Optimal control of a space shuttle and numerical simulations

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OPTIMAL CONTROL OF A SPACE SHUTTLE, AND NUMERICAL SIMULATIONS

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Abstract. We study the Earth re-entry problem of a space shuttle where the control is the angle of bank, the cost is the total amount of thermal flux, and the system is subject to state constraints on the thermal flux, the normal acceleration and the dynamic pressure. The optimal solution is approximated by a concatenation of bang and boundary arcs, and is numerically computed using a multiple-shooting code.

1. Introduction and presentation of the atmospheric arc problem.

1.1. **General meaning.** The aim is to present numerical simulations of the optimal trajectory of a space shuttle during its re-entry phase in the atmosphere. This problem was studied and solved in [3, 4] in which a geometric framework was introduced in order to solve this optimal control problem.

This project was set out by the CNES, and is motivated by the fact that it is important to control these aero-capture techniques in order to apply them to:

- problems of guidance of aeroassisted orbital transfers,
- development of reprocessable satellite launchers (this is an important financial stake),
- problems of re-entry in the atmosphere: this is the subject of the famous project *Mars Sample Return developped by the CNES.*

Roughly speaking, the role of the atmospheric arc is:

- to reduce sufficiently the kinetic energy by friction with the atmosphere,
- to steer the spacecraft from a precise initial point (position and speed) to a prescribed target,
- moreover we have to take into account some state-constraints on the thermal flux, on the normal acceleration and on the dynamic pressure,
- finally we aim to minimize an optimization criterion: the total thermal flux of the spacecraft.

Our control is the aerodynamic configuration of the shuttle, and we first examine the following question: can the aerodynamic forces be employed in order to adequately slow down the spacecraft ? Actually if the altitude is too large, more than about 120 km, then it is physically impossible to generate aerodynamic forces sufficiently intense to this task, because the air density is too small. At the contrary if the altitude is too small, less than about 20 km, the air density is too large and this would lead to violate the constraints on the thermal flux or the dynamic pressure.

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Indeed the traversing of the atmosphere is done at very high velocities. Anyway in the range where the altitude is between 20 km and 120 km a compromise may be achieved. So our goal is to analyse existing solutions of the problem, and to try to improve them using optimal control theory.

In the atmospheric phase the shuttle behaves as a *glider*, that is, there is no *thrust*. Hence it is only subject to the force of gravity and to the aerodynamic force. Our control is the bank angle which represents, roughly speaking, the angle between the wings and a plane containing the shuttle. Finally, the optimization criterion we try to minimize is the total thermal flux of the spacecraft.

More precisely the model is as follows.

1.2. **The model.** Let 0 be the center of the planet, $K = NS$ is the axis of rotation, Ω is the angular velocity. We denote by $E = (e_1, e_2, e_3)$ with $e_3 = K$, an inertial frame with center 0. The reference frame is the quasi-inertial frame $R_1 = (I, J, K)$ with origin 0, rotating around K, with angular speed Ω and I is chosen to intersect the Greenwich meridian. Let R be the radius of the planet, G the center of mass of the shuttle. We note (r, l, L) the spherical coordinates of G, $r \geq R$ being the distance OG, $h = r - R$ is the altitude, l is the longitude and L is the latitude. We note R_1', a moving frame with center G, where e_r is the local vertical,(e_l, e_r) is the local horizontal plane and e_L is pointing to the north. The spherical coordinates have a singularity at the poles.

Let $\xi : t \mapsto (x(t), y(t), z(t))$ be the trajectory of G measured in the quasi-inertial frame attached to the planet and let $\overrightarrow{v} = \dot{x}I + \dot{y}J + \dot{z}K$ be the relative velocity. The vector $\overrightarrow{v}$ is represented by its modulus v and two angles:

- γ: *path inclination* which is the angle with respect to the horizontal plane,
- χ: *azimuth angle* which is the angle of the projection of $\overrightarrow{v}$ in the horizontal plane measured with respect to the axis e_L.

