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Non subanalyticity of sub-Riemannian Martinet spheres

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Abstract. Consider the sub-Riemannian Martinet structure (M, Δ, g) where $M = \mathbb{R}^3$, $\Delta = \text{Ker}(dz - \frac{y^2}{2}dx)$ and g is the general gradated metric of order 0 : $g = (1 + \alpha y)^2 dx^2 + (1 + \beta x + \gamma y)^2 dy^2$. We prove that if $\alpha \neq 0$ then the sub-Riemannian spheres $S(0, r)$ with small radii are not subanalytic. © Académie des Sciences/Elsevier, Paris

Non sous-analyticité des sphères sous-Riemanniennes de Martinet

Résumé. *Considérons la structure sous-Riemannienne de Martinet (M, Δ, g) où $M = \mathbb{R}^3$, $\Delta = \text{Ker}(dz - \frac{y^2}{2}dx)$ et g est la métrique générale graduée d'ordre 0 : $g = (1 + \alpha y)^2 dx^2 + (1 + \beta x + \gamma y)^2 dy^2$. On montre que si $\alpha \neq 0$, les sphères sous-Riemanniennes $S(0, r)$ de petit rayon ne sont pas sous-analytiques.* © Académie des Sciences/Elsevier, Paris

Version française abrégée

Une *structure sous-Riemannienne* (M, Δ, g) est constituée d'une variété M de dimension finie, d'une distribution Δ sur M , et d'une métrique g sur Δ . Elle est dite analytique si tous les objets sont analytiques. Une courbe γ sur M est dite *horizontale* si elle est tangente en tout point à la distribution Δ . Soient x_0, x_1 des points de M , et $T > 0$. Le *problème sous-Riemannien* est de déterminer une courbe horizontale joignant x_0 à x_1 en temps T et minimisant la longueur (au sens de la métrique g). La *distance sous-Riemannienne* $d_{SR}(x_0, x_1)$ entre x_0 et x_1 est la borne inférieure des longueurs des chemins horizontaux joignant x_0 à x_1 en temps T . La *sphère sous-Riemannienne* $S(x_0, r)$ centrée en x_0 , de rayon r , est l'ensemble des points x de M tels que $d_{SR}(x_0, x) = r$.

Reformulons le problème sous-Riemannien dans le cadre de la théorie du contrôle optimal. Notre point de vue étant local, on peut supposer que $M = \mathbb{R}^n$ et $x_0 = 0$. Supposons que la distribution Δ est engendrée par m champs de vecteurs X_1, \dots, X_m . Sans perte de généralité, on peut supposer que $T = 1$. Le problème sous-Riemannien est alors équivalent au problème de *contrôle optimal* pour le système (voir par exemple [5]) :

$$\dot{x}(t) = \sum_{i=1}^m u_i(t) X_i(x(t)), \quad x(0) = 0$$

Note présentée par ???

où la fonction $u = (u_1, \dots, u_m)$, appelée le *contrôle*, doit minimiser la *longueur*

$$l(u) = \int_0^1 \sqrt{g(\dot{x}(t), \dot{x}(t))} dt$$

L'application $E : u \in L^2([0, 1], \mathbb{R}^m) \rightarrow x_u(1) \in \mathbb{R}^n$, qui à un contrôle u associe l'extrémité de la solution correspondante x_u , est appelée *application entrée/sortie*. C'est une application lisse.

Si un contrôle u minimise la longueur l , alors il existe un *multiplieur de Lagrange* $(\psi, \psi^0) \in \mathbb{R}^n \times \mathbb{R}$ défini à scalaire près tel que

$$\psi \cdot dE(u) = -\psi^0 dl(u)$$

Si $\psi^0 \neq 0$, le contrôle (ou la trajectoire associée) est dit *normal* ; sinon il est dit *anormal*. Un contrôle anormal est une singularité de l'application entrée/sortie.

Dans [1] il est prouvé que s'il n'existe pas de trajectoire anormale minimisante non triviale partant de 0 alors la distance sous-Riemannienne à 0, $d_{SR}(0, \cdot)$, est sous-analytique en dehors de 0, et donc les sphères sous-Riemanniennes $S(0, r)$ de petit rayon sont sous-analytiques. Cette situation est générique pour des distributions de rang ≥ 3 , voir [3]. Cependant les anormales minimisantes apparaissent génériquement pour des distributions de rang 2, et on conjecture que dans ce cas les sphères sous-Riemanniennes ne sont jamais sous-analytiques. Remarquons que dans [3] les auteurs prouvent que l'existence d'une anormale minimisante est responsable de la non sous-analyticité du germe en 0 de $d_{SR}(0, \cdot)$. Ici la conjecture concerne les sphères sous-Riemanniennes.

