (M \otimes N)(n) &= \bigoplus_{p+q=n} (M(p) \otimes N(q)) \otimes_{S_p \times S_q} \mathbf{k}[S_n] \\ &= \bigoplus_{p+q=n} (M(p) \otimes M(q)) \otimes \mathbf{k}[\text{Sh}_{p,q}]. \end{aligned}$$

Since a (p, q) -shuffle is uniquely determined by an ordered partition $I \sqcup J$ of $[p+q]$, an element in $(M \otimes N)(p+q)$ can be written $m \otimes n \otimes (I, J)$. For the sequel $m \otimes n$ denotes the element $m \otimes n \otimes ([p], p + [q])$ of $(M \otimes N)(p+q)$. The right action of the symmetric group is given by

$$(m \otimes n \otimes (I, J)) \cdot \sigma = m \cdot \sigma|_{\sigma^{-1}(I)} \otimes n \cdot \sigma|_{\sigma^{-1}(J)} \otimes (\sigma^{-1}(I), \sigma^{-1}(J)) \quad (1)$$

The unit for the tensor product is the \mathbb{S} -module $\mathbf{1}$ given by

$$\mathbf{1}(n) = \begin{cases} \mathbf{k}, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The symmetry isomorphism $\tau : M \otimes N \rightarrow N \otimes M$ is given by

$$\tau(m \otimes n \otimes (I, J)) = n \otimes m \otimes (J, I).$$

For any $\sigma \in S_k$, the symmetry isomorphism induces an isomorphism τ_σ of \mathbb{S} -modules from $M_1 \otimes \dots \otimes M_k$ to $M_{\sigma^{-1}(1)} \otimes \dots \otimes M_{\sigma^{-1}(k)}$ given by

$$\begin{aligned} \tau_\sigma(m_1 \otimes \dots \otimes m_k \otimes (I_1, \dots, I_k)) &= \\ &= m_{\sigma^{-1}(1)} \otimes \dots \otimes m_{\sigma^{-1}(k)} \otimes (I_{\sigma^{-1}(1)}, \dots, I_{\sigma^{-1}(k)}). \end{aligned} \quad (2)$$

As a consequence τ_σ induces a left S_k -action on $M^{\otimes k}$.

When it is necessary to distinguish the tensor products, we write \otimes_g for the tensor product in \mathbf{grVect} and $\otimes_{\mathbb{S}}$ for the one in $\mathbf{S-mod}$.

The forgetful functor \mathcal{O} does not preserve the tensor product, but one can define two maps

$$\pi_{M,N}^{\mathcal{O}}, \iota_{M,N}^{\mathcal{O}} : \mathcal{O}(M \otimes_{\mathbb{S}} N) \rightarrow \mathcal{O}(M) \otimes_g \mathcal{O}(N)$$

which are functorial in M and N by

$$\begin{aligned} \pi_{M,N}^{\mathcal{O}}(m \otimes n \otimes (I, J)) &= \begin{cases} m \otimes n, & \text{if } (I, J) = \text{Id}, \\ 0, & \text{otherwise,} \end{cases} \\ \iota_{M,N}^{\mathcal{O}}(m \otimes n \otimes (I, J)) &= m \otimes n. \end{aligned}$$

In case a \mathbb{S} -module M is concentrated in degree 0 one can consider M in the category of vector spaces then of graded vector spaces. The tensor

product of such two \mathbb{S} -modules is the tensor product in the category of vector spaces and in this case $\pi_{M,N}^{\mathcal{O}}$ and $\iota_{M,N}^{\mathcal{O}}$ are the identity map.

1.3. Endofunctors induced by a \mathbb{S} -module.

1.3.1. *Endofunctors in $\mathbf{S}\text{-mod}$.* The category of \mathbb{S} -modules is endowed with another monoidal structure (which is not symmetric): the *plethysm* \circ .

$$(M \circ N)(n) := \bigoplus_{k \geq 0} M(k) \otimes_{S_k} (N^{\otimes k})(n),$$

where S_k acts on the left on $(N^{\otimes k})$ by formula (2). The left and right unit for the plethysm is the \mathbb{S} -module I given by

$$I(n) = \begin{cases} \mathbf{k}, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence any \mathbb{S} -module M defines a functor

$$\begin{array}{ccc} F_M : \mathbf{S}\text{-mod} & \rightarrow & \mathbf{S}\text{-mod} \\ N & \mapsto & M \circ N \end{array}$$

satisfying

$$\begin{cases} F_I = & \text{Id} \\ F_{M \circ M'} = & F_M F_{M'} \end{cases}$$

1.3.2. *Endofunctors in \mathbf{grVect} .* For $M \in \mathbf{S}\text{-mod}$ and $A \in \mathbf{grVect}$, one can use the same definition for the plethysm:

$$(M \circ A)(n) := \bigoplus_{k \geq 0} M(k) \otimes_{S_k} (A^{\otimes k})(n).$$

where the tensor product $A^{\otimes k}$ is taken in \mathbf{grVect} . Similarly any \mathbb{S} -module M defines a functor

$$\begin{array}{ccc} F_M^g : \mathbf{grVect} & \rightarrow & \mathbf{grVect} \\ A & \mapsto & M \circ A \end{array}$$

satisfying

$$\begin{cases} F_I^g = & \text{Id} \\ F_{M \circ M'}^g = & F_M^g F_{M'}^g \end{cases}$$

1.3.3. *Example.* Here is an example that emphasises the fact that the two functors are different even if evaluated at the same underlying vector space. Consider the \mathbb{S} -module $\mathcal{C}om(n) = \mathbf{k}$ with the trivial S_n -action, and let V be a vector space. The vector space V can be considered either as a graded vector space concentrated in degree 1 or as a \mathbb{S} -module concentrated in degree 1. One has

$$\begin{aligned} F_{\mathcal{C}om}^g(V) &= \bigoplus_{n \geq 0} \mathbf{k} \otimes_{S_n} V^{\otimes_g n} = S(V), \\ F_{\mathcal{C}om}(V) &= \bigoplus_{n \geq 0} \mathbf{k} \otimes_{S_n} V^{\otimes_{\mathbb{S}} n} = T(V). \end{aligned}$$

1.4. **Proposition.** *Let M, N be two \mathbb{S} -modules. The map*

$$\begin{aligned} M \circ \mathcal{O}(N) &\rightarrow \mathcal{O}(M \circ N) \\ m \otimes n_1 \otimes \dots \otimes n_k &\mapsto \sum_{(T_1, \dots, T_k)} m \otimes n_1 \otimes \dots \otimes n_k \otimes (T_1, \dots, T_k), \end{aligned}$$

where $n_i \in N(l_i)$ and (T_1, \dots, T_k) is an ordered partition of $[l_1 + \dots + l_k]$ with $|T_i| = l_i$, defines a natural transformation

$$\psi_M : F_M^g \mathcal{O} \rightarrow \mathcal{O} F_M$$

functorial in $M \in \mathbf{S}\text{-mod}$. Furthermore the following diagram commutes

$$\begin{array}{ccc} F_{M \circ N}^g \mathcal{O} & \xrightarrow{\psi_{M \circ N}} & \mathcal{O} F_{M \circ N} \\ & \searrow^{F_M^g \psi_N} & \nearrow^{\psi_M F_N} \\ & F_M^g \mathcal{O} F_N & \end{array}$$

Proof. The relation (2) implies that

$$\begin{aligned} m \cdot \sigma \otimes n_1 \otimes \dots \otimes n_k \otimes (T_1, \dots, T_k) = \\ m \otimes n_{\sigma^{-1}(1)} \otimes \dots \otimes n_{\sigma^{-1}(k)} \otimes (T_{\sigma^{-1}(1)}, \dots, T_{\sigma^{-1}(k)}). \end{aligned}$$

Since the sum is taken over all ordered partitions (T_1, \dots, T_k) , the image of $m \cdot \sigma \otimes n_1 \otimes \dots \otimes n_k$ is

$$\sum_{(U_1, \dots, U_k)} m \otimes n_{\sigma^{-1}(1)} \otimes \dots \otimes n_{\sigma^{-1}(k)} \otimes (U_1, \dots, U_k)$$

with $|U_i| = l_{\sigma^{-1}(i)}$, which is the image of $m \otimes n_{\sigma^{-1}(1)} \otimes \dots \otimes n_{\sigma^{-1}(k)}$. As a consequence the map is well defined.

The known formula

$$\begin{aligned} \text{Sh}_{p_1^1, \dots, p_1^{l_1}, \dots, p_k^1, \dots, p_k^{l_k}} = \\ (\text{Sh}_{p_1^1, \dots, p_1^{l_1}} \times \dots \times \text{Sh}_{p_k^1, \dots, p_k^{l_k}}) \text{Sh}_{p_1^1 + \dots + p_1^{l_1}, \dots, p_k^1 + \dots + p_k^{l_k}} \quad (3) \end{aligned}$$

yields the commutativity of the diagram. \square

2. ALGEBRAS OVER AN OPERAD

In this section we give the definitions of operads and algebras over an operad and we refer to Fresse [11] for more general theory on operads. We further state the main results of the section: the underlying graded vector space of an algebra over an operad \mathcal{P} in $\mathbf{S}\text{-mod}$ is always a \mathcal{P} -algebra and when \mathcal{P} is regular there exists a second \mathcal{P} -algebra structure.

2.1. Operads. An operad is a monoid in the category of \mathbb{S} -modules with respect to the plethysm. Namely, an operad is a \mathbb{S} -module \mathcal{P} together with a product $\mu_{\mathcal{P}} : \mathcal{P} \circ \mathcal{P} \rightarrow \mathcal{P}$ and a unit $\eta_{\mathcal{P}} : I \rightarrow \mathcal{P}$ satisfying

$$\begin{aligned}\mu_{\mathcal{P}}(\mathcal{P} \circ \mu_{\mathcal{P}}) &= \mu_{\mathcal{P}}(\mu_{\mathcal{P}} \circ \mathcal{P}) \\ \mu_{\mathcal{P}}(\mathcal{P} \circ \eta_{\mathcal{P}}) &= \mu_{\mathcal{P}}(\eta_{\mathcal{P}} \circ \mathcal{P}) = \mathcal{P}.\end{aligned}$$

As a consequence the functors $F_{\mathcal{P}}$ and $F_{\mathcal{P}}^g$ are monads in the category **S-mod** and **grVect**.

The product $\mu_{\mathcal{P}}$ expresses as maps called compositions

$$\begin{aligned}\mathcal{P}(n) \otimes \mathcal{P}(l_1) \otimes \dots \otimes \mathcal{P}(l_n) &\rightarrow \mathcal{P}(l_1 + \dots + l_n) \\ \mu \otimes \nu_1 \otimes \dots \otimes \nu_n &\mapsto \mu(\nu_1, \dots, \nu_n)\end{aligned}$$

which are morphisms of $S_{l_1+\dots+l_n}$ -modules and which are S_n -equivariant.

The operad $\mathcal{A}s$ is the \mathbb{S} -module $(\mathbf{k}[S_n])_{n \geq 0}$. For $\sigma \in S_n$, $\tau_i \in S_{l_i}$ the composition $\mu_{\mathcal{A}s}(\sigma; \tau_1, \dots, \tau_n)$ is the permutation of $S_{l_1+\dots+l_n}$ obtained by substituting the block $\tau_i + l_{\sigma^{-1}(1)} + l_{\sigma^{-1}(2)} \dots + l_{\sigma^{-1}(\sigma(i)-1)}$ for the integer σ_i . For instance

$$\mu_{\mathcal{A}s}((3, 2, 1, 4); (2, 1), (1, 3, 2), (1), (2, 3, 1)) = (\underbrace{6, 5}_{\tau_1+4}, \underbrace{2, 4, 3}_{\tau_2+1}, \underbrace{1}_{\tau_3}, \underbrace{8, 9, 7}_{\tau_4+6}).$$

2.2. Algebras over an operad. Let \mathbf{C} denotes either the category of \mathbb{S} -modules or the category of graded vector spaces. For any \mathbb{S} -module M , the functor $F_M^{\mathbf{C}}$ denotes the functor F_M or F_M^g .

Let \mathcal{P} be an operad. A \mathcal{P} -algebra or an algebra over \mathcal{P} is an algebra over the monad $F_{\mathcal{P}}^{\mathbf{C}}$, that is an object M of \mathbf{C} together with a product $\mu_M : F_{\mathcal{P}}^{\mathbf{C}}(M) \rightarrow M$ such that the following diagrams commute:

$$\begin{array}{ccc} F_{\mathcal{P} \circ \mathcal{P}}^{\mathbf{C}}(M) & \xrightarrow{F_{\mathcal{P}}^{\mathbf{C}}(\mu_M)} & F_{\mathcal{P}}^{\mathbf{C}}(M) \\ F_{\mu_{\mathcal{P}}}^{\mathbf{C}}(M) \downarrow & & \downarrow \mu_M \\ F_{\mathcal{P}}^{\mathbf{C}}(M) & \xrightarrow{\mu_M} & M \end{array} \quad \begin{array}{ccc} F_I^{\mathbf{C}}(M) & \xrightarrow{=} & M \\ F_{\eta_{\mathcal{P}}}^{\mathbf{C}}(M) \downarrow & \nearrow \mu_M & \\ F_{\mathcal{P}}^{\mathbf{C}}(M) & & \end{array}$$

For $p \in \mathcal{P}(n)$ and $m_1, \dots, m_n \in M$ the product $\mu_M(p \otimes m_1 \otimes \dots \otimes m_n)$ is usually written

$$p(m_1, \dots, m_n) \in M.$$

In the category of graded vector spaces one gets the usual definition of an algebra over an operad. In the category of \mathbb{S} -modules the notion of a \mathcal{P} -algebra is related to some other terminology:

- *left module over \mathcal{P}* in the terminology of Fresse [11],
- if $\mathcal{P} = \mathcal{A}s$ or $\mathcal{P} = \mathcal{L}ie$, *twisted associative or twisted Lie algebra* in the terminology of Barratt [3],

- *twisted \mathcal{P} -algebra* in the terminology of Livernet and Patras [12].

In the sequel we dedicate the word *twisted* to the only case $\mathcal{P} = \mathcal{A}s$: a *twisted algebra* is an algebra over the operad $\mathcal{A}s$ in the category $\mathbf{S}\text{-mod}$.

Any free \mathcal{P} -algebra in the category \mathbf{C} writes $F_{\mathcal{P}}^{\mathbf{C}}(M)$ for some $M \in \mathbf{C}$. As a consequence \mathcal{P} is the free \mathcal{P} -algebra in $\mathbf{S}\text{-mod}$ generated by the \mathbf{S} -module I .

2.3. Relating \mathcal{P} -algebras in $\mathbf{S}\text{-mod}$ and in \mathbf{grVect} .

2.3.1. *Theorem.* Let $M \in \mathbf{S}\text{-mod}$ be an algebra over an operad \mathcal{P} . The graded vector space $\mathcal{O}(M)$ is a \mathcal{P} -algebra for the product $\hat{\mu}_{\mathcal{O}(M)}$ given by the composition

$$F_{\mathcal{P}}^g \mathcal{O}(M) \xrightarrow{\psi_{\mathcal{P}}(M)} \mathcal{O} F_{\mathcal{P}}(M) \xrightarrow{\mathcal{O}(\mu_M)} \mathcal{O}(M) .$$

Concretely

$$\hat{\mu}_{\mathcal{O}(M)}(p \otimes m_1 \otimes \dots \otimes m_n) = \mu_M(p \otimes m_1 \otimes \dots \otimes m_n) \cdot q_{l_1, \dots, l_n} \quad (4)$$

with $p \in \mathcal{P}(n)$, $m_i \in M(l_i)$ and where q_{l_1, \dots, l_n} is the sum of all (l_1, \dots, l_n) -shuffles.