We denote by (i, j, k) the orthonormal frame defined by $i = \overrightarrow{v}/v$, j is the unitary vector in the plane (i, e_r) perpendicular to i and oriented by $j.e_r > 0$ and $k = i \wedge j$.

The system is written in the coordinates $q = (r, v, \gamma, L, l, \chi)$. The forces acting on the vehicle are the gravitational force $\overrightarrow{P} = m\overrightarrow{g}$ and the aerodynamic force which decomposes into

- a drag force: $\overrightarrow{D} = \left(\frac{1}{2}\rho S C_D v^2\right) i$ opposite to $\overrightarrow{v}$,
- a lift force: $\overrightarrow{L} = \left(\frac{1}{2}\rho S C_L v^2\right) (j \cos\mu + k \sin\mu)$ perpendicular to $\overrightarrow{v}$,

where μ is the angle of bank, $\rho = \rho(r)$ is the air density, S is the reference area and C_D, C_L are respectively the lift and drag coefficients depending on the angle of attack α of the vehicle and the Mach number. The coefficients C_D and C_L are tabulated and their values are given in Section 2. For the air density we take an exponential model: $\rho(r) = \rho_0 \exp(-(r - r_T)/h_s)$. Due to the choice of a non inertial frame to represent the system, the spacecraft is submitted to the Coriolis force $2m\overrightarrow{\Omega} \wedge \dot{q}$ and to the centripetal force $m\overrightarrow{\Omega} \wedge (\overrightarrow{\Omega} \wedge q)$. The control is the angle of bank $\mu(t) \in \mathbb{R}$.

From [18] and [9] the equations of the system are:

$$\begin{aligned}
\frac{dr}{dt} &= v \sin\gamma \\
\frac{dv}{dt} &= -g \sin\gamma - \frac{1}{2}\rho \frac{SC_D}{m} v^2 + \Omega^2 r \cos L(\sin\gamma \cos L - \cos\gamma \sin L \cos\chi)
\end{aligned}$$

$$
\begin{aligned}
\frac{d\gamma}{dt} &= \cos\gamma\left(-\frac{g}{v}+\frac{v}{r}\right)+\frac{1}{2}\rho\frac{SC_L}{m}v\cos\mu \\
&\quad + 2\Omega\cos L\sin\chi+\Omega^2\frac{r}{v}\cos L(\cos\gamma\cos L+\sin\gamma\sin L\cos\chi) \\
\frac{dL}{dt} &= \frac{v}{r}\cos\gamma\cos\chi \\
\frac{dl}{dt} &= \frac{v}{r}\frac{\cos\gamma\sin\chi}{\cos L} \\
\frac{d\chi}{dt} &= \frac{1}{2}\rho\frac{SC_L}{m}\frac{v}{\cos\gamma}\sin\mu+\frac{v}{r}\cos\gamma\tan L\sin\chi+2\Omega(\sin L-\tan\gamma\cos L\cos\chi) \\
&\quad + \Omega^2\frac{r}{v}\frac{\sin L\cos L\sin\chi}{\cos\gamma}
\end{aligned}
\tag{1}
$$

where μ is the *bank angle* (it is our control), S is the reference area and C_L, C_D are respectively the lift and drag coefficients depending upon the *angle of attack* α and the Mach number. They are tabuled by the CNES and their values are given in the appendix. The air density is ρ and we take an exponential model: $\rho(r)=\rho_0 e^{-\beta r}$.

1.3. **Optimal control.** The problem is to steer the vehicle from an initial manifold M_0 to a terminal manifold M_1. The terminal time t_f is free and the boundary conditions are given in Table 1.