Cette conjecture a été prouvée dans [2] dans le cas dit de *Martinet plat*, mais il n'est pas générique. Ici on montre que cette conjecture est vraie pour une métrique générale graduée d'ordre 0. Le cas Martinet est le cas sous-Riemannien le plus simple pour lequel une anormale minimisante apparaît génériquement. La situation est la suivante. Considérons dans \mathbb{R}^3 la distribution $\Delta = \text{Ker}(dz - \frac{y^2}{2} dx)$; elle est engendrée par les deux champs de vecteurs

$$X = \frac{\partial}{\partial x} + \frac{y^2}{2} \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}$$

Les variables x, y sont de poids 1, et z est de poids 3 (voir [5]). Soit g une métrique analytique sur Δ ; d'après [2] une forme normale générale graduée d'ordre 0 est :

$$g = (1 + \alpha y)^2 dx^2 + (1 + \beta x + \gamma y)^2 dy^2$$

où $\alpha, \beta, \gamma \in \mathbb{R}$. C'est une perturbation du cas plat où $\alpha = \beta = \gamma = 0$. L'unique direction anormale correspond à l'axe x ; elle est minimisante [4].

THÉORÈME 1. – *Si $\alpha \neq 0$ alors les sphères sous-Riemanniennes $S(0, r)$ de petit rayon ne sont pas sous-analytiques.*

1. Introduction

1.1. Sub-Riemannian geometry

A *sub-Riemannian structure* (M, Δ, g) is a triple, where M is a finite dimensional manifold, Δ is a distribution on M , and g is a metric on Δ . It is said analytic if all objects are analytic. A curve γ on M is called *horizontal* if it is tangent at any point to the distribution Δ . Let x_0, x_1 in M ,

and $T > 0$. The *sub-Riemannian problem* is to determine an horizontal curve joining x_0 and x_1 in time T and minimizing the length (in the sense of the metric g). The *sub-Riemannian distance* $d_{SR}(x_0, x_1)$ between x_0 and x_1 is the infimum of lengths of horizontal paths joining x_0 and x_1 in time T . The *sub-Riemannian sphere* $S(x_0, r)$ centered at x_0 , with radius r , is the set of points x in M such that $d_{SR}(x_0, x) = r$.

1.2. Necessary optimality conditions

In order to determine optimality conditions for an horizontal curve, we shall reformulate the sub-Riemannian problem in optimal control theory. Our point of view is local, hence we can suppose that $M = \mathbb{R}^n$ et $x_0 = 0$. Assume that the distribution Δ is spanned by m vector fields X_1, \dots, X_m . Without loss of generality, we may suppose that $T = 1$. Then the sub-Riemannian problem is equivalent to the *optimal control problem* for the system (see for instance [5]) :

$$\dot{x}(t) = \sum_{i=1}^m u_i(t)X_i(x(t)), \quad x(0) = 0$$

where the function $u = (u_1, \dots, u_m)$, called the *control*, has to minimize the *length*

$$l(u) = \int_0^1 \sqrt{g(\dot{x}(t), \dot{x}(t))} dt$$

The mapping $E : u \in L^2([0, 1], \mathbb{R}^m) \rightarrow x_u(1) \in \mathbb{R}^n$, which to a control u associates the extremity of the corresponding solution x_u , is called *end-point mapping*. It is smooth.

If a control u minimizes the length l , then there exists a *Lagrange multiplier* $(\psi, \psi^0) \in \mathbb{R}^n \times \mathbb{R}$ defined up to a scalar such that

$$\psi \cdot dE(u) = -\psi^0 dl(u)$$

Moreover from the *Pontryagin's Maximum Principle* (see [9]) the trajectory x corresponding to this minimizing control u is the projection of an *extremal*, i.e. a solution of the Hamiltonian system :

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, u), \quad \dot{p} = -\frac{\partial H}{\partial x}(x, p, u), \quad \frac{\partial H}{\partial u}(x, p, u) = 0$$

where $H(x, p, u) = \langle p, \sum_{i=1}^m u_i X_i(x) \rangle + p^0 g(\dot{x}(t), \dot{x}(t))$ is the Hamiltonian function, p is an absolutely continuous function on $[0, 1]$ called *adjoint vector*, and p^0 is a constant. Moreover we have, up to a scalar : $(p(1), p^0) = (\psi, \psi^0)$. If $p^0 \neq 0$, the extremal is said to be *normal*, and we can normalize to $p^0 = -\frac{1}{2}$; otherwise it is said *abnormal*. An abnormal control is a singularity of the end-point mapping. An abnormal trajectory is said to be *strict* if it is not the projection of a normal extremal.

