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## ASYMPTOTICS OF ACCESSIBILITY SETS ALONG AN ABNORMAL TRAJECTORY

E. TRÉLAT<sup>1</sup>

**Abstract.** We describe precisely, under generic conditions, the contact of the accessibility set at time  $T$  with an abnormal direction, first for a single-input affine control system with constraint on the control, and then as an application for a sub-Riemannian system of rank 2. As a consequence we obtain in sub-Riemannian geometry a new splitting-up of the sphere near an abnormal minimizer  $\gamma$  into two sectors, bordered by the first Pontryagin's cone along  $\gamma$ , called the  $L^\infty$ -sector and the  $L^2$ -sector. Moreover we find again necessary and sufficient conditions of optimality of an abnormal trajectory for such systems, for any optimization problem.

**Résumé.** On décrit précisément, sous des conditions génériques, le contact des ensembles accessibles en temps  $T$  avec une direction anormale, tout d'abord pour un système de contrôle affine mono-entrée avec contrainte sur le contrôle, puis comme application pour un système sous-Riemannien de rang 2. Comme conséquence on obtient en géométrie sous-Riemannienne un nouveau découpage de la sphère au voisinage d'une anormale minimisante  $\gamma$  en deux secteurs appelés secteur  $L^\infty$  et secteur  $L^2$ , délimités par le premier cône de Pontryagin le long de  $\gamma$ . De plus on retrouve des conditions nécessaires et suffisantes d'optimalité pour une trajectoire anormale de tels systèmes, pour un problème d'optimisation quelconque.

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### 1. INTRODUCTION

#### 1.1. Abnormal trajectories

Consider a control system on  $\mathbb{R}^n$  :

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \tag{1}$$

where  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is smooth,  $x_0 \in \mathbb{R}^n$ , and the set of admissible controls  $\mathcal{U}$  is made of measurable bounded functions  $u : [0, T(u)] \rightarrow \Omega \subset \mathbb{R}^m$ .

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**Definition 1.1.** Let  $T > 0$ . The *end-point mapping* at time  $T$  of system (1) is the mapping

$$E_T : \begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathbb{R}^n \\ u & \longmapsto & x_u(T) \end{array}$$

where  $x_u$  is the trajectory associated to  $u$ .

It is a very classical fact that  $E_T$  is smooth in the  $L^\infty$  topology, if  $\mathcal{U} \subset L^\infty([0, T])$ .

**Definition 1.2.** The *time $\times$ end-point mapping* of system (1) is the mapping

$$F : \begin{array}{ccc} \mathbb{R}^+ \times \mathcal{U} & \longrightarrow & \mathbb{R}^n \\ (T, u) & \longmapsto & E_T(u) \end{array}$$

**Definition 1.3.** A control  $u$  (or the corresponding trajectory  $x_u$ ) is said to be *abnormal* on  $[0, T]$  if  $(T, u)$  is a singular point of the mapping  $F$ .

An equivalent definition may be given using the well-known *Pontryagin Maximum Principle* (see [21]), which parametrizes the previous condition :

**Definition 1.4.** A control  $u$  on  $[0, T]$  (or the corresponding trajectory  $x_u$ ) is said to be *abnormal* if there exists a non trivial absolutely continuous function  $p(\cdot) : [0, T] \longrightarrow \mathbb{R}^n \times \mathbb{R}$  called adjoint vector, such that the trajectory  $x$  satisfies almost everywhere the system :

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \frac{\partial H}{\partial u} = 0 \quad (2)$$

where  $H(x, p, u) = \langle p, f(x, u) \rangle$  is the Hamiltonian of the system, and moreover :

$$H(x, p, u) = 0 \quad \text{a.e. on } [0, T] \quad (3)$$

**Remark 1.5.** If a control  $u$  is abnormal on  $[0, T]$  then it is abnormal on  $[0, t]$  for any  $t \in [0, T]$ .

**Definition 1.6.** An abnormal control is said to be *of corank 1* if the adjoint vector  $p(\cdot)$  is defined uniquely up to a scalar multiplier.

**Definition 1.7.** Let  $u$  be an abnormal control on  $[0, T]$ , and  $x_u$  its associated trajectory. The subspace  $\text{im } dE_t(u)$  is called the *first Pontryagin's cone* at  $x_u(t)$ .

An abnormal control is of corank 1 on  $[0, T]$  if the subspace  $\text{im } dE_T(u)$  has codimension 1 in  $\mathbb{R}^n$ .

**Remark 1.8.** An abnormal control on  $[0, T]$  is in particular a singular point of the end-point mapping  $E_T$ . Conversely if  $u$  is a singularity of the end-point mapping and if moreover the trajectory  $x_u$  associated to  $u$  is almost everywhere tangent to its first Pontryagin's cone then  $u$  is abnormal.

## 1.2. Accessibility sets

**Definition 1.9.** Consider the control system (1), and let  $T > 0$ . The *accessibility set at time  $T$* , denoted by  $\text{Acc}(T)$ , is the set of points that can be reached from  $x_0$  in time  $T$  by solutions of system (1), i.e. this is the image of the end-point mapping  $E_T$ .

Let  $\gamma$  be a reference trajectory on  $[0, T]$ , solution of (1), associated to a control  $u$ . Our aim is to describe  $\text{Acc}(T)$  near  $\gamma(T)$ . If  $u$  is not a singular point of the end-point mapping, then obviously  $\text{Acc}(T)$  is open near  $\gamma(T)$ . The situation when  $u$  is a critical point of  $E_T$  has to be analyzed.

### 1.3. Single-input affine systems

Consider a smooth *single-input affine system* in  $\mathbb{R}^n$  with constraint on the control :

$$\begin{aligned} \dot{x}(t) &= X(x(t)) + u(t)Y(x(t)), & |u(t)| &\leq \eta \\ x(0) &= x_0 \end{aligned} \tag{4}$$

where  $X, Y$  are smooth vector fields in  $\mathbb{R}^n$ , and  $u$  is a scalar measurable function. Suppose that the trajectory  $\gamma$  associated to the control  $u = 0$  is abnormal and of corank 1 on  $[0, T]$ . Let  $p(\cdot)$  be an adjoint vector associated to  $\gamma$ . Then the condition  $H = 0$  is equivalent to :

$$\langle p(t), X(\gamma(t)) \rangle = 0 \quad \text{a.e.}$$

i.e.  $X \in \text{im } dE_t(u)$  along  $\gamma$ .

The aim of this article is to describe precisely the accessibility set  $\text{Acc}(T)$  at time  $T$  near  $\gamma(T)$ .

The basic object we have to study is the so-called *intrinsic second-order derivative* of the end-point mapping :

**Definition 1.10.** The *intrinsic second-order derivative* along  $\gamma$  is the real quadratic form :

$$E_u''(v) = p(T) \cdot d^2 E_T(u) \cdot (v, v)$$

where  $v \in \ker dE_T(u)$ .

We shall make a spectral analysis of the end-point mapping along such an abnormal trajectory. Using the formalism and normal forms of [9] we shall represent the intrinsic second-order derivative by an explicit differential operator along the abnormal trajectory  $\gamma$  ; then a spectral analysis of this operator shall lead to a precise description of the boundary of  $\text{Acc}(T)$  near  $\gamma(T)$  (Theorem 2.10). In Section 2.2 we apply this result to sub-Riemannian systems of rank 2 (Theorem 2.21), and obtain a new splitting-up of the sub-Riemannian sphere near an abnormal minimizer  $\gamma$  into two sectors, in which the behaviours of minimizing trajectories near  $\gamma$  are topologically different (Section 2.2.2). On the other part this theory on accessibility sets leads to find again some well-known results on optimality of abnormal trajectories, that we recall and improve slightly in Section 3.

## 2. ASYMPTOTICS OF THE ACCESSIBILITY SETS

In this Section we describe precisely the boundary of accessibility sets for a single-input affine system with constraint on the input near a reference abnormal trajectory. Then we apply our results to the sub-Riemannian case of rank 2 in order to get the *contact* of the sphere with the abnormal direction. As a consequence we obtain a *splitting-up of the sphere into two sectors near the abnormal minimizer*.

### 2.1. Single-input affine control systems

Consider a smooth *single-input affine control system* in  $\mathbb{R}^n$ ,  $n \geq 3$  :

$$\dot{x}(t) = X(x(t)) + u(t)Y(x(t)), \quad x(0) = 0 \tag{5}$$

with the *constraint* on the control

$$|u(t)| \leq \eta \tag{6}$$

Let  $\text{Acc}^\eta(T)$  denote the *accessibility set* at time  $T$  for this affine system with constraint  $\eta$  on the control. Let  $\gamma$  be a reference trajectory defined on  $[0, T]$ . In the sequel we make the following assumptions along  $\gamma$  :

$(H_0)$   $\gamma$  is injective, associated to  $u = 0$  on  $[0, T]$ .

(H<sub>1</sub>)  $\forall t \in [0, T]$   $K(t) = \text{Vect} \{ad^k X.Y(\gamma(t)) / k \in \mathbb{N}\}$  (first Pontryagin's cone along  $\gamma$ ) has codimension 1, and is spanned by the first  $n - 1$  vectors, i.e. :

$$K(t) = \text{Vect} \{ad^k X.Y(\gamma(t)) / k = 0 \dots n - 2\}$$

(H<sub>2</sub>)  $\forall t \in [0, T]$   $ad^2 Y.X(\gamma(t)) \notin K(t)$ .

(H<sub>3</sub>)  $\forall t \in [0, T]$   $X(\gamma(t)) \notin \text{Vect} \{ad^k X.Y(\gamma(t)) / k = 0 \dots n - 3\}$ .

(H<sub>4</sub>)  $\forall t \in [0, T]$   $X(\gamma(t)) \in K(t)$ .

In these conditions  $\gamma$  is *abnormal and of corank 1*. Actually assumptions (H<sub>1</sub> – H<sub>3</sub>) are generic, see [10]. Moreover we get normal forms in which the intrinsic second-order derivative may be represented by an explicit differential operator (see [9]), in the following way.

**Lemma 2.1.** [9] *Under the previous assumptions, the system  $(X, Y)$  is in a  $C^0$ -neighborhood of  $\gamma$  feedback-equivalent to :*

$$\begin{aligned} f_0 &= \frac{\partial}{\partial x_1} + \sum_{i=1}^{n-2} x_{i+1} \frac{\partial}{\partial x_i} + \sum_{i,j=2}^n a_{ij}(x_1) x_i x_j \frac{\partial}{\partial x_n} + \sum_{i=1}^{n-1} x_n f_i(x_1) \frac{\partial}{\partial x_i} + \sum_{i=1}^n Z_i \frac{\partial}{\partial x_i} \\ f_1 &= \frac{\partial}{\partial x_{n-1}} \end{aligned} \quad (7)$$

where  $a_{n-1, n-1}(t) > 0$  on  $[0, T]$ , and the 1-jet (resp. 2-jet) of  $Z_i, i = 1 \dots n - 1$  (resp.  $Z_n$ ) along  $\gamma$  is equal to 0. Moreover the feedback  $(\varphi, \alpha, \beta)$  satisfies :

$$(\varphi, \alpha, \beta).(X, Y) = (f_0, f_1)$$

where  $f_0 = \varphi_*(X + \alpha Y)$ ,  $f_1 = \varphi_*(\beta Y)$ , and  $\varphi$  is a germ of diffeomorphism along  $\gamma$  such that :

(i)  $\varphi(x_1, 0, \dots, 0) = (x_1, 0, \dots, 0)$

(ii)  $\frac{\partial \varphi}{\partial x_{n-1}} = (0, \dots, 0, *, 0)$

and  $\alpha, \beta$  are real functions defined in a neighborhood of  $\gamma$  such that  $\beta$  does not vanish along  $\gamma$  and  $\alpha|_\gamma = 0$ .

The  $k$ -jet is defined as follows :

**Definition 2.2.** Let  $V = \sum_{k=1}^n V_k \frac{\partial}{\partial x_k}$  a vector field. Since  $\gamma$  is given by  $x_1(t) = t$  and  $x_i(t) = 0, i = 2 \dots n$ , the

component  $V_k$  can be written in a neighborhood of  $\gamma$  as  $\sum_{p=1}^{+\infty} j_p V_k$ , where  $j_0 V = V|_\gamma$ , and :

$$j_1 V_k = \sum_{i=2}^n a_i^k(x_1) x_i, \quad j_2 V_k = \sum_{i,j=2}^n b_{ij}^k(x_1) x_i x_j, \quad \dots$$

Set  $j_i V = \sum_{k=1}^n j_i V_k \frac{\partial}{\partial x_k}$ . Then  $\sum_{i=0}^k j_i V$  is called *the  $k$ -jet of  $V$  along  $\gamma$* .