	Initial conditions	Terminal conditions
altitude (h)	119.82 km	15 km
velocity (v)	7404.95 m/s	445 m/s
path inclination (γ)	-1.84 deg	free
latitude (L)	0	10.99 deg
longitude (l)	free or fixed to 116.59 deg	166.48 deg
azimuth (χ)	free	free

TABLE 1. Boundary conditions

The state constraints are of the form $c_i(q)\le 0$, for $i=1,2,3$ and are:

- constraint on the *thermal flux*:
$$\varphi = C_q\sqrt{\rho}v^3 \le \varphi^{max}\ ,$$
where C_q is a constant,
- constraint on the *normal acceleration*
$$\gamma_n = \gamma_{n_0}\rho v^2 \le \gamma_n^{max}\ ,$$
- constraint on the *dynamic pressure*
$$\frac{1}{2}\rho v^2 \le P^{max}\ .$$

They are represented on Fig. 1 in the flight domain, in terms of the drag $d=\frac{1}{2}\frac{SC_D}{m}\rho v^2$ and v.

The optimal control problem is to minimize the total amount of the thermal flux:

$$J(\mu)=\int_0^{t_f} c_q\sqrt{\rho}v^3 dt \tag{2}$$

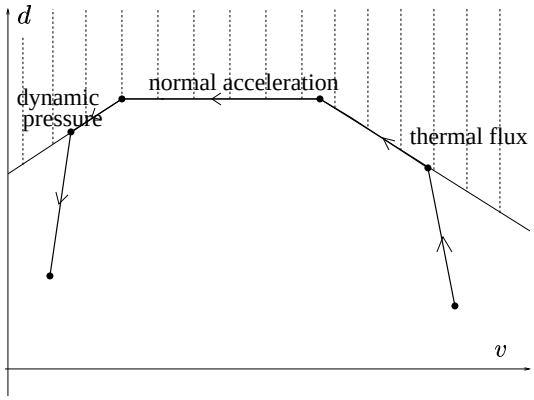

FIGURE 1. Constraints, and Harpold/Graves strategy

1.4. **Harpold and Graves strategy** [12]**.** If we use the approximation $\dot{v} \simeq -d$, the cost can be written:

$$J(\mu) = K \int_{v_0}^{v_f} \frac{v^2}{\sqrt{d}} dv,\ K > 0$$

and the optimal strategy is to maximize during the flight the drag d. This is the policy described in [12], which reduces the problem to find a system trajectory tracking the boundary of the domain in the following order: thermal flux $\rightarrow$ normal acceleration $\rightarrow$ dynamic pressure, see Fig. 1.

The advantage of this method is that along the boundary arcs the control may be expressed in a *closed-loop form*, i.e. it is a feedback control, in function of the state. This form is well-adapted to a real-time system governing the movement of the spacecraft.

Anyway this method is *not optimal* for our optimization criterion.

2. Results and numerical simulations.

2.1. **Quasi-optimal strategy.** We recall the following result, which was proved in [4] by a careful geometric analysis of the extremals given by a maximum principle with state constraints.

Proposition 1. *We distinguish between two problems, see Table 1:*

1. *If the initial longitude is not fixed, the optimal strategy is approximated by a policy $\gamma_-\gamma_+\gamma_{flux}\gamma_+\gamma_{acc}\gamma_+$, where γ_+ (resp. γ_-) is an arc corresponding to the control $\mu = 0$ (resp. $\mu = \pi$), and γ_{flux} (resp. γ_{acc}) denotes a boundary arc corresponding to the constraint on the thermal flux (resp. on the normal acceleration).*
2. *If the initial longitude is fixed, the optimal strategy is approximated by a policy $\gamma_-\gamma_+\gamma_{flux}\gamma_+\gamma_-$.*

2.2. **Multiple shooting algorithm.** This numerical algorithm is standard, and a classical reference is [21]. We briefly recall a description of this method. Consider a general optimal control problem, where the state $x \in \mathbb{R}^n$ may be subject to state constraints. It is well known that the maximum principle reduces the problem to a

boundary value problem of the following type:

$$\dot{z}(t) = F(z(t),t) = \begin{cases} F_0(t,z(t)) & \text{if } t_0 \le t < t_1 \\ F_1(t,z(t)) & \text{if } t_1 \le t < t_2 \\ \vdots & \\ F_s(t,z(t)) & \text{if } t_s \le t \le t_f \end{cases} \tag{3}$$

where $z = (q,p) \in \mathbb{R}^{2n}$ (p is the adjoint vector) and $t_1, t_2, \ldots, t_s \in [t_0, t_f]$ may be *switching times*, *junction times*, i.e. times at which the trajectory joins a boundary arc, or *contact times*, i.e. times at which the trajectory only touches the boundary. Moreover at these points there hold *continuity conditions* on the state and costate at switching points, or *continuity conditions* on the state, *jump conditions* on the costate, and *conditions on the constraint* c, at junction and contact points, see [15, 17, 7, 18, 3, 4]. We further have *boundary conditions* on the state, on the costate, and on the Hamiltonian if the final time is not fixed.