1.3. Abnormal minimizers and subanalyticity

In [1] it is proved that if there exists no (non trivial) abnormal minimizer starting from 0 then the sub-Riemannian distance to 0, $d_{SR}(0, \cdot)$, is subanalytic outside 0, and hence sub-Riemannian spheres $S(0, r)$ with small radii are subanalytic. This happens generically for distributions of rank ≥ 3 , see [3]. However abnormal minimizers may appear generically for rank 2 distributions, and we conjecture that in this case sub-Riemannian spheres are never subanalytic. Note that in [3]

authors prove that the existence of an abnormal minimizer is responsible for non subanalyticity of the germ at 0 of $d_{SR}(0, \cdot)$. Here the conjecture concerns sub-Riemannian spheres.

This conjecture was proved in [2] in the so-called *Martinet flat case*, but this case is not generic. Here we prove that this conjecture is true for a general gradated metric of order 0. The Martinet case is the easier sub-Riemannian case where an abnormal minimizer appears generically. The situation is the following. Consider in \mathbb{R}^3 the distribution $\Delta = \text{Ker}(dz - \frac{y^2}{2}dx)$; it is spanned by the two vector fields

$$X = \frac{\partial}{\partial x} + \frac{y^2}{2} \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}$$

Variables x, y have weight 1, and z has weight 3 (see [5]). Let g be an analytic metric on Δ ; from [2] a general gradated normal form of order 0 is :

$$g = adx^2 + cdy^2$$

where $a = (1 + \alpha y)^2$, $c = (1 + \beta x + \gamma y)^2$, and α, β, γ are real parameters. It is a perturbation of the flat case where $\alpha = \beta = \gamma = 0$. There exists an unique abnormal direction, corresponding to the axis x ; it is minimizing, see [4]. Moreover it is *strict* if and only if $\alpha \neq 0$ (see [2]).

THEOREM 1. – *If $\alpha \neq 0$ (i.e. in the strict case), the sub-Riemannian spheres $S(0, r)$ with small radii are not subanalytic.*

The main idea of the proof is the following. The interest of the previous metric of order 0 is all in the fact that the problem, formulated in the cotangent space of \mathbb{R}^3 , projects onto the phase plane of a *one-parameter pendulum*. This fact was already used in [2], [6]. In [2], in order to prove that the sphere in the Martinet flat case is not subanalytic, the authors make a direct computation of the sphere. But this is impossible in the general case because equations of extremals given by the Maximum Principle *are no more integrable*. Actually such computations lead to asymptotic expansions of the sphere near the abnormal direction (see [6]), but these expansions are not precise enough to check non subanalyticity. The new idea here is rather *indirect* and consists in using Lagrange multipliers : then proving the theorem amounts to estimating precisely a *first return time near a saddle point* of this one-parameter pendulum and proving that it is not subanalytic.

2. Proof of Theorem 1

Consider for the system $\dot{q} = uX(q) + vY(q)$, $q(0) = 0$, the abnormal control $u = -r, v = 0$. It is always minimizing, see [4]. In order to prove the theorem we shall prove that *the intersection of the sphere $S(0, r)$ with the plane $y = 0$ is not a subanalytic curve near the abnormal direction (Ox)* . Let $A = (-r, 0, 0)$ be the extremity of the abnormal extremal associated to $u = -r, v = 0$; it belongs to the sub-Riemannian sphere $S(0, r)$ with radius r . Consider near A the curve $\mathcal{C} = S(0, r) \cap (y = 0) \cap (z \leq 0)$. If $\alpha \neq 0$, i.e. if the abnormal is *strict*, this curve is tangent at A to $z = 0$ with a contact of order 2 (see [6] and [11] for a general result). Consider the set of Lagrange multipliers $(\psi(q), \psi^0(q))$ associated to points $q \in \mathcal{C} \setminus \{A\}$. Since these points are reached by normal minimizers, and since Lagrange multipliers are defined up to a scalar, we may suppose that their component ψ^0 is equal to $-1/2$. On the other part we easily get from the definition of Lagrange multipliers that vectors $(\psi_x(q), \psi_z(q))$, $q \in \mathcal{C}$ (where ψ_x, ψ_z denote respectively the x and z -components of ψ), are normal to the curve \mathcal{C} in the plane $y = 0$. Hence to prove that the germ at A of the curve \mathcal{C} is not subanalytic, it is enough to prove that the set $\mathcal{L} = \{(\psi_x(q), \psi_z(q)) / q \in \mathcal{C}\}$ is not subanalytic.

