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# RIGOROUS ASYMPTOTICS FOR THE ELECTRIC FIELD IN TM MODE AT MID-FREQUENCY IN A BIDIMENSIONAL MEDIUM WITH A THIN LAYER

CLAIR POIGNARD

**ABSTRACT.** Consider an ambient medium and a heterogeneous entity composed of a bidimensional material surrounded by a thin membrane. The electromagnetic constants of these materials are different. By analogy with biological cells, we call this entity a cell. We study the asymptotic behavior of the electric field in the transverse magnetic (TM) mode, when the thickness of the membrane tends to zero. The membrane is of thickness of  $hf(\theta)$ , with  $\theta$  a curvilinear coordinate. We provide a rigorous derivation of the first two terms of the asymptotic expansion for  $h$  tending to zero. In the membrane, these terms are given explicitly in local coordinates in terms of the boundary data and of the function  $f$ , while outside the membrane they are the solutions of a scalar Helmholtz equation with appropriate boundary and transmissions conditions given explicitly in terms of the boundary data and of the above function  $f$ . We prove that the remainder terms are of order  $O(h^{3/2})$ . In addition, if the complex dielectric permittivity in the membrane, denoted by  $z_m$ , tends to zero faster than  $h$ , we give the difference between the exact solution and the above asymptotic with  $z_m = 0$ ; it is of order  $O(h^{3/2} + |z_m|)$ .

## INTRODUCTION

We study in this paper the behavior of the solution of Helmholtz equation in a bidimensional medium in transverse magnetic (TM) mode (see Balanis and Constantine [3]). The medium is made out of three materials: a central region surrounded by a thin membrane of thickness  $hf(\theta)$ , with  $\theta$  curvilinear coordinate and a third material, which is not assumed to be thin; see Fig. 1. This assemblage is submitted to a field of pulsation  $\omega$ ; after proper scalings,  $\omega$  is included in the complex dielectric permittivity, which may be different in the three materials. By analogy with the biological cell, we call this entity a cell in an environment. In this article, we show that as the thickness of the membrane tends to zero, *i.e* as  $h$  tends to zero, the electric field tends to the solution of a Helmholtz equation with an appropriate transmission condition at the boundary between the cell and the ambient medium. This work is a sequel to the author's former article in the static case [10].

Let us give now the precise notations. Let  $\Omega$  be a bidimensional bounded domain composed of three subdomains: a bounded domain  $\mathcal{O}_c$  surrounded by a thin membrane  $\mathcal{O}_h$  with small thickness  $hf$ , and an exterior domain  $\mathcal{O}_{e,h}$ :

$$\Omega = \mathcal{O}_c \cup \mathcal{O}_h \cup \mathcal{O}_{e,h}.$$

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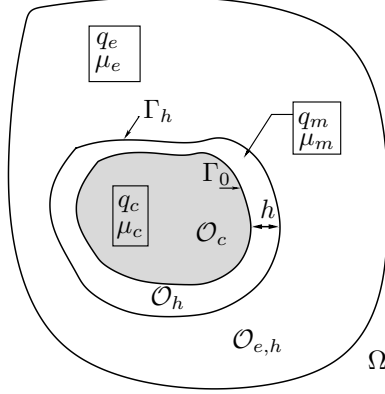


FIGURE 1. Geometric and dielectric data.

We suppose that the cell is strictly embedded in the ambient domain, that is:

$$\overline{(\mathcal{O}_c \cup \mathcal{O}_h)} \cap \partial\Omega = \emptyset.$$

The closed curves  $\Gamma_0$  and  $\Gamma_h$  are supposed to be of class  $\mathcal{C}^\infty$ . We denote by  $\Gamma_0$  and  $\Gamma_h$  respectively the boundaries of  $\partial\mathcal{O}_c$  and of  $\partial\mathcal{O}_h \cap \partial\mathcal{O}_{e,h}$ :

$$\begin{aligned}\Gamma_0 &= \partial\mathcal{O}_c, \\ \Gamma_h &= \partial\mathcal{O}_h \cap \partial\mathcal{O}_{e,h}.\end{aligned}$$

Let  $\mu_e$ ,  $\mu_m$  and  $\mu_c$  be the magnetic permittivities: they are constant and strictly positive. Let  $q_e$ ,  $q_m$  and  $q_c$  be three complex numbers with strictly negative imaginary part; they are non dimensionalized complex permittivities ( see [10] or [11] for a description of the non dimensionalization). We define piecewise constant functions  $\mu$  and  $q$  on  $\Omega$  as follows:

$$\mu = \begin{cases} \mu_e, & \text{in } \mathcal{O}_{e,h}, \\ \mu_m, & \text{in } \mathcal{O}_h, \\ \mu_c, & \text{in } \mathcal{O}_c, \end{cases} \quad q = \begin{cases} q_e, & \text{in } \mathcal{O}_{e,h}, \\ q_m, & \text{in } \mathcal{O}_h, \\ q_c, & \text{in } \mathcal{O}_c. \end{cases}$$

To simplify, we denote by  $z$  the product  $\mu q$ , and  $z_e$ ,  $z_m$  and  $z_c$  designate the restrictions of  $z$  respectively to the domains  $\mathcal{O}_{e,h}$ ,  $\mathcal{O}_m$  and  $\mathcal{O}_c$ . We summarize our hypotheses in Fig 1.

Let  $\phi$  be a given function of  $\partial\Omega$ . Its regularity will be chosen later on. We consider the electric field  $u$  solution of the following Helmholtz equation with Neumann boundary condition:

$$(1a) \quad \operatorname{div} \left( \frac{1}{\mu} \operatorname{grad} u \right) + qu = 0, \text{ in } \Omega,$$

$$(1b) \quad \partial_n u|_{\partial\Omega} = \phi, \text{ in } \Omega,$$

where  $\partial_n$  denotes the exterior normal derivative. Let us denote by  $u^e$ ,  $u^h$  and  $u^c$  the restrictions of  $u$  respectively to the domains  $\mathcal{O}_{e,h}$ ,  $\mathcal{O}_m$  and  $\mathcal{O}_c$ . These restrictions

satisfy the following transmission conditions:

$$\begin{aligned}
 (2a) \quad & \frac{1}{\mu_c} \partial_n u^c|_{\Gamma_0} = \frac{1}{\mu_m} \partial_n u^h|_{\Gamma_0}, \\
 (2b) \quad & \frac{1}{\mu_e} \partial_n u^e|_{\Gamma_h} = \frac{1}{\mu_m} \partial_n u^h|_{\Gamma_h}, \\
 (2c) \quad & u^c|_{\Gamma_0} = u^h|_{\Gamma_0}, \\
 (2d) \quad & u^e|_{\Gamma_h} = u^h|_{\Gamma_h}.
 \end{aligned}$$

We would like to understand the behavior for  $h$  tending to zero of the solution  $u$  of Problem (1).