Remark 1. A priori the final time t_f is unknown. On the other part in the multi-shooting method *the number s of switchings has to be fixed* and must be deduced from a geometric analysis of the problem.

The multi-shooting method consists in subdividing the interval $[t_0, t_f]$ in N subintervals, where the value of $z(t)$ at the beginning of each subinterval is unknown. More precisely, let $t_0 < \sigma_1 < \cdots < \sigma_k < t_f$ be a *fixed* subdivision of the interval $[t_0, t_f]$. At each point σ_j the function z is *continuous*. We can consider σ_j as a fixed switching point at which the following conditions hold:

$$\begin{cases} z(\sigma_j^+) = z(\sigma_j^-), \\ \sigma_j = \sigma_j^* \quad \text{fixed.} \end{cases}$$

Now introduce the *nodes* :

$$\{x_1, \ldots, x_m\} \;=\; \{t_0, t_f\} \cup \{\sigma_1, \ldots, \sigma_k\} \cup \{t_1, \ldots, t_s\}. \tag{4}$$

We arrive at the following *boundary value problem*:

$$\begin{aligned} &\bullet\ \dot{z}(t) = F(t,z(t)) = \begin{cases} F_1(t,z(t)) & \text{if } x_1 \le t < x_2 \\ F_2(t,z(t)) & \text{if } x_2 \le t < x_3 \\ \vdots & \\ F_{m-1}(t,z(t)) & \text{if } x_{m-1} \le t \le x_m \end{cases} \\ &\bullet\ \forall j \in \{2, \ldots, m-1\} \quad r_j(x_j, z(x_j^-), z(x_j^+)) = 0 \\ &\bullet\ r_m(x_m, z(x_1), z(x_m)) = 0 \end{aligned} \tag{5}$$

where $x_1 = t_0$ is fixed, $x_m = t_f$, and the r_j's represent boundary or interior conditions as explained above.

Remark 2. The stability of the method can be improved by increasing the number of nodes. Indeed the principle of the method is to overcome the unstability of a simple shooting method where the influence of inaccurate initial data can grow exponentially with the length t_f-t_0, see [21].

Set $z_j^+ = z(x_j^+)$, and let $z(t, x_{j-1}, z_{j-1}^+)$ denote the solution of the Cauchy problem:

$$\dot{z}(t) = F(t,z(t)),\ z(x_{j-1}) = z_{j-1}^+.$$

We have:

$$z(x_j^-) = z(x_j^-, x_{j-1}, z_{j-1}^+).$$

The interior and boundary conditions can be rewritten as:

$$\begin{aligned} \forall j \in \{2, \dots, m-1\} \quad r_j(x_j, z(x_j^-, x_{j-1}, z_{j-1}^+), z_j^+) = 0, \\ r_m(x_m, z_1^+, z(x_m^-, x_{m-1}, z_{m-1}^+)) = 0. \end{aligned} \tag{6}$$

Now set:

$$Z = (z_1^+, x_m, z_2^+, x_2, \dots, z_{m-1}^+, x_{m-1})^T \in \mathbb{R}^{(2n+1)(m-1)}$$

(where $z \in \mathbb{R}^{2n}$). Then the previous conditions (6) hold if:

$$G(Z) = \begin{pmatrix} r_m(x_m, z_1^+, z(x_m^-, x_{m-1}, z_{m-1}^+)) \\ r_2(x_2, z(x_2^-, x_1, z_1^+), z_2^+) \\ \vdots \\ r_{m-1}(x_m, z(x_{m-1}^-, x_{m-2}, z_{m-2}^+), z_{m-1}^+) \end{pmatrix} = 0. \tag{7}$$

The problem is now reduced to *determine a zero* of the function G which is defined on a space vector whose dimension is proportional to the number of switching points and points of the subdivision. The equation $G = 0$ can be solved iteratively using a Newton type method. We refer to [8, 13] for more details on numerical methods.

