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Description of accessibility sets near an abnormal trajectory and consequences

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Abstract. We describe precisely, under generic conditions, the contact of the accessibility set at time T with an abnormal direction, first for a single-input affine control system with constraint on the control, and then as an application for a sub-Riemannian system of rank 2. As a consequence we obtain in sub-Riemannian geometry a new splitting-up of the sphere near an abnormal minimizer γ into two sectors, bordered by the first Pontryagin's cone along γ , called the L^∞ -sector and the L^2 -sector. Moreover we find again necessary and sufficient conditions of optimality of an abnormal trajectory for such systems, for any optimization problem.

1 Introduction

Consider a smooth control system on \mathbb{R}^n :

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \quad (1)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is smooth, $x_0 \in \mathbb{R}^n$, and the set of admissible controls \mathcal{U} is made of measurable bounded functions $u : [0, T(u)] \rightarrow \Omega \subset \mathbb{R}^m$.

Definition 1. Let $T > 0$. The *end-point mapping* at time T of system (1) is the mapping

$$E_T : \begin{array}{l} \mathcal{U} \rightarrow \mathbb{R}^n \\ u \mapsto x_u(T) \end{array}$$

where x_u is the trajectory associated to u .

The application E_T is smooth in the L^∞ topology if $\mathcal{U} \subset L^\infty([0, T])$.

Definition 2. A control u (or the corresponding trajectory x_u) is said to be *abnormal* on $[0, T]$ if it is a singular point of the mapping E_T and if moreover the Hamiltonian of the system $H = \langle p, f(x, u) \rangle$ is equal to 0 along the trajectory x_u .

Remark 1. If a control u is abnormal on $[0, T]$ then it is abnormal on $[0, t]$ for any $t \in [0, T]$.

Definition 3. Let u be an abnormal control on $[0, T]$, and x_u its associated trajectory. The subspace $\text{Im } dE_t(u)$ is called the *first Pontryagin's cone* at $x_u(t)$.

Definition 4. Consider the control system (1), and let $T > 0$. The *accessibility set at time T* , denoted by $\text{Acc}(T)$, is the set of points that can be reached from x_0 in time T by solutions of system (1), i.e. this is the image of the end-point mapping E_T .

Let γ be a reference abnormal trajectory on $[0, T]$, solution of (1), associated to a control u . Our aim is to describe $\text{Acc}(T)$ near $\gamma(T)$.

2 Asymptotics of the accessibility sets

In this Section we describe precisely the boundary of accessibility sets for a single-input affine system with constraint on the input near a reference abnormal trajectory.

Consider a smooth *single-input affine control system* in \mathbb{R}^n , $n \geq 3$:

$$\dot{x}(t) = X(x(t)) + u(t)Y(x(t)), \quad x(0) = 0 \quad (2)$$

with the *constraint* on the control

$$|u(t)| \leq \eta \quad (3)$$

Let $\text{Acc}^\eta(T)$ denote the *accessibility set* at time T for this affine system with constraint η on the control. Let γ be a reference trajectory defined on $[0, T]$. In the sequel we make the following assumptions along γ :

(H_0) γ is injective, associated to $u = 0$ on $[0, T]$.

(H_1) $\forall t \in [0, T]$ $K(t) = \text{Vect} \{ad^k X.Y(\gamma(t)) / k \in \mathbb{N}\}$ (first Pontryagin's cone along γ) has codimension 1, and is spanned by the first $n-1$ vectors, i.e. :

$$K(t) = \text{Vect} \{ad^k X.Y(\gamma(t)) / k = 0 \dots n-2\}$$

(H_2) $\forall t \in [0, T]$ $ad^2 Y.X(\gamma(t)) \notin K(t)$.

(H_3) $\forall t \in [0, T]$ $X(\gamma(t)) \notin \text{Vect} \{ad^k X.Y(\gamma(t)) / k = 0 \dots n-3\}$.

(H_4) $\forall t \in [0, T]$ $X(\gamma(t)) \in K(t)$.