Proof. One has to prove the commutativity of the following two diagrams

$$\begin{array}{ccc} F_{\mathcal{P} \circ \mathcal{P}}^g \mathcal{O}(M) & \xrightarrow{F_{\mathcal{P}}^g \hat{\mu}_{\mathcal{O}(M)}} & F_{\mathcal{P}}^g \mathcal{O}(M) \\ F_{\mu_{\mathcal{P}}}^g \mathcal{O}(M) \downarrow & & \downarrow \hat{\mu}_{\mathcal{O}(M)} \\ F_{\mathcal{P}}^g \mathcal{O}(M) & \xrightarrow{\hat{\mu}_{\mathcal{O}(M)}} & \mathcal{O}(M) \end{array} \quad \begin{array}{ccc} F_I^g \mathcal{O}(M) & \xrightarrow{=} & \mathcal{O}(M) \\ F_{\eta_{\mathcal{P}}}^g \mathcal{O}(M) \downarrow & \nearrow \hat{\mu}_{\mathcal{O}(M)} & \\ F_{\mathcal{P}}^g \mathcal{O}(M) & & \end{array}$$

The second diagram is commutative because ψ_N is functorial in N , so

$$\begin{aligned} \psi_I &= \text{Id}, \\ \psi_{\mathcal{P}}(F_{\eta_{\mathcal{P}}}^g \mathcal{O}) &= (\mathcal{O} F_{\eta_{\mathcal{P}}}) \psi_I, \end{aligned}$$

and because $\mu_M(F_{\eta_{\mathcal{P}}}(M)) = \text{Id}_M$.

Since M is a \mathcal{P} -algebra $\mu_M(F_{\mu_{\mathcal{P}}} M) = \mu_M(F_{\mathcal{P}} \mu_M)$.

The commutativity of the first diagram is a consequence of the computation

$$\begin{aligned} & \hat{\mu}_{\mathcal{O}(M)} F_{\mu_{\mathcal{P}}}^g \mathcal{O}(M) \\ &= \mathcal{O}(\mu_M) \psi_{\mathcal{P}}(M) (F_{\mu_{\mathcal{P}}}^g \mathcal{O}(M)) && \text{by definition,} \\ &= \mathcal{O}(\mu_M) (\mathcal{O} F_{\mu_{\mathcal{P}}}) \psi_{\mathcal{P} \circ \mathcal{P}}(M) && \text{by functoriality of } \psi, \\ &= \mathcal{O}(\mu_M) \mathcal{O}(F_{\mathcal{P}} \mu_M) \psi_{\mathcal{P} \circ \mathcal{P}}(M) && M \text{ is a } \mathcal{P}\text{-algebra,} \\ &= \mathcal{O}(\mu_M) \mathcal{O}(F_{\mathcal{P}} \mu_M) (\psi_{\mathcal{P}} F_{\mathcal{P}}) (F_{\mathcal{P}}^g \psi_{\mathcal{P}}) && \text{by proposition 1.4,} \\ &= \mathcal{O}(\mu_M) \psi_{\mathcal{P}}(M) (F_{\mathcal{P}}^g \mathcal{O}(\mu_M)) (F_{\mathcal{P}}^g \psi_{\mathcal{P}}) && \psi \text{ is a natural transformation,} \\ &= \hat{\mu}_{\mathcal{O}(M)} (F_{\mathcal{P}}^g \hat{\mu}_{\mathcal{O}(M)}) && \text{by definition.} \quad \square \end{aligned}$$

As pointed out in section 2.2 any free \mathcal{P} -algebra in $\mathbf{S}\text{-mod}$ is a \mathcal{P} -algebra and satisfies the conditions of the theorem. Hence any free \mathcal{P} -algebra in $\mathbf{S}\text{-mod}$ gives rise to a \mathcal{P} -algebra in \mathbf{grVect} . In particular the graded vector space $\bigoplus_{n \geq 0} \mathcal{P}(n)$ is a \mathcal{P} -algebra.

2.3.2. *Example.* We apply formula (4) for the examples of the commutative operad and the associative operad.

The commutative operad \mathcal{Com} is the trivial S_n -module \mathbf{k} for all n . Let e_n be a generator of $\mathcal{Com}(n)$. The composition is

$$\mu_{\mathcal{Com}}(e_n \otimes e_{l_1} \otimes \dots \otimes e_{l_n}) = e_{l_1 + \dots + l_n}.$$

The graded vector space $\bigoplus_n \mathcal{Com}(n)$ is isomorphic to $\mathbf{k}[X]$. The commutative product on $\mathbf{k}[X]$ induced by the composition $\mu_{\mathcal{Com}}$ is

$$X^n \cdot X^m = \binom{n+m}{n} X^{n+m}, \quad (5)$$

since the number of (n, m) -shuffles is $\binom{n+m}{n}$.

The associative operad was defined in section 2.1. The associative product on the space $\bigoplus_n \mathbf{k}[S_n]$ induced by the composition $\mu_{\mathcal{As}}$ is

$$\sigma \hat{*} \tau = \sum_{\xi \in \text{Sh}_{p,q}} (\sigma \times \tau) \cdot \xi, \quad (6)$$

where $\sigma \in S_p$ and $\tau \in S_q$. This is the product defined by Malvenuto and Reutenauer in [18] (see also paragraphs 2.4.4 and 5.2).

2.3.3. *Remark.* Let \mathcal{P} be an operad and M be a \mathcal{P} -algebra in $\mathbf{S}\text{-mod}$ such that the action of S_n on $M(n)$ is trivial. There is another \mathcal{P} -algebra structure on $\mathcal{O}(M)$ given by

$$\mu_{\mathcal{O}(M)}^{t,g}(p \otimes m_1 \otimes \dots \otimes m_n) = \mu_M(p \otimes m_1 \otimes \dots \otimes m_n), \quad (7)$$

since the formula (2) together with the trivial action imply the S_n -invariance.

If $\mathcal{P} = \mathcal{Com}$ then $\mathbf{k}[X]$ is a commutative algebra for the product

$$X^n \cdot X^m = X^{n+m}.$$

In characteristic 0 the two commutative products on $\mathbf{k}[X]$ are isomorphic but it is no more the case in characteristic p .

2.4. **Regular Operads.** In this section we prove that any algebra over a regular operad \mathcal{P} gives rise to two structures of \mathcal{P} -algebra on its underlying graded vector space. This is the generalization to operads of the result of Patras and Reutenauer in [21] in the associative case. Note that this generalization holds only for regular operads.

2.4.1. *Definition.* The forgetful functor $\mathcal{O} : \mathbf{S}\text{-mod} \rightarrow \mathbf{grVect}$ has a left adjoint, the symmetrization functor $\mathcal{S} : \mathbf{grVect} \rightarrow \mathbf{S}\text{-mod}$ which associates to a graded vector space $(V_n)_n$ the \mathbb{S} -module $(V_n \otimes \mathbf{k}[S_n])_n$, where the action of the symmetric group is the right multiplication. A \mathbb{S} -module M is *regular* if there exists a graded vector space \tilde{M} such that $M = \mathcal{S}\tilde{M}$. For instance, the \mathbb{S} -module I is regular, since $I = \mathcal{S}I$. Let $\mathbf{S}\text{-mod}_{\mathbf{r}}$ be the full subcategory of $\mathbf{S}\text{-mod}$ of regular modules. A *regular operad* $\mathcal{P} = \mathcal{S}\tilde{\mathcal{P}}$ is an operad in the category $\mathbf{S}\text{-mod}_{\mathbf{r}}$, i.e. $\mu(\nu_1, \dots, \nu_k) \in \tilde{\mathcal{P}}$ as soon as $\mu, \nu_i \in \tilde{\mathcal{P}}$.

Indeed, there is also a plethysm in the category \mathbf{grVect} :

$$V \circ^g W = \bigoplus_k V_k \otimes W^{\otimes k}.$$

Note that

$$\mathcal{S}V \circ \mathcal{S}W = \mathcal{S}(V \circ^g W).$$

A *non-symmetric operad* is a monoid in the category \mathbf{grVect} with respect to the plethysm \circ^g . The operad $\mathcal{S}\tilde{\mathcal{P}}$ is regular if and only if $\tilde{\mathcal{P}}$ is a non-symmetric operad.

2.4.2. *Proposition.* Let $M = \mathcal{S}\tilde{M}$ be a regular module and N be a \mathbb{S} -module. The map

$$\begin{array}{ccc} \tilde{M} \circ \mathcal{O}(N) & \rightarrow & \mathcal{O}(M \circ N) \\ m \otimes n_1 \otimes \dots \otimes n_k & \mapsto & m \otimes n_1 \otimes \dots \otimes n_k \end{array}$$

defines a natural transformation

$$\psi_M^r : F_M^g \mathcal{O} \rightarrow \mathcal{O} F_M$$

functorial in $M \in \mathbf{S}\text{-mod}_{\mathbf{r}}$. Furthermore for M and N in $\mathbf{S}\text{-mod}_{\mathbf{r}}$ the following diagram commutes

$$\begin{array}{ccc} F_{M \circ N}^g \mathcal{O} & \xrightarrow{\psi_{M \circ N}^r} & \mathcal{O} F_{M \circ N} \\ \searrow F_M^g \psi_N^r & & \nearrow \psi_M^r F_N \\ & F_M^g \mathcal{O} F_N & \end{array}$$

Proof. Since any element in M writes $m \cdot \sigma$ for $m \in \tilde{M}$, define $\psi_M^r(m \cdot \sigma \otimes n_1 \otimes \dots \otimes n_k)$ to be $m \otimes n_{\sigma^{-1}(1)} \otimes \dots \otimes n_{\sigma^{-1}(k)}$. It is straightforward to verify the statement with this definition of ψ_M^r . \square

Adapting the proof of theorem 2.3.1 by using $\psi_{\mathcal{P}}^r$ in place of $\psi_{\mathcal{P}}$ we prove the following theorem.

2.4.3. *Theorem.* Let $M \in \mathbf{S}\text{-mod}$ be an algebra over a regular operad \mathcal{P} . The graded vector space $\mathcal{O}(M)$ is a \mathcal{P} -algebra for the product $\hat{\mu}_{\mathcal{O}(M)}$ given by the composition

$$F_{\mathcal{P}}^g \mathcal{O}(M) \xrightarrow{\psi_{\mathcal{P}}^r(M)} \mathcal{O} F_{\mathcal{P}}(M) \xrightarrow{\mathcal{O}(\mu_M)} \mathcal{O}(M).$$

Hence $\hat{\mu}_{\mathcal{O}(M)}$ and $\bar{\mu}_{\mathcal{O}(M)}$ endow $\mathcal{O}(M)$ with two structures of \mathcal{P} -algebra. \square

If M is a trivial \mathbb{S} -module $\bar{\mu}_{\mathcal{O}(M)}$ coincides with $\mu_{\mathcal{O}(M)}^{t,g}$.

Since any free \mathcal{P} -algebra is a \mathcal{P} -algebra, the theorem holds for any free \mathcal{P} -algebra $F_{\mathcal{P}}(M)$. In particular the graded vector space $\bigoplus_{n \geq 0} \mathcal{P}(n)$ is endowed with two structures of \mathcal{P} -algebra.

2.4.4. *Example.* Let (M, m) be a twisted algebra (see definition 2.2). Patras and Reutenauer proved in [21] that $\mathcal{O}(M)$ is endowed with two structures of associative algebra, which are exactly the ones described here. For $a, b \in \mathcal{O}(M)$, $a \in M(r)$ and $b \in M(s)$, one has

$$\begin{aligned} a\bar{*}b &= m(a, b), \\ a\hat{*}b &= m(a, b) \cdot q_{r,s}. \end{aligned}$$

For $M = \bigoplus_n \mathcal{A}s(n) = \bigoplus_n \mathbf{k}[S_n]$, $\sigma \in S_p, \tau \in S_q$, the two associative products are

$$\begin{aligned} \sigma\bar{*}\tau &= \sigma \times \tau, \\ \sigma\hat{*}\tau &= \sum_{\xi \in \text{Sh}_{p,q}} (\sigma \times \tau) \cdot \xi. \end{aligned}$$

2.5. **Multiplicative operads.** An operad \mathcal{P} is *multiplicative* if there exists a morphism of operads $\mathcal{A}s \rightarrow \mathcal{P}$. Any algebra in $\mathbf{S}\text{-mod}$ over a multiplicative operad \mathcal{P} is a twisted algebra and thus its underlying graded vector space is endowed with two associative products. It holds in particular for the graded vector space $\bigoplus_n \mathcal{P}(n)$. For instance Com is a multiplicative operad and we recover the two associative (and commutative) structures found in example 2.3.2 and remark 2.3.3.

3. HOPF ALGEBRAS OVER A HOPF OPERAD

In this section, we generalize the results of Patras and Reutenauer in [21] obtained in the associative case to any Hopf operad \mathcal{P} . After defining the notion of Hopf \mathcal{P} -algebra when \mathcal{P} is a Hopf operad, we prove that any Hopf \mathcal{P} -algebra in $\mathbf{S}\text{-mod}$ yields a Hopf \mathcal{P} -algebra in \mathbf{grVect} and two Hopf \mathcal{P} -algebras in \mathbf{grVect} if \mathcal{P} is regular.

3.1. **Hopf operads.** In [19], Moerdijk defined the notion of *Hopf monad* in a tensor category (\mathbf{C}, \otimes, U) . It is a monad (S, μ, η) equipped with maps $\tau_{X,Y} : S(X \otimes Y) \rightarrow S(X) \otimes S(Y)$ natural in X and Y and $\theta : S(U) \rightarrow U$ compatible with the tensor structure of \mathbf{C} and with the monad structure of S .

For now on \mathbf{C} , is either the category \mathbf{grVect} or the category $\mathbf{S}\text{-mod}$ and $U = \mathbf{1}$.

3.1.1. *Definition.* Let **Coalg** be the category of coassociative counital coalgebras, that is vector spaces V endowed with a coassociative coproduct $\Delta : V \rightarrow V \otimes V$ and a counit $\epsilon : V \rightarrow \mathbf{k}$. A *Hopf operad* \mathcal{P} is an operad in **Coalg**, i.e. $\mu_{\mathcal{P}}$ and $\eta_{\mathcal{P}}$ are morphisms of coassociative counital coalgebras.