Having reduced the problem to determine a quasi-optimal policy among a concatenation of bang and boundary arcs, either $\gamma_- \gamma_+^T \gamma_{flux} \gamma_+^T \gamma_-$ or $\gamma_- \gamma_+^T \gamma_{flux} \gamma_+^B \gamma_{acc} \gamma_+^T$, the length of the bang or boundary arcs are computed using the multiple-shooting algorithm, see [21, 8, 13]. The algorithm is written in Fortran and simulations were lead with Matlab[1].

Our reduction procedure avoids implementing the complete extremal equations with the adjoint vector and this is essential in order to improve the convergence of the algorithm. The results are the following, we distinguish between both problems.

2.3. **Problem 1: initial longitude not fixed.** Switching times and initial values of longitude and azimuth are computed using the multiple-shooting method. More precisely:

- The first switching time, from γ_- to γ_+, allows to adjust the entry in the iso-flux phase, which is characterized by $\varphi = \varphi^{\max}, \dot{\varphi} = 0$.
- The third switching time, from γ_{flux} to γ_+, is used to adjust the entry in the iso-normal acceleration phase.
- The fifth switching time, from γ_{acc} to γ_+, permits to adjust the final velocity $v(t_f)$.
- The initial azimuth $\chi(0)$ is used to adjust the terminal latitude $L(t_f)$.

On the other part the final time is determined by the final altitude.

Results are drawn on Fig. 2, 3.

2.4. **Problem 2: initial longitude fixed.** Switching times and the initial value of azimuth are computed by the multi-shooting method. More precisely:

- The first switching time, from γ_- to γ_+, allows to adjust the entry in the iso-flux phase.
- The third switching time, from γ_{flux} to γ_+, permits to adjust the final velocity $v(t_f)$.

[1]The fortran code was developped for academic research by Pr. Hiltmann from Munchen University, and is available on his web page, at http://www-m2.ma.tum.de/Software/mumus.en.html

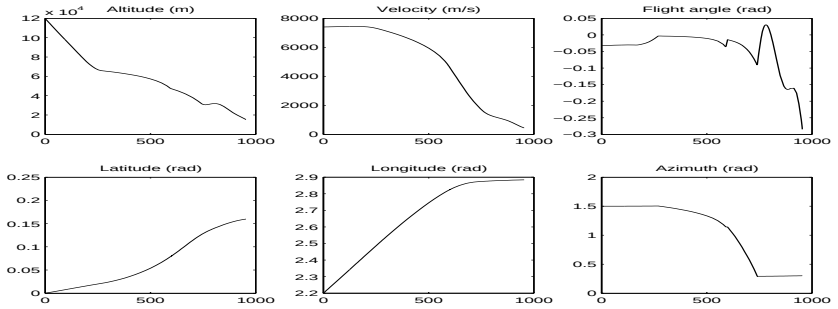

FIGURE 2. State coordinates in problem 1.

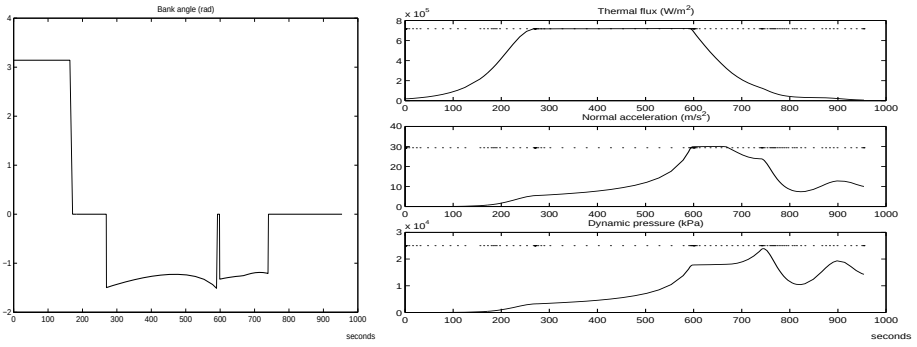

FIGURE 3. Bank angle, and state constraints in problem 1.

