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# Description of accessibility sets near an abnormal trajectory and consequences

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**Abstract.** We describe precisely, under generic conditions, the contact of the accessibility set at time  $T$  with an abnormal direction, first for a single-input affine control system with constraint on the control, and then as an application for a sub-Riemannian system of rank 2. As a consequence we obtain in sub-Riemannian geometry a new splitting-up of the sphere near an abnormal minimizer  $\gamma$  into two sectors, bordered by the first Pontryagin's cone along  $\gamma$ , called the  $L^\infty$ -sector and the  $L^2$ -sector. Moreover we find again necessary and sufficient conditions of optimality of an abnormal trajectory for such systems, for any optimization problem.

## 1 Introduction

Consider a smooth control system on  $\mathbb{R}^n$  :

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \quad (1)$$

where  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is smooth,  $x_0 \in \mathbb{R}^n$ , and the set of admissible controls  $\mathcal{U}$  is made of measurable bounded functions  $u : [0, T(u)] \rightarrow \Omega \subset \mathbb{R}^m$ .

**Definition 1.** Let  $T > 0$ . The *end-point mapping* at time  $T$  of system (1) is the mapping

$$E_T : \begin{array}{l} \mathcal{U} \rightarrow \mathbb{R}^n \\ u \mapsto x_u(T) \end{array}$$

where  $x_u$  is the trajectory associated to  $u$ .

The application  $E_T$  is smooth in the  $L^\infty$  topology if  $\mathcal{U} \subset L^\infty([0, T])$ .

**Definition 2.** A control  $u$  (or the corresponding trajectory  $x_u$ ) is said to be *abnormal* on  $[0, T]$  if it is a singular point of the mapping  $E_T$  and if moreover the Hamiltonian of the system  $H = \langle p, f(x, u) \rangle$  is equal to 0 along the trajectory  $x_u$ .

*Remark 1.* If a control  $u$  is abnormal on  $[0, T]$  then it is abnormal on  $[0, t]$  for any  $t \in [0, T]$ .

**Definition 3.** Let  $u$  be an abnormal control on  $[0, T]$ , and  $x_u$  its associated trajectory. The subspace  $\text{Im } dE_t(u)$  is called the *first Pontryagin's cone* at  $x_u(t)$ .

**Definition 4.** Consider the control system (1), and let  $T > 0$ . The *accessibility set at time  $T$* , denoted by  $\text{Acc}(T)$ , is the set of points that can be reached from  $x_0$  in time  $T$  by solutions of system (1), i.e. this is the image of the end-point mapping  $E_T$ .

Let  $\gamma$  be a reference abnormal trajectory on  $[0, T]$ , solution of (1), associated to a control  $u$ . Our aim is to describe  $\text{Acc}(T)$  near  $\gamma(T)$ .

## 2 Asymptotics of the accessibility sets

In this Section we describe precisely the boundary of accessibility sets for a single-input affine system with constraint on the input near a reference abnormal trajectory.

Consider a smooth *single-input affine control system* in  $\mathbb{R}^n$ ,  $n \geq 3$  :

$$\dot{x}(t) = X(x(t)) + u(t)Y(x(t)), \quad x(0) = 0 \quad (2)$$

with the *constraint* on the control

$$|u(t)| \leq \eta \quad (3)$$

Let  $\text{Acc}^\eta(T)$  denote the *accessibility set* at time  $T$  for this affine system with constraint  $\eta$  on the control. Let  $\gamma$  be a reference trajectory defined on  $[0, T]$ . In the sequel we make the following assumptions along  $\gamma$  :

( $H_0$ )  $\gamma$  is injective, associated to  $u = 0$  on  $[0, T]$ .

( $H_1$ )  $\forall t \in [0, T]$   $K(t) = \text{Vect} \{ad^k X.Y(\gamma(t)) / k \in \mathbb{N}\}$  (first Pontryagin's cone along  $\gamma$ ) has codimension 1, and is spanned by the first  $n-1$  vectors, i.e. :

$$K(t) = \text{Vect} \{ad^k X.Y(\gamma(t)) / k = 0 \dots n-2\}$$

( $H_2$ )  $\forall t \in [0, T]$   $ad^2 Y.X(\gamma(t)) \notin K(t)$ .

( $H_3$ )  $\forall t \in [0, T]$   $X(\gamma(t)) \notin \text{Vect} \{ad^k X.Y(\gamma(t)) / k = 0 \dots n-3\}$ .