Assume \mathcal{P} is a Hopf operad. There is a coproduct $\delta(n) : \mathcal{P}(n) \rightarrow \mathcal{P}(n) \otimes \mathcal{P}(n)$ and a counit $\epsilon(n) : \mathcal{P}(n) \rightarrow \mathbf{k}$ for each n . We use the Sweedler's notation that is to say

$$\delta(\mu) = \sum_{(1),(2)} \mu_{(1)} \otimes \mu_{(2)}.$$

Define in the category **S-mod** the map

$$\tau_{M,N} : F_{\mathcal{P}}(M \otimes N) \rightarrow F_{\mathcal{P}}(M) \otimes F_{\mathcal{P}}(N)$$

to be

$$\begin{aligned} \tau_{M,N}(\mu \otimes m_1 \otimes n_1 \otimes \cdots \otimes m_k \otimes n_k \otimes (A_1, B_1, \dots, A_k, B_k)) = \\ \sum_{(1),(2)} (\mu_{(1)} \otimes m_1 \dots m_k \otimes \text{st}(A_1, \dots, A_k)) \otimes (\mu_{(2)} \otimes n_1 \dots n_k \otimes \text{st}(B_1, \dots, B_k)) \\ \otimes (\cup A_i, \cup B_i). \end{aligned} \quad (8)$$

Define in the category **grVect** the map

$$\tau_{X,Y}^g : F_{\mathcal{P}}^g(X \otimes Y) \rightarrow F_{\mathcal{P}}^g(X) \otimes F_{\mathcal{P}}^g(Y)$$

to be

$$\begin{aligned} \tau_{X,Y}^g(\mu \otimes x_1 \otimes y_1 \otimes \cdots \otimes x_k \otimes y_k) = \\ \sum_{(1),(2)} (\mu_{(1)} \otimes x_1 \otimes \cdots \otimes x_k) \otimes (\mu_{(2)} \otimes y_1 \otimes \cdots \otimes y_k). \end{aligned} \quad (9)$$

Define

$$\theta^{\mathbf{C}} : F_{\mathcal{P}}^{\mathbf{C}}(\mathbf{1}) = \bigoplus_{l \geq 0} \mathcal{P}(l) / S_l \rightarrow \mathbf{1}$$

by $\theta^{\mathbf{C}}(\mu) = \epsilon(\mu)$.

The maps $\tau^{\mathbf{C}}$ and $\theta^{\mathbf{C}}$ endow $F_{\mathcal{P}}^{\mathbf{C}}$ with a structure of Hopf monad, as pointed out in [19]. As a consequence if M and N are \mathcal{P} -algebras in the category **C**, then $M \otimes N$ is a \mathcal{P} -algebra for the following product

$$F_{\mathcal{P}}^{\mathbf{C}}(M \otimes N) \xrightarrow{\tau_{M,N}^{\mathbf{C}}} F_{\mathcal{P}}^{\mathbf{C}}(M) \otimes F_{\mathcal{P}}^{\mathbf{C}}(N) \xrightarrow{\mu_M \otimes \mu_N} M \otimes N.$$

The object $\mathbf{1}$ is a \mathcal{P} -algebra and then a unit for the tensor product of \mathcal{P} -algebras.

These results are summed up in the following proposition:

3.1.2. *Proposition.* ([19]) *Let \mathcal{P} be a Hopf operad. The monads $F_{\mathcal{P}}$ and $F_{\mathcal{P}}^g$ are Hopf monads. As a consequence, the category of \mathcal{P} -algebras is a tensor category.*

Note that *a priori* we are dealing with bi-monads instead of Hopf monads since we are not considering the antipode. Further we will consider connected objects and it is known that in this context the two notions are equivalent.

3.2. **Hopf algebras over a Hopf operad.** In this section a Hopf operad \mathcal{P} is given.

3.2.1. *Definition.* A \mathcal{P} -algebra M is a Hopf \mathcal{P} -algebra if M is endowed with a coassociative coproduct and a counit

$$\Delta_M : M \rightarrow M \otimes M \quad \epsilon_M : M \rightarrow \mathbf{1}$$

which are morphisms of \mathcal{P} -algebras. If $\mathcal{P} = \mathcal{A}s$ then a Hopf \mathcal{P} -algebra is a twisted bialgebra.

3.2.2. *Theorem.* *The underlying graded vector space of any Hopf \mathcal{P} -algebra M in $\mathbf{S-mod}$ is a Hopf \mathcal{P} -algebra in \mathbf{grVect} . More precisely, the \mathcal{P} -algebra product on $\mathcal{O}(M)$ is $\hat{\mu}_{\mathcal{O}(M)}$ and the coproduct*

$$\bar{\Delta}_{\mathcal{O}(M)} : \mathcal{O}(M) \xrightarrow{\mathcal{O}(\Delta_M)} \mathcal{O}(M \otimes M) \xrightarrow{\pi_{M,M}^{\mathcal{O}}} \mathcal{O}(M) \otimes \mathcal{O}(M)$$

is a morphism of \mathcal{P} -algebras. This Hopf \mathcal{P} -algebra is denoted \bar{M} and named the symmetrized Hopf \mathcal{P} -algebra associated to M .

In particular if for $m \in M(p)$ one writes

$$\Delta(m) = \sum_{\substack{(1),(2) \\ S \sqcup T = [p]}} m_{(1)}^{(S,T)} \otimes m_{(2)}^{(S,T)} \otimes (S, T)$$

then

$$\bar{\Delta}(m) = \sum_{(1),(2)} \sum_{i=0}^p m_{(1)}^{([i], i+[p-i])} \otimes m_{(2)}^{([i], i+[p-i])} \quad (10)$$

Proof. One has to prove that the following diagram is commutative:

$$\begin{array}{ccc} F_{\mathcal{P}}^g \mathcal{O}(M) & \xrightarrow{\hat{\mu}_{\mathcal{O}(M)}} & \mathcal{O}(M) \\ F_{\mathcal{P}}^g \bar{\Delta}_{\mathcal{O}(M)} \downarrow & & \downarrow \bar{\Delta}_{\mathcal{O}(M)} \\ F_{\mathcal{P}}^g (\mathcal{O}(M) \otimes \mathcal{O}(M)) & & \\ \tau_{\mathcal{O}(M), \mathcal{O}(M)}^g \downarrow & & \\ F_{\mathcal{P}}^g \mathcal{O}(M) \otimes F_{\mathcal{P}}^g \mathcal{O}(M) & \xrightarrow{\hat{\mu}_{\mathcal{O}(M)} \otimes \hat{\mu}_{\mathcal{O}(M)}} & \mathcal{O}(M) \otimes \mathcal{O}(M) \end{array}$$

Recall that

$$\begin{aligned}\bar{\Delta}_{\mathcal{O}(M)} &= \pi_{M,M}^{\mathcal{O}} \mathcal{O}(\Delta_M), \\ \hat{\mu}_{\mathcal{O}(M)} &= \mathcal{O}(\mu_M) \psi_{\mathcal{P}}(M), \\ \Delta_M \mu_M &= (\mu_M \otimes \mu_M) \tau_{M,M} F_{\mathcal{P}} \Delta_M.\end{aligned}$$

As a consequence using the functoriality and naturality of $\pi^{\mathcal{O}}$ and $\psi_{\mathcal{P}}$, one has the following

$$\begin{aligned}\bar{\Delta}_{\mathcal{O}(M)} \hat{\mu}_{\mathcal{O}(M)} &= \pi_{M,M}^{\mathcal{O}} \mathcal{O}(\Delta_M) \mathcal{O}(\mu_M) \psi_{\mathcal{P}}(M) \\ &= \pi_{M,M}^{\mathcal{O}} \mathcal{O}(\mu_M \otimes \mu_M) \mathcal{O}(\tau_{M,M}) \mathcal{O}(F_{\mathcal{P}} \Delta_M) \psi_{\mathcal{P}}(M) \\ &= (\mathcal{O}(\mu_M) \otimes \mathcal{O}(\mu_M)) \pi_{F_{\mathcal{P}}(M), F_{\mathcal{P}}(M)}^{\mathcal{O}} \mathcal{O}(\tau_{M,M}) \psi_{\mathcal{P}}(M \otimes M) F_{\mathcal{P}}^g \mathcal{O}(\Delta_M).\end{aligned}$$

Then the commutativity of the previous diagram is a consequence of the commutativity of the following diagram

$$\begin{array}{ccc} F_{\mathcal{P}}^g \mathcal{O}(M \otimes M) & \xrightarrow{\psi_{\mathcal{P}}(M \otimes M)} & \mathcal{O} F_{\mathcal{P}}(M \otimes M) \\ F_{\mathcal{P}}^g(\pi_{M,M}^{\mathcal{O}}) \downarrow & & \downarrow \mathcal{O} \tau_{M,M} \\ F_{\mathcal{P}}^g(\mathcal{O}(M) \otimes \mathcal{O}(M)) & & \mathcal{O}(F_{\mathcal{P}}(M) \otimes F_{\mathcal{P}}(M)) \\ \tau_{\mathcal{O}(M), \mathcal{O}(M)}^g \downarrow & & \downarrow \pi_{F_{\mathcal{P}}(M), F_{\mathcal{P}}(M)}^{\mathcal{O}} \\ F_{\mathcal{P}}^g \mathcal{O}(M) \otimes F_{\mathcal{P}}^g \mathcal{O}(M) & \xrightarrow{\psi_{\mathcal{P}}(M) \otimes \psi_{\mathcal{P}}(M)} & \mathcal{O} F_{\mathcal{P}}^g(M) \otimes \mathcal{O} F_{\mathcal{P}}^g(M) \end{array}$$

Let us compute the composition $R = \pi_{F_{\mathcal{P}}(M), F_{\mathcal{P}}(M)}^{\mathcal{O}} \mathcal{O} \tau_{M,M} \psi_{\mathcal{P}}(M \otimes M)$ applied to

$$X = \mu \otimes (m_1 \otimes n_1 \otimes (A_1, B_1)) \otimes \dots \otimes (m_k \otimes n_k \otimes (A_k, B_k)) \in F_{\mathcal{P}}^g \mathcal{O}(M \otimes M),$$

where $m_i \in M(l_i)$ and $n_i \in M(r_i)$ and $A_i \sqcup B_i$ is an ordered partition of $[l_i + r_i]$.

$$Y := \psi_{\mathcal{P}}(M \otimes M)(X) = \sum_{(T_1, \dots, T_k)} X \otimes (T_1, \dots, T_k),$$

where the sum is taken over all ordered partitions of $[l_1 + r_1 + \dots + l_k + r_k]$ such that $|T_i| = l_i + r_i$. For $U_i = T_i(A_i)$ and $V_i = T_i(B_i)$ one has

$$((A_1, B_1) \times \dots \times (A_k, B_k))(T_1, \dots, T_k) = (U_1, V_1, \dots, U_k, V_k).$$

As a consequence

$$Y = \sum_{(U_1, V_1, \dots, U_k, V_k)} \mu \otimes m_1 \otimes n_1 \otimes \dots \otimes m_k \otimes n_k \otimes (U_1, V_1, \dots, U_k, V_k),$$

where the sum is taken over all ordered partitions of $[l_1 + r_1 + \dots + l_k + r_k]$ such that $|U_i| = l_i, |V_i| = r_i$ and $\text{st}(U_i) = A_i, \text{st}(V_i) = B_i$. Denote by

$\bar{m} = m_1 \otimes \dots \otimes m_k$ and $\bar{n} = n_1 \otimes \dots \otimes n_k$. By the definition of τ (see (8)),

$$Z := \mathcal{O}\tau_{M,M}(Y) = \sum_{(U_1, V_1, \dots, U_k, V_k)} (\mu_{(1)} \otimes \bar{m} \otimes \text{st}(U_1, \dots, U_k)) \otimes (\mu_{(2)} \otimes \bar{n} \otimes \text{st}(V_1, \dots, V_k)) \otimes (\cup U_i, \cup V_i).$$

Furthermore $\pi_{F_{\mathcal{P}}(M), F_{\mathcal{P}}(M)}^{\mathcal{O}}(Z) = 0$ except in case $(\cup U_i, \cup V_i) = \text{Id}$. But

$$(\cup U_i, \cup V_i) = \text{Id} \Leftrightarrow (A_i, B_i) = \text{Id}, \forall i.$$

This is due to the fact that if $(\cup U_i, \cup V_i) = \text{Id}$ then $T_i(A_i) < T_i(B_i)$ and $A_i < B_i$ which is equivalent to $(A_i, B_i) = \text{Id}$.

As a consequence if $\forall i, (A_i, B_i) = \text{Id}$, then

$$R(X) = \sum_{\substack{(U_1, \dots, U_k) \\ (V_1, \dots, V_k)}} (\mu_{(1)} \otimes \bar{m} \otimes (U_1, \dots, U_k)) \otimes (\mu_{(2)} \otimes \bar{n} \otimes (V_1, \dots, V_k)),$$

where the sum is taken over all shuffles (U_1, \dots, U_k) and (V_1, \dots, V_k) . If there exists i such that $(A_i, B_i) \neq \text{Id}$, then $R(X) = 0$.

The composition $L = (\psi_{\mathcal{P}}(M) \otimes \psi_{\mathcal{P}}(M))\tau_{\mathcal{O}(M), \mathcal{O}(M)}^g F_{\mathcal{P}}^g(\pi_{M,M}^{\mathcal{O}})$ is easier to compute. First of all, if there exists i such that $(A_i, B_i) \neq \text{Id}$, then $L(X) = 0$. Assume that $(A_i, B_i) = \text{Id}, \forall i$. Then

$$W = \tau_{\mathcal{O}(M), \mathcal{O}(M)}^g F_{\mathcal{P}}^g(\pi_{M,M}^{\mathcal{O}})(X) = \sum_{(1), (2)} (\mu_{(1)} \otimes \bar{m}) \otimes (\mu_{(2)} \otimes \bar{n})$$

and $(\psi_{\mathcal{P}}(M) \otimes \psi_{\mathcal{P}}(M))(W) = R(X)$. Thus $R(X) = L(X), \forall X$ and the diagram is commutative. \square

3.2.3. Connected operads and connected Hopf operads. An operad is *connected* if $\mathcal{P}(0) = \mathbf{k}$ and $\mathcal{P}(1) = \mathbf{k}$. Let 1_0 denote the unit of $\mathbf{k} \in \mathcal{P}(0)$. If \mathcal{P} is a connected operad, for any subset S of $[n]$, there exists a map

$$\begin{aligned} \mathcal{P}(n) &\rightarrow \mathcal{P}(|S|) \\ \mu &\mapsto \mu|_S = \mu(x_1, \dots, x_n) \end{aligned}$$

where

$$\begin{cases} x_i = 1_1 & \text{if } i \in S, \\ x_i = 1_0 & \text{if } i \notin S. \end{cases}$$

For $\mathcal{P} = \mathcal{A}s$ one recovers the definition given in section 1.1 for the symmetric group.

A *connected Hopf operad* is a Hopf operad which is connected and such that $\epsilon(0) : \mathbf{k} = \mathcal{P}(0) \rightarrow \mathbf{k}$ is the identity isomorphism. As a consequence

$$\epsilon(\mu) = \mu|_{\emptyset}.$$

Recall from [12] that if \mathcal{P} is a connected Hopf operad then \mathcal{P} is a Hopf \mathcal{P} -algebra for the coproduct given by

$$\begin{aligned} \Delta(\mu) &= \sum_{\substack{(1),(2) \\ S \sqcup T = [n]}} \mu_{(1)}|_S \otimes \mu_{(2)}|_T \otimes (S, T) \\ &= 1 \otimes \mu + \mu \otimes 1 + \sum_{\substack{S \sqcup T = [n] \\ S, T \neq \emptyset}} \mu_{(1)}|_S \otimes \mu_{(2)}|_T \otimes (S, T). \end{aligned}$$

This is a consequence of proposition 3.1.2 and the freeness of \mathcal{P} as a \mathcal{P} -algebra in **S-mod**: the map Δ is the unique \mathcal{P} -algebra morphism such that $\Delta(1_1) = 1_0 \otimes 1_1 + 1_1 \otimes 1_0$.