Set  $x_1 = t + \xi$ . In these conditions, the controllable part of the system is  $(\xi, x_2, \dots, x_{n-1})$ , the reference abnormal trajectory is  $\gamma(t) = (t, 0, \dots, 0)$ , and the intrinsic second-order derivative  $E_0''(v)$  along  $\gamma$  is identified to :

$$E_0''(v) = \int_0^T \sum_{i,j=2}^{n-1} a_{ij}(t) \xi_i(t) \xi_j(t) dt, \quad \text{where :}$$

$$\dot{\xi}_1 = \xi_2, \dots, \dot{\xi}_{n-2} = \xi_{n-1}, \dot{\xi}_{n-1} = v, \quad \text{and} \quad \xi_i(0) = \xi_i(T) = 0, i = 1 \dots, n-1.$$

Integrating by parts, it can be written into two different ways, namely either as a quadratic form in  $\xi_1$  or as a quadratic form in  $\xi_2$  :

(1) It is equal to  $Q_{1/G_1}(\xi_1)$ , where :

$$Q_1(\xi_1) = \int_0^T q_1(\xi_1) dt \quad \text{and} \quad q_1(\xi_1) = \sum_{i,j=1}^{n-2} b_{ij} \xi_1^{(i)} \xi_1^{(j)}$$

with  $b_{i-1,j-1} = \frac{a_{ij} + a_{ji}}{2}$ , and where  $G_1$  is the following space corresponding to the kernel  $\ker dE_T(0)$  of the first derivative of the end-point mapping :

$$G_1 = \{ \xi_1 / \xi_1^{(2(n-2))} \in L^2([0, T]), \xi_1^{(i)}(0) = \xi_1^{(i)}(T) = 0, i = 0 \dots n-2 \}$$

**Lemma 2.3.** *The quadratic form  $Q_1$  is represented on  $G_1$  by the operator  $D_1$  so that :*

$$Q_1(\xi_1) = (\xi, D_1 \xi)_{L^2}$$

where  $(\cdot, \cdot)$  is the usual scalar product in  $L^2([0, T])$ , and :

$$D_1 = \frac{1}{2} \sum_{i=1}^{n-2} (-1)^i \frac{d^i}{dt^i} \frac{\partial q_1}{\partial y^{(i)}} = \sum_{i,j=1}^{n-2} (-1)^j \frac{d^j}{dt^j} b_{ij} \frac{d^i}{dt^i} \quad (8)$$

(2) It is equal to  $Q_{2/G_2}(\xi_2)$ , where :

$$Q_2(\xi_2) = \int_0^T q_2(\xi_2) dt \quad \text{and} \quad q_2(\xi_2) = \sum_{i,j=0}^{n-3} b_{i+1,j+1} x_2^{(i)} x_2^{(j)}$$

and where  $G_2$  is the space corresponding to the kernel of the first derivative :

$$G_2 = \{ \xi_2 / \xi_2^{(2(n-3))} \in L^2([0, T]), \xi_2^{(i)}(0) = \xi_2^{(i)}(T) = 0, i = 0 \dots n-3, \text{ and } \int_0^T \xi_2 dt = 0 \}$$

**Lemma 2.4.** *The quadratic form  $Q_2$  is represented on  $G_2$  by the operator  $D_2$  so that :*

$$Q_2(\xi_2) = (\xi_2, D_2 \xi_2)_{L^2}$$

where

$$D_2 = \frac{1}{2} \sum_{i=0}^{n-3} (-1)^i \frac{d^i}{dt^i} \frac{\partial q_2}{\partial y^{(i)}} = \sum_{i,j=0}^{n-3} (-1)^j \frac{d^j}{dt^j} b_{i+1,j+1} \frac{d^i}{dt^i} \quad (9)$$

Note that  $Q_1(\xi) = Q_2(\dot{\xi})$  and  $D_1 = -\frac{d}{dt} D_2 \frac{d}{dt}$ .

Our aim is to make a spectral analysis of these operators  $D_1, D_2$ . Unfortunately the spectrum of  $D_i$  on  $G_i$  is empty. Hence we shall enlarge the Sobolev space  $G_i$  so that the spectrum is not trivial and that Representation Lemmas 2.3 and 2.4 are still valid. We set :

- $F_1 = \{ \xi_1 / \xi_1^{(n-2)} \in L^2([0, T]), \xi_1^{(i)}(0) = \xi_1^{(i)}(T) = 0, i = 0 \dots n-3 \}$  for the operator  $D_1$ . Endowed with the norm  $\|\xi_1\|_{F_1} = \|\xi_1^{(n-2)}\|_{L^2}$ ,  $F_1$  is a Sobolev space.

- $F_2 = \{\xi_2 / \xi_2^{(n-3)} \in L^2([0, T]), \xi_2^{(i)}(0) = \xi_2^{(i)}(T) = 0, i = 0 \dots n-4\}$  for  $D_2$  if  $n \geq 4$  (if  $n = 3$ , no condition is imposed). Endowed with the norm  $\|\xi_2\|_{F_2} = \|\xi_2^{(n-3)}\|_{L^2}$ ,  $F_2$  is a Sobolev space.

**Definition 2.5.** Let  $i = 1$  or  $2$ . We call  $T$  a *conjugate time* of  $Q_i$  along  $\gamma$  if there exists  $\xi_i \in F_i \setminus \{0\}$  such that  $\xi_i^{(2(n-1-i))} \in L^2([0, T])$  and  $D_i \xi_i = 0$ .

**Lemma 2.6.** (see for instance [13]) For any  $f \in L^2([0, T])$ , if  $T$  is not a conjugate time, there exists  $\xi_i \in F_i$  unique such that  $\xi_i^{(2(n-1-i))} \in L^2([0, T])$  and  $D_i \xi_i = f$ . Let  $L$  denote the operator  $f \mapsto \xi_i$  considered as an operator from  $L^2([0, T])$  into  $L^2([0, T])$ ; it is selfadjoint and compact.

The following Lemma is an improvement of [9], where only a non strict inequality is proved :

**Lemma 2.7.** Let  $t_c$  (resp.  $t_{cc}$ ) denote the first conjugate time of  $Q_1$  on  $F_1$  (resp.  $Q_2$  on  $F_2$ ). We have :  $0 < t_{cc} < t_c$ .

*Proof.* This proof can be found in [9], where only a non strict inequality is proved. It is included only for convenience of the reader.

Let  $y \in F_1$  such that  $y^{(2(n-2))} \in L^2([0, T])$ , and  $z = \dot{y}$ ; then  $Q_1(y) = Q_2(z)$ . Let  $\lambda$  (resp.  $\mu$ ) denote the smallest eigenvalue of  $D_1$  (resp.  $D_2$ ). From spectral properties of compact selfadjoint operators, we have :

$$\lambda = \inf_{y \in F_1} \frac{Q_1(y)}{(y, y)} \quad \text{and} \quad \mu = \inf_{z \in F_2} \frac{Q_2(z)}{(z, z)}$$

By definition :

$$\frac{Q_1(y)}{(y, y)} = \frac{Q_2(z)}{\int_0^T (\int_0^t z(s) ds)^2 dt}$$

where  $z \in F_2$  and  $\int_0^T z(t) dt = 0$ . Hence :

$$\lambda \geq \inf \left\{ \frac{Q_2(z)}{\int_0^T (\int_0^t z(s) ds)^2 dt} / z \in F_2 \setminus \{0\} \right\}$$

Moreover we get from the Cauchy-Schwarz inequality :

$$\int_0^T \left( \int_0^t z(s) ds \right)^2 dt \leq \int_0^T t \int_0^t z(s)^2 ds$$

and integrating by parts :

$$\int_0^T t \int_0^t z(s)^2 ds dt \leq \frac{T^2}{2} \int_0^T z(t)^2 dt - \frac{1}{2} \int_0^T t^2 z(t)^2 dt < \frac{T^2}{2} \int_0^T z(t)^2 dt$$

Therefore :

$$\lambda > \frac{2}{T^2} \inf \left\{ \frac{Q_2(z)}{\int_0^T z(t)^2 dt} / z \in F_1 \right\} = \frac{2}{T^2} \mu$$

From [22] we know that eigenvalues of  $D_1$  (or  $D_2$ ) are continuous and decreasing functions of  $T$ . Hence if  $T = t_c$  then  $\lambda = 0$ , thus  $\mu < 0$  and consequently  $t_{cc} < t_c$ .  $\square$

**Remark 2.8.** If  $n = 3$ , we have  $t_{cc} = +\infty$  provided assumptions  $(H_0 - H_3)$  are fulfilled on  $\mathbb{R}^+$ .

**Remark 2.9.** The notion of conjugate time *does not depend on the constraint* on the control. It comes from the fact that the abnormal reference control belongs to the *interior* of the domain of constraints.

The main result is the following.

**Theorem 2.10.** *Consider the affine system (5) with the constraint (6), and suppose that assumptions  $(H_0-H_4)$  are fulfilled along the reference abnormal trajectory  $\gamma$  on  $[0, T]$ . Let  $t_{cc}$  and  $t_c$  denote the first conjugate times associated to  $\gamma$ . Then :*

- (1) *There exist coordinates  $(x_1, \dots, x_n)$  locally along  $\gamma$  such that in these coordinates :  $\gamma(t) = (t, 0, \dots, 0)$ , and the first Pontryagin's cone along  $\gamma$  is :  $K(t) = \text{Vect} \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}} \right\}_{|\gamma}$ .*
- (2) *If  $T$  is small enough then for any point  $(x_1, \dots, x_n)$  of  $\text{Acc}^\eta(T) \setminus \{\gamma(T)\}$  close to  $\gamma(T)$  we have :  $x_n > 0$  (see Fig. 1).*

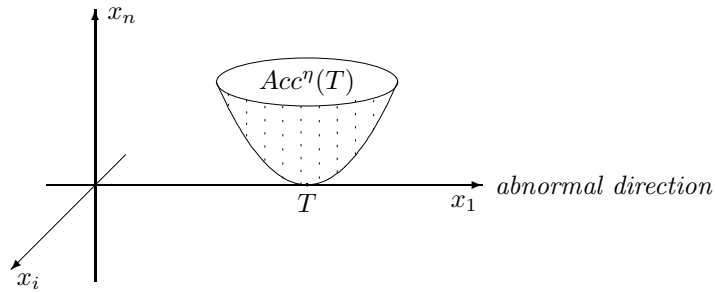


FIGURE 1. Shape of  $\text{Acc}^\eta(T)$ ,  $T$  small

- (3) *If  $T < t_c$ , then in the plane  $(x_1, x_n)$ , near the point  $(T, 0)$ , the boundary of  $\text{Acc}^\eta(T)$  does not depend on  $\eta$ , is a curve of class  $C^2$  tangent to the abnormal direction, and its first term is :*

$$x_n = A_T(x_1 - T)^2 + o((x_1 - T)^2)$$

*The function  $T \mapsto A_T$  is continuous and strictly decreasing on  $[0, t_c[$ . It is positive on  $[0, t_{cc}[$  and negative on  $]t_{cc}, t_c[$ .*

*Moreover, if  $\eta$  depends on  $x_1 - T$  then the result is still valid providing :  $x_1 - T = o(\eta)$  as  $x_1 \rightarrow T$ .*

- (4) *If  $T > t_c$  then  $\text{Acc}^\eta(T)$  is open near  $\gamma(T)$ .*

The evolution in function of  $T$  of the intersection of  $\text{Acc}^\eta(T)$  with the plane  $(x_1, x_n)$  is represented on Fig. 2. The contact with the abnormal direction is of order 2 ; the coefficient  $A_T$  describes the *concavity* of the curve. Beyond  $t_c$  the accessibility set is open.