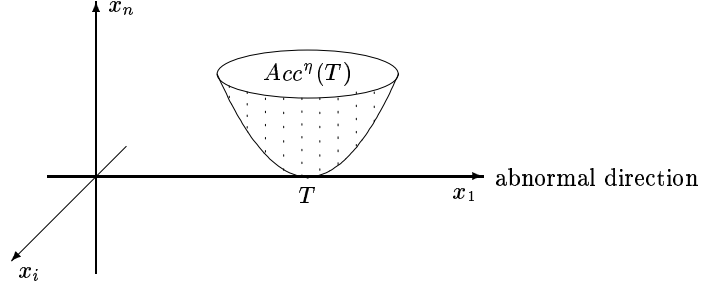
( $H_4$ )  $\forall t \in [0, T]$   $X(\gamma(t)) \in K(t)$ .

In these conditions  $\gamma$  is *abnormal* and its first Pontryagin's cone  $K(t)$  is an hyperplane in  $\mathbb{R}^n$ . Actually assumptions ( $H_1 - H_3$ ) are generic along  $\gamma$ , see [4].

The following Theorem is founded on a very precise spectral analysis of the intrinsic second-order derivative of the end-point mapping along the abnormal direction  $\gamma$  (initialised in [3]), which leads actually to a contact theory of accessibility sets (see [7]).

**Theorem 1.** Consider the affine system (2) with the constraint (3), and suppose that assumptions  $(H_0 - H_4)$  are fulfilled along the reference abnormal trajectory  $\gamma$  on  $[0, T]$ . Then there exist coordinates  $(x_1, \dots, x_n)$  locally along  $\gamma$  such that in these coordinates :

1.  $\gamma(t) = (t, 0, \dots, 0)$ , and the first Pontryagin's cone along  $\gamma$  is :  $K(t) = \text{Vect} \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}} \right\} |_{\gamma}$ .
2. If  $T$  is small enough then for any point  $(x_1, \dots, x_n)$  of  $\text{Acc}^\eta(T) \setminus \{\gamma(T)\}$  close to  $\gamma(T)$  we have :  $x_n > 0$  (see Fig. 1).



**Fig. 1.** Shape of  $\text{Acc}^\eta(T)$ ,  $T$  small

3. There exist two times  $t_{cc}, t_c$  such that  $0 < t_{cc} < t_c$ , called conjugate times or bifurcation times along  $\gamma$ , and such that the following holds.  
If  $T < t_c$ , then in the plane  $(x_1, x_n)$ , near the point  $(T, 0)$ , the boundary of  $\text{Acc}^\eta(T)$  does not depend on  $\eta$ , is a curve of class  $C^2$  tangent to the abnormal direction, and its first term is :

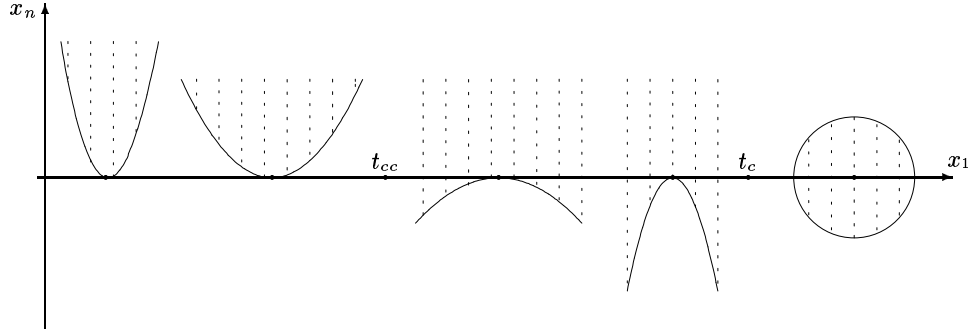
$$x_n = A_T(x_1 - T)^2 + o((x_1 - T)^2)$$

The function  $T \mapsto A_T$  is continuous and strictly decreasing on  $[0, t_c[$ . It is positive on  $[0, t_{cc}[$  and negative on  $]t_{cc}, t_c[$ .

4. If  $T > t_c$  then  $\text{Acc}^\eta(T)$  is open near  $\gamma(T)$ .

The evolution in function of  $T$  of the intersection of  $\text{Acc}^\eta(T)$  with the plane  $(x_1, x_n)$  is represented on Fig. 2. The contact with the abnormal direction is of order 2 ; the coefficient  $A_T$  describes the *concavity* of the curve. Beyond  $t_c$  the accessibility set is open.