In our proof, it suffices to assume that  $\mu_m$  and  $q_m$  are given constants;  $\mu_c$ ,  $\mu_e$ ,  $q_c$  and  $q_e$  could be continuous functions of the spatial coordinates with the imaginary part of  $q_c$  and  $q_e$  bounded away from zero.

Beretta and Francini have worked on a similar problem in [4]. They considered the thin dielectric material  $\mathcal{O}_h$  in an ambient medium, and they dealt with a Helmholtz equation with Dirichlet boundary condition. They compared, on the boundary of the domain  $\partial\Omega$ , the exact solution to the so-called background solution defined by replacing the material of the membrane by the interior material. The difference between these two solutions restricted to the boundary  $\partial\Omega$  has then been given through an integral involving the polarization tensor defined for instance in [1], [2], [5], [6], [7], plus some remainder terms. The remainder terms are estimated in terms of the measure of the inhomogeneity. In this paper, we do not use this approach since we are definitely interested in the transmembranar potential (see Fear and Stuchly [8]), and in the behavior of the field in the whole domain. We work with bidimensional domains and we expect that the same analysis could be performed in higher dimensions.

The heuristics of this work are the same as these performed in [10]. A change of coordinates in the membrane  $\mathcal{O}_h$  is performed, so as to parameterize it by local coordinates  $(\eta, \theta)$ , which vary in a domain independently of  $h$ ; in particular, if we denote by  $L$  the length of  $\partial\mathcal{O}_c$ , the variables  $(\eta, \theta)$  belong to  $[0, 1] \times \mathbb{R}/L\mathbb{Z}$ . This change of coordinates leads to an expression of the Laplacian in the membrane, which depends on  $h$ . Once the transmission conditions of the new problem are derived, we perform a formal asymptotic expansion of the solution of (1) in terms of  $h$ . It remains to validate this expansion.

This paper is structured as follows. First, we suppose that the electromagnetic constants  $q_c$ ,  $q_m$ ,  $q_e$ ,  $\mu_c$ ,  $\mu_m$  and  $\mu_e$  are given constants such that  $z_c$ ,  $z_m$  and  $z_e$  have strictly negative imaginary parts. In Section 1, we make precise our geometric conventions. We perform the change of variables in the membrane described above. We refer the reader to [10] for more information on the local coordinates. In Section 3, we derive formally the first two terms of the asymptotic expansion of the solution of our problem in terms of  $h$ . Section 5 is devoted to estimating the error.

In addition, in Remark 5.4, we give the first two terms of the asymptotic expansion of the electric field for a thin inhomogeneity attached to the boundary of the domain, and in Remark 5.6, we consider the case  $z_m = 0$ . We do not give the proofs of these asymptotics since they are very close to the one we performed in Section 5.

In the case of a biological cell,  $\mu$  is identically equal to 1 and the ratio  $|z_m/z_c|$  and  $|z_m/z_e|$  are very small, while  $|z_e/z_c|$  is of order 1. In Section 6, we show that if

the ratio  $|z_m|/|z_c|$  and  $|z_m|/|z_e|$  are small compared to  $h$ , we just have to replace  $z_m$  by 0 in the asymptotics found in Section 3 to obtain the electric field in the whole domain  $\Omega$  with an error in  $O(h^{3/2} + |z_m|)$ .

### 1. GEOMETRY STATEMENT

The boundary of the domain  $\mathcal{O}_c$  is assumed to be smooth. The boundary  $\Gamma_0$  is counterclockwise oriented, and we denote by  $\partial_t$  the tangential derivative along  $\mathcal{O}_c$ . To simplify, we suppose that the length of  $\Gamma_0$  is equal to  $2\pi$ . We denote by  $\mathbb{T}$  the flat torus:

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$

Since  $\Gamma_0$  is of class  $\mathcal{C}^\infty$ , we can parameterize it by a smooth function  $\Psi$  from  $\mathbb{T}$  to  $\mathbb{R}^2$  satisfying:

$$\forall \theta \in \mathbb{T}, \quad |\Psi'(\theta)| = 1.$$

We suppose that there exists a function  $f$  of class  $C^\infty(\mathbb{T})$  strictly positive, such that the membrane is of thickness  $hf$ .

The following identities hold:

$$\Gamma_0 = \{\Psi(\theta), \theta \in \mathbb{T}\},$$

and

$$\Gamma_h = \{\Psi(\theta) + hf(\theta)n(\theta), \theta \in \mathbb{T}\}.$$

Here  $n(\theta)$  is the unitary exterior normal at  $\Psi(\theta)$  to  $\Gamma_0$ . We parameterize the membrane  $\mathcal{O}_h$  as follows:

$$\mathcal{O}_h = \{\Psi(\theta) + hf(\theta)\eta n(\theta), (\eta, \theta) \in ]0, 1[ \times \mathbb{T}\}.$$

We define now:

$$\Phi(\eta, \theta) = \Psi(\theta) + hf(\theta)\eta n(\theta).$$

We denote by  $\kappa$  the curvature of the  $\Gamma_0$ . Let  $h_0 \in (0, 1)$  satisfy:

$$(3) \quad h_0 < \frac{1}{\|f\kappa\|_\infty}.$$

Then, for all  $h$  in  $[0, h_0]$ , there exists an open interval  $I$  containing  $(0, 1)$  such that  $\Phi$  is a diffeomorphism of class  $\mathcal{C}^\infty$  from  $I \times \mathbb{R}/2\pi\mathbb{Z}$  to its image, which is a neighborhood of the membrane. The metric in  $\mathcal{O}_h$  is:

$$(4) \quad h^2 f(\theta)^2 d\eta^2 + (1 + hf(\theta)\eta\kappa(\theta))^2 d\theta^2.$$