It happens in many examples that \mathcal{P} is not connected and $\mathcal{P}(0) = 0$. Nevertheless it is sometimes possible to define a \mathcal{P} -algebra structure on $(\mathcal{P}_+ \otimes \mathcal{P}_+)_-$ where

$$\begin{cases} \mathcal{P}_+(0) = \mathbf{k} \\ \mathcal{P}_+(n) = \mathcal{P}(n), n > 0 \end{cases} \quad \text{and} \quad \begin{cases} (\mathcal{P}_+ \otimes \mathcal{P}_+)_-(0) = 0 \\ (\mathcal{P}_+ \otimes \mathcal{P}_+)_-(n) = (\mathcal{P}_+ \otimes \mathcal{P}_+)(n), n > 0. \end{cases}$$

(see for instance [13]). In that case, Δ is defined as the unique \mathcal{P} -algebra map such that $\Delta(1_1) = 1_0 \otimes 1_1 + 1_1 \otimes 1_0$. These kind of operads will be treated as connected operads.

If \mathcal{P} is connected, one has two examples of Hopf \mathcal{P} -algebras in **grVect**:

- $F_{\mathcal{P}}^g(V)$ for V in **grVect** where the product is given by $F_{\mu_{\mathcal{P}}}^g(V)$ and the coproduct is given by $F_{\Delta}^g(V)$.
- For any \mathbb{S} -module M the free \mathcal{P} -algebra $F_{\mathcal{P}}(M)$ is a Hopf \mathcal{P} -algebra in **S-mod**. The symmetrized Hopf \mathcal{P} -algebra $\overline{F_{\mathcal{P}}(M)}$ is a Hopf \mathcal{P} -algebra in **grVect** by theorem 3.2.2.

3.3. Regular Hopf operads. Let $\mathcal{P} = \mathcal{S}\tilde{\mathcal{P}}$ be a regular operad. Assume (\mathcal{P}, δ) is a Hopf operad. The operad \mathcal{P} is a *regular Hopf operad* if $\delta(\tilde{\mathcal{P}}) \subset \tilde{\mathcal{P}} \otimes \tilde{\mathcal{P}}$. For instance $\mathcal{A}s$ is a regular Hopf operad. We prove that for any regular Hopf operad a Hopf \mathcal{P} -algebra in the category **S-mod** gives rise to two structures of Hopf \mathcal{P} -algebra in the category **grVect**. This is a generalization to regular operads of a theorem announced by Stover in [24] and proved by Patras and Reutenauer in [21] in the context of twisted bialgebras. Note that the hypothesis regular is needed to obtain two structures of Hopf \mathcal{P} -algebras.

3.3.1. Theorem. *Let \mathcal{P} be a regular Hopf operad. Let (M, μ_M, Δ_M) be a Hopf \mathcal{P} -algebra in **S-mod**. The product $\bar{\mu}_{\mathcal{O}(M)}$ together with the coproduct $\hat{\Delta}_{\mathcal{O}(M)}$ defined as the composite*

$$\mathcal{O}(M) \xrightarrow{\mathcal{O}(\Delta_M)} \mathcal{O}(M \otimes M) \xrightarrow{i_{M,M}^g} \mathcal{O}(M) \otimes \mathcal{O}(M)$$

endows $\mathcal{O}(M)$ with a structure of Hopf \mathcal{P} -algebra which is cocommutative if (M, Δ_M) is. This Hopf \mathcal{P} -algebra is denoted \hat{M} and named the cosymmetrized Hopf \mathcal{P} -algebra associated to M .

Note that the coproduct $\hat{\Delta}_{\mathcal{O}(M)}$ is always defined: for $m \in M(p)$, if the coproduct in M writes

$$\Delta_M(m) = \sum_{\substack{(1),(2) \\ S \sqcup T = [p]}} m_{(1)}^{(S,T)} \otimes m_{(2)}^{(S,T)} \otimes (S, T),$$

then the induced coproduct in $\mathcal{O}(M)$ writes

$$\hat{\Delta}_{\mathcal{O}(M)}(m) = \sum_{\substack{(1),(2) \\ S \sqcup T = [p]}} m_{(1)}^{(S,T)} \otimes m_{(2)}^{(S,T)}. \quad (11)$$

Proof. The proof is similar to the proof of theorem 3.2.2. The commutativity of the diagram

$$\begin{array}{ccc} F_{\mathcal{P}}^g \mathcal{O}(M) & \xrightarrow{\bar{\mu}_{\mathcal{O}(M)}} & \mathcal{O}(M) \\ F_{\mathcal{P}}^g \hat{\Delta}_{\mathcal{O}(M)} \downarrow & & \downarrow \hat{\Delta}_{\mathcal{O}(M)} \\ F_{\mathcal{P}}^g (\mathcal{O}(M) \otimes \mathcal{O}(M)) & & \mathcal{O}(M) \otimes \mathcal{O}(M) \\ \tau_{\mathcal{O}(M), \mathcal{O}(M)}^g \downarrow & & \downarrow \\ F_{\mathcal{P}}^g \mathcal{O}(M) \otimes F_{\mathcal{P}}^g \mathcal{O}(M) & \xrightarrow{\bar{\mu}_{\mathcal{O}(M)} \otimes \bar{\mu}_{\mathcal{O}(M)}} & \mathcal{O}(M) \otimes \mathcal{O}(M) \end{array}$$

is a consequence of the commutativity of the diagram

$$\begin{array}{ccc} F_{\mathcal{P}}^g \mathcal{O}(M \otimes M) & \xrightarrow{\psi_{\mathcal{P}}^r(M \otimes M)} & \mathcal{O} F_{\mathcal{P}}(M \otimes M) \\ F_{\mathcal{P}}^g (\iota_{M, M}^{\mathcal{O}}) \downarrow & & \downarrow \mathcal{O} \tau_{M, M} \\ F_{\mathcal{P}}^g (\mathcal{O}(M) \otimes \mathcal{O}(M)) & & \mathcal{O}(F_{\mathcal{P}}(M) \otimes F_{\mathcal{P}}(M)) \\ \tau_{\mathcal{O}(M), \mathcal{O}(M)}^g \downarrow & & \downarrow \iota_{F_{\mathcal{P}}(M), F_{\mathcal{P}}(M)}^{\mathcal{O}} \\ F_{\mathcal{P}}^g \mathcal{O}(M) \otimes F_{\mathcal{P}}^g \mathcal{O}(M) & \xrightarrow{\psi_{\mathcal{P}}^r(M) \otimes \psi_{\mathcal{P}}^r(M)} & \mathcal{O} F_{\mathcal{P}}^g(M) \otimes \mathcal{O} F_{\mathcal{P}}^g(M) \end{array}$$

We first compute $R = \iota_{F_{\mathcal{P}}(M), F_{\mathcal{P}}(M)}^{\mathcal{O}} \mathcal{O} \tau_{M, M} \psi_{\mathcal{P}}^r(M \otimes M)$ on an element

$$X = \mu \otimes (m_1 \otimes n_1 \otimes (A_1, B_1)) \otimes \dots \otimes (m_k \otimes n_k \otimes (A_k, B_k)) \in F_{\mathcal{P}}^g \mathcal{O}(M \otimes M),$$

where $\mu \in \tilde{\mathcal{P}}(k)$, $m_i \in M(l_i)$, $n_i \in M(r_i)$ and $A_i \sqcup B_i$ is an ordered partition of $[l_i + r_i]$. Let us write $\tilde{m} = m_1 \otimes \dots \otimes m_k$, $\tilde{n} = n_1 \otimes \dots \otimes n_k$ and

$$(A_1, B_1) \times \dots \times (A_k, B_k) = (U_1, V_1, \dots, U_k, V_k).$$

Using the definitions one gets

$$R(X) = \sum_{(1),(2)} (\mu_{(1)} \otimes \bar{m} \otimes \text{st}(U_1, \dots, U_k)) \otimes (\mu_{(2)} \otimes \bar{n} \otimes \text{st}(V_1, \dots, V_k)),$$

and $\text{st}(U_1, \dots, U_k) = \text{Id}$ and $\text{st}(V_1, \dots, V_k) = \text{Id}$.

Let L be the composition $(\psi_{\mathcal{P}}^r(M) \otimes \psi_{\mathcal{P}}^r(M))\tau_{\mathcal{O}(M), \mathcal{O}(M)}^g F_{\mathcal{P}}^g(\iota_{M,M}^{\mathcal{O}})$. Applying the definitions one gets

$$L(X) = \sum_{(1),(2)} (\mu_{(1)} \otimes \bar{m}) \otimes (\mu_{(2)} \otimes \bar{n}).$$

Thus $R(X) = L(X), \forall X$ and the diagram is commutative.

It is clear that if Δ_M is cocommutative so is $\hat{\Delta}_{\mathcal{O}(M)}$. \square

As a consequence, if \mathcal{P} is a regular Hopf operad then any Hopf \mathcal{P} -algebra M in $\mathbf{S-mod}$ gives rise to two structures of Hopf \mathcal{P} -algebra in \mathbf{grVect} . In particular this result holds for $\oplus_n \mathcal{P}(n)$ and for the underlying graded vector space of any free \mathcal{P} -algebra $F_{\mathcal{P}}(M)$.

3.4. Application to multiplicative Hopf operads. In a first step we establish that the corresponding Hopf structures in case $\mathcal{P} = \mathcal{A}s$ coincide with the ones found by Stover [24] and proved by Patras and Reutenauer in [21]. In a second step we apply the above results to multiplicative Hopf operads.

3.4.1. The associative case. Recall that the operad $\mathcal{A}s$ is a regular Hopf operad. Hence the underlying graded vector space of a twisted bialgebra is endowed with two structures of Hopf algebra. Let M be a twisted bialgebra with product m and coproduct Δ .

The Hopf algebra $\hat{M} = (\mathcal{O}(M), \hat{m}_{\mathcal{O}(M)}, \bar{\Delta}_{\mathcal{O}(M)})$ is described for $a \in M(p), b \in M(q)$ by

$$\begin{aligned} \hat{m}_{\mathcal{O}(M)}(a, b) &= m(a, b) \cdot q_{a,b}, \\ \bar{\Delta}_{\mathcal{O}(M)}(a) &= \sum_{i=0}^p a_{(1)}^{([i], i+[p-i])} \otimes a_{(2)}^{([i], i+[p-i])}, \end{aligned}$$

which is the *symmetrized bialgebra* associated to the twisted bialgebra M as in [21, proposition 15].

The Hopf algebra $\hat{M} = (\mathcal{O}(M), \bar{m}, \hat{\Delta}_{\mathcal{O}(M)})$ is described for $a \in M(p), b \in M(q)$ by

$$\begin{aligned} \bar{m}_{\mathcal{O}(M)}(a, b) &= m(a, b), \\ \hat{\Delta}_{\mathcal{O}(M)}(a) &= \sum_{S \sqcup T = [p]} a_{(1)}^{(S,T)} \otimes a_{(2)}^{(S,T)}, \end{aligned}$$

which is the *cosymmetrized bialgebra* associated to the twisted bialgebra M as in [21, definition 8].

A *multiplicative Hopf operad* is a Hopf operad \mathcal{P} together with an operad morphism $\mathcal{A}s \rightarrow \mathcal{P}$ which commutes with the Hopf structure. As a consequence any Hopf \mathcal{P} -algebra is a Hopf $\mathcal{A}s$ -algebra. The result below is a consequence of the previous sections.

3.4.2. *Corollary.* *Let \mathcal{P} be a multiplicative Hopf operad. The underlying graded vector space of any Hopf \mathcal{P} -algebra is endowed with two different structures of Hopf algebra.* \square

4. UNITAL INFINITESIMAL \mathcal{P} -BIALGEBRAS

In this section we give some comparison between $\bar{\Delta}$ and $\bar{\mu}$ when the operad is regular, in view of generalizing the theory of *unital infinitesimal bialgebra* developed by Loday and Ronco in [17]. This yields the definition of unital infinitesimal \mathcal{P} -bialgebras. As a consequence we prove that any Hopf algebra over a multiplicative Hopf operad is isomorphic to a cofree coassociative algebra. Moreover, if \mathcal{P} is regular then this isomorphism respects the \mathcal{P} -algebra structure. We study the associative case in detail.

From now on a connected Hopf operad \mathcal{P} is given.

4.1. **Unital infinitesimal \mathcal{P} -bialgebras.** After some definitions, we prove that the underlying graded vector space of a Hopf \mathcal{P} -algebra is a unital infinitesimal \mathcal{P} -bialgebra in theorem 4.1.2. We prove also in theorem 4.1.3 that the same result holds for $F_{\mathcal{P}}^g(V)$ when V is a graded vector space such that $V(0) = 0$.

A *connected coalgebra* M in **S-mod** or **grVect** is a coalgebra such that $M(0) = \mathbf{k}$ and such that the counit $\epsilon : \mathbf{k} = M(0) \rightarrow \mathbf{k}$ is the identity isomorphism. Concretely for $M \in \mathbf{S-mod}$ the coproduct writes

$$\Delta(m) = 1 \otimes m + m \otimes 1 + \sum_{\substack{S \sqcup T = [p] \\ S, T \neq \emptyset}} m_{(1)}^{(S,T)} \otimes m_{(2)}^{(S,T)} \otimes (S, T),$$

and for $V \in \mathbf{grVect}$ it writes

$$\Delta(v) = 1 \otimes v + v \otimes 1 + \sum_{\substack{p+q=|v| \\ p, q > 0}} m_{(1),p} \otimes m_{(2),q}.$$

4.1.1. *Definition.* Assume $\mathcal{P} = \mathcal{S}\tilde{\mathcal{P}}$ is a regular operad. A *unital infinitesimal \mathcal{P} -bialgebra* M is a \mathcal{P} -algebra in **grVect** endowed with a connected coalgebra structure $\Delta : M \rightarrow M \otimes M$ satisfying the *infinitesimal relation*:

$$\begin{aligned} \Delta\mu(m_1, \dots, m_k) &= \sum_{j=1}^k \mu(m_1 \otimes 1, \dots, m_{j-1} \otimes 1, \Delta(m_j), 1 \otimes m_{j+1}, \dots, 1 \otimes m_k) \\ &\quad - \sum_{j=1}^{k-1} \mu(m_1 \otimes 1, \dots, m_j \otimes 1, 1 \otimes m_{j+1}, \dots, 1 \otimes m_k), \end{aligned} \quad (12)$$

for $\mu \in \tilde{\mathcal{P}}(k)$. Note that the operad needs to be regular since the infinitesimal relation is not S_k -equivariant.

For instance if $\mathcal{P} = \mathcal{A}s$, a unital infinitesimal $\mathcal{A}s$ -bialgebra is the definition of Loday and Ronco in [17] of a *unital infinitesimal bialgebra* since the previous relation restricts to

$$\Delta(ab) = \Delta(a)(1 \otimes b) + (a \otimes 1)\Delta(b) - a \otimes b.$$

Let M be a Hopf \mathcal{P} -algebra in $\mathbf{S}\text{-mod}$ with \mathcal{P} regular. Theorems 3.2.2 and 3.3.1 assert that the underlying graded vector space of M is endowed with two structures of Hopf \mathcal{P} -algebras in \mathbf{grVect} . One is given by $(\hat{\mu}, \hat{\Delta})$ and the other one by $(\bar{\mu}, \bar{\Delta})$. The next theorem explores the relation between $\bar{\mu}$ and $\bar{\Delta}$.