*Remark 2.* The coefficient  $A_T$  can be explicitly computed. It is an invariant of the system, see [7]. Moreover the bifurcation times  $t_{cc}$  and  $t_c$  can be computed using an algorithm, see [3].



**Fig. 2.** Evolution of  $Acc^n(T)$  in function of  $T$

### 3 Applications

#### 3.1 Application to the optimality status of an abnormal trajectory

In this Section we apply our previous theory on accessibility sets to studying optimality of abnormal trajectories ; this leads us to find again and to improve slightly some well-known results. Consider the single-input affine system (2) with constraint (3), and suppose assumptions  $(H_0 - H_4)$  are fulfilled along a reference abnormal trajectory  $\gamma$ . We first investigate the time-optimal problem, and then the problem of minimizing any cost.

**Time optimality** The trajectory  $\gamma$  is said  $C^0$ -time-minimal on  $[0, T]$  if there exists a  $C^0$ -neighborhood of  $\gamma$  such that  $T$  is the minimal time to steer  $\gamma(0)$  to  $\gamma(T)$  among the solutions of the system (2) with the constraint (3) that are entirely contained in this neighborhood.

We have the following result (compare with [1–3]) :

**Theorem 2.** *Under assumptions of Theorem 1, the trajectory  $\gamma$  is  $C^0$ -time-minimal if and only if  $T < t_{cc}$ .*

**Optimization of any cost** Let us now consider the problem of minimizing some cost  $C(T, u)$ , also denoted by  $C_T(u)$ , where  $C$  is a smooth function satisfying the following additional assumption along the reference abnormal trajectory  $\gamma$  :

$$(H_5) \quad \forall T \quad \text{rank} (dE_T(0), dC_T(0)) = n$$

We distinguish between two optimization problems.

**1. Final time not fixed** The trajectory  $\gamma$  is said to be  $C^0$ -cost-minimal on  $[0, T]$  if there exists a  $C^0$ -neighborhood of  $\gamma$  such that for any trajectory  $q$  contained in this neighborhood, with  $q(0) = \gamma(0)$  and  $q(T) = \gamma(T)$ , we have :  $C(t, v) \geq C(T, 0)$ , where  $v$  is the control associated to  $q$ .

We have the following (compare with [2]) :

**Theorem 3.** *Under assumptions  $(H_0 - H_5)$ , the trajectory  $\gamma$  is  $C^0$ -cost-minimal if and only if it is  $C^0$ -time-minimal, i.e. if and only if  $T < t_{cc}$ .*

**2. Final time fixed** The trajectory  $\gamma$  is said to be  $C^0$ -cost-minimal on  $[0, T]$  if there exists a  $C^0$ -neighborhood of  $\gamma$  such that for any trajectory  $q$  contained in this neighborhood, with  $q(0) = \gamma(0)$  and  $q(T) = \gamma(T)$ , we have :  $C_T(v) \geq C_T(0)$ , where  $v$  is the control associated to  $q$ .

**Theorem 4.** *The trajectory  $\gamma$  is  $C^0$ -cost-minimal if and only if  $T < t_c$ .*

Hence in this case the time-optimal problem is not equivalent to the problem of minimizing some cost. The trajectory  $\gamma$  ceases to be  $C^0$ -time-optimal before it ceases to be  $C^0$ -cost-optimal (since  $t_{cc} < t_c$ ).

### 3.2 Application to the sub-Riemannian case

Consider a smooth sub-Riemannian structure  $(M, \Delta, g)$  where  $M$  is a Riemannian  $n$ -dimensional manifold,  $n \geq 3$ ,  $\Delta$  is a rank 2 distribution on  $M$ , and  $g$  is a metric on  $\Delta$ . Let  $x_0 \in M$  ; our point of view is local and we can assume that  $M = \mathbb{R}^n$  and  $x_0 = 0$ . Suppose there exists a smooth injective abnormal trajectory  $\gamma$  passing through 0. Up to changing coordinates and reparametrizing we can assume that :

- $\gamma(t) = (t, 0, \dots, 0)$ ,
- $\Delta = \text{Span} \{X, Y\}$  where  $X, Y$  are  $g$ -orthonormal,
- $\gamma$  is the integral curve of  $X$  passing through 0.

Under these assumptions, the sub-Riemannian problem is equivalent to the *time-optimal problem* for the system :

$$\dot{x} = vX(x) + uY(x), \quad x(0) = 0 \quad (4)$$

where the controls  $v, u$  satisfy the *constraint* :

$$v^2 + u^2 \leq 1 \quad (5)$$

The reference abnormal trajectory  $\gamma$  corresponds to the control :  $v = 1, u = 0$ .