**Remark 2.11.** The coefficient  $A_T$  can be computed in the following way (see [9], and the Proof just below). Actually there exists a function  $J$  of class  $C^2(n-2)$  on  $[0, T]$  such that  $D_1 J = 0$  and satisfying the limit conditions :

$$\forall k \in \{0, \dots, n-3\} \quad J^{(k)}(0) = 0, \quad J^{(k)}(T) = \delta_0^k$$

Then :

$$A_T = Q_1(J) \tag{10}$$

**Remark 2.12.** Let us give a geometric interpretation of the role of the quadratic forms  $Q_1, Q_2$ .

- (1) Role of  $Q_1$ . On the one part note that  $G_1$  is dense in  $F_1$  and  $Q_1$  is continuous on  $F_1$  for the norm  $\| \cdot \|_{F_1}$  ; hence the sign of  $Q_1$  on  $G_1$  is the same as on  $F_1$ . On the other part, from the definition of the first conjugate time  $t_c$  and the extremal properties of selfadjoint compact operators, we get that  $Q_{1/F_1}$  is positive definite if  $T < t_c$ , and indefinite if  $T > t_c$ . Hence the same goes for  $Q_{1/G_1}$ .



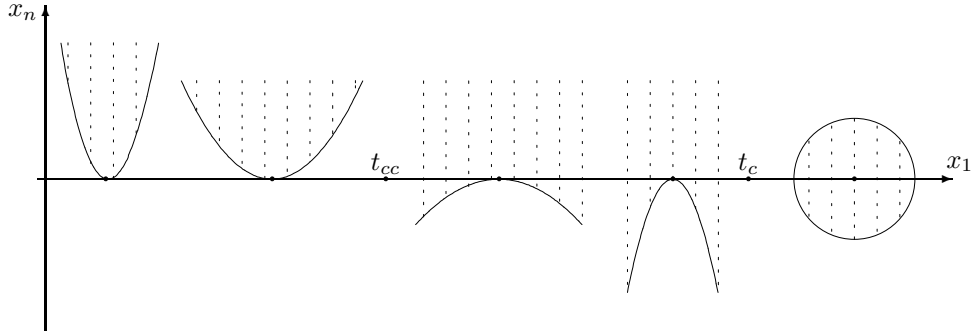


FIGURE 2.

Now the interpretation of  $Q_{1/G_1}$  is the following : it is an equivalent of the coordinate  $x_n(T)$  as all others coordinates are fixed :  $x_1(T) = T, x_2(T) = \dots = x_{n-1}(T) = 0$ .

As a consequence if  $T < t_c$  then points  $(T, 0, \dots, 0, \varepsilon)$ , where  $\varepsilon \geq 0$  are attainable at time  $T$ . If  $T > t_c$ , then the same goes with  $\varepsilon \geq 0$  or  $\varepsilon \leq 0$ .

(2) Role of  $Q_2$ . Contrarily to the previous case, the subspace  $G_2$  is a strict subspace of  $F_2$ . However, set :

$$H_2 = \{\xi_2 / \xi_2^{(2(n-3))} \in L^2([0, T]), \xi_2^{(i)}(0) = \xi_2^{(i)}(T) = 0, i = 0 \dots n-3\}$$

Then  $H_2$  is dense in  $F_2$ , and  $Q_2$  is continuous on  $F_2$  for the norm  $\|\cdot\|_{F_2}$ .

Hence using the same reasoning we obtain the following : if  $T < t_{cc}$  then  $Q_{2/H_2}$  is positive definite ; if  $T > t_{cc}$  then  $Q_{2/H_2}$  is indefinite.

Now note that  $Q_{2/H_2}$  is an equivalent of the coordinate  $x_n(T)$  as coordinates  $x_2(T), \dots, x_{n-1}(T)$  are fixed to 0, but *the coordinate  $x_1(T)$  is let free*.

As a consequence if  $T < t_{cc}$  then points  $(T \pm \delta, 0, \dots, 0, \varepsilon)$ , where  $\varepsilon, \delta \geq 0$  are attainable at time  $T$ . If  $T > t_{cc}$ , then the same goes with  $\varepsilon \geq 0$  or  $\varepsilon \leq 0$ .

This gives us the qualitative shape of  $Acc(T)$ . Actually this interpretation is enough to study the time-optimality of the abnormal trajectory  $\gamma$ , see Section 3.1.1. Here Theorem 2.10 gives more : it describes the contact of  $Acc(T)$  with the abnormal direction.

*Proof of Theorem 2.10.* We shall recall briefly the formalism used by [9], which leads actually to a very precise decomposition of the intrinsic second-order derivative. It is based on normal forms of Lemma 2.1.

Let  $D_1^t$  denote the operator (8) and  $D_2^t$  the operator (9).

**Lemma 2.13.** *For all  $t \in [0, T]$  there exists an Hilbertian basis  $(e_n^t)_{n \in \mathbb{N}}$  of  $L^2([0, T])$  such that*

- $\forall n \in \mathbb{N} \quad e_n^t$  is  $C^2(n-2)$  and  $e_n^t \in F_1$ .
- $\forall n \in \mathbb{N} \quad D_1^t e_n^t = \lambda_n^t e_n^t$ .
- $\lambda_1^t \leq \lambda_2^t \leq \dots \leq \lambda_n^t \leq \dots$

*The eigenvalues  $\lambda_n^t$  are continuous and decreasing functions of  $t$ , and  $\lambda_n^t \xrightarrow{n \rightarrow +\infty} +\infty$ . Moreover, if  $t_c$  denotes the first conjugate time of  $D_1^t$ , then :*

- if  $0 \leq t < t_c$  then  $\lambda_1^t > 0$ ,
- if  $t > t_c$  then  $\lambda_1^t < 0$ .

**Lemma 2.14.** *Let  $0 < t < t_c$ . Then there exist  $J_i^t, \bar{J}_i^t$  in  $C^2(n-2)([0, t])$ ,  $i = 1 \dots n-2$ , uniquely defined by the following equations :*

$$\begin{aligned} \text{(i)} \quad & D_1^t J_i^t = D_1^t \bar{J}_i^t = 0, \quad i = 1 \dots n-2 \\ \text{(ii)} \quad & J_i^{(k)}(0) = \bar{J}_i^{(k)}(t) = 0 \\ & J_i^{(k)}(t) = \bar{J}_i^{(k)}(0) = \delta_{i-1}^k, \quad k = 0 \dots n-3 \end{aligned}$$

Every  $\xi$  in  $C^2(n-2)([0, t])$  can be expanded in a uniformly convergent series :

$$\xi = \sum_{i=1}^{n-2} \alpha_i J_i^t + \sum_{i=1}^{n-2} \beta_i \bar{J}_i^t + \sum_{n=1}^{\infty} y_n e_n^t$$

where  $y_n \in \mathbb{R}$ ,  $\alpha_i = \xi^{(i-1)}(0)$ ,  $\beta_i = \xi^{(i-1)}(t)$ .

We shall use this lemma in the following way. In first approximation the system written in the normal form of Lemma 2.1 is given by :

$$\dot{\xi} = x_2, \quad \dot{x}_2 = x_3, \quad \dots, \quad \dot{x}_{n-1} = u$$

except the last coordinate, which is given at order 2 by :

$$\dot{x}_n = \sum_{i,j=2}^{n-1} a_{ij} x_i x_j$$

Now the meaning of the  $J_i, \bar{J}_i$ 's is clear : the functions  $\bar{J}_i$  (resp.  $J_i$ ) represent the initial (resp. final) conditions of the  $n-2$  first coordinates. More precisely we have, see [9] :

**Lemma 2.15.** *Let  $T < t_c$ , and  $\xi = \sum_{i=1}^{\infty} v_i e_i^T + \sum_{i=1}^{n-2} \alpha_i J_i^T$ . We have :*

- $E_i^T(\xi^{(n-1)}) = \delta_1^i T + \alpha_i + o(\|\xi^{(n-1)}\|_{L^\infty})$ ,  $i = 1 \dots n-2$ .
- $E_n^T(\xi^{(n-1)}) = Q_1^T(\xi) + o(\|\xi^{(n-1)}\|_{L^\infty}^2)$ , where

$$Q_1^T(\xi) = \sum_{i,j=1}^{n-2} A_{ij}^T \alpha_i \alpha_j + \sum_{i=1}^{\infty} \lambda_i^T v_i^2$$

and  $A_{ij}^T = \check{Q}_1^T(J_i^T, J_j^T)$ , where  $\check{Q}_1^T$  denotes the symmetric bilinear form associated to the quadratic form  $Q_1^T$ . In fact :  $Q_1^T(\xi) = d^2 E_n^T(0).(u, u)$ , where  $u \sim \xi^{(n-1)}$ , i.e.

$$Q_1^T(\xi) = \int_0^T \sum_{i,j=1}^{n-2} b_{ij}(t) \xi^{(i)}(t) \xi^{(j)}(t) dt \quad (11)$$

where  $b_{n-2, n-2}$  is continuous and strictly positive on  $[0, T]$ .

Hence with this formalism we study the image of the end-point mapping by constructing directly the trajectories (and not by choosing controls). Actually once a function  $\xi$  has been determined, the corresponding control is  $u = \xi^{(n-1)} + o(\|\xi^{(n-1)}\|_{L^\infty})$ .

First of all using expression (11) it is easy to prove point 2 of Theorem 2.10. Indeed to study the accessibility set at time  $T$  from 0 we have to consider functions  $\xi$  such that :

$$\xi^{(i)}(0) = 0, \quad i = 1 \dots n-2$$

Hence using repeatedly Poincaré's inequality<sup>1</sup> we check that *if  $T$  is small enough* then there exists  $\beta > 0$  such that :

$$Q_1^T(\xi) \geq \beta \int_0^T (\xi^{(n-2)}(t))^2 dt$$

We shall deduce the result by proving that actually  $E_n^T(u) > 0$  for the corresponding control  $u$ . The result would be immediate if  $E_n^T$  were  $C^2$  in  $H^{-1}$ , i.e. in  $L^2$  topology for coordinate  $x_{n-1}$ . But this is wrong (for instance the function  $f \mapsto \int_0^T (f^2 + f^3)dt$  is not  $C^2$  on  $L^2$ ). Anyway  $E_n^T$  is a little better than  $C^1$ . Precisely we have (see [4], Theorem 7.1) :

**Lemma 2.16.**  $E_n^T(u) = Q_1^T(\xi) + R(\xi)$  where  $|R(\xi)| \leq C \|\xi^{(n-2)}\|_{L^\infty} \|\xi^{(n-2)}\|_{L^2}^2$ .

And hence if  $T$  is small enough then for any point  $(x_1, \dots, x_n)$  belonging to  $Acc^\eta(T) \setminus \{(T, 0, \dots, 0)\}$  and to the  $C^0$ -neighborhood of  $\gamma$  of Lemma 2.1, we have :  $x_n > 0$ .

So Lemma 2.15 describes the  $n - 2$  first coordinates and the last coordinate. Anyway [9] do not control the coordinate  $E_{n-1}^T(u)$ . Indeed their method consists in replacing the control  $u \in L^2$  by the control  $x_{n-1} \in L^2$ , which is called *Goh transformation* and consists in fact in immersing  $L^2$  into the space  $H^{-1}$ , see [5]. Such trajectories are called *generalized trajectories*, and this corresponds to add to the set of controls  $L^2([0, T])$  Dirac measures at 0 and  $T$ .