In these conditions γ is *abnormal* and its first Pontryagin's cone $K(t)$ is an hyperplane in \mathbb{R}^n . Actually assumptions ($H_1 - H_3$) are generic along γ , see [4].

The following Theorem is founded on a very precise spectral analysis of the intrinsic second-order derivative of the end-point mapping along the abnormal direction γ (initialised in [3]), which leads actually to a contact theory of accessibility sets (see [7]).

Theorem 1. Consider the affine system (2) with the constraint (3), and suppose that assumptions $(H_0 - H_4)$ are fulfilled along the reference abnormal trajectory γ on $[0, T]$. Then there exist coordinates (x_1, \dots, x_n) locally along γ such that in these coordinates :

1. $\gamma(t) = (t, 0, \dots, 0)$, and the first Pontryagin's cone along γ is : $K(t) = \text{Vect} \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}} \right\} |_{\gamma}$.
2. If T is small enough then for any point (x_1, \dots, x_n) of $\text{Acc}^\eta(T) \setminus \{\gamma(T)\}$ close to $\gamma(T)$ we have : $x_n > 0$ (see Fig. 1).

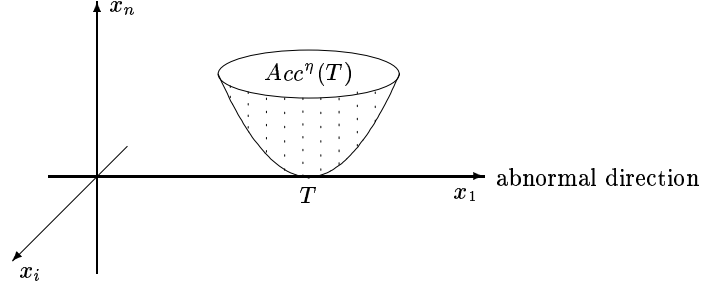


Fig. 1. Shape of $\text{Acc}^\eta(T)$, T small

3. There exist two times t_{cc}, t_c such that $0 < t_{cc} < t_c$, called conjugate times or bifurcation times along γ , and such that the following holds.
If $T < t_c$, then in the plane (x_1, x_n) , near the point $(T, 0)$, the boundary of $\text{Acc}^\eta(T)$ does not depend on η , is a curve of class C^2 tangent to the abnormal direction, and its first term is :

$$x_n = A_T(x_1 - T)^2 + o((x_1 - T)^2)$$

The function $T \mapsto A_T$ is continuous and strictly decreasing on $[0, t_c[$. It is positive on $[0, t_{cc}[$ and negative on $]t_{cc}, t_c[$.

4. If $T > t_c$ then $\text{Acc}^\eta(T)$ is open near $\gamma(T)$.

The evolution in function of T of the intersection of $\text{Acc}^\eta(T)$ with the plane (x_1, x_n) is represented on Fig. 2. The contact with the abnormal direction is of order 2 ; the coefficient A_T describes the *concavity* of the curve. Beyond t_c the accessibility set is open.

Remark 2. The coefficient A_T can be explicitly computed. It is an invariant of the system, see [7]. Moreover the bifurcation times t_{cc} and t_c can be computed using an algorithm, see [3].