For convenience, we denote by  $f$  the following function corresponding to  $f$  written in Euclidean coordinates on  $\partial\mathcal{O}_c$ :

$$\forall x \in \partial\mathcal{O}_c, \quad f(x) = f \circ \Psi^{-1}(x).$$

We use two systems of coordinates, depending on the domains  $\mathcal{O}_{e,h}$ ,  $\mathcal{O}_c$  and  $\mathcal{O}_h$ : in the interior and exterior domains  $\mathcal{O}_{e,h}$  and  $\mathcal{O}_c$ , we use Euclidean coordinates  $(x, y)$  and in the membrane  $\mathcal{O}_h$ , we use local  $(\eta, \theta)$  coordinates with metric (4).

## 2. STATEMENT OF THE PROBLEM

In this section, we write Problem (1) in local coordinates. It is convenient to write:

$$\forall \theta \in \mathbb{T}, \quad \Phi_0(\theta) = \Phi(0, \theta), \quad \Phi_1(\theta) = \Phi(1, \theta).$$

Let us denote by  $u^e$  and  $u^c$  the electric field in  $\mathcal{O}_{e,h}$  and in  $\mathcal{O}_c$  respectively, written in Euclidean coordinates, and by  $u^m$  the electric field in  $\mathcal{O}_h$  in the local coordinates:

$$\begin{aligned} u^e &= u, \text{ in } \mathcal{O}_{e,h}, \\ u^c &= u, \text{ in } \mathcal{O}_c, \\ u^m &= u \circ \Phi, \text{ in } [0, 1] \times \mathbb{T}. \end{aligned}$$

We have shown in [10] that the Laplacian is given in the local coordinates  $(\eta, \theta)$  by:

$$(5) \quad \Delta|_{\Phi(\eta, \theta)} = \frac{1}{hf(1+hf\eta\kappa)} \left( \partial_\eta \left( \frac{1+hf\eta\kappa}{hf} \partial_\eta \right) + \partial_\theta \left( \frac{hf}{1+hf\eta\kappa} \partial_\theta \right) \right).$$

Therefore, we rewrite Problem (1) as follows:

$$(6a) \quad \Delta u^e + z_e u^e = 0, \text{ in } \mathcal{O}_{e,h},$$

$$(6b) \quad \Delta u^c + z_c u^c = 0, \text{ in } \mathcal{O}_c,$$

$$\forall (\eta, \theta) \in [0, 1] \times \mathbb{T},$$

$$(6c) \quad \partial_\eta \left( \frac{1+hf\eta\kappa}{hf} \partial_\eta u^m \right) + \partial_\theta \left( \frac{hf}{1+hf\eta\kappa} \partial_\theta u^m \right) + z_m hf(1+hf\eta\kappa) u^m = 0,$$

with the transmission conditions (2) expressed in local coordinates at  $\eta = 0$ :

$$(6d) \quad \frac{1}{\mu_c} \partial_n u^c \circ \Phi_0 = \frac{1}{hf\mu_m} \partial_\eta u^m \Big|_{\eta=0},$$

$$(6e) \quad u^c \circ \Phi_0 = u^m|_{\eta=0},$$

at  $\eta = 1$ :

$$(6f) \quad \frac{1}{\mu_e} \partial_n u^e \circ \Phi_1 = \frac{1+hf\kappa}{hf\mu_m} \partial_\eta u^m \Big|_{\eta=1},$$

$$(6g) \quad u^e \circ \Phi_1 = u^m|_{\eta=1},$$

and with the boundary condition

$$(6h) \quad \partial_n u^e|_{\partial\Omega} = \phi.$$

## 3. FORMAL ASYMPTOTIC EXPANSION

In this section, we derive asymptotic expansions of the electric field  $(u^e, u^c, u^m)$  solution of (6) in terms of the parameter  $h$ . We multiply (6c) by  $hf(1+hf\eta\kappa)^2$  and

we order in powers of  $h$  in order to obtain the partial differential equation satisfied by  $u^m$ :

$$\begin{aligned}
& \forall(\eta, \theta) \in [0, 1] \times \mathbb{T}, \\
& \partial_\eta^2 u^m + hf\kappa \{3\eta \partial_\eta^2 u^m + \partial_\eta u^m\} + h^2 f \left\{ 3\eta^2 f \kappa^2 \partial_\eta^2 u^m \right. \\
(7) \quad & \left. + 2\eta f \kappa^2 \partial_\eta u^m + f \partial_\theta^2 u^m + z_m f u^m + f' \partial_\theta u^m \right\} \\
& + h^3 f^3 \{ \eta^3 \kappa^3 \partial_\eta^2 u^m + \eta^2 \kappa^3 \partial_\eta u^m + \eta \kappa \partial_\theta^2 u^m \\
& - \eta \kappa' \partial_\theta u^m + 3z_m \eta \kappa u^m \} + 3h^4 f^4 z_m \eta^2 \kappa^2 u^m + h^5 f^5 \eta^3 \kappa^3 z_m u^m = 0
\end{aligned}$$

We write the following ansatz:

$$(8a) \quad u^e = u_0^e + hu_1^e + \dots,$$

$$(8b) \quad u^c = u_0^c + hu_1^c + \dots,$$

$$(8c) \quad u^m = u_0^m + hu_1^m + \dots.$$

We are now ready to derive the first two terms of the asymptotic expansions of  $u^e$ ,  $u^c$  and  $u^m$  by identifying the terms of the same power in  $h$ .

We want to obtain transmission conditions on  $\Gamma_0$  between  $u^e$  and  $u^c$  so as to replace the membrane. Therefore, we extend formally  $u^e$  to  $\Omega \setminus \mathcal{O}_c$ .