Here, in order to take into consideration the constraint  $|u| \leq \eta$  and to control the coordinate  $E_{n-1}(u)$ , the method used in [9] has to be *adapted*. The aim is to describe the boundary of the intersection of  $Acc^\eta(T)$  with the plane  $(x_1, x_n)$ , close to the point  $(T, 0)$  (corresponding to  $\gamma(T)$ ). Let  $x$  be a real close to  $T$ . We have to solve equations :

$$E_1^T(u) = x, \quad E_2^T(u) = \dots = E_{n-1}^T(u) = 0 \quad (12)$$

in the domain  $|u| \leq \eta$ , and then among such solutions we have to minimize the last coordinate  $E_n^T(u)$  (in order to describe the boundary of the accessibility set). We proceed in the following way. Set  $x_1(t) = t + \xi(t)$  ; we shall first determine  $\xi$  such that the control  $u = \xi^{(n-1)}$  satisfies (12). From Lemma 2.14,  $\xi$  can be expanded in :

$$\xi = \sum_{i=1}^{n-2} \alpha_i J_i^T + \sum_{i=1}^{\infty} v_i e_i^T \quad (13)$$

From Lemma (2.15), the first  $n - 2$  coordinates are, if  $u \sim \xi^{(n-1)} = \sum_{i=1}^{n-2} \alpha_i J_i^{T(n-1)} + v$  :

$$\begin{aligned} E_1^T(u) &= T + \alpha_1 + o(\alpha_1, \dots, \alpha_{n-2}, \|v\|_{L^\infty}) \\ E_2^T(u) &= \alpha_2 + o(\alpha_1, \dots, \alpha_{n-2}, \|v\|_{L^\infty}) \\ &\vdots \\ E_{n-2}^T(u) &= \alpha_{n-2} + o(\alpha_1, \dots, \alpha_{n-2}, \|v\|_{L^\infty}) \end{aligned}$$

Let us solve equations (12). We get from the *Implicit Function Theorem* :

$$\begin{aligned} \alpha_1 &= x - T + o(x - T, \|v\|_{L^\infty}) \\ \alpha_2 &= o(x - T, \|v\|_{L^\infty}) \\ &\vdots \\ \alpha_{n-2} &= o(x - T, \|v\|_{L^\infty}) \end{aligned}$$

as  $x \rightarrow T$  and  $v \xrightarrow{L^\infty} 0$ . Further we will modify  $u$  in order to get also :

$$x_{n-1}(0) = x_{n-1}(T) = 0 \quad \text{and} \quad |u| \leq \eta$$

---

<sup>1</sup> $f(0) = 0 \implies \|f\|_{L^\infty([0, T])} \leq \sqrt{T} \|f'\|_{L^2([0, T])}$

Let us first study the last coordinate. We get from Lemma (2.15) :

$$E_n^T(u) = A_T(x - T)^2 + \sum_{i=1}^{\infty} \lambda_i^T v_i^2 + o\left((x - T + \|v\|_{L^\infty})^2\right)$$

as  $x \rightarrow T$  and  $v \xrightarrow{L^\infty} 0$ . In this expression we can see that the minimum is reached at  $v = 0$  up to a  $o((x - T)^2)$ , for if  $T < t_c$  then all  $\lambda_i^T$ 's are positive (Lemma 2.13).

**Remark 2.17.** If we do not neglect any term then the minimum is not necessarily reached at  $v = 0$  because of terms of order 3 in  $(x - T)^2 v_i$ .

Therefore, *at order 2*, the minimum is reached at  $u \sim \sum_{i=1}^{n-1} \alpha_i J_i^{T(n-1)}$  and equals :

$$E_n^T(u) = A_T(x - T)^2 + o((x - T)^2) \quad \text{as } x \rightarrow T$$

We shall now prove that it is possible to modify  $u$ , taking into account the constraint  $|u| \leq \eta$ , in order to obtain  $x_{n-1}(0) = x_{n-1}(T) = 0$ , without changing the previous results on the other coordinates.

First of all, if  $x$  is close enough to  $T$ , then the  $\alpha_i$ 's are small and thus the constraint  $|u| \leq \eta$  is satisfied.

Let us modify  $u$  (and hence  $x_{n-1}$ ) in the following way. Set  $u(t) = \eta$  if  $x_{n-1}(0) > 0$  or  $-\eta$  if  $x_{n-1}(0) < 0$  on  $[0, t_1]$  (same construction on  $[t_2, T]$ ), where  $t_1$  and  $t_2$  are the coinciding times, i.e. the times at which  $x_{n-1}$  coincides with its initial graph, see Fig. 3.

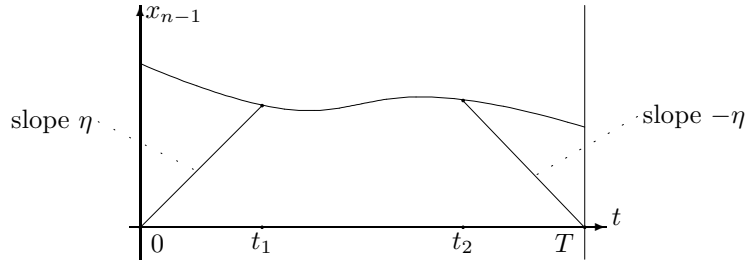


FIGURE 3.

Precisely :

- if  $0 \leq t \leq t_1$  :  $|u(t)| = \eta$ ,  $|x_{n-1}(t)| = \eta t$ , and the coinciding time  $t_1$  is such that  $\eta t_1 \sim \left| \sum_{i=1}^{n-2} \alpha_i J_i^{T(n-2)}(t_1) \right|$ .
- if  $t_1 \leq t \leq t_2$  :  $u(t) \sim \xi^{(n-1)}(t)$ ,  $x_{n-1}(t) = \sum_{i=1}^{n-2} \alpha_i J_i^{T(n-2)}(t)$ .
- if  $t_2 \leq t \leq T$  :  $|u(t)| = \eta$ ,  $|x_{n-1}(t)| = \eta(T - t)$ , where  $t_2$  is such that  $\eta(T - t_2) \sim \left| \sum_{i=1}^{n-2} \alpha_i J_i^{T(n-2)}(t_2) \right|$ .

We shall now check that if

$$x - T = o(\eta) \quad \text{as } x \rightarrow T$$

then all previous results are still valid.

Clearly :  $\eta t_1 = O(x - T)$  and  $\eta(t_2 - T) = O(x - T)$  as  $x \rightarrow T$ . Moreover, at order 1, the system is :

$$\dot{x}_1 = 1 + x_2, \quad \dot{x}_2 = x_3, \quad \dots, \quad \dot{x}_{n-1} = u$$

Therefore :

$$x(t_1) - t_1 \sim \eta \frac{t_1^{n-1}}{(n-1)!} \quad \text{and} \quad x_1(t_2) - t_2 - x_1(T) + T \sim \eta \frac{t_2^{n-1}}{(n-1)!}$$

Now if  $n \geq 3$  and  $x - T = o(\eta)$  then these terms are negligibly small in comparison to  $x - T$ . As concerns the last coordinate, we obtain :

$$x_n(t_1) = O(\eta^2 t_1^3) = o((x - T)^2) \quad \text{and} \quad x_n(T) - x_n(t_2) = O(\eta^2 t_2^3) = o((x - T)^2)$$

In these conditions, all our previous construction is still valid. Hence in the plane  $(x_1, x_n)$  the boundary of  $Acc^n(T)$  is a curve of class  $C^2$ , independant of the constraint, such that  $x_n \sim A_T(x_1 - T)^2$ , which proves Theorem 2.10. Moreover it results from [9] and [22] that the function  $T \mapsto A_T$  is continuous and decreasing on  $[0, t_c[$ , positive on  $[0, t_{cc}[$  and negative on  $]t_{cc}, t_c[$ .  $\square$

## 2.2. Application to the sub-Riemannian case

### 2.2.1. Asymptotics of the sub-Riemannian sphere along an abnormal direction

Consider a smooth sub-Riemannian structure  $(M, \Delta, g)$  where  $M$  is a Riemannian  $n$ -dimensional manifold,  $n \geq 3$ ,  $\Delta$  is a rank 2 distribution on  $M$ , and  $g$  is a metric on  $\Delta$ . Let  $x_0 \in M$  ; our point of view is local and we can assume that  $M = \mathbb{R}^n$  and  $x_0 = 0$ . Suppose there exists a smooth injective abnormal trajectory  $\gamma$  passing through 0. Up to changing coordinates and reparametrizing we can assume that :

- $\gamma(t) = (t, 0, \dots, 0)$ ,
- $\Delta = \text{Span} \{X, Y\}$  where  $X, Y$  are  $g$ -orthonormal,
- $\gamma$  is the integral curve of  $X$  passing through 0.

Under these assumptions, the sub-Riemannian problem is equivalent to the *time-optimal problem* for the system :

$$\dot{x} = vX(x) + uY(x), \quad x(0) = 0 \tag{14}$$

where the controls  $v, u$  satisfy the *constraint* :

$$v^2 + u^2 \leq 1 \tag{15}$$

The reference abnormal trajectory  $\gamma$  corresponds to the control :  $v = 1, u = 0$ .

Let us now define a notion of *constrained accessibility set* :

**Definition 2.18.** Let  $0 < \alpha < 1$ . We denote by  $Acc_{SR}^\alpha(T)$  the accessibility set at time  $T$  for the sub-Riemannian system (14) with the additional constraint on the control :

$$v^2 + u^2 \leq 1, \quad 1 - \alpha \leq v \leq 1, \quad |u| \leq \alpha$$

(see Fig. 4)

Note that controls steering 0 to points of  $Acc_{SR}^\alpha(T)$  are in a  $\alpha$ -neighborhood in  $L^\infty$  metric of the abnormal reference control  $v = 1, u = 0$ .

**Definition 2.19.** We call *affine system associated to the sub-Riemannian system* (14) the following system :

$$\dot{x} = X(x) + wY(w) \tag{16}$$

where the control  $w$  satisfies a constraint of the form :  $|w| \leq \eta$ .

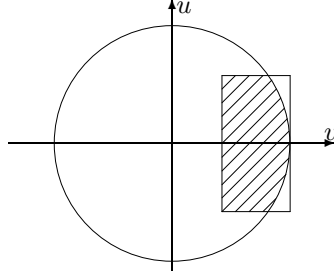


FIGURE 4.

Let  $Acc_A^\eta(T)$  denote the accessibility set at time  $T$  for this affine system with the constraint :  $|w| \leq \eta$ . The reference trajectory  $\gamma$  corresponds to  $w = 0$ , and is also *abnormal* for this affine system.

The following lemma gives a precise comparison of constrained accessibility sets of systems (14) and (16) :

**Lemma 2.20.** (1)  $\forall \alpha \in ]0, 1[ \quad Acc_{SR}^\alpha(T) \subset \bigcup_{(1-\alpha)T \leq s \leq T} Acc_A^{\frac{\alpha}{1-\alpha}}(s)$

(2)  $\bigcup_{T_0 \leq s \leq \frac{T}{\sqrt{1+\eta^2}}} Acc_A^\eta(s) \subset Acc_{SR}^\alpha(T) \quad \text{where } \alpha = \max\left(1 - \frac{T_0}{T}, \frac{\eta}{\sqrt{1+\eta^2}}\right).$

*Proof.* Let us prove the first inclusion. If  $x_1 \in Acc_{SR}^\alpha(T)$  then there exists a control  $(v, u)$  such that  $v^2 + u^2 \leq 1$ ,  $1 - \alpha \leq v \leq 1$ ,  $|u| \leq \alpha$ , and such that the corresponding trajectory satisfies :

$$\dot{x} = vf_0 + uf_1, \quad x(0) = 0, \quad x(T) = x_1$$

As  $\alpha < 1$ ,  $v$  does not vanish, the following reparametrizing holds :  $\frac{ds}{dt} = v$ . Set  $y(s) = x(t)$ ,  $w(s) = \frac{u(t)}{v(t)}$ , and  $S = \int_0^T v$ . Then :  $x_1 = x(T) = y(S)$ , and  $S$  is such that :

$$(1 - \alpha)T \leq S \leq T$$

Moreover :

$$\frac{dy}{ds} = f_0 + \frac{u}{v}f_1 = f_0 + wf_1$$

where  $|w| \leq \frac{\alpha}{1-\alpha}$ . Therefore :

$$x_1 \in \bigcup_{(1-\alpha)T \leq s \leq T} Acc_A^{\frac{\alpha}{1-\alpha}}(s)$$

which proves the first part of the lemma.

Let us now check the second inclusion. Let  $S \in [T_0, \frac{T}{\sqrt{1+\eta^2}}]$  and  $x_1 \in Acc_A^\eta(S)$ . There exists a control  $w$  such that  $|w| \leq \eta$  and the corresponding trajectory satisfies :

$$\frac{dy}{ds} = f_0 + wf_1, \quad y(0) = 0, \quad y(S) = x_1$$

Let  $\varepsilon > 0$  such that  $T = \frac{S}{1-\varepsilon}$ . Let us make the reparametrizing :  $\frac{ds}{dt} = 1 - \varepsilon$ , and set :  $x(t) = y(s)$ ,  $v(t) = 1 - \varepsilon$ ,  $u(t) = (1 - \varepsilon)w(s)$ , where  $t \in [0, T]$ . Then :

$$\dot{x} = vf_0 + uf_1, \quad x(0) = 0, \quad x(T) = y(S) = x_1$$

Let us now check the constraint on the control  $(v, u)$ . By definition :  $T_0 \leq S \leq \frac{T}{\sqrt{1+\eta^2}}$  and  $S = (1 - \varepsilon)T$ .