4.1.2. Theorem. Let \mathcal{P} be a connected regular Hopf operad and M be a connected Hopf \mathcal{P} -algebra in $\mathbf{S}\text{-mod}$. The product $\bar{\mu} := \bar{\mu}_{\mathcal{O}(M)}$ and coproduct $\bar{\Delta} = \bar{\Delta}_{\mathcal{O}(M)}$ endow $\mathcal{O}(M)$ with a structure of unital infinitesimal \mathcal{P} -bialgebra.

Proof. Recall that

$$\begin{aligned}\bar{\Delta} &= \pi_{M,M}^{\mathcal{O}} \mathcal{O}(\Delta) \\ \bar{\mu} &= \mathcal{O}(\mu_M) \psi_{\mathcal{P}}^r(M).\end{aligned}$$

Following the proof of theorem 3.2.2 one has

$$\bar{\Delta} \bar{\mu} = (\mathcal{O} \mu_M \otimes \mathcal{O} \mu_M) \pi_{F_{\mathcal{P}}(M), F_{\mathcal{P}}(M)}^{\mathcal{O}} \mathcal{O} \tau_{M,M} \psi_{\mathcal{P}}^r(M \otimes M) F_{\mathcal{P}}^g \mathcal{O} \Delta.$$

Let $X = \mu \otimes m_1 \otimes \dots \otimes m_k \in F_{\mathcal{P}}^g \mathcal{O}(M)$ with $m_i \in M(h_i)$.

$$\begin{aligned}Y &:= \psi_{\mathcal{P}}^r(M \otimes M) F_{\mathcal{P}}^g \mathcal{O} \Delta(X) = \\ &\sum \mu \otimes (m_{1(1)}^{(S_1, T_1)} \otimes m_{1(2)}^{(S_1, T_1)} \otimes (S_1, T_1)) \otimes \dots \otimes (m_{k(1)}^{(S_k, T_k)} \otimes m_{k(2)}^{(S_k, T_k)} \otimes (S_k, T_k)),\end{aligned}$$

where the sum is taken over all ordered partitions $S_i \sqcup T_i$ of $[h_i]$ for all i . In order to compute $\mathcal{O} \tau_{M,M}(Y)$, we write $(S_1, T_1) \times \dots \times (S_k, T_k)$ as $(U_1, V_1, \dots, U_k, V_k)$ which is an ordered partition of $[h_1 + \dots + h_k]$ and $U_i = |S_i|$, $V_i = |T_i|$. It is obvious that $\text{st}(U_1, \dots, U_k) = \text{Id}$ and that the same equality holds for the sequence of V_i 's. Furthermore if $S = \cup U_i$ and $T = \cup V_i$, then $S = S_1 \times \dots \times S_k$ and $T = T_1 \times \dots \times T_k$. As a consequence

$$\begin{aligned}\mathcal{O} \tau_{M,M}(Y) &= \\ &\sum_{(S,T)} (\mu_{(1)} \otimes (m_{1(1)}^{(S_1, T_1)} \otimes \dots \otimes m_{k(1)}^{(S_k, T_k)}) \otimes (\mu_{(2)} \otimes (m_{1(2)}^{(S_1, T_1)} \otimes \dots \otimes m_{k(2)}^{(S_k, T_k)})) \otimes (S, T),\end{aligned}$$

where the sum is taken over all ordered partitions (S, T) of $[h_1 + \dots + h_k]$ and where $S = S_1 \times \dots \times S_k$ and $T = T_1 \times \dots \times T_k$ with $S_i, T_i \subset [h_i]$. The map $\pi_{F_{\mathcal{P}}(M), F_{\mathcal{P}}(M)}^{\mathcal{O}}$ is non zero on an ordered partition (S, T) if and only if there exists r such that $S = [r]$. For any such r there exists j such that $S_k = [h_k]$

for $k < j$ and $S_k = \emptyset$ for $k > j$. Since M is connected $m_{(1)}^{\emptyset, [h]} \otimes m_{(2)}^{\emptyset, [h]} = 1 \otimes m$. As a consequence

$$\begin{aligned} (\mathcal{O} \mu_M \otimes \mathcal{O} \mu_M) \pi_{F\mathcal{P}(M), F\mathcal{P}(M)}^{\mathcal{O}} \mathcal{O} \tau_{M, M}(Y) = \\ \sum_{j=1}^k \sum_{\alpha=1}^{h_j-1} \mu_{(1)}(m_1, \dots, m_{j-1}, m_{j(1)}^{([\alpha], \alpha + [h_j - \alpha])}, 1, \dots, 1) \\ \otimes \mu_{(2)}(1, \dots, 1, m_{j(2)}^{([\alpha], \alpha + [h_j - \alpha])}, m_{j+1}, \dots, m_k) \\ + \sum_{j=0}^k \mu_{(1)}(m_1, \dots, m_j, 1, \dots, 1) \otimes \mu_{(2)}(1, \dots, 1, m_{j+1}, \dots, m_k). \end{aligned}$$

On the other hand let us compute the right hand side of the equation (12):

$$\begin{aligned} \sum_{j=1}^k \mu(m_1 \otimes 1, \dots, m_{j-1} \otimes 1, \Delta(m_j), 1 \otimes m_{j+1}, \dots, 1 \otimes m_k) \\ - \sum_{j=1}^{k-1} \mu(m_1 \otimes 1, \dots, m_j \otimes 1, 1 \otimes m_{j+1}, \dots, 1 \otimes m_k) = \\ \sum_{j=1}^k \mu(m_1 \otimes 1, \dots, m_{j-1} \otimes 1, \Delta'(m_j), 1 \otimes m_{j+1}, \dots, 1 \otimes m_k) + \\ \sum_{j=1}^k \mu(m_1 \otimes 1, \dots, m_{j-1} \otimes 1, 1 \otimes m_j + m_j \otimes 1, 1 \otimes m_{j+1}, \dots, 1 \otimes m_k) \\ - \sum_{j=1}^{k-1} \mu(m_1 \otimes 1, \dots, m_j \otimes 1, 1 \otimes m_{j+1}, \dots, 1 \otimes m_k) \end{aligned}$$

where

$$\Delta'(m_j) = \sum_{\alpha=1}^{h_j-1} m_{j(1)}^{([\alpha], \alpha + [h_j - \alpha])} \otimes m_{j(2)}^{([\alpha], \alpha + [h_j - \alpha])}.$$

Thus the left and right hand sides of the equation (12) are equal and the theorem is proved. \square

4.1.3. Theorem. *Let $\mathcal{P} = \mathcal{S}\tilde{\mathcal{P}}$ be a connected regular Hopf operad. Let V be a graded vector space with $V(0) = 0$. The free \mathcal{P} -algebra in \mathbf{grVect} $F_{\mathcal{P}}^g(V)$ is a unital infinitesimal \mathcal{P} -bialgebra. The product is given by the usual product on free \mathcal{P} -algebras and the coproduct is given for $x = \mu \otimes v_1 \otimes \dots \otimes v_k \in F_{\mathcal{P}}^g(V)$ with $\mu \in \tilde{\mathcal{P}}(k)$ by*

$$\bar{\Delta}(x) = 1 \otimes x + x \otimes 1 + \sum_{i=1}^{k-1} (\mu_{(1)}|_{[i]} \otimes v_1 \dots \otimes v_i) \otimes (\mu_{(2)}|_{i+[k-i]} \otimes v_{i+1} \dots \otimes v_k).$$

Proof. When \mathcal{P} is regular $F_{\mathcal{P}}^g(V) = \bigoplus_n \tilde{\mathcal{P}}(n) \otimes V^{\otimes_g n}$, hence it is enough to prove the formula (12) for

$$\bar{\Delta}\mu(\nu_1 \otimes \bar{v}_1, \dots, \nu_k \otimes \bar{v}_k)$$

with $\mu, \nu_i \in \tilde{\mathcal{P}}$ and $\bar{v}_i \in V^{\otimes l_i}$. The computation is straightforward. \square

4.2. Rigidity for twisted bialgebras. Loday and Ronco proved a theorem of rigidity for unital infinitesimal bialgebras. Recall from [17] that the fundamental example of a unital infinitesimal bialgebra is given by $T^{fc}(V) = F_{\mathcal{A}s}^g(V)$ where V is a graded vector space concentrated in degree 1 and where the product is given by the concatenation and the coproduct is given by the deconcatenation. Recall also that for a connected coalgebra C , with a coproduct Δ and a counit ϵ , the space of primitive elements is defined by

$$\text{Prim}_{\Delta}(C) = \{x \in \text{Ker } \epsilon \mid \Delta(x) = 1 \otimes x + x \otimes 1\}.$$

Here is the statement of the theorem:

4.2.1. Theorem. [17] *Any connected unital infinitesimal Hopf bialgebra H is isomorphic to $T^{fc}(\text{Prim}(H))$.*

Let (A, m, Δ) be a connected twisted bialgebra and $\bar{A} = (A, \hat{m}, \bar{\Delta})$ the symmetrized bialgebra and $\hat{A} = (A, \bar{m}, \hat{\Delta})$ the cosymmetrized bialgebra as in paragraph 3.4.1. The triplet $(A, \bar{m}, \bar{\Delta})$ is a unital infinitesimal bialgebra by theorem 4.1.2 and then, by theorem 4.2.1 is isomorphic to $T^{fc}(\text{Prim}_{\bar{\Delta}}(A))$. Hence \hat{A} is a free associative algebra and \bar{A} is a cofree coassociative coalgebra. Assume furthermore that \mathbf{k} is of characteristic 0 and Δ is cocommutative. Then \hat{A} is a cocommutative Hopf algebra, and by the theorem of Cartier-Milnor-Moore, it is the universal enveloping algebra of its primitive elements. If each A_n is finite dimensional, since \hat{A} is free as an associative algebra, by lemma 22 in [21] the space of primitive elements is a free Lie algebra.

These results are summed up in the following theorem

4.2.2. Theorem. *Let (A, m, Δ) be a connected twisted bialgebra. The associated symmetrized bialgebra \bar{A} is a cofree coassociative algebra. The associated cosymmetrized bialgebra \hat{A} is a free associative algebra.*

If \mathbf{k} is of characteristic 0, if Δ is cocommutative and if A_n is finite dimensional for all n , there is an isomorphism of Lie algebras

$$\text{Prim}_{\hat{\Delta}}(A) = F_{\text{Lie}}^g(\text{Prim}_{\bar{\Delta}}(A)).$$

This isomorphism is functorial in A . \square

Using the results of Loday and Ronco we have improved the results of Patras and Reutenauer. Furthermore, if \mathcal{P} is a connected multiplicative Hopf operad then it provides connected twisted bialgebras: indeed, any Hopf \mathcal{P} -algebra in $\mathbf{S}\text{-mod}$ is a twisted bialgebra. For instance \mathcal{P} and more

generally $F_{\mathcal{P}}(M)$ with M a \mathbb{S} -module such that $M(0) = 0$ are connected twisted bialgebras.

4.2.3. Remark. If (A, m, Δ) is a connected twisted bialgebra then $(A, \hat{m}, \bar{m}, \bar{\Delta})$ is a connected 2-associative bialgebra in the terminology of Loday and Ronco in [17], that is $(A, \hat{m}, \bar{\Delta})$ is a Hopf algebra and $(A, \bar{m}, \bar{\Delta})$ is a unital infinitesimal bialgebra. By the structure theorem in [17], one gets that $\text{Prim}_{\bar{\Delta}}(A)$ is a B_{∞} -algebra and A is the enveloping 2-as bialgebra of its primitive elements.

Assume \mathcal{P} and V satisfies the conditions of theorem 4.1.3. Assume \mathcal{P} is multiplicative and $A = F_{\mathcal{P}}^g(V)$ is finite dimensional in each degree. Then $A2 = (A^*, {}^t\Delta, {}^t\bar{\Delta}, {}^t m)$ where m is the associative product induced by the multiplicative structure of \mathcal{P} is also a 2-associative bialgebra. If $A2$ is connected then it is the enveloping 2-as bialgebra of its primitive elements.

4.3. Rigidity for unital infinitesimal \mathcal{P} -bialgebras. Let \mathcal{P} be a connected regular Hopf operad which is multiplicative. Any connected unital infinitesimal \mathcal{P} -bialgebra $(M, \mu_M, \bar{\Delta})$ is a unital infinitesimal bialgebra. Applying the theorem 4.2.1 of Loday and Ronco, there is an isomorphism of unital infinitesimal bialgebras between M and $T^{fc}(\text{Prim}_{\bar{\Delta}} M)$. We prove in this section that for any vector space V , the space $T^{fc}(V)$ is endowed with a unique structure of unital infinitesimal \mathcal{P} -bialgebra and that the latter isomorphism is a morphism of unital infinitesimal \mathcal{P} -bialgebras.

4.3.1. Proposition. *Let \mathcal{P} be a connected regular multiplicative Hopf operad and let V be a vector space. There is a unique \mathcal{P} -algebra structure*

$$\mu : F_{\mathcal{P}}^g(T(V)) \rightarrow T(V)$$

on $T(V)$ such that the composition with the natural transformation $F_{\mathcal{A}s}^g \rightarrow F_{\mathcal{P}}^g$ applied at $T(V)$ gives the concatenation product and such that the deconcatenation

$$\Delta_c : T(V) \rightarrow T(V) \otimes T(V)$$

together with μ endow $T(V)$ with a structure of unital infinitesimal \mathcal{P} -bialgebra.

Proof. One has to build a unique map $F_{\mathcal{P}}^g(T(V)) \rightarrow T(V)$ which satisfies the three conditions

- i) The graded vector space $T(V)$ is a \mathcal{P} -algebra.
- ii) The map restricts to the concatenation product on $\mathcal{A}s \subset \mathcal{P}$.
- iii) The map together with the deconcatenation product satisfies the infinitesimal condition.

In order to do so one needs to define $x = \mu(\bar{v}_1, \dots, \bar{v}_k)$ for any $\mu \in \tilde{\mathcal{P}}(k)$ and $\bar{v}_i = v_i^1 \otimes \dots \otimes v_i^{l_i} \in V^{\otimes g^i}$. Note that \bar{v}_i is the concatenation

$$\bar{v}_i = \text{Id}_i(v_i^1, \dots, v_i^{l_i}).$$

Condition i) and ii) impose that

$$x = \mu(\text{Id}_{l_1}, \dots, \text{Id}_{l_k})(v_1^1, \dots, v_1^{l_1}, \dots, v_k^1, \dots, v_k^{l_k}).$$

Since \mathcal{P} is regular $\mu(\text{Id}_{l_1}, \dots, \text{Id}_{l_k})$ still lives in $\tilde{\mathcal{P}}$. As a consequence it is enough to prove the existence and unicity of $\mu(v_1, \dots, v_k) \in V^{\otimes k}$ for $v_i \in V$ satisfying the required conditions.