Moreover, we suppose that  $\phi$  is as regular as we need, so that we can apply Taylor formulae at any orders to the function  $u^e \circ \Phi$ :

$$\begin{aligned}
& \forall \theta \in \mathbb{T}, \forall \eta \in [0, 1], \\
& u^e \circ \Phi(\eta, \theta) = u^e \circ \Phi_0(\theta) + hf \partial_n u^e \circ \Phi_0(\theta) + \dots, \\
& \partial_n u^e \circ \Phi(\eta, \theta) = \partial_n u^e \circ \Phi_0(\theta) + hf \partial_n^2 u^e \circ \Phi_0(\theta) + \dots.
\end{aligned}$$

In particular, transmission conditions (6f)–(6g) are rewritten formally as follows:

$$(9a) \quad \frac{hf}{\mu_e} \left( \partial_n u^e \circ \Phi_0 + hf \partial_n^2 u^e \circ \Phi_0 + \dots \right) = \frac{1 + hf\kappa}{\mu_m} \partial_\eta u^m \Big|_{\eta=1},$$

$$(9b) \quad u^e \circ \Phi_0 + hf \partial_n u^e \circ \Phi_0 + \dots = u^m|_{\eta=1}.$$

**First step.** Substituting into (7) the potential  $u^m$  by its expansion (8c), and using the transmission conditions (6d) and (9a), we obtain:

$$\begin{cases} \partial_\eta^2 u_0^m = 0, \\ \partial_\eta u_0^m|_{\eta=1} = \partial_\eta u_0^m|_{\eta=0} = 0, \end{cases}$$

thus,

$$(10) \quad \forall(\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad u_0^m(\eta, \theta) = u_0^m(\theta).$$

In particular, we have :

$$(11) \quad u_0^c \circ \Phi_0 = u_0^e \circ \Phi_0.$$

We will determine  $u_0^m$  later on.

**Second step.** Substituting into (7) the potential  $u^m$  by its expansion (8c), and using the transmission conditions (6d)–(9a) and equality (10), we obtain:

$$\begin{aligned}\partial_\eta^2 u_1^m &= 0, \\ \partial_\eta u_1^m|_{\eta=0} &= \frac{f\mu_m}{\mu_c} \partial_n u_0^c \circ \Phi_0, \\ \partial_\eta u_1^m|_{\eta=1} &= \frac{f\mu_m}{\mu_e} \partial_n u_0^e \circ \Phi_1.\end{aligned}$$

According to transmission conditions (6e), by integrating by parts with respect to  $\eta$ , the following equality must hold:

$$(12) \quad u_1^m = \eta f \frac{\mu_m}{\mu_c} \partial_n u_0^c \circ \Phi_0 + u_1^c \circ \Phi_0.$$

thus

$$(13) \quad \frac{1}{\mu_c} \partial_n u_0^c \circ \Phi_0 = \frac{1}{\mu_e} \partial_n u_0^e \circ \Phi_0.$$

Remark that we must have also :

$$u_1^m = (\eta - 1) f \frac{\mu_m}{\mu_c} \partial_n u_0^c \circ \Phi_0 + u_1^e \circ \Phi_0 + f \partial_n u_0^e \circ \Phi_0.$$

Therefore, according to (11)–(13) and substituting into (6a)–(6b) respectively the potentials  $u^e$  and  $u^c$  by their expansion (8b), we infer that they satisfy the following PDE in  $\mathcal{O}_c \cup \mathcal{O}_{e,h}$ :

$$(14a) \quad \begin{cases} \Delta u_0^e + z_e u_0^e = 0, & \text{in } \mathcal{O}_{e,h}, \\ \Delta u_0^c + z_c u_0^c = 0, & \text{in } \mathcal{O}_c, \end{cases}$$

with the transmission conditions

$$(14b) \quad u_0^c|_{\Gamma_0} = u_0^e|_{\Gamma_0},$$

$$(14c) \quad \frac{1}{\mu_c} \partial_n u_0^c|_{\Gamma_0} = \frac{1}{\mu_e} \partial_n u_0^e|_{\Gamma_0},$$

with Neumann boundary condition:

$$(14d) \quad \partial_n u_0^e|_{\partial\Omega} = \phi.$$

According to the transmission condition (6e),  $u_0^m$  is equal to:

$$(15) \quad \forall (\eta, \theta) \in [0, L] \times \mathbb{T}, \quad u_0^m(\eta, \theta) = u_0^c \circ \Phi_0(\theta).$$

We have determined  $u_0^e$ ,  $u_0^c$  and  $u_0^m$ .

**Third step.** As in the previous paragraph, substituting into (7) the potential  $u^m$  by its expansion (8c), we obtain:

$$(16) \quad \partial_\eta^2 u_2^m + f\kappa \partial_\eta u_1^m + f^2 \partial_\theta^2 u_0^m + z_m f^2 u_0^m + f' f \partial_\theta u_0^m = 0,$$

and hence integrating (16) with respect to  $\eta$  and using the transmission conditions (6d) the following equality holds

$$(17) \quad \partial_\eta u_2^m = -\eta f (\kappa \partial_\eta u_1^m + f \partial_\theta^2 u_0^m + z_m f u_0^m + f' \partial_\theta u_0^m) + f \frac{\mu_m}{\mu_c} \partial_n u_1^c \circ \Phi_0.$$



We also have

$$(18) \quad \begin{aligned} \partial_\eta u_2^m &= (1 - \eta) f (\kappa \partial_\eta u_1^m + f \partial_\theta^2 u_0^m + z_m f u_0^m + f' \partial_\theta u_0^m) \\ &+ f \left( \frac{\mu_m}{\mu_e} (\partial_n u_1^e \circ \Phi_0 + f \partial_n^2 u_0^e \circ \Phi_0) - \kappa \partial_\eta u_1^m|_{\eta=1} \right). \end{aligned}$$