Hence :

$$1 - \varepsilon \leq \frac{1}{\sqrt{1+\eta^2}} \quad \text{and} \quad 1 - \varepsilon \geq \frac{T_0}{T}$$

and thus :

$$1 - \left(1 - \frac{T_0}{T}\right) \leq v \leq 1 \quad \text{and} \quad |u| \leq \frac{\eta}{\sqrt{1+\eta^2}}$$

Moreover :

$$v^2 + u^2 \leq (1 - \varepsilon)^2(1 + \eta^2) \leq 1$$

Therefore we can conclude that :

$$x_1 \in \text{Acc}_{SR}^\alpha(T) \quad \text{where} \quad \alpha = \max\left(1 - \frac{T_0}{T}, \frac{\eta}{\sqrt{1+\eta^2}}\right)$$

□

Using the previous lemma and Theorem 2.10 we can prove the following :

**Theorem 2.21.** *Suppose assumptions  $(H_0 - H_3)$  are fulfilled along the reference abnormal trajectory  $\gamma$  for the system  $(X, Y)$ . Let  $t_{cc}$  and  $t_c$  denote the first conjugate times of  $\gamma$  for the associated affine system. Let  $\alpha \in ]0, 1[$ . Then :*

- (1) *There exist coordinates  $(x_1, \dots, x_n)$  locally along  $\gamma$  such that in these coordinates :  $\gamma(t) = (t, 0, \dots, 0)$ , and the first Pontryagin's cone along  $\gamma$  is :  $K(t) = \text{Vect} \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}} \right\} |_\gamma$ .*
- (2) *If  $T$  is small enough then for any point  $(x_1, \dots, x_n)$  of  $\text{Acc}_{SR}^\alpha(T)$  close to  $\gamma(T)$  we have  $x_n \geq 0$  (see Fig. 1).*
- (3) *If  $T < t_{cc}$ , then in the plane  $(x_1, x_n)$ , close to the point  $(T, 0)$ , the boundary of  $\text{Acc}_{SR}^\alpha(T)$  does not depend on  $\alpha$ , is a curve of class  $C^2$  outside  $(T, 0)$ , tangent to the abnormal direction, whose first term is :*
  - *if  $x_1 \leq T$  then  $x_n = 0$ .*
  - *if  $x_1 \geq T$  then  $x_n = A_T(x_1 - T)^2 + o((x_1 - T)^2)$ .**The function  $T \mapsto A_T$  is the same as in Theorem 2.10.*
- (4) *If  $T > t_{cc}$  then  $\text{Acc}_{SR}^\alpha(T)$  is open near  $\gamma(T)$ .*

Figure 5 represents the evolution of  $\text{Acc}_{SR}^\alpha(T)$  in function of  $T$  in the plane  $(x_1, x_n)$ . It is open in a neighborhood of  $\gamma(T)$  if  $T > t_{cc}$ , contrarily to the affine case where it becomes open only beyond  $t_c$ .

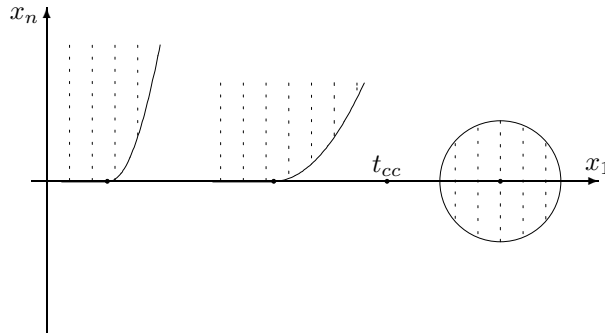


FIGURE 5.

**Remark 2.22.** To compare the system (14) with its associated affine system (16) we need the following reparametrizing (see Proof of Lemma 2.20) :

$$\frac{ds}{dt} = v$$

which only holds if  $v$  does not vanish. This condition is satisfied when the control  $(v, u)$  is in a  $\alpha$ -neighborhood in  $L^\infty$  metric of the abnormal reference control  $(1, 0)$ , for in this case  $v$  is close to 1 in  $L^\infty$ . Hence using this method it is only possible to describe a constrained accessibility set, i.e. in a  $\alpha$ -neighborhood in  $L^\infty$  metric of the reference abnormal control.

*Proof of Theorem 2.21.* The aim is to compare precisely systems (14) and (16) using Lemma 2.20 and to apply Theorem 2.10. In order to do this we first have to *normalize* the affine system (16) using Lemma 2.1. We denote by  $Acc_A^\beta(T)$  the accessibility set at time  $T$  with constraint  $\beta$  of the affine system (16).

The system (7),  $\dot{x} = f_0 + uf_1$ , is called *normalized affine system* and will be referred as (AN). Let  $Acc_{AN}^\eta(T)$  denote the accessibility set at time  $T$  for this system with the constraint :  $|u| \leq \eta$ . Due to the particular forms of the feedback  $(\varphi, \alpha, \beta)$  and of the system, we have :

**Lemma 2.23.**  $\varphi^{-1}(Acc_{AN}^\eta(T)) = Acc_A^\beta(T)$  where  $\beta = O(\eta)$  as  $\eta \rightarrow 0$ .

We know from Lemma 2.20 that :

$$Acc_{SR}^\alpha(T) \subset \bigcup_{s \leq T} Acc_A^{\frac{\alpha}{1-\alpha}}(s)$$

Hence in the normalized coordinates we get :

$$\varphi(Acc_{SR}^\alpha(T)) \subset \bigcup_{s \leq T} Acc_{AN}^\beta(s) \quad (17)$$

We shall use Theorem 2.10, which describes the boundary of  $Acc_{AN}^\beta(s)$  in the plane  $(x_1, x_n)$ , to study the boundary of  $\bigcup_{s \leq T} Acc_{AN}^\beta(s)$ . Using the fact that the function  $t \mapsto A_T$  is continuous and decreasing on  $[0, t_{cc}[$ ,

we can assert that the boundary of  $\bigcup_{s \leq T} Acc_{AN}^\beta(s)$  is given, in the plane  $(x_1, x_n)$ , close to the point  $(T, 0)$ , by the following curve, see Fig. 6 :

- if  $x_1 \leq T$  then  $x_n = 0$ .
- if  $x_1 \geq T$  then  $x_n = A_T(x_1 - T)^2 + o((x_1 - T)^2)$

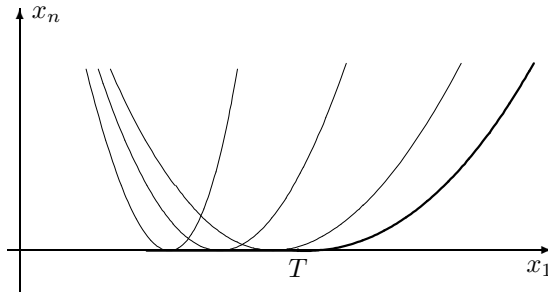


FIGURE 6.



Let  $x_n = f(x_1)$  denote the curve parametrizing the boundary of  $\varphi(\text{Acc}_{SR}^\alpha(T))$  in the plane  $(x_1, x_n)$ . From inclusion (17) we get :

- if  $x_1 \leq T$  then  $f(x_1) \geq 0$ ,
- if  $x_1 \geq T$  then  $f(x_1) \geq A_T(x_1 - T)^2 + o((x_1 - T)^2)$ .

Let us now prove the converse inequality. To this aim we shall use varying constraints depending on  $x_1 - T$ . We proceed in the following way. From Lemma 2.23 :

$$\varphi^{-1}\left(\text{Acc}_{AN}^\eta\left(\frac{T}{\sqrt{1+\eta^2}}\right)\right) \subset \text{Acc}_A^\beta\left(\frac{T}{\sqrt{1+\eta^2}}\right)$$

where  $\beta = O(\eta)$ . Without loss of generality we can assume  $\beta \leq \eta$ , and thus :

$$\text{Acc}_A^\beta\left(\frac{T}{\sqrt{1+\eta^2}}\right) \subset \text{Acc}_A^\eta\left(\frac{T}{\sqrt{1+\eta^2}}\right)$$

Now from Lemma 2.20, we get that for any  $\eta > 0$  small enough :

$$\text{Acc}_A^\eta\left(\frac{T}{\sqrt{1+\eta^2}}\right) \subset \text{Acc}_{SR}^\alpha(T)$$

And thus in the normalized coordinates :

$$\text{Acc}_{AN}^\eta\left(\frac{T}{\sqrt{1+\eta^2}}\right) \subset \varphi(\text{Acc}_{SR}^\alpha(T)) \quad (18)$$

Let  $f_{AN}(x_1)$  be the function parametrizing the boundary of  $\text{Acc}_{AN}^\eta\left(\frac{T}{\sqrt{1+\eta^2}}\right)$  in the plane  $(x_1, x_n)$ . We know from Theorem 2.10 that :

$$f_{AN}(x_1) = A \frac{T}{\sqrt{1+\eta^2}} \left(x_1 - \frac{T}{\sqrt{1+\eta^2}}\right)^2 + o\left(\left(\frac{T}{\sqrt{1+\eta^2}}\right)^2\right)$$

provided  $x_1 - \frac{T}{\sqrt{1+\eta^2}} = o(\eta)$ . This latter condition is fulfilled if  $\eta = (x_1 - T)^{\frac{3}{4}}$ , and in this case we have moreover :  $\eta^2 = o(x_1 - T)$ . On the other part, from the continuity of  $t \mapsto A_t$  :

$$A \frac{T}{\sqrt{1+\eta^2}} = A_T + o(1) \quad \text{as } x_1 \rightarrow T$$

We obtain :  $f_{AN}(x_1) = A_T(x_1 - T)^2 + o((x_1 - T)^2)$ . Finally, from inclusion (18) we conclude :

$$f(x_1) \leq A_T(x_1 - T)^2 + o((x_1 - T)^2)$$

which ends the proof. □

### 2.2.2. Splitting-up of the sphere near an abnormal direction

Let  $T > 0$  small enough so that properties 2 and 3 of Theorem 2.21 are satisfied. In particular the reference abnormal trajectory  $\gamma$  is minimizing, see Section 3.1. Then  $A = \gamma(T)$  belongs to the sub-Riemannian sphere  $S(0, T)$  with radius  $T$ . If controls steering 0 to points of the boundary of  $\text{Acc}_{SR}^\alpha(T)$  in  $x_n > 0$  (that are  $L^\infty$ -optimal) are actually *globally optimal*, then this boundary is included in the sphere  $S(0, T)$ . In this case the sphere splits into *two sectors* near  $\gamma(T)$ , bordered by the first Pontryagin's cone  $x_n = 0$  :

- sector  $x_n > 0$  corresponding to the previous description,

- sector  $x_n < 0$ .

According to the previous results, final points at time  $T$  associated to controls which are  $L^\infty$ -close to the reference abnormal control are in the first sector :  $x_n > 0$ . Obviously due to controllability of the system the sector  $x_n < 0$  is accessible. In fact a basic calculus shows :

**Lemma 2.24.** *For any neighborhood  $V$  of the point  $A$  in  $\mathbb{R}^n$  we have :*

$$S(0, T) \cap V \cap (x_n < 0) \neq \emptyset$$

These points in  $(x_n < 0)$  are reached by controls which are close to the reference control in  $L^2$  metric but not in  $L^\infty$  metric. More precisely :

**Lemma 2.25.** *Let  $M_n = E_T(u_n) \in S(0, T)$  whose last coordinate  $x_n$  is strictly negative. Let  $u$  denote the abnormal reference control. We suppose that  $M_n$  converges to  $A = E(u)$  in  $\mathbb{R}^n$ . Then  $u_n$  converges to  $u$  in  $L^2([0, T])$  but not in  $L^\infty([0, T])$ .*

Hence near the abnormal direction the sphere is splits into two sectors :

- the  $L^\infty$ -sector :  $(x_n > 0) \cap S(0, T)$  (described by Theorem 2.21), made of end-points of minimizing trajectories associated to controls  $L^\infty$ -close to the abnormal reference control,
- the  $L^2$ -sector :  $(x_n < 0) \cap S(0, T)$ , made of points reached by minimizing controls  $L^2$ -close, but not  $L^\infty$ -close to the abnormal reference control.