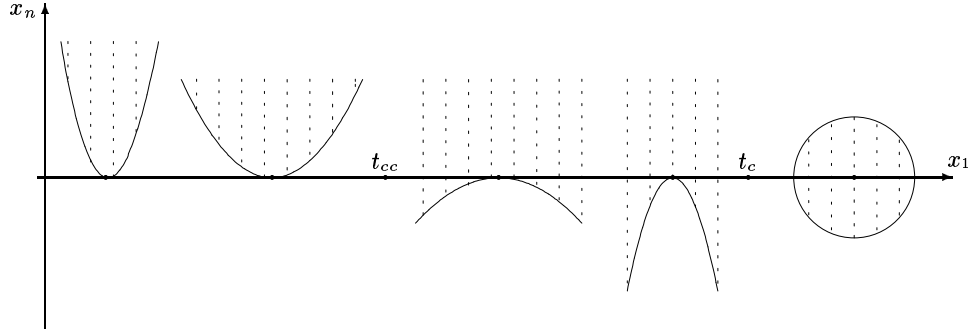


Fig. 2. Evolution of $Acc^n(T)$ in function of T

3 Applications

3.1 Application to the optimality status of an abnormal trajectory

In this Section we apply our previous theory on accessibility sets to studying optimality of abnormal trajectories ; this leads us to find again and to improve slightly some well-known results. Consider the single-input affine system (2) with constraint (3), and suppose assumptions $(H_0 - H_4)$ are fulfilled along a reference abnormal trajectory γ . We first investigate the time-optimal problem, and then the problem of minimizing any cost.

Time optimality The trajectory γ is said C^0 -time-minimal on $[0, T]$ if there exists a C^0 -neighborhood of γ such that T is the minimal time to steer $\gamma(0)$ to $\gamma(T)$ among the solutions of the system (2) with the constraint (3) that are entirely contained in this neighborhood.

We have the following result (compare with [1–3]) :

Theorem 2. *Under assumptions of Theorem 1, the trajectory γ is C^0 -time-minimal if and only if $T < t_{cc}$.*

Optimization of any cost Let us now consider the problem of minimizing some cost $C(T, u)$, also denoted by $C_T(u)$, where C is a smooth function satisfying the following additional assumption along the reference abnormal trajectory γ :

$$(H_5) \quad \forall T \quad \text{rank} (dE_T(0), dC_T(0)) = n$$

We distinguish between two optimization problems.

1. Final time not fixed The trajectory γ is said to be C^0 -cost-minimal on $[0, T]$ if there exists a C^0 -neighborhood of γ such that for any trajectory q contained in this neighborhood, with $q(0) = \gamma(0)$ and $q(T) = \gamma(T)$, we have : $C(t, v) \geq C(T, 0)$, where v is the control associated to q .

We have the following (compare with [2]) :

Theorem 3. *Under assumptions $(H_0 - H_5)$, the trajectory γ is C^0 -cost-minimal if and only if it is C^0 -time-minimal, i.e. if and only if $T < t_{cc}$.*

2. Final time fixed The trajectory γ is said to be C^0 -cost-minimal on $[0, T]$ if there exists a C^0 -neighborhood of γ such that for any trajectory q contained in this neighborhood, with $q(0) = \gamma(0)$ and $q(T) = \gamma(T)$, we have : $C_T(v) \geq C_T(0)$, where v is the control associated to q .

Theorem 4. *The trajectory γ is C^0 -cost-minimal if and only if $T < t_c$.*

Hence in this case the time-optimal problem is not equivalent to the problem of minimizing some cost. The trajectory γ ceases to be C^0 -time-optimal before it ceases to be C^0 -cost-optimal (since $t_{cc} < t_c$).

3.2 Application to the sub-Riemannian case

Consider a smooth sub-Riemannian structure (M, Δ, g) where M is a Riemannian n -dimensional manifold, $n \geq 3$, Δ is a rank 2 distribution on M , and g is a metric on Δ . Let $x_0 \in M$; our point of view is local and we can assume that $M = \mathbb{R}^n$ and $x_0 = 0$. Suppose there exists a smooth injective abnormal trajectory γ passing through 0. Up to changing coordinates and reparametrizing we can assume that :

- $\gamma(t) = (t, 0, \dots, 0)$,
- $\Delta = \text{Span} \{X, Y\}$ where X, Y are g -orthonormal,
- γ is the integral curve of X passing through 0.

Under these assumptions, the sub-Riemannian problem is equivalent to the *time-optimal problem* for the system :

$$\dot{x} = vX(x) + uY(x), \quad x(0) = 0 \quad (4)$$

where the controls v, u satisfy the *constraint* :

$$v^2 + u^2 \leq 1 \quad (5)$$

The reference abnormal trajectory γ corresponds to the control : $v = 1, u = 0$.