The coassociative coproduct δ of the Hopf operad \mathcal{P} can be iterated. We use Sweedler's notation:

$$\begin{aligned} \delta^k(n) : \mathcal{P}(n) &\rightarrow \mathcal{P}(n)^{\otimes k} \\ \mu &\mapsto \sum \mu_{(1)} \otimes \dots \otimes \mu_{(k)}. \end{aligned}$$

Let us prove by induction on k that

$$\mu(a_1, \dots, a_k) = \left(\prod_{i=1}^k \mu_{(i)}|_{\{i\}} \right) a_1 \otimes \dots \otimes a_k.$$

Note that this formula makes sense because $\mathcal{P}(1) = \mathbf{k}$ and $\mu|_{\{i\}} \in \mathbf{k}$. If $k = 1$, then the formula holds. Assume that the formula holds for any $1 \leq l \leq k - 1$. $\Delta_c \mu(a_1, \dots, a_k)$ has to verify the infinitesimal relation (12), that is

$$\begin{aligned} \Delta_c \mu(a_1, \dots, a_k) &= \sum_{j=1}^k \mu(a_1 \otimes 1, \dots, a_{j-1} \otimes 1, \Delta_c(a_j), 1 \otimes a_{j+1}, \dots, 1 \otimes a_k) \\ &\quad - \sum_{j=1}^{k-1} \mu(a_1 \otimes 1, \dots, a_j \otimes 1, 1 \otimes a_{j+1}, \dots, 1 \otimes a_k). \end{aligned}$$

By definition of the deconcatenation coproduct $\Delta_c(a_j) = 1 \otimes a_j + a_j \otimes 1$. As a consequence, the inductive hypothesis implies

$$\begin{aligned} \Delta_c \mu(a_1, \dots, a_k) &= \sum_{j=0}^k \mu_{(1)}(a_1, \dots, a_j, 1, \dots, 1) \otimes \mu_{(2)}(1, \dots, 1, a_{j+1}, \dots, a_k) \\ &= 1 \otimes \mu(a_1, \dots, a_k) + \mu(a_1, \dots, a_k) \otimes 1 + \\ &\quad \left(\prod_{i=1}^k \mu_{(i)}|_{\{i\}} \right) \sum_{i=1}^k a_1 \dots a_i \otimes a_{i+1} \dots a_k. \end{aligned}$$

The element $\mu(a_1, \dots, a_k) - \left(\prod_{i=1}^k \mu_{(i)}|_{\{i\}} \right) a_1 \otimes \dots \otimes a_k$ is primitive for the deconcatenation and lies in $V^{\otimes k}$, $k > 1$ so is null and the formula is proved.

It remains to prove that $T(V)$ is a \mathcal{P} -algebra. Once again it is enough to prove it for elements in V . Let $\mu \in \mathcal{P}(k), \nu_i \in \mathcal{P}(l_i), 1 \leq i \leq k$. Let

$r = \sum_i l_i, v_j \in V$ and let $\bar{v} = v_1 \otimes \dots \otimes v_r \in V^{\otimes r}$. On the one hand,

$$\mu(\nu_1, \dots, \nu_k)(v_1, \dots, v_r) = \underbrace{\prod_{u=1}^r (\mu(\nu_1, \dots, \nu_k))_{(u)}|_{\{u\}}}_{\alpha} \bar{v},$$

and on the other hand

$$\begin{aligned} \mu(\nu_1(v_1, \dots, v_{l_1}), \dots, \nu_k(v_{l_1+\dots+l_{k-1}+1}, \dots, v_r)) &= \\ \prod_{j=1}^k \prod_{u=1}^{l_j} (\nu_j)_{(u)}|_{\{u\}} \mu(\text{Id}_{l_1}, \dots, \text{Id}_{l_k})(v_1, \dots, v_r) &= \\ \underbrace{\prod_{j=1}^k \prod_{u=1}^{l_j} (\nu_j)_{(u)}|_{\{u\}} \prod_{s=1}^r (\mu(\text{Id}_{l_1}, \dots, \text{Id}_{l_k}))_{(s)}|_{\{s\}}}_{\beta} \bar{v}. \end{aligned}$$

Since \mathcal{P} is connected, using Sweedler's notation one has

$$\sum \mu_{(1)} \otimes \dots \otimes \mu_{(i)}|_{\emptyset} \otimes \dots \otimes \mu_{(k)} = \mu_{(1)} \otimes \dots \otimes \mu_{(k-1)}.$$

So

$$\alpha = \prod_{j=1}^k \prod_{u=1}^{l_j} (\nu_j)_{(u)}|_{\{u\}} \mu_{(l_1+\dots+l_{j-1}+u)}|_{\{j\}} = \beta,$$

and $T(V)$ is a \mathcal{P} -algebra satisfying the three conditions. \square

As a consequence the following theorem holds.

4.3.2. Theorem. *Let \mathcal{P} be a connected regular multiplicative operad. Any connected unital infinitesimal \mathcal{P} -bialgebra $(M, \mu_M, \bar{\Delta})$ is isomorphic as a unital infinitesimal \mathcal{P} -bialgebra to $T^{fc}(\text{Prim}_{\bar{\Delta}} M)$.* \square

5. APPLICATION TO COMBINATORIAL HOPF ALGEBRAS

In this section, we would like to apply our previous results to combinatorial Hopf algebras. The idea is the following: given a graded vector space $H = \bigoplus_n H(n)$, how does a Hopf algebra structure arise on H ? We present two cases coming from the two examples detailed in section 3.2.3.

Case 1. The space $H(n)$ is endowed with a right S_n -action. We denote by $H^{\mathbb{S}}$ the associated \mathbb{S} -module. Assume there exists a connected multiplicative Hopf operad structure on $\mathcal{P}_H = H^{\mathbb{S}}$. From section 3.2.3, we obtain our first result: there exists a \mathcal{P}_H -algebra product μ and a coalgebra product Δ such that $(H^{\mathbb{S}}, \mu, \Delta)$ is a Hopf \mathcal{P}_H -algebra. The graded vector space H has a Hopf \mathcal{P} -algebra structure which is the symmetrized Hopf \mathcal{P} -algebra $\overline{H^{\mathbb{S}}}$ by theorem 3.2.2.

The second result is a direct consequence of theorem 4.2.2: since the operad \mathcal{P}_H is multiplicative, there is a twisted product $m : H^{\mathbb{S}} \otimes H^{\mathbb{S}} \rightarrow H^{\mathbb{S}}$.

As a consequence $(H^{\mathbb{S}}, m, \Delta)$ is a twisted bialgebra. The associated symmetrized Hopf algebra $(H, \hat{m}, \bar{\Delta})$ is cofree and the associated cosymmetrized Hopf algebra $\bar{H} = (H, \bar{m}, \hat{\Delta})$ is free. In case Δ is cocommutative, under the hypothesis of theorem 4.2.2, \bar{H} is the enveloping algebra of the free Lie algebra generated by $\text{Prim}_{\bar{\Delta}}(H)$. Furthermore, by remark 4.2.3 the 2-associative bialgebra $(H, \hat{m}, \bar{m}, \bar{\Delta})$ is the 2-associative enveloping bialgebra of its primitive elements: $\text{Prim}_{\bar{\Delta}}(A)$ is endowed with a B_{∞} -structure.

The **Case 1** applies also when H is a free \mathcal{P} -algebra in **S-mod** generated by a \mathbb{S} -module M , with \mathcal{P} a multiplicative Hopf operad and $M(0) = 0$.

Case 2. Assume $\mathcal{P}_H^r = \mathcal{S}H$ is a connected regular Hopf operad. The graded vector space H is the free graded \mathcal{P}_H^r -algebra generated by the graded vector space I . As a consequence, the graded vector space (H, μ, Δ) is a Hopf \mathcal{P}_H^r -algebra, where μ is the \mathcal{P}_H^r -product and where

$$\Delta(h) = \sum_{S \sqcup T = [|h|]} h_{(1)}|_S \otimes h_{(2)}|_T$$

comes from the regular Hopf operad \mathcal{P}_H^r . Also $(H, \mu, \bar{\Delta})$ with

$$\bar{\Delta}(h) = \sum_{i=0}^{|h|} h_{(1)}|_{[i]} \otimes h_{(2)}|_{i+[|h|-i]}$$

is a unital infinitesimal \mathcal{P}_H^r -bialgebra by theorem 4.1.3. If \mathcal{P}_H^r is multiplicative then $(H, \mu, \bar{\Delta})$ is isomorphic as a unital infinitesimal \mathcal{P}_H^r -bialgebra to $T^{fc}(\text{Prim}_{\bar{\Delta}}(H))$ by theorem 4.3.2.

Again, by remark 4.2.3, if H_n is finite dimensional then $(H^*, {}^t\Delta, {}^t\bar{\Delta}, {}^t m)$ where m is the associative product, is a 2-associative bialgebra, which is the 2-associative enveloping bialgebra of its primitive elements.

The **Case 2** applies also when H is a free \mathcal{P} -algebra in **grVect** generated by a graded vector space V , with \mathcal{P} a Hopf regular operad and $V(0) = 0$.

We illustrate by some examples that many combinatorial Hopf algebras arise either from case 1 or from case 2.

5.1. The Hopf algebra $T(V)$. Let us apply **Case 1** for $H = F_{\text{Com}}(V)$ where V is considered as a \mathbb{S} -module concentrated in degree 1. That is $H = T(V)$. As a twisted bialgebra, $T(V)$ is endowed with the concatenation product and with the following coproduct

$$\Delta(v_1 \otimes \dots \otimes v_k) = \sum_{S \sqcup T = [k]} \bar{v}|_S \otimes \bar{v}|_T \otimes (S, T)$$

where $\bar{v}|_S = v_{s_1} \otimes \dots \otimes v_{s_j}$ for $S = \{s_1 < \dots < s_j\}$. It is cocommutative. The symmetrized Hopf algebra structure on $T(V)$ is the shuffle product together with the deconcatenation, whereas the cosymmetrized Hopf algebra structure on $T(V)$ is the dual structure: the product is the concatenation

and the coproduct is the unshuffle coproduct. In characteristic 0 it is the enveloping algebra of the free Lie algebra generated by V .

5.2. The Malvenuto-Reutenauer Hopf algebra. This Hopf algebra, denoted H_{MR} has been extensively studied in [18], in [9] under the name of free quasisymmetric functions or in [1]. The graded vector space considered is $A = \bigoplus_n \mathbf{k}[S_n]$. It is the underlying graded vector space of the operad $\mathcal{A}s$ and **Case 1** applies. Recall that the operad $\mathcal{A}s$ gives rise to a cocommutative twisted bialgebra:

$$\begin{aligned} m(\sigma, \tau) &= \sigma \times \tau, \\ \Delta(\sigma) &= \sum_{S \sqcup T = [n]} \sigma|_S \otimes \sigma|_T \otimes (S, T). \end{aligned}$$

The Hopf algebra H_{MR} is the symmetrized Hopf algebra $(A, \hat{m}, \bar{\Delta})$. Concretely, for $\sigma \in S_n, \tau \in S_m$

$$\begin{aligned} \hat{m}(\sigma, \tau) &= \sum_{\xi \in \text{Sh}_{p,q}} (\sigma \times \tau) \cdot \xi, \\ \bar{\Delta}(\sigma) &= \sum_{i=0}^n \text{st}(\sigma_1, \dots, \sigma_i) \otimes \text{st}(\sigma_{i+1}, \dots, \sigma_n). \end{aligned}$$

which is not commutative nor cocommutative.

The cosymmetrized Hopf algebra $\hat{A} = (A, \bar{m}, \hat{\Delta})$ is given by

$$\begin{aligned} \bar{m}(\sigma, \tau) &= \sigma \times \tau, \\ \hat{\Delta}(\sigma) &= \sum_{S \sqcup T = [n]} \sigma|_S \otimes \sigma|_T. \end{aligned}$$

The latter Hopf algebra is different from the former one or its dual since it is a cocommutative Hopf algebra.

From **Case 1** we get that H_{MR} is cofree and that \hat{A} is free as an associative algebra: it is generated by the *connected permutations* that is permutations which don't write $\sigma \times \tau$ for $\sigma \in S_n, \tau \in S_m, n, m > 0$. In characteristic 0, \hat{A} is the enveloping algebra of the free Lie algebra generated by the connected permutations (compare with theorems 20 and 21 in [21]). Furthermore, H_{MR} together with \bar{m} is a 2-associative bialgebra and it is the 2-associative enveloping bialgebra generated by the connected permutations: in [9] and in [1] a basis of the space of primitive elements of H_{MR} , indexed by the connected permutations is given.

In paragraph 5.3.4 we prove that H_{MR} is free as an associative algebra, without using the selfduality of H_{MR} .

5.3. Hopf algebra structures on the faces of the permutohedron.

Recall that \mathcal{Com} is a Hopf operad. Let $\overline{\mathcal{Com}}$ denote the kernel of the counit: it coincides with the \mathbb{S} -module \mathcal{Com} except in degree 0 where $\overline{\mathcal{Com}}(0) = 0$.

The \mathbb{S} -module $\mathcal{Comp} = \mathcal{As} \circ \overline{\mathcal{Com}}$ has for linear basis the faces of the n -permutohedra. Indeed

$$\begin{aligned} \mathcal{As} \circ \overline{\mathcal{Com}}(n) &= \bigoplus_{k \geq 0} \mathcal{As}(k) \otimes_{S_k} (\overline{\mathcal{Com}})^{\otimes k}(n) \\ &= \sum_{(I_1, \dots, I_k)=[n]} \mathbf{k} \end{aligned}$$

where the sum is taken over all *set compositions* (or *ordered set partitions*) (I_1, \dots, I_k) of $[n]$ such that $I_j \neq \emptyset, \forall 1 \leq j \leq k$. The action of the symmetric group is given, for $\sigma \in S_n$, by

$$(I_1, \dots, I_k) \cdot \sigma = (\sigma^{-1}(I_1), \dots, \sigma^{-1}(I_k)).$$

Chapoton described Hopf algebra structures on the graded vector space $\mathcal{O}(\mathcal{Comp})$ in [5] and in [6], whereas Patras and Schocker described a twisted bialgebra structure on \mathcal{Comp} in [22]. Chapoton described a (differential graded) operad structure on \mathcal{Comp} in [8] and Loday described a (filtered) one in [14].

The aim of this section is to apply our operadic point of view **Case 1** in order to relate these structures.

A set composition of $[n]$ is written as a word in the alphabet $\{, \} \cup \{i, 1 \leq i \leq n\}$. For instance $(14, 2, 35)$ is the set composition $(\{1, 4\}, \{2\}, \{3, 5\})$ of [5].

5.3.1. *Operad structures on the faces of the permutohedron.* Both operads built by Loday in [14] and Chapoton in [8] are quadratic binary operads. They are generated by the commutative operation represented by the set composition (12) and by the operation represented by the set composition (1, 2) in $\mathcal{Comp}(2)$. Let $w_f = (12) + (1, 2) + (2, 1)$ and $w_g = (1, 2) + (2, 1)$. The composition in the operad \mathcal{CTD} described by Loday is given by the following inductive formula

$$\begin{aligned} (12)(\emptyset, P) &= 0, & (12)(P, \emptyset) &= 0, \\ (1, 2)(\emptyset, P) &= 0, & (1, 2)(P, \emptyset) &= P, \end{aligned}$$

$$\begin{aligned} (12)(P, Q) &= (P_1 \cup Q_1, w_f((P_2, \dots, P_k), (Q_2, \dots, Q_l))), \\ (1, 2)(P, Q) &= (P_1, w_f((P_2, \dots, P_k), Q)), \end{aligned}$$

where $P = (P_1, \dots, P_k)$ is a set composition of $[n]$ and $Q = (Q_1, \dots, Q_l)$ is a set composition of $[m]$ considered as a set composition of $n + [m]$. The degree of the set composition P is $n - k$. Set compositions of degree 0 are in 1-to-1 correspondance with permutations.