The method is the following. First note that $\psi(q)$ is equal to $p(1)$ where $p(\cdot)$ is the adjoint vector associated to the normal minimizer steering 0 to q . Hence to compute Lagrange multipliers we have

to estimate adjoint vectors at final time $t = 1$. The Maximum Principle gives us a parametrization of normal extremals and leads to the following differential system in the Martinet case :

$$\begin{aligned} \dot{x} &= \frac{1}{a}(p_x + p_z \frac{y^2}{2}), \quad \dot{y} = \frac{p_y}{c}, \quad \dot{z} = \frac{y^2}{2a}(p_x + p_z \frac{y^2}{2}) \\ \dot{p}_x &= \frac{p_y^2 c_x}{2c^2} + \frac{(p_x + p_z \frac{y^2}{2})^2}{2a^2} a_x, \quad \dot{p}_y = \frac{p_y^2 c_y}{2c^2} + \frac{(p_x + p_z \frac{y^2}{2})^2}{2a^2} a_y - \frac{(p_x + p_z \frac{y^2}{2})}{a} p_z y, \quad \dot{p}_z = 0 \end{aligned} \quad (1)$$

We investigate the Lagrange multipliers $(p_x(1), p_z(1))$ as $y(1) = 0$ and $z(1) \leq 0$. Note that p_z is constant along an extremal. Near the point A , it happens that $p_z \rightarrow +\infty$; it is a phenomenon of *non-properness* due to the existence of an abnormal minimizer, see [11]. From [5], extremal controls satisfy $au^2 + cv^2 = r^2$. Then we can set : $r \cos \theta = \sqrt{a}u = \frac{p_x + \frac{y^2}{2} p_z}{\sqrt{a}}$, $r \sin \theta = \sqrt{c}v = \frac{p_y}{\sqrt{c}}$. In particular we introduce θ which is an angular coordinate of the adjoint vector. This (*analytic*) cylindric change of coordinates on the adjoint vector is classical in sub-Riemannian geometry.

Now proving that the set \mathcal{L} is not subanalytic amounts to exhibiting a non subanalytic relation between $\theta(1)$ and p_z as $p_z \rightarrow +\infty$. We proceed as follows. Reparametrizing with $ds = \frac{r\sqrt{p_z}}{\sqrt{a}\sqrt{c}} dt$, first note that, due to the particular form of the metric, the previous equations (1) project onto a *pendulum equation* :

$$\frac{d^2\theta}{ds^2} + \sin \theta + \frac{\alpha^2}{p_z} \sin \theta \cos \theta - \frac{\alpha\beta}{p_z} \sin^2 \theta + \frac{\beta}{\sqrt{p_z}} \cos \theta \frac{d\theta}{ds} = 0 \quad (2)$$

It is a *pendulum perturbed* by the small parameter $\varepsilon = \frac{1}{\sqrt{p_z}}$. In this pendulum representation the plane $y = 0$ projects onto the curve Σ :

$$\frac{d\theta}{ds} = \frac{\alpha}{\sqrt{p_z}} \cos \theta - \frac{\beta}{\sqrt{p_z}} \sin \theta \quad (3)$$

and moreover the abnormal direction projects onto the *saddle point* $\theta = -\pi$, $\frac{d\theta}{ds} = 0$. Near this saddle point we shall now compute a *first return time* t_f for the section Σ , in function of $\theta(t_f)$ and of the parameter p_z . Then, imposing $t_f = 1$ shall give a relation between p_z and $\theta(1)$, and we shall prove that this relation is not subanalytic.

Remark 2.1. – In the pendulum representation the branch \mathcal{C} corresponds to a *local* computation near a saddle point, whereas the branch $S(0, r) \cap (y = 0) \cap (z \geq 0)$ corresponds to a *global* computation of return mapping near the separatrices of the pendulum. The interest of this branch \mathcal{C} is all in the fact that computations are *localized near a saddle*, hence we shall use *normal forms* according to the following Lemma.