The following result is a consequence of Theorem 1 and of a general statement proved in [6].

Theorem 5. Consider the sub-Riemannian problem for the system $\dot{x} = vX(x) + uY(x)$. Let γ be an abnormal reference trajectory. Suppose assumptions $(H_0 - H_4)$ hold along γ . Then there exist coordinates (x_1, \dots, x_n) locally along γ in which, if T is small enough :

- $\gamma(t) = (t, 0, \dots, 0)$,
- the first Pontryagin's cone along γ is $K_\gamma(t) = (x_n = 0)$,
- The sub-Riemannian sphere $S(0, T)$ splits into two sectors near $\gamma(T)$:
 1. the L^∞ -sector : $(x_n > 0) \cap S(0, T)$, made of end-points of minimizing trajectories associated to controls which are close to the abnormal reference control in L^∞ -topology. Hence minimizing trajectories steering 0 to these points are close to γ in C^1 -topology. Moreover in the plane (x_1, x_n) , its graph is :

$$x_1 \geq T, \quad x_n \sim A_T \cdot (x_1 - T)^2$$

where $T \mapsto A_T$ is continuous, positive and decreasing.

2. the L^2 -sector : $(x_n < 0) \cap S(0, T)$, made of end-points associated to minimizing controls which are close to the abnormal reference control in L^2 -topology, but not in L^∞ -topology. Hence trajectories steering 0 to these points are close to γ in C^0 -topology, but not in C^1 -topology. This sector is tangent to the abnormal direction.

These two sectors are separated by the first Pontryagin's cone $x_n = 0$ along γ (see Fig. 3).

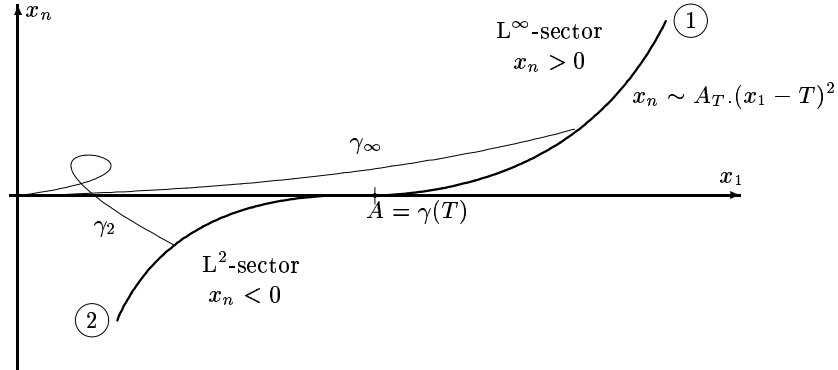


Fig. 3.

Remark 3. The abnormal trajectory γ is optimal for the sub-Riemannian problem if and only if $T < t_{cc}$.

Typical example : the Martinet case.

Consider the two following vector fields in \mathbb{R}^3 :

$$X = \frac{\partial}{\partial x} + \frac{y^2}{2} \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}$$

and endow the distribution spanned by these vector fields with an analytic metric g of the type :

$$g = adx^2 + cdy^2$$

where $a = (1+\alpha y)^2$ and $c = (1+\beta x+\gamma y)^2$. The abnormal reference control for the sub-Riemannian system $\dot{x} = vX(x) + uY(x)$ with constraint $v^2 + u^2 \leq 1$ is $v = 1, u = 0$, and corresponds to the trajectory $\gamma : x(t) = t, y(t) = z(t) = 0$. We have, see [5] :

Lemma 1. *Assumptions ($H_0 - H_4$) are fulfilled along γ if and only if $\alpha \neq 0$. In this case branches 1 and 2 (see Fig. 3 with $x_1 = x, x_n = z$) have the following contacts with the abnormal direction :*

- branch 1 : $x \geq T, z = \frac{1}{2T\alpha^2}(x - T)^2 + o((x - T)^2)$
- branch 2 : $x \leq T, z \sim \frac{1}{6}(1 + O(T))(x - T)^3$

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