The operad \mathcal{CTD} is filtered by the degree of set compositions but not graded. It is not regular and algebras (in the category of vector spaces) over this operad are named *commutative tridendriform algebras* by Loday, that

is vector spaces endowed with a product \prec and a commutative product \cdot satisfying the relations

$$\begin{aligned}(x \prec y) \prec z &= x \prec (y \prec z + z \prec y + y \cdot z), \\ (x \cdot y) \prec z &= x \cdot (y \prec z), \\ (x \cdot y) \cdot z &= x \cdot (y \cdot z).\end{aligned}$$

The composition in the operad Π described by Chapoton has the same definition when using w_f in place of w_g . It is graded by the degree of set compositions. Algebras (in the category of graded vector spaces) over Π are described in [8].

These operads are not connected in the strict sense, since the composition with $\emptyset \in \text{Comp}(0)$ is not always defined. The equalities involving the emptyset above, are needed for an inductive definition and are needed in order to build a coproduct $\text{Comp} \rightarrow \text{Comp} \otimes \text{Comp}$, as was explained in the paragraph 3.2.3 on connected operads.

5.3.2. Proposition. The \mathbb{S} -module Comp is a CTD-Hopf algebra for the coproduct

$$\Delta(P_1, \dots, P_k) = \sum_{l=0}^k \text{st}(P_1, \dots, P_l) \otimes \text{st}(P_{l+1}, \dots, P_k) \otimes (\cup_{1 \leq j \leq l} P_j, \cup_{l+1 \leq h \leq k} P_h).$$

The \mathbb{S} -module Comp is a Π -Hopf algebra for the same coproduct.

Proof. Let \mathcal{X} denotes either the operad CTD or the operad Π . Let w denote either $w_f \in \text{CTD}(2)$ or $w_g \in \Pi(2)$. Note that for any set composition P

$$w(P, \emptyset) = w(\emptyset, P) = P. \quad (13)$$

One needs first to define the \mathcal{X} -algebra structure on $\text{Comp} \otimes \text{Comp}$. This trick is due to Loday and Ronco: for $x \in \mathcal{X}(2)$,

$$\begin{aligned}x(P_1 \otimes P_2, Q_1 \otimes Q_2) &= \\ &\begin{cases} \emptyset \otimes x(P_2, Q_2), & \text{if } P_1 = Q_1 = \emptyset, \\ x(P_1, Q_1) \otimes w(P_2, Q_2) \otimes (P_1 \cup Q_1, P_2 \cup Q_2), & \text{otherwise.} \end{cases} \quad (14)\end{aligned}$$

The coproduct $\Delta : \text{Comp} \rightarrow \text{Comp} \otimes \text{Comp}$ is the unique \mathcal{X} -algebra morphism mapping (1) to (1) \otimes \emptyset + \emptyset \otimes (1).

Let $(I_{l_1}, \dots, I_{l_k})$ be the set composition of $[l_1 + \dots + l_k]$ defined by

$$I_j = l_1 + \dots + l_{j-1} + [l_j], \forall 1 \leq j \leq k.$$

Let $n = l_1 + \dots + l_k$. We prove the formula for such a set composition by induction on k . If $k = 1$, the set composition is just (n) . For $n = 1$ the

formula is proved. If $n > 1$ then $(n) = (12)((1), (n-1))$. By induction one has

$$\begin{aligned} \Delta(n) &= (12)(\Delta(1), \Delta(n-1)) = \\ &= (12)((1) \otimes \emptyset + \emptyset \otimes (1), (n-1) \otimes \emptyset + \emptyset \otimes (n-1)) = (n) \otimes \emptyset + \emptyset \otimes (n), \end{aligned}$$

because $(12)(P, Q) = 0$ if P or Q is empty and because of relation (14) and (13).

If $k > 1$, then $X = (I_{l_1}, \dots, I_{l_k}) = (1, 2)((I_{l_1}), (I_{l_2}, \dots, I_{l_k}))$. By induction

$$\begin{aligned} \Delta(X) &= (1, 2)(\Delta(I_{l_1}), \Delta(I_{l_2}, \dots, I_{l_k})) = \\ &= (1, 2)(I_{l_1} \otimes \emptyset + \emptyset \otimes I_{l_1}, \sum_{j=1}^k (I_{l_2}, \dots, I_{l_j}) \otimes (I_{l_{j+1}}, \dots, I_{l_k})) = \\ &= \emptyset \otimes X + \sum_{j=1}^k (I_{l_1}, I_{l_2}, \dots, I_{l_j}) \otimes (I_{l_{j+1}}, \dots, I_{l_k}), \end{aligned}$$

because $(1, 2)(P, \emptyset) = P$ and $(1, 2)(\emptyset, P) = 0$ and because of relations (14) and (13).

For any set composition $P = (P_1, \dots, P_k)$ of $[n]$, there exists $\sigma \in S_n$ such that

$$(P_1, \dots, P_k) = (I_{l_1}, \dots, I_{l_k}) \cdot \sigma.$$

One can choose for σ the shuffle associated to the set composition P . The coproduct Δ is a morphism of \mathbb{S} -modules and applying the formula (1) we get the result. \square

In [12] we proved that the space of primitive elements with respect to Δ is a suboperad of the initial operad. The space of primitive elements is clearly the vector space generated by the set compositions (n) , for $n > 0$. Then $\text{Prim}_\Delta(\mathcal{CTD})$ is the operad Com (compare with [14]).

5.3.3. Twisted bialgebras associated to the faces of the permutohedron. The operation w_f (resp. w_g) is associative and commutative. As a consequence, the operads \mathcal{CTD} and Π are Hopf multiplicative operads and give rise to twisted connected commutative (non cocommutative) bialgebras $H_f = (\text{Comp}, w_f, \Delta)$ and $H_g = (\text{Comp}, w_g, \Delta)$.

i) The twisted bialgebra H_f . Patras and Schocker [22] defined a twisted bialgebra structure on Comp denoted $\mathcal{T} = (\text{Comp}, *, \delta)$ which is the following. The product $*$ is the concatenation of set compositions and the coproduct δ is defined for a set composition P of $[n]$ by

$$\delta(P) = \sum_{S \sqcup T = [n]} \text{st}(P \cap S) \otimes \text{st}(P \cap T) \otimes (S, T),$$

where $(P_1, \dots, P_k) \cap S = (P_1 \cap S, \dots, P_k \cap S)$ and if $P_i \cap S = \emptyset$ the i -th term is omitted. For instance $(14, 2, 35) \cap \{1, 3, 5\} = (1, 35)$. It is clear that

* is the dual of Δ and one can check that δ is the dual of w_f . It gives an operadic interpretation of their structure:

*The dual of the twisted bialgebra defined by Patras and Schocker is the free commutative tridendriform algebra on one generator in the category **S-mod**.*

Applying **Case 1** one gets that the symmetrized Hopf algebra $\bar{\mathcal{T}}$ associated to \mathcal{T} is cofree, and that the associated cosymmetrized Hopf algebra $\hat{\mathcal{T}}$ is free generated by *reduced set compositions*: a set composition which is non reduced is the concatenation of two non trivial set compositions. For instance $(13, 24, 6, 5)$ is non reduced since it is the concatenation of $(13, 24)$ and $(2, 1)$. Moreover if the field \mathbf{k} is of characteristic 0, then $\hat{\mathcal{T}}$ is the enveloping algebra of the free Lie algebra generated by reduced set compositions. (Compare with proposition 10 and corollary 13 in [22]). Applying remark 4.2.3 one gets that $(\mathcal{T}, \bar{\star}, \hat{\star}, \bar{\Delta})$ is the 2-associative enveloping bialgebra on its primitive elements.

The Hopf algebra structure given by Chapoton in [5] is $(\text{Comp}, \bar{w}_f, \hat{\Delta})$ which is the dual of the symmetrized Hopf algebra $(\mathcal{T}, \hat{\star}, \bar{\delta})$. It is also the Hopf algebra *NCQSym* of Bergeron et Zabrocky in [4] and we recover that it is a free algebra.

ii) The twisted bialgebra H_g . This twisted bialgebra gives rise to two Hopf algebras, which are $(\text{Comp}, \hat{w}_g, \bar{\Delta})$ and $(\text{Comp}, \bar{w}_g, \hat{\Delta})$. One can check that the latter Hopf algebra is the one described by Chapoton in [6]. Again it is a free associative algebra because $(\text{Comp}, \bar{w}_g, \bar{\Delta})$ is a unital infinitesimal bialgebra. The space of primitive elements $\text{Prim}_{\bar{\Delta}}(\text{Comp})$ is generated by reduced set compositions. One can check by an inductive argument, that the Hopf algebra described by Chapoton is a free associative algebra generated by reduced set compositions.

5.3.4. *From set compositions to permutations.* Let Comp_0 be the sub \mathbb{S} -module of Comp of set compositions of degree 0. The vector space $\text{Comp}_0(n)$ is isomorphic to S_n but the right S_n -action is given by $\sigma \cdot \tau = \tau^{-1}\sigma$. The \mathbb{S} -module Comp_0 is a sub-operad of Π , it is, as noticed by Chapoton in [8], the operad \mathcal{Zin} : An algebra (in the category of vector spaces) over \mathcal{Zin} is a Zinbiel algebra that is a vector space Z together with a product \prec satisfying the relation

$$(x \prec y) \prec z = x \prec (y \prec z + z \prec y), \quad \forall x, y, z \in Z.$$

As a consequence there are surjective morphisms of Hopf operads

$$\text{CTD} \rightarrow \mathcal{Zin}, \quad \Pi \rightarrow \mathcal{Zin}.$$

The operad \mathcal{Zin} is consequently a multiplicative operad and Comp_0 is a commutative twisted bialgebra. The product and coproduct are given, for

$\sigma \in \mathcal{Comp}_0(p)$ and $\tau \in \mathcal{Comp}_0(q)$ by

$$m_{\mathcal{Z}}(\sigma, \tau) = \sum_{x \in \text{Sh}_{p,q}} (\sigma \times \tau)x$$

$$\Delta_{\mathcal{Z}}(\sigma) = \sum_{i=0}^p \sigma|_{[i]} \otimes \sigma|_{i+[p-i]} \cdot (\cup_{1 \leq j \leq i} \{\sigma(j)\}, \cup_{i+1 \leq k \leq p} \{\sigma(k)\})$$

The morphisms above induce surjective morphisms of twisted bialgebras

$$H_f \rightarrow \mathcal{Comp}_0, \quad H_g \rightarrow \mathcal{Comp}_0.$$

The cosymmetrized Hopf algebra associated to \mathcal{Comp}_0 is clearly H_{MR} , and since it is a cosymmetrized algebra associated to a twisted bialgebra it is free on $\text{Prim}_{\bar{\Delta}_{\mathcal{Z}}}(\mathcal{Comp}_0)$. But $\bar{\Delta}_{\mathcal{Z}}(\sigma) = \sum_{\rho \times \tau = \sigma} \rho \otimes \tau$. As a consequence, H_{MR} is free generated by the connected permutations and cofree (see section 5.2). We recover the results obtained in e.g. [23], [9] and [1].

Considering the graded linear duals, one has an embedding of cocommutative twisted bialgebras

$$(\mathcal{Comp}_0)^* \hookrightarrow (H_f)^* = \mathcal{T}.$$

The symmetrized Hopf algebra associated to $(\mathcal{Comp}_0)^*$ is the dual of the Malvenuto-Reutenauer Hopf algebra H_{MR}^* (which is isomorphic to H_{MR}). By functoriality in theorem 4.2.2, we obtain in characteristic 0 an embedding of enveloping algebras at the level of associated cosymmetrized algebras (compare with Theorem 17 in [22]).

5.4. Hopf algebra structures on the faces of the associahedron.

In his thesis, Chapoton considers Hopf algebra structures on the faces of the associahedra, or Stasheff polytopes, filtered in [5], graded in [6]. He considered also filtered and graded operad structures on these objects in [7]. The filtered operadic structure coincides with the one defined by Loday and Ronco in [16], under the name of *tridendriform operad*. In this section, we apply **Case 2** to obtain Hopf algebra structures on the faces of the associahedra.

5.4.1. *Planar trees.* The set of planar trees with $n + 1$ vertices is denoted by T_n . The set $T_n = \cup_{k=0}^{n-1} T_{n,k}$ is graded by k where $n - k$ is the number of internal vertices. For instance $T_{n,0}$ is the set of planar binary trees. The Stasheff polytope of dimension $n - 1$ has its faces of dimension $0 \leq k \leq n - 1$ indexed by $T_{n,k}$. The aim of this section is to provide the vector space $\oplus \mathbf{k}[T_n]$ with Hopf structures.

Given some planar trees t_1, \dots, t_k the planar tree $\vee(t_1, \dots, t_k)$ is the one obtained by joining the roots of the trees t_1, \dots, t_k to an extra root, from left to right. If t_i has degree l_i then $\vee(t_1, \dots, t_k)$ has degree $l_1 + \dots + l_k + k - 1$.

One can label the n sectors delimited by a tree t in T_n from left to right as in the following example:

$$t = \begin{array}{c} \begin{array}{ccc} & / & \backslash \\ & \mathbf{1} & \mathbf{2} \\ & \backslash & / \\ & & \mathbf{3} \end{array} \quad \begin{array}{ccc} & / & \backslash \\ & \mathbf{5} & \mathbf{6} \\ & \backslash & / \\ & & \mathbf{4} \end{array} \\ \backslash & & / \\ & & \mathbf{3} \quad \mathbf{4} \\ / & & \backslash \end{array} = \vee(\begin{array}{c} \backslash / \\ | \\ \backslash / \end{array}, \begin{array}{c} \backslash / \\ | \\ \backslash / \end{array}).$$

5.4.2. *The operad of tridendriform algebras.* ([16], [7]) The operad $\mathcal{T}\text{riDend}$ is a regular operad whose underlying \mathbb{S} -module is $\mathcal{T}\text{riDend}(n) = \mathcal{S}\mathbf{k}[T_n]$. It is a quadratic binary operad generated by 3 operations

$$\begin{aligned} \prec &:= \begin{array}{c} \backslash / \\ | \\ \backslash / \end{array}, \\ \succ &:= \begin{array}{c} / \backslash \\ | \\ / \backslash \end{array}, \\ \cdot &:= \begin{array}{c} \backslash / \\ \backslash / \\ / \backslash \end{array}, \end{aligned}$$

satisfying the relations

$$\begin{cases} (x \prec y) \prec z = x \prec (y * z) = \begin{array}{c} \backslash / \\ \backslash / \\ / \backslash \end{array} + \begin{array}{c} \backslash / \\ \backslash / \\ / \backslash \end{array} + \begin{array}{c} \backslash / \\ | \\ \backslash / \end{array}, \\ (x \succ y) \prec z = x \succ (y \prec z) = \begin{array}{c} \backslash / \\ | \\ \backslash / \end{array}, \\ (x * y) \succ z = x \succ (y \succ z) = \begin{array}{c} \backslash / \\ \backslash / \\ / \backslash \end{array} + \begin{array}{c} \backslash / \\ \backslash / \\ / \backslash \end{array} + \begin{array}{c} \backslash / \\ | \\ \backslash / \end{array}, \\ (x \succ y) \cdot z = x \succ (y \cdot z) = \begin{array}{c} \backslash / \\ \backslash / \\ / \backslash \end{array}, \\ (x \prec z) \cdot z = x \cdot (y \succ z) = \begin{array}{c} \backslash / \\ \backslash / \\ / \backslash \end{array}, \\ (x \cdot y) \prec z = x \cdot (y \prec z) = \begin{array}{c} \backslash / \\ \backslash / \\ / \backslash \end{array}, \\ (x \cdot y) \cdot z = x \cdot (y \cdot z) = \begin{array}{c} \backslash / \\ \backslash / \\ / \backslash \\ \backslash / \\ / \backslash \end{array}, \end{cases}$$

where

$$x * y = x \cdot y + x \prec y + x \succ y \quad (15)$$

is associative.