By the transmission condition (6d) and equality (12),  $u_1^e$  and  $u_1^c$  are the solution of:

$$(19a) \quad \begin{cases} \Delta u_1^e + z_e u_1^e = 0, \text{ in } \mathcal{O}_{e,h}, \\ \Delta u_1^c + z_c u_1^c = 0, \text{ in } \mathcal{O}_c, \\ \partial_n u_1^e|_{\partial\Omega} = 0, \end{cases}$$

with transmission conditions

$$(19b) \quad \begin{aligned} \frac{\mu_m}{\mu_c} \partial_n u_1^c|_{\Gamma_0} - \frac{\mu_m}{\mu_e} \partial_n u_1^e|_{\Gamma_0} &= f \partial_t^2 u_0^c|_{\Gamma_0} + z_m f u_0^c|_{\Gamma_0} + f' \partial_t u_0^c|_{\Gamma_0} \\ &+ \frac{\mu_m}{\mu_e} f \partial_n^2 u_0^e|_{\Gamma_0}, \end{aligned}$$

$$(19c) \quad u_1^c|_{\Gamma_0} - u_1^e|_{\Gamma_0} = \left(1 - \frac{\mu_m}{\mu_e}\right) f \partial_n u_0^e|_{\Gamma_0}.$$

Therefore,  $u_1^m$  given by equality (12) is entirely determined.

Let us summarize the first two terms of the asymptotics we obtained formally.

- *The 0<sup>th</sup> order terms.* The electric fields  $u_0^e$  and  $u_0^c$  are solution of the following problem in  $\Omega$ :

$$(20a) \quad \begin{cases} \Delta u_0^e + z_e u_0^e = 0, \text{ in } \Omega \setminus \mathcal{O}_c, \\ \Delta u_0^c + z_c u_0^c = 0, \text{ in } \mathcal{O}_c, \end{cases}$$

with transmission conditions

$$(20b) \quad u_0^c|_{\Gamma_0} = u_0^e|_{\Gamma_0},$$

$$(20c) \quad \frac{1}{\mu_c} \partial_n u_0^c|_{\Gamma_0} = \frac{1}{\mu_e} \partial_n u_0^e|_{\Gamma_0},$$

and with Neumann boundary condition:

$$(20d) \quad \partial_n u_0^e|_{\partial\Omega} = \phi.$$

In the membrane, the field  $u_0^m$  is equal to:

$$(21) \quad \forall (\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad u_0^m = u_0^c \circ \Phi_0(\theta).$$

- *The first order terms.* The fields  $u_1^e$  and  $u_1^c$  are solution of the following problem in  $\Omega$ :

$$(22a) \quad \begin{cases} \Delta u_1^e + z_e u_1^e = 0, \text{ in } \Omega \setminus \mathcal{O}_c, \\ \Delta u_1^c + z_c u_1^c = 0, \text{ in } \mathcal{O}_c, \\ \partial_n u_1^e|_{\partial\Omega} = 0, \end{cases}$$

with the following transmission conditions

$$(22b) \quad \frac{1}{\mu_c} \partial_n u_1^c|_{\Gamma_0} - \frac{1}{\mu_e} \partial_n u_1^e|_{\Gamma_0} = \frac{1}{\mu_m} (\mathbf{f} \partial_t^2 u_0^c|_{\Gamma_0} + z_m \mathbf{f} u_0^c|_{\Gamma_0} + (\partial_t \mathbf{f}) \partial_t u_0^c|_{\Gamma_0}) \\ + \frac{\mathbf{f}}{\mu_e} \partial_n^2 u_0^e|_{\Gamma_0},$$

$$(22c) \quad u_1^c|_{\Gamma_0} - u_1^e|_{\Gamma_0} = \frac{\mu_e - \mu_m}{\mu_c} \mathbf{f} \partial_n u_0^c|_{\Gamma_0}.$$

In the membrane, we have:

$$(23) \quad \forall (\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad u_1^m = \eta \mathbf{f} \frac{1}{\mu_c} \partial_n u_0^c \circ \Phi_0 + u_1^c \circ \Phi_0.$$

**Remark 3.1.** Remark that we can write  $\partial_n^2 u_0^e|_{\Gamma_0}$  in terms of  $\partial_n u_0^e|_{\Gamma_0}$ , of  $u_0^e|_{\Gamma_0}$  and of its tangential derivatives. Actually, we perform the change in local coordinates in a neighborhood of  $\partial \mathcal{O}_c$ .

according to (20), the following identity holds along  $\Gamma_0$ :

$$\partial_n^2 u_0^e|_{\Gamma_0} = -\kappa \circ \Phi_0^{-1} \partial_n u_0^e|_{\Gamma_0} - \partial_t^2 u_0^e|_{\Gamma_0} - z_e u_0^e|_{\Gamma_0},$$

thus we can rewrite transmission condition (22b) as follows:

$$(24) \quad \frac{1}{\mu_c} \partial_n u_1^c|_{\Gamma_0} - \frac{1}{\mu_e} \partial_n u_1^e|_{\Gamma_0} = \left( \frac{\mathbf{f}}{\mu_m} \partial_t^2 u_0^c - \frac{\mathbf{f}}{\mu_e} \partial_t^2 u_0^e \right) + \mathbf{f} (q_m u_0^c - q_e u_0^e) \\ + \frac{\partial_t \mathbf{f}}{\mu_m} \partial_t u_0^c - \frac{\kappa \circ \Phi_0^{-1}}{\mu_e} \mathbf{f} \partial_n u_0^e|_{\Gamma_0}.$$

We have given the first two terms of the asymptotic expansion of  $u^e$ ,  $u^c$  and  $u^m$ . It remains to prove that the remainder terms are small. First we need to study the regularity of  $u_0^c$  and  $u_1^e$  in a neighborhood of  $\Gamma_0$ .

#### 4. REGULARITY RESULT

In this section, we study the regularity of the solution of Helmholtz equation with non usual transmission condition. This result is required to prove our main theorem in Section 5, which gives the errors estimates between the exact solution and the asymptotics. The following result is well-known, and such a work has already been done by Y.Li and M.Vogelius in [9] in the Appendix page 147. However, they do not give exactly the result we need. We thanks very warmly Michael Vogelius for his suggestions on the reflection principle.