The contact of the first sector is known, but not the second one a priori. Anyway according to the *Tangency Theorem* (see [23]), under some nice stratification assumptions, this  $L^2$ -sector *ramifies tangently* to the Pontryagin cone  $x_n = 0$ , see Fig. 7.

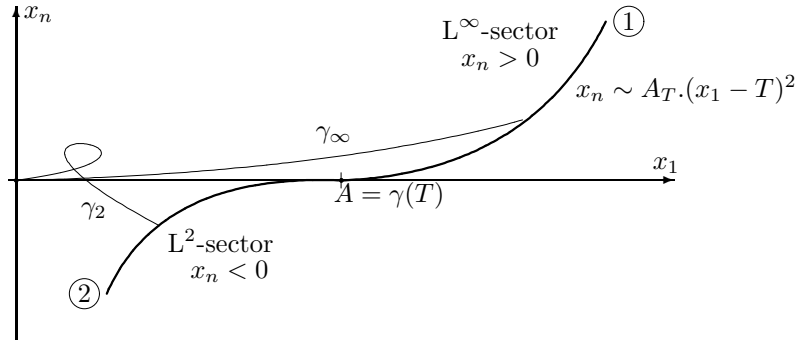


FIGURE 7.

**Remark 2.26.** In particular, minimizing trajectories  $\gamma_\infty$  joining 0 to points of the sphere in the  $L^\infty$ -sector are  $C^1$ -close to the reference abnormal trajectory  $\gamma$ . Minimizing trajectories  $\gamma_2$  joining 0 to points of the sphere in the  $L^2$ -sector are  $C^0$ -close, but not  $C^1$ -close to  $\gamma$ , see Fig. 7.

**Typical example : the Martinet case.** Consider the two following vector fields in  $\mathbb{R}^3$  :

$$X = \frac{\partial}{\partial x} + \frac{y^2}{2} \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}$$

and endow the distribution spanned by these vector fields with an analytic metric  $g$  of the type :

$$g = adx^2 + cdy^2$$

where  $a = (1 + \alpha y)^2$  and  $c = (1 + \beta x + \gamma y)^2$ . The abnormal reference control for the sub-Riemannian system  $\dot{x} = vX(x) + uY(x)$  with constraint  $v^2 + u^2 \leq 1$  is  $v = 1, u = 0$ , and corresponds to the trajectory  $\gamma : x(t) = t, y(t) = z(t) = 0$ . We have, see [11] :

**Lemma 2.27.** *Assumptions  $(H_0 - H_3)$  are fulfilled along  $\gamma$  if and only if  $\alpha \neq 0$ . In this case branches 1 and 2 (see Fig. 7 with  $x_1 = x, x_n = z$ ) have the following contacts with the abnormal direction :*

- branch 1 :  $x \geq T, z = \frac{1}{2T\alpha^2}(x - T)^2 + o((x - T)^2)$
- branch 2 :  $x \leq T, z \sim \frac{1}{6}(1 + O(T))(x - T)^3$

**Remark 2.28.** The coefficient  $A_T$  of the first branch can be computed directly or using formula (10) (see Remark 2.11).

As we are in dimension 3, results of Theorem 2.21 are in fact available on  $\mathbb{R}^+$ , see Remark 2.8. The  $L^2$ -sector is  $z < 0$  and the  $L^\infty$ -sector is  $z > 0$ .

*Proof of Lemma 2.24.* Consider the sub-Riemannian system (14) in the coordinates  $(x_1, \dots, x_n)$  of Theorem 2.21 :

$$\begin{aligned} \dot{x}_1 &= v(1 + x_2 + x_n f_1(x_1) + Z_1) \\ \dot{x}_2 &= v(x_3 + x_n f_2(x_1) + Z_2) \\ &\vdots \\ \dot{x}_{n-1} &= u + x_n f_{n-1}(x_1) + Z_{n-1} \\ \dot{x}_n &= v \sum_{i=2}^n a_{ij}(x_1) x_i x_j + Z_n \end{aligned}$$

where the 1-jet (resp. the 2-jet) of  $Z_1, \dots, Z_{n-1}$  (resp.  $Z_n$ ) along  $u = 0$  is equal to 0.

Our goal is to construct a control close to the reference abnormal control in  $L^2$  metric, whose associated final point at time  $T$  is close to  $(T, 0, \dots, 0)$  and is such that  $x_n < 0$ . Let  $\varepsilon > 0$ . Consider the following control  $v$  (see Fig. 8) :

- if  $0 \leq t \leq \frac{T-\varepsilon}{2}$  then  $v(t) = 1$ ,
- if  $\frac{T-\varepsilon}{2} \leq t \leq \frac{T+\varepsilon}{2}$  then  $v(t) = -1$ ,
- if  $\frac{T+\varepsilon}{2} \leq t \leq T$  then  $v(t) = 1$ .

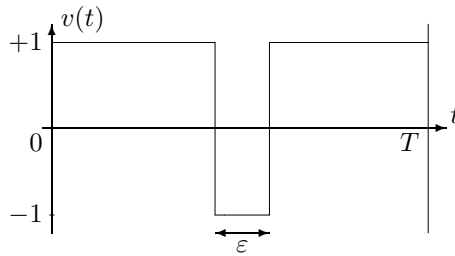


FIGURE 8.

Set  $u = 0$ . It is clear that  $(v, u)$  is abnormal (but not minimizing). Consider the following perturbation (see Fig. 9) :

- if  $0 \leq t \leq \frac{T-\varepsilon}{2}$  then  $\delta v(t) = \delta u(t) = 0$ ,

- if  $\frac{T-\varepsilon}{2} < t \leq \frac{T}{2}$  then  $\delta v(t) = 1 - \sqrt{1 - \varepsilon^2}$ ,  $\delta u(t) = \varepsilon$ ,
- if  $\frac{T}{2} < t \leq \frac{T+\varepsilon}{2}$  then  $\delta v(t) = 1 - \sqrt{1 - \varepsilon^2}$ ,  $\delta u(t) = -\varepsilon$ ,
- if  $\frac{T+\varepsilon}{2} < t \leq T$  then  $\delta v(t) = \delta u(t) = 0$ .

It is clear that  $(v + \delta v)^2 + (\delta u)^2 = 1$ . Moreover :  $\|\delta v\|_{L^1} = O(\varepsilon^3)$  and  $\|\delta u\|_{L^1} = O(\varepsilon^2)$ .

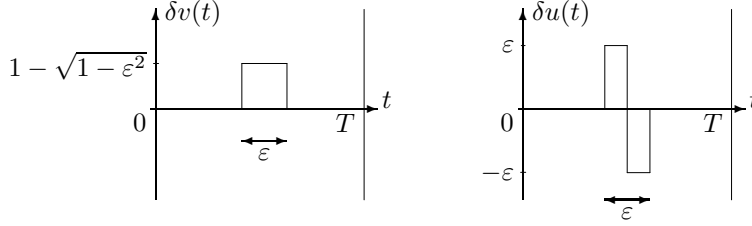


FIGURE 9.

The end-point mapping  $E_T$  being  $C^\infty$  in  $L^1$  topology (see for instance [17]), we have :

$$\forall i \in \{1, \dots, n-1\} \quad E_T^i(v + \delta v, \delta u) = E_T^i(v, 0) + dE_T^i(v, 0) \cdot (\delta v, \delta u) + O(\varepsilon^4) \quad (19)$$

and

$$E_T^n(v + \delta v, \delta u) = E_T^n(v, 0) + dE_T^n(v, 0) \cdot (\delta v, \delta u) + \frac{1}{2} d^2 E_T^n(v, 0) \cdot (\delta v, \delta u)^{[2]} + O(\varepsilon^6) \quad (20)$$

Moreover we have  $E_T^i(v, 0) = \delta_i^1(T - \varepsilon)$  for  $i = 1, \dots, n$ . On the other part the linearized system along  $(v, u)$  is :

$$\dot{y}_1 = \delta v + v y_2, \quad \dot{y}_2 = v y_3, \quad \dots, \quad \dot{y}_{n-2} = v y_{n-1}, \quad \dot{y}_{n-1} = \delta u, \quad \dot{y}_n = 0$$

Let us calculate  $y_{n-1}(t)$  :

- if  $0 \leq t \leq \frac{T-\varepsilon}{2}$  then  $y_{n-1}(t) = 0$ ,
- if  $\frac{T-\varepsilon}{2} < t \leq \frac{T}{2}$  then  $y_{n-1}(t) = \varepsilon(t - \frac{T-\varepsilon}{2})$ ,
- if  $\frac{T}{2} < t \leq \frac{T+\varepsilon}{2}$  then  $y_{n-1}(t) = -\varepsilon(t - \frac{T+\varepsilon}{2})$ ,
- if  $\frac{T+\varepsilon}{2} < t \leq T$  then  $y_{n-1}(t) = 0$ .

In particular :  $\|y_{n-1}\|_{L^\infty} = O(\varepsilon^2)$ , and  $y_{n-1}$  is equal to 0 outside an interval of length  $\varepsilon$ . We get easily :

$$\|y_{n-2}\|_{L^\infty} = \dots = \|y_1\|_{L^\infty} = O(\varepsilon^3) \quad (21)$$

Hence using (19) :

$$\begin{aligned} E_1^T(v + \delta v, \delta u) &= T - \varepsilon + O(\varepsilon^3) \\ E_2^T(v + \delta v, \delta u) &= O(\varepsilon^3) \\ &\vdots \\ E_{n-2}^T(v + \delta v, \delta u) &= O(\varepsilon^3) \\ E_{n-1}^T(v + \delta v, \delta u) &= O(\varepsilon^2) \end{aligned}$$

Let us now compute the last coordinate. We have to calculate  $z_n$ , where :

$$\dot{z}_n = v \sum_{i,j=2}^{n-1} a_{ij}(t) y_i y_j$$

It represents the intrinsic second-order derivative. From (21) we get :

$$\dot{z}_n(t) = v(t)a_{n-1,n-1}(t)y_{n-1}^2(t) + y_{n-1}(t)O(\varepsilon^3) + O(\varepsilon^6)$$

As  $y_{n-1}$  is equal to 0 outside  $[\frac{T-\varepsilon}{2}, \frac{T+\varepsilon}{2}]$ , we have :

$$\int_0^T y_{n-1}(t)O(\varepsilon^3)dt = O(\varepsilon^6)$$

Moreover the coefficient  $a_{n-1,n-1}$  is continuous and does not vanish on  $[0, T]$ , hence there exists  $\alpha > 0$  such that  $a_{n-1,n-1}(t) \geq \alpha$  on  $[0, T]$ . We get :

$$z_n(T) \leq -\alpha \int_{\frac{T-\varepsilon}{2}}^{\frac{T+\varepsilon}{2}} y_{n-1}^2(t)dt + O(\varepsilon^6) \leq -\frac{\alpha\varepsilon^5}{12} + O(\varepsilon^6)$$

Therefore from (20) :

$$E_n^T(v + \delta v, \delta u) \leq -\frac{\alpha\varepsilon^5}{12} + O(\varepsilon^6)$$

Hence for any neighborhood  $V$  of  $(T, 0, \dots, 0)$ ,  $Acc(T) \cap V$  contains points such that  $x_n < 0$ , and hence the same goes for  $S(0, T) \cap V$  since the abnormal reference trajectory is minimizing. By construction, controls steering to these points are  $L^2$ -close but not  $L^\infty$ -close to the reference abnormal control.  $\square$

*Proof of Lemma 2.25.* As  $M_n$  belongs to the sphere  $S(0, T)$ , we have :  $\|u_n\|_{L^2} = \|u\|_{L^2} = T$ , thus the sequence  $(u_n)_{n \in \mathbb{N}}$  is bounded in  $L^2$ . Hence up to a subsequence we can assume that  $u_n$  converges weakly to  $v \in L^2$  (denoted by  $u_n \rightharpoonup v$ ). From the continuity of the end-point mapping  $E_T$  in the weak topology on  $L^2$  (see [23]), we can assert that  $M_n = E_T(u_n)$  converges to  $E_T(v)$ , and thus  $E_T(v) = E_T(u)$ . The assumptions on the reference abnormal trajectory imply that  $v = u$ . Hence  $u_n \rightharpoonup u$ , and on the other part :  $\|u_n\|_{L^2} = \|u\|_{L^2}$ , therefore  $u_n$  converges (strongly) towards  $u$  in  $L^2$ .