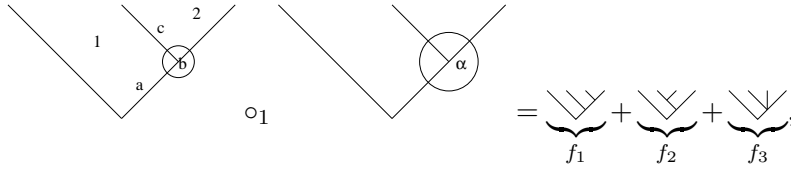
One can also give an inductive formula for the composition in $\mathcal{T}\text{riDend}$ by the following

$$\begin{aligned} \begin{array}{c} \backslash / \\ | \\ \backslash / \end{array} (|, y) &= y, & \begin{array}{c} \backslash / \\ | \\ \backslash / \end{array} (x, |) &= 0, \\ \begin{array}{c} \backslash / \\ | \\ \backslash / \end{array} (|, y) &= 0, & \begin{array}{c} \backslash / \\ | \\ \backslash / \end{array} (x, |) &= 0, \\ \begin{array}{c} \backslash / \\ \backslash / \\ / \backslash \end{array} (|, y) &= 0, & \begin{array}{c} \backslash / \\ \backslash / \\ / \backslash \end{array} (x, |) &= x, \\ \begin{array}{c} \backslash / \\ \backslash / \\ / \backslash \end{array} (\vee(x_1, \dots, x_k), y) &= \vee(x_1, \dots, x_{k-1}, x_k * y), \\ \begin{array}{c} \backslash / \\ \backslash / \\ / \backslash \end{array} (\vee(x_1, \dots, x_k), \vee(y_1, \dots, y_l)) &= \vee(x_1, \dots, x_{k-1}, x_k * y_1, y_2, \dots, y_l), \\ \begin{array}{c} \backslash / \\ \backslash / \\ / \backslash \end{array} (x, \vee(y_1, \dots, y_l)) &= \vee(x * y_1, y_2, \dots, y_l). \end{aligned}$$

In [7], Chapoton describes the composition $x \circ_i y$ for trees x and y :

$$x \circ_i y = \sum_{(f_l, f_r)} x \circ_i^{(f_l, f_r)} y,$$

where y is inserted in the sector i of x following the maps (f_l, f_r) : the left (right) edge of the sector i of x is a set of edges and vertices ordered from bottom to top and denoted by $x_l^i, (x_r^i)$. The left (right) most edge of y has several vertices: the ordered set of these vertices is denoted by $y_l (y_r)$. The map $f_l (f_r)$ is an increasing map from $y_l (y_r)$ to $x_l^i (x_r^i)$. For instance



where

$$\begin{aligned} x_l^1 &= y^l = \emptyset, \\ x_r^1 &= \{a < b < c\}, \quad y_r = \{\alpha\}, \\ f_1 &= (\text{Id}, \alpha \mapsto a), \quad f_2 = (\text{Id}, \alpha \mapsto c) \quad \text{and} \quad f_3 = (\text{Id}, \alpha \mapsto b). \end{aligned}$$

Moreover one can define a Hopf structure on this operad by the same technics as developed in proposition 5.3.2: for $x \in \mathcal{T}riDend(2)$,

$$x(t_1 \otimes t_2, s_1 \otimes s_2) = \begin{cases} | \otimes x(t_2, s_2), & \text{if } t_1 = s_1 = |, \\ x(t_1, s_1) \otimes t_2 * s_2, & \text{otherwise.} \end{cases}$$

By induction on the degree of a tree t one can prove that

5.4.3. *Proposition.* *The \mathbb{S} -module $(\mathbf{k}[T_n] \otimes \mathbf{k}[S_n])_n$ is a tridendriform Hopf algebra for the coproduct*

$$\begin{aligned} \Delta(\vee(t_1, \dots, t_k)) &= | \otimes \vee(t_1, \dots, t_k) + \\ &\sum_{S_i \sqcup T_i = [l_i]} \vee(t_{1(1)}^{S_1}, \dots, t_{k(1)}^{S_k}) \otimes t_{1(2)}^{T_1} * \dots * t_{k(2)}^{T_k} \otimes (S_1 \bar{\times} \dots \bar{\times} S_k, T_1 \bar{\times} \dots \bar{\times} T_k) \end{aligned}$$

where $|t_i| = l_i$,

$$\Delta(t_i) = \sum_{S_i \sqcup T_i = [l_i]} t_{i(1)}^{S_i} \otimes t_{i(2)}^{T_i} \otimes (S_i, T_i)$$

and for any $U_i \subset [l_i]$,

$$U_1 \bar{\times} \dots \bar{\times} U_k = \{l_1 + \dots + l_i + i, 1 \leq i \leq k-1\} \cup_{1 \leq i \leq k} U_i + l_1 + \dots + l_{i-1} + i - 1. \quad \square$$

5.4.4. *Hopf structures.* Let us apply **Case 2** to the connected Hopf operad $\mathcal{T}\text{riDend}$ which is multiplicative with the product $*$ introduced in equation (15).

The graded vector space $\mathcal{T}\text{ree} = \bigoplus_n \mathbf{k}[T_n]$ is the free graded tridendriform algebra over one generator, hence it is a Hopf tridendriform algebra in \mathbf{grVect} . Let μ be the tridendriform product and Δ be the coproduct. By theorem 4.1.3, $(\mathcal{T}\text{ree}, \mu, \bar{\Delta})$ is a unital tridendriform bialgebra and it is isomorphic as a unital tridendriform bialgebra to $T^{fc}(\text{Prim}_{\bar{\Delta}}(\mathcal{T}\text{ree}))$. The description of Δ gives the description of $\bar{\Delta}$: if $\bar{\Delta}(t_k) = \sum t_{k(a)} \otimes t_{k(b)}$, then

$$\bar{\Delta}(\vee(t_1, \dots, t_k)) = |\otimes \vee(t_1, \dots, t_k) + \sum \vee(t_1, \dots, t_{k-1}, t_{k(a)}) \otimes t_{k(b)}.$$

As a consequence a basis of $\text{Prim}_{\bar{\Delta}}(\mathcal{T}\text{ree})$ is given by the planar trees of type $\vee(t_1, \dots, t_{k-1}, |)$.

*The Hopf algebra $(\mathcal{T}\text{ree}, *, \Delta)$ is the free associative algebra spanned by the set of trees of the form $\vee(t_1, \dots, t_{k-1}, |)$.*

Recall that the product $*$ is defined by induction: for $x = \vee(x_1, \dots, x_k)$ and $y = \vee(y_1, \dots, y_l)$

$$x * y = \vee(x_1, \dots, x_{k-1}, x_k * y) + \vee(x * y_1, y_2, \dots, y_l) + \vee(x_1, \dots, x_{k-1}, x_k * y_1, y_2, \dots, y_l).$$

and

$$\Delta(t) = |\otimes t + \sum \vee(t_{1(1)}, \dots, t_{k(1)}) \otimes t_{1(2)} * \dots * t_{k(2)}.$$

The Hopf structure defined by Chapoton in [5] is essentially the same: the product is the same and the coproduct is $\tau\Delta$, where τ is the symmetry isomorphism. Hence it is a free associative algebra spanned by the set of trees of the form $\vee(|, t_2, \dots, t_k)$.

The graded linear dual of $\mathcal{T}\text{ree}$ is a 2-associative bialgebra: it is free as an associative algebra for the product ${}^t\bar{\Delta}$, cofree as a coalgebra for t* and it is the enveloping 2-as bialgebra of its primitive elements (with respect to t*). The product of two trees ${}^t\bar{\Delta}(t, s)$ is the tree obtained by gluing the tree s on the right most leaf of t . it is usually denoted $t \setminus s$. For instance

5.4.5. *Some operad morphisms.* There are morphisms of Hopf operads

$$\begin{array}{ccc} \mathcal{T}\text{riDend} & \xrightarrow{\psi} & \mathcal{CTD} \\ \pi_{\mathcal{T}\text{riDend}} \downarrow & & \downarrow \pi_{\mathcal{CTD}} \\ \mathcal{Dend} & \xrightarrow{\psi_0} & \mathcal{Zin} \end{array} \quad (16)$$

The vertical maps are projection onto cells of degree 0. The map $\pi_{\mathcal{CTD}}$ has been explained in paragraph 5.3.4. The map $\pi_{\mathcal{T}\text{riDend}}$ is the projection onto

the dendriform operad, which is a regular operad generated by planar binary trees (see e.g. [15]). The morphism ψ sends \prec to the set composition $(1, 2)$, \succ to the set composition $(2, 1)$ and \cdot to the set composition (12) . Indeed we can describe ψ at the level of trees. There is a map ϕ from set compositions to trees, described by induction as follows. Let $P = (P_1, \dots, P_k)$ be a set composition of $[n]$. If $P_1 = \{l_1 < \dots < l_j\}$ then it shares $[n]$ into $j + 1$ intervals I_s possibly empty: for $0 \leq s \leq j$, $I_s = [l_s, l_{s+1}]$ with $l_0 = 1$ and $l_{j+1} = n$. The map ϕ is defined by

$$\begin{cases} \phi(\emptyset) = |, \\ \phi(P) = \vee(\phi(P \cap I_0), \dots, \phi(P \cap I_j)). \end{cases}$$

For instance if $P = (34, 1, 56, 2)$ then $I_0 = \{1, 2\}$, $I_1 = \emptyset$ and $I_2 = \{5, 6\}$ and

$$\phi(34, 1, 56, 2) = \vee(\phi(1, 2), \phi(\emptyset), \phi(12)) = \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \quad | \\ \diagdown \quad \diagup \end{array} .$$

Note that the function θ from the permutohedra to associahedra defined by Tonks in [25] (see also [5] and [22]) satisfies

$$\theta(P_1, \dots, P_k) = \phi(P_k, P_{k-1}, \dots, P_2, P_1).$$

The morphism ψ is the transpose of ϕ . It is an operad morphism, whereas the transpose of θ is not an operad morphism.

The morphism ψ_0 is the transpose of ϕ_0 which is an operad morphism. Loday and Ronco defined in [15] a function from $\mathbf{k}[Y_n]$ (the vector space generated by planar binary trees with n vertices) to $\mathbf{k}[S_n]$ in order to embed $\mathbf{k}[Y_n]$ as a Hopf subalgebra of the (graded linear dual of the) Malvenuto-Reutenauer Hopf algebra. It is also a transpose of a set morphism $S_n \rightarrow Y_n$. If $\alpha : S_n \rightarrow S_n$ is the involution defined by

$$\alpha(\sigma_1, \dots, \sigma_n) = (\sigma_n, \dots, \sigma_1)^{-1},$$

then the set morphism defined by Loday and Ronco is $\phi_0 \alpha$.

5.4.6. Consequences on Hopf algebra morphisms.

The operad $\mathcal{T}riDend$ is a regular operad. A *tridendriform 2-bialgebra* is a 4-uple (H, μ, Δ, δ) where (H, μ, Δ) is a Hopf tridendriform bialgebra in \mathbf{grVect} and (H, μ, δ) is a unital infinitesimal tridendriform bialgebra. The diagram (16) is a diagram of tridendriform 2-bialgebras:

- The operad $\mathcal{T}riDend$ induces the tridendriform 2-bialgebra $(\mathcal{T}ree := \bigoplus_n \mathbf{k}[T_n], \mu_T, \Delta_T, \hat{\Delta}_T)$ explained in paragraph 5.4.4.
- Using the surjective operad morphism $\Pi_{\mathcal{T}riDend} : \mathcal{T}riDend \rightarrow \mathcal{D}end$ one gets the tridendriform 2-bialgebra structure on the vector space

spanned by planar binary trees denoted by

$$(PBT := \oplus_n \mathbf{k}[Y_n], \mu_Y, \Delta_Y, \bar{\Delta}_Y),$$

where Y_n is the set of planar binary trees with $n + 1$ leaves as in [15].

- The underlying \mathbb{S} -modules of the operads \mathcal{CTD} and \mathcal{Zin} are tridendriform algebras in $\mathbf{S}\text{-mod}$ then by theorems 3.2.2 and 4.1.2 one gets tridendriform 2-bialgebras on the underlying graded vector spaces. These structures are denoted respectively $(\mathcal{Comp}, \mu_C, \Delta_C, \bar{\Delta}_C)$ and $(\oplus_n \mathbf{k}[S_n], \mu_S, \Delta_S, \bar{\Delta}_S)$.

As a consequence we obtain a diagram of Hopf algebras (and unital infinitesimal bialgebras as well):

$$\begin{array}{ccc} (\oplus_n \mathbf{k}[T_n], *_{T}, \Delta_T) & \xrightarrow{\psi} & (\mathcal{Comp}, *_{C}, \Delta_C) = NCQSym \\ \pi_{\text{TriDend}} \downarrow & & \downarrow \pi_{\mathcal{CTD}} \\ (\oplus_n \mathbf{k}[Y_n], *_{Y}, \Delta_Y) & \xrightarrow{\psi_0} & (\oplus_n \mathbf{k}[S_n], *_{S}, \Delta_S) = H_{MR} \end{array}$$

where the horizontal arrows are injective morphisms of Hopf algebras and vertical arrows are surjective morphisms of Hopf algebras. Note that the Hopf algebra structure on the planar binary tree $(\oplus_n \mathbf{k}[Y_n], *_{Y}, \tau\Delta_Y)$ where τ is the symmetry isomorphism is the one described by Loday and Ronco in [15]. Note also that the graded linear dual of this diagram is a diagram of 2-associative bialgebras.

5.4.7. Conclusion. For the last decade, many results of freeness and cofreeness of combinatorial Hopf algebras have appeared in the literature (see the references cited throughout the paper and recently [2], [10], [20]). The present paper illustrates that these freeness results are a consequence of an operadic structure on the Hopf algebra H itself or its symmetrization $\mathcal{S}H$. Namely, either the Hopf algebra H is a \mathbb{S} -module and one can find an operad structure on H in order to apply **Case 1**; or the Hopf algebra is not a \mathbb{S} -module and one can find an operad structure on $\mathcal{S}H$ in order to apply **Case 2**.

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