- The fourth switching time, from γ_+ to γ_-, is used to adjust the final longitude $l(t_f)$.
- The initial azimuth $\chi(0)$ allows to adjust the terminal latitude $L(t_f)$.

Results are drawn on Fig. 4, 5.

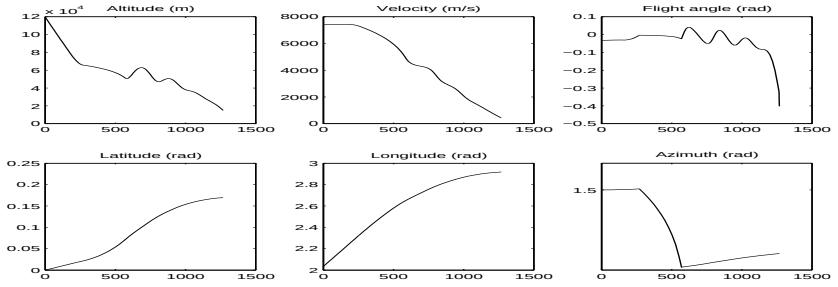

FIGURE 4. State coordinates in problem 2.

2.5. **Appendix: numerical data.**

- General data

  Earth radius: $r_T = 6378139$ m.

  Earth rotation velocity: $\Omega = 7.292115853608596.10^{-5}$ rad.s^{-1}.

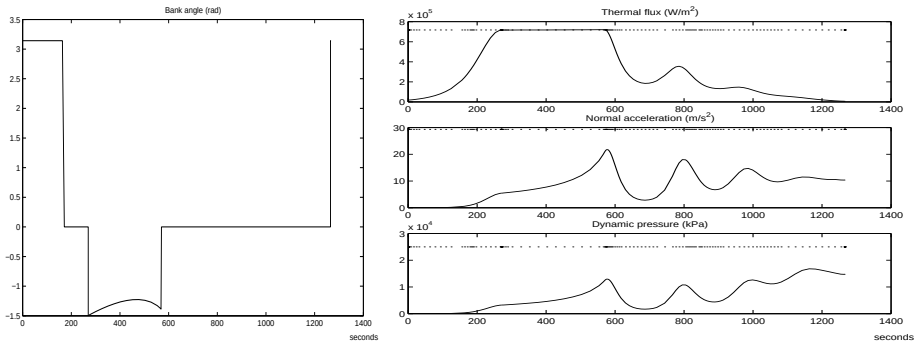

FIGURE 5. Bank angle and state constraints in problem 2.

Gravity model: $g(r) = \dfrac{g_0}{r^2}$ with $g_0 = 3.9800047.10^{14}$ m^3.s^{-2}.

- Atmospheric density model: $\rho(r) = \rho_0 \exp\big(-\dfrac{1}{h_s}(r - r_T)\big)$ with $\rho_0 = 1.225$ kg.m^{-3} and $h_s = 7143$ m.
- Sound velocity model: $v_{\rm son}(r) = \displaystyle\sum_{i=0}^{5} a_i r^i$, with

$$a_5 = -1.880235969632294.10^{-22}, \quad a_4 = 6.074073670669046.10^{-15},$$
$$a_3 = -7.848681398343154.10^{-8}, \quad a_2 = 5.070751841994340.10^{-1},$$
$$a_1 = -1.637974278710277.10^{6}, \quad a_0 = 2.116366606415128.10^{12}.$$

- Mach number: $Mach(v, r) = v/v_{\rm son}(r)$.
- Shuttle data

  Mass: $m = 7169.602$ kg.
  Reference surface: $S = 15.05$ m^2.
  Drag coefficient: $k = \dfrac{1}{2}\dfrac{SC_D}{m}$.
  Lift coefficient: $k' = \dfrac{1}{2}\dfrac{SC_L}{m}$.
- Aerodynamic coefficients:

Table of $C_D(Mach, incidence)$

	0.00	10.00	15.00	20.00	25.00	30.00	35.00	40.00	45.00	50.00	55.00 deg
0.00	0.231	0.231	0.269	0.326	0.404	0.500	0.613	0.738	0.868	0.994	1.245
2.00	0.231	0.231	0.269	0.326	0.404	0.500	0.613	0.738	0.868	0.994	1.245
2.30	0.199	0.199	0.236	0.292	0.366	0.458	0.566	0.688	0.818	0.948	1.220
2.96	0.159	0.159	0.195	0.248	0.318	0.405	0.509	0.628	0.757	0.892	1.019
3.95	0.133	0.133	0.169	0.220	0.288	0.373	0.475	0.592	0.721	0.857	0.990
4.62	0.125	0.125	0.160	0.211	0.279	0.363	0.465	0.581	0.710	0.846	0.981
10.00	0.105	0.105	0.148	0.200	0.269	0.355	0.458	0.576	0.704	0.838	0.968
20.00	0.101	0.101	0.144	0.205	0.275	0.363	0.467	0.586	0.714	0.846	0.970
30.00	0.101	0.101	0.144	0.208	0.278	0.367	0.472	0.591	0.719	0.849	0.972
50.00	0.101	0.101	0.144	0.208	0.278	0.367	0.472	0.591	0.719	0.849	0.972
Mach											

Table of $C_L(Mach, incidence)$

0.00	0.00	10.00	15.00	20.00	25.00	30.00	35.00	40.00	45.00	50.00	55.00 deg
0.00	0.000	0.185	0.291	0.394	0.491	0.578	0.649	0.700	0.729	0.734	0.756
2.00	0.000	0.185	0.291	0.394	0.491	0.578	0.649	0.700	0.729	0.734	0.756
2.30	0.000	0.172	0.269	0.363	0.454	0.535	0.604	0.657	0.689	0.698	0.723
2.96	0.000	0.154	0.238	0.322	0.404	0.481	0.549	0.603	0.639	0.655	0.649
3.95	0.000	0.139	0.215	0.292	0.370	0.445	0.513	0.569	0.609	0.628	0.626
4.62	0.000	0.133	0.206	0.281	0.358	0.433	0.502	0.559	0.600	0.620	0.618
10.00	0.000	0.103	0.184	0.259	0.337	0.414	0.487	0.547	0.591	0.612	0.609
20.00	0.000	0.091	0.172	0.257	0.336	0.416	0.490	0.552	0.596	0.616	0.612
30.00	0.000	0.087	0.169	0.258	0.338	0.418	0.493	0.555	0.598	0.619	0.613
50.00	0.000	0.087	0.169	0.258	0.338	0.418	0.493	0.555	0.598	0.619	0.613
Mach											

- Incidence profile imposed: if the Mach number is larger than 10 then the incidence is set to 40. If the Mach number is between 2 and 10 then the incidence is a linear function of the Mach number between the values 12 and 40. If the Mach number is less than 2 then the incidence is set to 12.
- State constraints:

  Constraint on the thermal flux: $\varphi = C_q\sqrt{\rho}v^3 \ \leq \ \varphi^{\max}$, where

  $$C_q = 1.705.10^{-4} \text{ S.I.} \quad \text{and} \quad \varphi^{\max} = 717300 \text{ W.m}^{-2}.$$

  Constraint on the normal acceleration:

  $$\gamma_n = \frac{S}{2m}\rho v^2 C_D\sqrt{1+\left(\frac{C_L}{C_D}\right)^2} \ \leq \ \gamma_n^{\max} = 29.34 \text{ m.s}^{-2}$$

  Constraint on the dynamic pressure: $P = \frac{1}{2}\rho v^2 \ \leq \ P^{\max} = 25000$ kPa.
- Initial and terminal conditions: see Table 1.

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