Moreover results stated by Theorem 2.21 imply that  $u_n$  does not converge towards  $u$  in  $L^\infty$  (because it would imply that  $x_n \geq 0$ ).  $\square$

### 3. APPLICATION : OPTIMALITY OF ABNORMAL TRAJECTORIES

In this Section we apply our previous theory on accessibility sets to studying optimality of abnormal trajectories ; this leads us to find again some well-known results. Indeed in the notations of Theorem 2.10, this Theorem implies in particular that if  $T < t_{cc}$  then  $\gamma$  is *isolated in  $C^0$ -topology* in the space of all trajectories which connect given end-points, and thus is *optimal for any cost* in this topology. This well-known property, called *rigidity*, was intensively studied. The main results concerning this analysis in a generic context were given first in [22] and [9] for single-input affine systems, then in [4, 14, 18, 25], for sub-Riemannian systems, and in [6] in general. Moreover these authors developed a Morse theory in order to characterize conjugate points, that is, points beyond which the abnormal trajectory is no more optimal.

Hence results given in this Section are not really new. However they are slightly different from the results cited above. Indeed on the one part in [9] were obtained necessary and sufficient conditions for  $C^0$ -time-optimality of abnormal trajectories of single-input affine systems without any constraint on the control. Here we improve their statement by adding a *constraint on the control* and studying the problem of minimizing *any cost*. On the other part, in [6] was made a general theory (i.e. for nonlinear systems) on optimality of abnormal trajectories in  $L^\infty$  topology on the controls. Results given here are valid in the  $C^0$ -topology on the trajectories (but only for single-input affine systems). Moreover we study the equivalence between the time-optimality problem and the problem of minimizing any cost, the final time being fixed or not. Finally Theorem 3.10, which concerns optimality of abnormal trajectories for sub-Riemannian systems of rank 2, makes a link between the works of [9] and [4, 5].

### 3.1. Optimality of abnormal trajectories for single-input affine systems

Consider the single-input affine system (5) with constraint (6), and suppose assumptions  $(H_0-H_4)$  are fulfilled along a reference abnormal trajectory  $\gamma$ . We first study the time-optimal problem, and then the problem of minimizing some cost.

#### 3.1.1. Time optimality

**Definition 3.1.** • The trajectory  $\gamma$  is said  *$C^0$ -time-minimal* on  $[0, T]$  if there exists a  $C^0$ -neighborhood of  $\gamma$  such that  $T$  is the minimal time to steer  $\gamma(0)$  to  $\gamma(T)$  among the solutions of the system (5) with the constraint (6) that are entirely contained in this neighborhood.

- Recall that  $\gamma$  is associated to the control  $u = 0$ . Let  $\delta > 0$ . The trajectory  $\gamma$  is said  *$L^\infty$ -time-minimal* on  $[0, T]$  if there exists a neighborhood of 0 in  $L^\infty([0, T + \delta])$  such that  $T$  is the minimal time to steer  $\gamma(0)$  to  $\gamma(T)$  among trajectories associated to controls of this neighborhood.

Obviously if  $\gamma$  is  $C^0$ -time-minimal then it is  $L^\infty$ -time-minimal.

**Theorem 3.2.** *Under assumptions of Theorem 2.10, the trajectory  $\gamma$  is  $C^0$ -time-minimal if and only if  $T < t_{cc}$ . Moreover  $\gamma$  is not  $L^\infty$ -time-minimal if  $T > t_{cc}$ .*

The proof is clear by inspecting Fig. 2 and Proof of Theorem 2.10.

**Remark 3.3.** If  $n = 3$ , we have  $t_{cc} = +\infty$  provided assumptions  $(H_0 - H_3)$  are fulfilled on  $\mathbb{R}^+$ . Hence in this case  $\gamma$  is  $C^0$ -time-minimal on  $\mathbb{R}^+$ .

#### 3.1.2. Optimality for some cost

Let us now consider the problem of minimizing some cost  $C(T, u)$ , also denoted by  $C_T(u)$ , where  $C$  is a smooth function satisfying the following additional assumption along the reference singular trajectory  $\gamma$  :

$$(H_5) \quad \forall T \quad \text{rank} (dE_T(0), dC_T(0)) = n$$

i.e. the singularity of the end-point mapping of the extended system has codimension 1, and in particular the cost is independant from the end-point mapping along  $\gamma$ . We consider several optimization problems :

- (1) final time not fixed : the aim is to steer the system from  $x_0$  to  $x_1$  in some time  $T$  (not preassigned) and minimizing the cost  $C$ .
- (2) final time fixed : let  $T > 0$  fixed ; the aim is to steer the system from  $x_0$  to  $x_1$  in time  $T$  and minimizing the cost  $C_T$ .

##### 1. Final time not fixed.

**Definition 3.4.** • The trajectory  $\gamma$  is said to be  *$C^0$ -cost-minimal* on  $[0, T]$  if there exists a  $C^0$ -neighborhood of  $\gamma$  such that for any trajectory  $q$  contained in this neighborhood, with  $q(0) = \gamma(0)$  and  $q(t) = \gamma(T)$ , we have :  $C(t, v) \geq C(T, 0)$ , where  $v$  is the control associated to  $q$ .

- Let  $\delta > 0$ . The trajectory  $\gamma$  is said to be  *$L^\infty$ -cost-minimal* on  $[0, T]$  if there exists a neighborhood of 0 in  $L^\infty([0, T + \delta])$  such that, for any trajectory  $q$  associated to a control  $v$  of this neighborhood, with  $q(0) = \gamma(0)$  and  $q(t) = \gamma(T)$ , we have :  $C(t, v) \geq C(T, 0)$ .

Obviously the  $C^0$ -cost-minimality implies the  $L^\infty$ -cost-minimality.

We have the following result (compare with [6]) :

**Theorem 3.5.** *Under assumptions  $(H_0 - H_5)$ , the trajectory  $\gamma$  is  $C^0$ -cost-minimal if and only if it is  $C^0$ -time-minimal. Actually,  $\gamma$  is  $C^0$ -cost-minimal if  $T < t_{cc}$ , and is not  $L^\infty$ -cost-minimal if  $T > t_{cc}$ .*

Hence if the final time is not fixed then *both problems of cost-minimization and time-minimization are equivalent.*

*Proof.* If  $T < t_{cc}$ , then the abnormal trajectory  $\gamma$  is isolated in a  $C^0$ -neighborhood, hence in particular is  $C^0$ -cost-minimal. If  $T > t_{cc}$ , we know that  $\gamma$  is no more time-minimal since the point  $(T, 0, \dots, 0)$  belongs to  $Acc(t)$  for some  $t < T$ , see fig. 10.

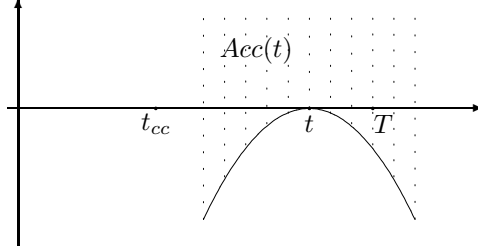


FIGURE 10.

To prove that  $\gamma$  is no more  $L^\infty$ -cost-minimal, we have to show that *the point  $(T, 0, \dots, 0)$  belongs to  $M_r(t)$  for some  $r < C_T(0)$* , where  $C_T(0)$  is the cost of  $\gamma$  at time  $T$ , and  $M_r(t)$  is the level set  $r$  at time  $t$  of the value function associated to the cost  $C$ .

In the coordinates  $(x_1, \dots, x_n)$  of Lemma 2.1, the end-point mapping is  $E_t = (E_1^t, \dots, E_n^t)$ , where :

$$\text{rank}(dE_1^t(0), \dots, dE_n^t(0)) = n - 1 \quad \text{and} \quad dE_n^t(0) = 0$$

Instead of working in  $L^2([0, T])$ , we choose as control space the Sobolev space  $H^1([0, T]) = \{f \in L^2([0, T]) / f' \in L^2([0, T])\}$ . Endowed with the norm  $\|f\|_{H^1} = \sqrt{\|f\|_{L^2}^2 + \|f'\|_{L^2}^2}$ , this is a Hilbert space. The reason to use it is the following : it can be immersed in a compact way in  $C^0([0, T])$ , and in the proof we shall need continuous controls.

The end-point mapping is still differentiable in  $H^1$  since it is differentiable in  $L^2$ . From assumption  $(H_4)$ , the end-point mapping is independant from the cost along  $\gamma$ , hence at time  $T$  there exist  $n$  independant vector fields  $e_0^T, e_1^T, \dots, e_{n-1}^T$  such that :

$$dC_T(0).e_0^T = 1 \quad \text{and} \quad dE_i^T(0).e_i^T = 1, \quad i = 1 \dots n - 1$$

We can decompose  $H^1$  in the following way :

$$H^1([0, T]) = \mathbb{R}e_1^T \oplus \mathbb{R}e_2^T \oplus \dots \oplus \mathbb{R}e_{n-1}^T \oplus \mathbb{R}e_0^T \oplus F$$

where  $\ker dE_i^T(0) = \bigoplus_{j \neq i} \mathbb{R}e_j^T \oplus F$  and  $\ker dC_T(0) = \bigoplus_{j \neq 0} \mathbb{R}e_j^T \oplus F$ .

Let  $Q_T(u) = d^2 E_n^T(0).(u, u)$  (as in Lemma 2.15). The kernel of the first derivative is :  $\ker dE_T(0) = \mathbb{R}e_0^T \oplus F$ , and hence the intrinsic second-order derivative at time  $T$  along  $\gamma$  is the quadratic form  $Q_{T / \mathbb{R}e_0^T \oplus F}$ . On the other part the space  $\mathbb{R}e_1^T \oplus \mathbb{R}e_0^T \oplus F$  represents the domain of the latter quadratic form where the limit condition on the first coordinate has been relaxed, hence it is the domain of the *reduced operator*  $D_1$ . Now by definition of the first conjugate times  $t_{cc}$  and  $t_c$ , we have :

- if  $T < t_{cc}$  :  $Q_{T / \mathbb{R}e_1^T \oplus \mathbb{R}e_0^T \oplus F}$  is positive definite,
- if  $t_{cc} < T < t_c$  :  $Q_{T / \mathbb{R}e_0^T \oplus F}$  is positive definite, and  $Q_{T / \mathbb{R}e_1^T \oplus \mathbb{R}e_0^T \oplus F}$  is indefinite,
- if  $T > t_c$  :  $Q_{T / \mathbb{R}e_0^T \oplus F}$  is indefinite.

From now on we assume that  $T > t_{cc}$ . Then there exist  $v, w$  in  $\mathbb{R}e_0^T \oplus F$  such that  $Q_T(e_1^T + v) < 0$  and  $Q_T(w) > 0$ .

**Remark 3.6.** Note that we do not have necessarily  $Q_T(e_1^T) < 0$ .

Let  $\delta > 0$  small. As  $v$  and  $w$  are continuous, it is possible to extend them on  $[T, T + \delta]$  respectively by  $v(T)$  and  $w(T)$  (here is the role of  $H^1$ ). Consider now the following control  $u \in H^1([0, T])$  :

$$u = a_1 e_1^T + a_2 e_2^T + \cdots + a_{n-1} e_{n-1}^T + a_1 v + a_n w$$

where  $a_1, \dots, a_n$  are real numbers. In the same way we can extend  $u$  on  $[0, T + \delta]$ . We shall prove that we can choose  $a_1, \dots, a_n$  small and  $t$  close to  $T$  such that the previous control  $u$  satisfies :

$$|u| \leq \eta, \quad t \leq T + \delta, \quad E_t(u) = \gamma(T), \quad C_t(u) < C_T(0)$$

i.e.  $\gamma(T)$  belongs to  $M_r(t)$  for some  $r < C_T(0)$ . To this aim we shall first use the *Implicit Function Theorem* on  $a_1, \dots, a_n$  (to take into account the  $n - 1$  first coordinates, i.e. the controllable part of the system), then the *Mean Value Theorem* on the last coordinate, and endly choose  $t$  in order to make the cost lower than  $C_T(0)$ . Note that in this method all reasonings are in finite dimension.

Let us expand, for  $t$  close to  $T$  and  $i = 1, \dots, n - 1$  :

$$E_i^t(u) = E_i^t(0) + dE_i^t(0).u + o(\|u\|_{H^1([0, t])})$$

Moreover :

- $E_1^t(0) = t$ , and for  $i = 2, \dots, n - 1$  :  $E_i^t(0) = 0$ ,
- $o(\|u\|_{H^1([0, t])}) = o(\|u\|_{H^1([0, T + \delta])}) = o(a_1, \dots, a_n)$ ,
- For  $i = 1, \dots, n - 1$  :  $dE_i^t(0) = dE_i^T(0) + O(t - T)$ .