The following result is a consequence of Theorem 1 and of a general statement proved in [6].

**Theorem 5.** Consider the sub-Riemannian problem for the system  $\dot{x} = vX(x) + uY(x)$ . Let  $\gamma$  be an abnormal reference trajectory. Suppose assumptions  $(H_0 - H_4)$  hold along  $\gamma$ . Then there exist coordinates  $(x_1, \dots, x_n)$  locally along  $\gamma$  in which, if  $T$  is small enough :

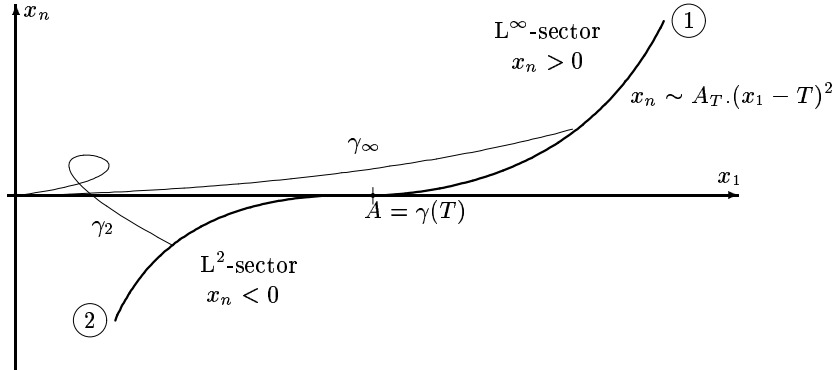
- $\gamma(t) = (t, 0, \dots, 0)$ ,
- the first Pontryagin's cone along  $\gamma$  is  $K_\gamma(t) = (x_n = 0)$ ,
- The sub-Riemannian sphere  $S(0, T)$  splits into two sectors near  $\gamma(T)$  :
  1. the  $L^\infty$ -sector :  $(x_n > 0) \cap S(0, T)$ , made of end-points of minimizing trajectories associated to controls which are close to the abnormal reference control in  $L^\infty$ -topology. Hence minimizing trajectories steering 0 to these points are close to  $\gamma$  in  $C^1$ -topology. Moreover in the plane  $(x_1, x_n)$ , its graph is :

$$x_1 \geq T, \quad x_n \sim A_T \cdot (x_1 - T)^2$$

where  $T \mapsto A_T$  is continuous, positive and decreasing.

2. the  $L^2$ -sector :  $(x_n < 0) \cap S(0, T)$ , made of end-points associated to minimizing controls which are close to the abnormal reference control in  $L^2$ -topology, but not in  $L^\infty$ -topology. Hence trajectories steering 0 to these points are close to  $\gamma$  in  $C^0$ -topology, but not in  $C^1$ -topology. This sector is tangent to the abnormal direction.

These two sectors are separated by the first Pontryagin's cone  $x_n = 0$  along  $\gamma$  (see Fig. 3).



**Fig. 3.**

*Remark 3.* The abnormal trajectory  $\gamma$  is optimal for the sub-Riemannian problem if and only if  $T < t_{cc}$ .

**Typical example : the Martinet case.**

Consider the two following vector fields in  $\mathbb{R}^3$  :

$$X = \frac{\partial}{\partial x} + \frac{y^2}{2} \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}$$

and endow the distribution spanned by these vector fields with an analytic metric  $g$  of the type :

$$g = adx^2 + cdy^2$$

where  $a = (1+\alpha y)^2$  and  $c = (1+\beta x+\gamma y)^2$ . The abnormal reference control for the sub-Riemannian system  $\dot{x} = vX(x) + uY(x)$  with constraint  $v^2 + u^2 \leq 1$  is  $v = 1, u = 0$ , and corresponds to the trajectory  $\gamma : x(t) = t, y(t) = z(t) = 0$ . We have, see [5] :

**Lemma 1.** *Assumptions ( $H_0 - H_4$ ) are fulfilled along  $\gamma$  if and only if  $\alpha \neq 0$ . In this case branches 1 and 2 (see Fig. 3 with  $x_1 = x, x_n = z$ ) have the following contacts with the abnormal direction :*

- branch 1 :  $x \geq T, z = \frac{1}{2T\alpha^2}(x - T)^2 + o((x - T)^2)$
- branch 2 :  $x \leq T, z \sim \frac{1}{6}(1 + O(T))(x - T)^3$

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