**Theorem 4.1.** Let  $G$  be in  $H^s(\Gamma_0)$ ,  $s \geq -1/2$ . Let  $(U^e, U^c)$  be the solution of the following problem:

$$\operatorname{div} \left( \frac{1}{\mu_c} \operatorname{grad} U^c \right) + q_c U^c = 0, \text{ in } \mathcal{O}_c, \\ \operatorname{div} \left( \frac{1}{\mu_e} \operatorname{grad} U^e \right) + q_e U^e = 0, \text{ in } \Omega \setminus \mathcal{O}_c,$$

with the following transmission condition:

$$U^e|_{\Gamma_0} = U^c|_{\Gamma_0}, \\ \frac{1}{\mu_e} \partial_n U^e|_{\Gamma_0} - \frac{1}{\mu_c} \partial_n U^c|_{\Gamma_0} = G,$$

with the Neumann boundary condition on  $\partial\Omega$

$$\partial_n U^e|_{\partial\Omega} = 0$$

Then we have:

$$U^e \in H^{s+3/2}(\Omega \setminus \mathcal{O}_c), \quad U^s \in H^{s+3/2}(\mathcal{O}_c).$$

*Proof.* Since  $\Gamma_0$  is smooth, we use local coordinates in a neighborhood of  $\Gamma_0$ . Actually, as in Section 1, there exists  $h_0$  such that:

$$\mathcal{V}_0 = \{\Psi(\theta) + h_0\eta n, (\eta, \theta) \in ]-1, 1[ \times \mathbb{T}\},$$

is an open neighborhood of  $\Gamma_0$ . We denote by  $g$  the function  $G$  written in local coordinates:

$$\forall \theta \in \mathbb{T}, \quad g(\theta) = G \circ \Psi(\theta).$$

We denote by  $D$  the unit disc  $[0, 1] \times \mathbb{T}$  and by  $H_m^1(D)$  the space of the functions  $\alpha$  defined on  $D$  such that:

$$\begin{aligned} & \int_0^1 \int_0^{2\pi} hf(1 + hf\eta\kappa) |\alpha(\eta, \theta)|^2 d\eta d\theta \\ & + \int_0^1 \int_0^{2\pi} \left( \frac{1 + hf\eta\kappa}{hf} |\partial_\eta \alpha(\eta, \theta)|^2 + \frac{hf}{1 + hf\eta\kappa} |\partial_\theta \alpha(\eta, \theta)|^2 \right) d\eta d\theta < +\infty. \end{aligned}$$

By partition of the unity, to prove Theorem 4.1, it is obvious that it suffices to prove that the solutions  $(V^e, V^c)$  of the following problem:

$$\begin{aligned} & \forall (\eta, \theta) \in [-1, 0] \times \mathbb{T}, \\ & \partial_\eta \left( \frac{1}{(1 + h_0\eta\kappa)\mu_c} \partial_\eta V^c \right) + \partial_\theta \left( \frac{1 + h_0\eta\kappa}{\mu_c} \partial_\theta V^c \right) + q_c(1 + h_0\eta\kappa)V^c = 0, \\ & \forall (\eta, \theta) \in [0, 1] \times \mathbb{T}, \\ & \partial_\eta \left( \frac{1}{(1 + h_0\eta\kappa)\mu_e} \partial_\eta V^e \right) + \partial_\theta \left( \frac{1 + h_0\eta\kappa}{\mu_e} \partial_\theta V^e \right) + q_e(1 + h_0\eta\kappa)V^e = 0, \end{aligned}$$

with Dirichlet boundaries conditions

$$V^c|_{\eta=-1} = 0, \quad V^e|_{\eta=1} = 0,$$

with transmission conditions

$$\begin{aligned} & V^c|_{\eta=0} = V^e|_{\eta=0}, \\ & \frac{1}{\mu_e} \partial_\eta V^e|_{\eta=0} - \frac{1}{\mu_c} \partial_\eta V^c|_{\eta=0} = g, \end{aligned}$$

have the same regularity respectively as the functions  $U^c$  and  $U^e$  restricted to  $\mathcal{V}_0$ . We use the reflection principle, suggested by Vogelius, which comes from an idea of Nirenberg (see [9] page 147). With this principle, we transform transmission conditions in boundary conditions.

We define  $V^r$  in  $[0, 1] \times \mathbb{T}$  by:

$$\forall (\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad V^r(\eta, \theta) = V^c(-\eta, \theta).$$

The functions  $V^e, V^r$  satisfy the following problem in  $[0, 1] \times \mathbb{T}$ :

$$\begin{aligned} & \forall(\eta, \theta) \in [0, 1] \times \mathbb{T}, \\ (25a) \quad & \partial_\eta \left( \frac{1 + h_0 \eta \kappa}{\mu_e} \partial_\eta V^e \right) + \partial_\theta \left( \frac{1}{(1 + h_0 \eta \kappa) \mu_e} \partial_\theta V^e \right) + q_e(1 + h_0 \eta \kappa) V^e = 0, \\ (25b) \quad & \partial_\eta \left( \frac{1 - h_0 \eta \kappa}{\mu_c} \partial_\eta V^r \right) + \partial_\theta \left( \frac{1}{(1 - h_0 \eta \kappa) \mu_c} \partial_\theta V^r \right) + q_c(1 - h_0 \eta \kappa) V^r = 0, \end{aligned}$$

with Dirichlet boundary conditions in  $\eta = 1$

$$(25c) \quad V^r|_{\eta=1} = 0, \quad V^e|_{\eta=1} = 0,$$

with boundary conditions in  $\eta = 0$ :

$$\begin{aligned} (25d) \quad & V^r|_{\eta=0} - V^e|_{\eta=0} = 0, \\ (25e) \quad & \frac{1}{\mu_c} \partial_\eta V^r|_{\eta=0} + \frac{1}{\mu_e} \partial_\eta V^e|_{\eta=0} = g, \end{aligned}$$