Therefore we get, for  $i = 1, \dots, n - 1$  :

$$dE_i^t(0).u = a_i + o(a_1, \dots, a_n, t - T)$$

and thus :

$$E_t(u) = (t + a_1 + o(a_1, \dots, a_n, t - T), a_2 + o(a_1, \dots, a_n, t - T), \dots, a_{n-1} + o(a_1, \dots, a_n, t - T), E_n^t(u))$$

Let us first solve the system of equations :

$$E_1^t(u) = T, \quad E_2^t(u) = \cdots = E_{n-1}^t(u) = 0$$

We get from the *Implicit Function Theorem* :

$$\forall i \in \{1, \dots, n - 1\} \quad a_i = f_i(T - t, a_n)$$

and moreover :

$$a_1 = T - t + o(T - t, a_n) \quad \text{and for } i = 2, \dots, n - 1 : a_i = o(T - t, a_n)$$

Let us now calculate the last coordinate :

$$E_n^t(u) = E_n^t(0) + dE_n^t(0).u + \frac{1}{2}d^2E_n^t(0).(u, u) + o(\|u\|_{H^1}^2) = Q_t(u) + o((T - t)^2, a_n^2)$$

Moreover :

$$Q_t(u) = Q_t(a_1(e_1^T + v) + a_2 e_2^T + \cdots + a_{n-1} e_{n-1}^T + a_n w)$$

Hence

$$\begin{aligned} E_n^t(u) &= (T - t)^2 Q_t(e_1^T + v) + a_n^2 Q_t(w) + a_n (T - t) \check{Q}_t(e_1^T + v, w) + o((T - t)^2, a_n^2) \\ &= g(t, a_n) \end{aligned}$$



where  $\tilde{Q}$  denotes the symmetric bilinear form associated to the quadratic form  $Q$ . On the one part :  $g(T, a_n) = a_n^2 Q_T(w) + o(a_n^2) > 0$  if  $a_n$  is small enough (by definition of  $w$ ). On the other part :  $g(t, 0) = (T - t)^2 Q_t(e_1^T + v) + o((T - t)^2)$ . From the definition of  $v$ , we have :  $Q_T(e_1^T + v) < 0$ . Hence if  $t$  is close enough to  $T$ , but not equal to  $T$ , we get :  $g(t, 0) < 0$ . Now applying the *Mean Value Theorem* to  $g$  we can assert that there exist  $t$  close to  $T$  and  $a_n$  small such that  $g(t, a_n) = 0$ , that is  $E_n^t(u) = 0$ . Actually there exist four such couples  $(t, a_n)$  with  $t > 0$  or  $t < 0$ , and  $a_n > 0$  or  $a_n < 0$ .

**Remark 3.7.** For such a couple  $(t, a_n)$  we can say more about the asymptotics of  $t$  with respect to  $a_n$ , which will be useful in the sequel. To this aim let us solve the following equation, where  $a_n$  is fixed :

$$g(t, a_n) = 0 \quad (22)$$

We have :

- $Q_t(e_1^T + v) = Q_T(e_1^T + v) + O(T - t)$ ,
- $Q_t(w) = Q_T(w) + O(T - t)$ ,
- $\tilde{Q}_t(e_1^T + v, w) = \tilde{Q}_T(e_1^T + v, w) + O(T - t)$ .

Hence equation (22) becomes :

$$Q_T(e_1^T + v)(T - t)^2 + a_n \tilde{Q}_T(e_1^T + v, w)(T - t) + a_n^2 Q_T(w) + o((T - t)^2, a_n^2) = 0$$

If  $a_n$  is fixed, close to 0, we easily find two solutions  $t_1 < T$  and  $t_2 > T$  such that both satisfy :

$$T - t = \beta a_n + o(a_n), \quad \beta \in \mathbb{R} \setminus \{0\}$$

To end the proof we have to study the cost  $C_t(u)$  and prove that  $t$  and  $a_n$  can be chosen so that this cost is lower than  $C_T(0)$ . Recall vectors  $v, w \in \mathbb{R}e_0^T \oplus F$  previously defined so that :

$$Q_T(e_1^T + v) < 0 \quad \text{and} \quad Q_T(w) > 0 \quad (23)$$

We can write :

$$v = \lambda e_0^T + v_1, \quad w = \mu e_0^T + w_1 \quad \text{where } v_1, w_1 \in F$$

Up to adding  $\varepsilon e_0^T$ ,  $\varepsilon$  small, we can assume that  $\mu \neq 0$ . This does not change anything in inequalities (23), nor in our previous reasoning. Let us denote  $C(t, u) = C_t(u)$ . We can expand :

$$C_t(u) = C_T(0) + \alpha(t - T) + \frac{\partial C}{\partial u}(T, 0).u + o(T - t, \|u\|)$$

where  $\alpha = \frac{\partial C}{\partial t}(T, 0)$ . Moreover by definition of  $u$  and  $e_i^T$  :

$$\frac{\partial C}{\partial u}(T, 0).u = dC_T(0).u = a_1 \lambda + a_n \mu$$

On the one part :  $a_1 = T - t + o(T - t, a_n)$ . On the other part from Remark 3.7 there exists  $\beta \neq 0$  such that  $a_n = \beta(t - T) + o(t - T)$ . Hence we obtain :

$$C_t(u) = C_T(0) + (t - T)(\alpha - \lambda + \beta\mu) + o(t - T)$$

Now among the four solutions of Remark 3.7, we can choose  $t$  and  $a_n$  such that  $\alpha - \lambda + \beta\mu \neq 0$  and  $(t - T)(\alpha - \lambda + \beta\mu) < 0$ . In these conditions :

$$C_t(u) < C_T(0)$$

which ends the proof. □

2. *Final time fixed.*

**Definition 3.8.** • The trajectory  $\gamma$  is said to be  $C^0$ -cost-minimal on  $[0, T]$  if there exists a  $C^0$ -neighborhood of  $\gamma$  such that for any trajectory  $q$  contained in this neighborhood, with  $q(0) = \gamma(0)$  and  $q(T) = \gamma(T)$ , we have :  $C_T(v) \geq C_T(0)$ , where  $v$  is the control associated to  $q$ .

• The trajectory  $\gamma$  is said to be  $L^\infty$ -cost-minimal on  $[0, T]$  if there exists a neighborhood of 0 in  $L^\infty([0, T])$  such that, for any trajectory  $q$  associated to a control  $v$  of this neighborhood, with  $q(0) = \gamma(0)$  and  $q(T) = \gamma(T)$ , we have :  $C_T(v) \geq C_T(0)$ .

We have the following :

**Theorem 3.9.** *The trajectory  $\gamma$  is  $C^0$ -cost-minimal if and only if  $T < t_c$ . Moreover,  $\gamma$  is not  $L^\infty$ -cost-minimal if  $T > t_c$  (whereas  $\gamma$  is  $C^0$ -time-minimal if and only if  $T < t_{cc}$ ), where  $t_{cc}$  and  $t_c$  denote the two types of first conjugate times of  $\gamma$ .*

Hence in this case, the times at which  $\gamma$  ceases to be minimizing are different in the time-optimal problem and cost-optimal problem :  $\gamma$  ceases to be  $C^0$ -time-optimal *before* it ceases to be  $C^0$ -cost-optimal (since  $t_{cc} < t_c$ , see Lemma 2.7).

*Proof.* The proof is quite similar to proof of Theorem 3.5 (but simpler) and is only sketched. In the coordinates  $(x_1, \dots, x_n)$  of Lemma 2.1, we write  $E_T = (E_1^T, \dots, E_n^T)$ , where  $d^2 E_n^T(0) = 0$ , and  $dE_1^T(0), \dots, dE_{n-1}^T(0), dC_T(0)$  are independant. We decompose :

$$L^2([0, T]) = \mathbb{R}e_1 \oplus \dots \oplus \mathbb{R}e_{n-1} \oplus \mathbb{R}e_0 \oplus F$$

where  $dE_i^T(0).e_i = 1$  and  $dC_T(0).e_0 = 1$ . By definition of  $t_c$  :

- if  $T < t_c$  :  $Q|_{\mathbb{R}e_0 \oplus F}$  is positive definite,
- if  $T > t_c$  :  $Q|_{\mathbb{R}e_0 \oplus F}$  is indefinite.

where  $Q = d^2 E_n^T(0)$ . Hence if  $T > t_c$  there exist  $v, w \in \mathbb{R}e_0 \oplus F$  such that  $Q(v) > 0$  and  $Q(w) < 0$ . Up to adding  $\varepsilon e_0$  we can assume that projections of  $v, w$  on  $\mathbb{R}e_0$  are not trivial. Consider the control :

$$u = a_1 e_1 + \dots + a_{n-1} e_{n-1} + \lambda v + \mu w$$

and try to solve equations :  $E_i(u) = \delta_i^1 T, i = 1 \dots n$ .

First from the Implicit Function Theorem we get, solving the  $n - 1$  first coordinates (i.e. the controllable part of the system) :

$$a_i = f_i(\lambda, \mu) = o(\lambda, \mu), i = 1 \dots n - 1$$

Now the last coordinate is :

$$\begin{aligned} E_n(u) &= Q(u) + o(\|u\|_{L^2}^2) \\ &= \lambda^2 Q(v) + \mu^2 Q(w) + \lambda\mu \check{Q}(v, w) + o(\lambda^2, \mu^2) \end{aligned}$$

Using the Mean Value Theorem we obtain :

- $\exists \lambda_1 < 0, \mu_1 > 0 / E_n(a_1 e_1 + \dots + a_{n-1} e_{n-1} + \lambda_1 v + \mu_1 w) = 0$
- $\exists \lambda_2 > 0, \mu_2 > 0 / E_n(a_1 e_1 + \dots + a_{n-1} e_{n-1} + \lambda_2 v + \mu_2 w) = 0$

There exists  $i = 1$  or  $2$  such that  $dC_T(0).(\lambda_i v + \mu_i w) \neq 0$ , and therefore there exists a trajectory steering 0 to  $\gamma(T)$  in time  $T$  with a cost strictly lower than  $C_T(0)$ .  $\square$

### 3.2. Optimality of abnormal trajectories for sub-Riemannian systems of rank 2

With the notations of Section 2.2 we have the following result, which can be proved in the same way as Theorem 2.21.

**Theorem 3.10.** *Under assumptions of Theorem 2.21, the abnormal reference trajectory  $\gamma$  is  $C^0$ -optimal for the sub-Riemannian system (14) if and only if it is  $C^0$ -time-minimal for its associated affine system (16). Moreover  $\gamma$  is abnormal for this affine system ; actually  $\gamma$  is  $C^0$ -optimal if  $T < t_{cc}$  and is not  $L^\infty$ -optimal if  $T > t_{cc}$ .*

In particular *conjugate times are the same along  $\gamma$  for the sub-Riemannian system (14) and its associated affine system (16)*. Therefore the whole formalism that was introduced for affine systems (the differential operators  $D_1, D_2$ ) is still valid in sub-Riemannian geometry. Hence the conjugate time of the sub-Riemannian problem can be computed using an algorithm. This result makes a link between works of [9] and [4], [5].

**Example 3.11.** The Martinet case (see Section 2.2.2) is in dimension 3, hence  $t_{cc} = +\infty$  (see Remark 3.3). The abnormal trajectory is optimal on  $\mathbb{R}^+$ .

**Remark 3.12.** As proved in [3] the  $C^0$ -optimality is in sub-Riemannian geometry equivalent to the optimality in the  $L^2$  topology on controls.

**Remark 3.13.** If  $T$  is small enough (depending on the choice of the Riemannian structure, and lower than  $t_{cc}$ ), then as first noted by [4]  $\gamma$  is moreover *globally* optimal among all sub-Riemannian trajectories steering 0 to  $\gamma(T)$ .

**Remark 3.14.** It should be noted that the loss of optimality holds in  $L^\infty$ . Hence using the definitions introduced in Section 2.2.2, the  $L^2$ -sector plays no role in the optimality of the reference abnormal trajectory. The loss of optimality holds in the  $L^\infty$ -sector.

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