LEMMA 2.1. – *Let $(X_\varepsilon)_{0 \leq \varepsilon < 1}$ be a one-parameter analytic family of vector fields in \mathbb{R}^2 near 0 such that for any ε : $X_\varepsilon(0) = 0$, and let (u, v) denote standard coordinates in \mathbb{R}^2 . Let $\mu(\varepsilon)$ and $\nu(\varepsilon)$ be the eigenvalues of $dX_\varepsilon(0)$. We suppose that $\mu(0) = -1$ and $\nu(0) = 1$. Then there exists an analytic germ at 0 from \mathbb{R}^3 to \mathbb{R}^2 :*

$$\varphi : (u, v, \varepsilon) \mapsto (u_1, v_1) = \varphi(u, v, \varepsilon) \quad \text{with } \varphi(0, 0, \varepsilon) = 0$$

such that in the new coordinates (u_1, v_1) the system $(\dot{u}, \dot{v}) = X_\varepsilon(u, v)$ can be written near 0 as :

$$\dot{u}_1 = \mu(\varepsilon)u_1(1 + o(\varepsilon)), \quad \dot{u}_2 = \nu(\varepsilon)u_2(1 + o(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0$$

This Lemma may be proved by adding parameters in the proof of [7], see for instance [8]. To fit in this situation, set $u = \theta + \pi$, $v = \frac{d\theta}{ds}$, and $\varepsilon = \frac{1}{\sqrt{p_z}}$. Then the pendulum equation (2) can be written as the system in \mathbb{R}^2 :

$$\frac{du}{ds} = v, \quad \frac{dv}{ds} = \sin u - \alpha^2 \varepsilon^2 \sin u \cos u + \alpha \beta \varepsilon^2 \sin^2 u + \beta \varepsilon v \cos u$$

Applying Lemma 2.1, we get in new coordinates (u_1, v_1) :

$$\frac{du_1}{ds} = \mu(\varepsilon)u_1(1 + o(\varepsilon)), \quad \frac{dv_1}{ds} = \nu(\varepsilon)v_1(1 + o(\varepsilon))$$

where $\mu(\varepsilon) = 1 + \frac{\beta\varepsilon}{2} + O(\varepsilon^2)$, $\nu(\varepsilon) = -1 + \frac{\beta\varepsilon}{2} + O(\varepsilon^2)$, and actually : $u = u_1 + v_1 + o(\varepsilon)$, $v = u_1 - v_1 + o(\varepsilon)$. Therefore in the new coordinates the section (3) is $\Sigma : v_1 = u_1 + \alpha\varepsilon + o(\varepsilon)$.

Now the return time s_f can be computed in a standard way :

$$s_f = \int_0^{s_f} ds = \int_{v_1(0)}^{v_1(s_f)} \frac{dv_1}{\nu(\varepsilon)v_1(1 + o(\varepsilon))} = \frac{1}{\nu(\varepsilon)}(1 + o(\varepsilon)) \ln \frac{v_1(s_f)}{v_1(0)} \quad (4)$$

On the one part : $v_1(0) = \alpha\varepsilon + o(\varepsilon)$; on the other part we get by integrating the system up to $O(\varepsilon^{\frac{3}{2}})$: $\frac{dt}{ds} = \frac{\varepsilon}{r}(1 + \alpha y)(1 + \beta x + \gamma y) = \frac{\varepsilon}{r}e^{-\beta\varepsilon s} + O(\varepsilon^{\frac{3}{2}})$. Now claiming that the final t_f is equal to 1, we get :

$$1 = \int_0^{s_f} \frac{dt}{ds} ds = \frac{1 - e^{-\beta\varepsilon s_f}}{\beta r} + O(\varepsilon^2)$$

And thus : $s_f = -\frac{\ln(1-\beta r)}{\beta\varepsilon} + O(\varepsilon)$. Putting into (4) we obtain finally :

$$v_1(s_f) = \frac{\alpha\varepsilon}{\sqrt{1-\beta r}} e^{\frac{\ln(1-\beta r)}{\beta\varepsilon}} (1 + o(1)) \quad \text{as } \varepsilon \rightarrow 0$$

In particular $v_1(s_f)$ is not an analytic function in ε as $\varepsilon \rightarrow 0$.

From Lemma 2.1 we know that $v_1(s_f) = An(u(s_f), v(s_f), \varepsilon) \sim \frac{u(s_f) - v(s_f)}{2}$, where $An(\cdot)$ denotes an analytic germ at 0. Moreover, on the section Σ , we have : $v(s_f) = -\alpha\varepsilon \cos u(s_f) + \beta\varepsilon \sin u(s_f)$. Hence : $v_1(s_f) = An(u(s_f), \varepsilon) = \frac{u(s_f)}{2} + \dots$. From the Implicit Function Theorem in the analytic class we get : $u(s_f) = An(v_1(s_f), \varepsilon)$. Therefore $u(s_f)$ is not an analytic function in ε , which ends the proof.

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