By multiplying (25a) by  $\overline{V^e}$  and (25b) by  $\overline{V^r}$  and by integrating by parts and by summing, we obtain:

$$\begin{aligned} & \int_0^1 \int_0^{2\pi} \left( \frac{1 + h_0 \eta \kappa}{\mu_e} |\partial_\eta V^e|^2 + \frac{1 - h_0 \eta \kappa}{\mu_c} |\partial_\eta V^r|^2 \right. \\ & + \frac{1}{(1 + h_0 \eta \kappa) \mu_e} |\partial_\theta V^e|^2 + \frac{1}{(1 - h_0 \eta \kappa) \mu_c} |\partial_\theta V^r|^2 - q_c(1 - h_0 \eta \kappa) |V^r|^2 \\ & \left. - q_e(1 + h_0 \eta \kappa) |V^e|^2 \right) d\eta d\theta = \int_0^{2\pi} \left( \frac{1}{\mu_e} \partial_\eta V^e|_{\eta=0} \overline{V^e}|_{\eta=0} + \frac{1}{\mu_c} \partial_\eta V^r|_{\eta=0} \overline{V^r}|_{\eta=0} \right) d\theta \end{aligned}$$

Using boundary conditions (25d)–(25e), we obtain easily

$$\int_0^{2\pi} \left( \frac{1}{\mu_e} \partial_\eta V^e|_{\eta=0} \overline{V^e}|_{\eta=0} + \frac{1}{\mu_c} \partial_\eta V^r|_{\eta=0} \overline{V^r}|_{\eta=0} \right) d\theta = \int_0^{2\pi} g \overline{V^e} d\theta.$$

This implies directly the uniqueness and then the existence of  $(V^e, V^r)$  in  $H_m^1(D)$ .

To obtain the regularity result, we just have to apply the method of freezing coefficients. Let  $\theta_0 \in \mathbb{T}$ , and denote by  $\kappa_0$  the value of  $\kappa$  in  $\theta_0$ . Actually, it is obvious that  $V^e$  and  $V^r$  have the same regularity respectively as  $V'$  and  $V''$  solutions of:

$$\begin{aligned} & \forall(\eta, \theta) \in [-1, 0] \times \mathbb{T}, \\ & \partial_\eta^2 V' + \frac{h_0 \kappa_0}{1 + h_0 \eta \kappa_0} \partial_\eta V' + \frac{\partial_\theta^2 V'}{(1 + h_0 \eta \kappa_0)^2} = 0, \\ & \forall(\eta, \theta) \in [0, 1] \times \mathbb{T}, \\ & \partial_\eta^2 V'' - \frac{h_0 \kappa_0}{1 - h_0 \eta \kappa_0} \partial_\eta V'' + \frac{\partial_\theta^2 V''}{(1 - h_0 \eta \kappa_0)^2} = 0, \end{aligned}$$

with Dirichlet boundaries conditions

$$V'|_{\eta=1} = 0, \quad V''|_{\eta=1} = 0,$$

with transmission conditions

$$\begin{aligned} V'|_{\eta=0} &= V''|_{\eta=0}, \\ \frac{1}{\mu_e} \partial_\eta V'|_{\eta=0} + \frac{1}{\mu_c} \partial_\eta V''|_{\eta=0} &= g, \end{aligned}$$

Then, writing this problem in Fourier coefficients, we obtain easily the result.  $\square$

## 5. ERROR ESTIMATES

We give an error estimate, which proves that the first two terms found in Section 3 by a formal argument are indeed the first terms, in the sense that the remainder is smaller.

**Remark 5.1.** *Let us denote by  $D$  the unit disc  $[0, 1] \times \mathbb{T}$ . We remember that the  $L^2$  norm of a 0-form  $\alpha$  in  $D$  with the metric (4), denoted by  $\|\alpha\|_{\Lambda^0 L_m^2(D)}$ , is equal to:*

$$\begin{aligned} \|\alpha\|_{\Lambda^0 L_m^2(D)}^2 &= \int_0^1 \int_0^{2\pi} h f (1 + h f \eta \kappa) |\alpha(\eta, \theta)|^2 d\eta d\theta, \\ &= \|\alpha \circ \Phi^{-1}\|_{L^2(\mathcal{O}_h)}^2, \end{aligned}$$

and the  $L^2$  norm of the exterior derivative  $d\alpha$  of  $\alpha$ , denoted by  $\|d\alpha\|_{\Lambda^1 L_m^2(D)}$  is equal to

$$\begin{aligned} \|d\alpha\|_{\Lambda^1 L_m^2(D)}^2 &= \int_0^1 \int_0^{2\pi} \frac{1 + h f \eta \kappa}{h f} |\partial_\eta \alpha(\eta, \theta)|^2 + \frac{h f}{1 + h f \eta \kappa} |\partial_\theta \alpha(\eta, \theta)|^2 d\eta d\theta, \\ &= \|\text{grad}(\alpha \circ \Phi^{-1})\|_{L^2(\mathcal{O}_h)}^2. \end{aligned}$$

We have the following theorem.

**Theorem 5.2.** *Let  $f$  be in  $C^\infty(\mathbb{T})$ , such that*

$$\forall \theta \in \mathbb{T}, \quad f(\theta) > 0,$$

and let  $h_0$  be in  $(0, 1)$  such that:

$$h_0 < \frac{1}{\|f\kappa\|_\infty}.$$

Let  $h$  be in  $(0, h_0)$  and let  $\phi$  be in  $H^{5/2}(\partial\Omega)$ .

We denote by  $u$  the solution of Problem (1), and  $(u_0^e, u_0^c)$  and  $(u_1^e, u_1^c)$ , are defined in Section 3 by equalities (20)–(23). Let  $(v^e, v^c)$  be the functions defined on  $\Omega$  by:

$$\begin{aligned} v^e &= u_0^e + h u_1^e, \text{ in } \Omega \setminus \mathcal{O}_c, \\ v^c &= u_0^c + h u_1^c, \text{ in } \mathcal{O}_c. \end{aligned}$$

Then, there exists a constant  $C > 0$  independant of  $h$  but depending on the domain  $\mathcal{O}_c$ , on the function  $f$  and on the dielectric parameters  $\mu_e, \mu_c, q_e, q_m$  and  $q_c$  of the material such that

$$\begin{aligned} \|u - v^c\|_{H^1(\mathcal{O}_c)} &\leq C h^{3/2} \|\phi\|_{H^{5/2}(\partial\Omega)}, \\ \|u - v^e\|_{\Lambda^0 L_m^2(D)} + \left\| \frac{1}{\mu_m} du - \frac{1}{\mu_e} dv^e \right\|_{\Lambda^1 L_m^2(D)} &\leq C h^{3/2} \|\phi\|_{H^{5/2}(\partial\Omega)}, \\ \|u - v^e\|_{H^1(\mathcal{O}_{e,h})} &\leq C h^{3/2} \|\phi\|_{H^{5/2}(\partial\Omega)}. \end{aligned}$$

*Proof.* First, remark that since  $\phi$  is in  $H^{5/2}(\partial\Omega)$ , it is clear, by the results of Section 4, that  $u_0^e$  is in  $H^4(\Omega \setminus \mathcal{O}_c)$ . Thus, for almost all  $\theta \in \mathbb{T}$ ,  $u_0^e \circ \Phi(\cdot, \theta)$  is in  $H^{7/2}([0, 1])$ . We infer also that  $u_1^e$  is in  $H^3(\Omega \setminus \mathcal{O}_c)$ .

To prove theorem (5.2), we need the following lemma.

**Lemma 5.3.** *Let  $h$  be in  $(0, h_0)$ .*

*Let  $u_0^e$ ,  $u_1^e$ ,  $u_0^m$  and  $u_1^m$  be defined by (20) and (22). We denote by  $v$  and  $\tilde{v}$  the following functions:*

$$\begin{aligned} \forall x \in \Omega \setminus \mathcal{O}_c, \quad v^e(x) &= u_0^e(x) + hu_1^e(x), \\ \forall (\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad \tilde{v}(\eta, \theta) &= u_0^m + hu_1^m. \end{aligned}$$

*Then, there exists a constant  $C > 0$  independant of  $h$  but depending on the domain  $\mathcal{O}_c$ , on the function  $f$  and on the dielectric parameters  $\mu_e$ ,  $\mu_m$ ,  $\mu_c$ ,  $q_e$ ,  $q_m$  and  $q_c$  of the material such that*

$$(26) \quad \|v^e \circ \Phi - \tilde{v}\|_{\Lambda^0 L^2(D)} \leq Ch^{3/2} \|\phi\|_{H^{5/2}(\partial\Omega)},$$

$$(27) \quad \left\| \frac{1}{\mu_e} d(v^e \circ \Phi) - \frac{1}{\mu_m} d\tilde{v} \right\|_{\Lambda^1 L^2(D)} \leq Ch^{3/2} \|\phi\|_{H^{5/2}(\partial\Omega)}$$

and

$$(28) \quad \|v^e \circ \Phi_1 - \tilde{v}|_{\eta=1}\|_{\Lambda^0 L^2(\mathbb{T})} \leq Ch^2 \|\phi\|_{H^{5/2}(\partial\Omega)}.$$

*Proof.* Since for almost all  $\theta \in \mathbb{T}$ ,  $u_0^e \circ \Phi(\cdot, \theta)$  is in  $H^{7/2}([0, 1])$ , we can apply Taylor formula. Let  $(\eta, \theta)$  be in  $[0, 1] \times \mathbb{T}$ , we have:

$$\begin{aligned} v^e \circ \Phi(\eta, \theta) &= u_0^e \circ \Phi_0(\theta) + hf\eta \partial_n u_0^e \circ \Phi_0(\theta) + hu_1^e \circ \Phi_0(\theta) \\ &\quad + h^2 f \int_0^\eta (\eta - t) \left( \partial_n u_1^e \circ \Phi(t, \theta) + \frac{(\eta - t)}{2} \partial_n^2 u_0^e \circ \Phi(t, \theta) \right) dt, \end{aligned}$$

and

$$\begin{aligned} \partial_\eta(v^e \circ \Phi)(\eta, \theta) &= hf \partial_n u_0^e \circ \Phi_0(\theta) + h^2 f \left( \partial_n u_1^e \circ \Phi(\eta, \theta) \right. \\ &\quad \left. + f \int_0^\eta (\eta - t) \partial_n^2 u_0^e \circ \Phi(t, \theta) dt \partial_n u_1^e \circ \Phi \right). \end{aligned}$$

Since we have:

$$\begin{aligned} \tilde{v}(\eta, \theta) &= u_0^c \circ \Phi_0(\theta) + h\eta f \frac{\mu_m}{\mu_c} \partial_n u_0^c \circ \Phi_0(\theta) + hu_1^c \circ \Phi_0(\theta), \\ \partial_\eta \tilde{v}(\eta, \theta) &= hf \frac{\mu_m}{\mu_c} \partial_n u_0^c \circ \Phi_0(\theta), \end{aligned}$$

using transmission condition (20b) we obtain:

$$\begin{aligned} v^e \circ \Phi - \tilde{v} &= h \left( \eta f(\theta) \left( 1 - \frac{\mu_m}{\mu_e} \right) \partial_n u_0^e \circ \Phi_0(\theta) + u_1^e \circ \Phi_0(\theta) - u_1^c \circ \Phi_0(\theta) \right. \\ &\quad \left. + hf(\theta) \int_0^\eta (\eta - t) \left( \partial_n u_1^e \circ \Phi(t, \theta) + \frac{(\eta - t)}{2} \partial_n^2 u_0^e \circ \Phi(t, \theta) \right) dt \right). \end{aligned}$$

This equality implies directly estimate (26). Moreover, using transmission conditions (22b) and (22c), we obtain:

$$v^e \circ \Phi_1(\theta) - \tilde{v}(1, \theta) = h^2 f \int_0^1 (1-t) \left( \partial_n u_1^e \circ \Phi(t, \theta) + \frac{(\eta-t)}{2} \partial_n^2 u_0^e \circ \Phi(t, \theta) \right) dt,$$

which implies (28). We also have:

$$\begin{aligned} \frac{1}{\mu_e} \partial_\eta (v^e \circ \Phi)(\eta, \theta) - \frac{1}{\mu_m} \partial_\eta \tilde{v}(\eta, \theta) &= \frac{h^2 f}{\mu_e}(\theta) \left( f(\theta) \int_0^\eta (\eta-t) \partial_n^2 u_0^e \circ \Phi(t, \theta) dt \right. \\ &\quad \left. + \partial_n u_1^e \circ \Phi(\eta, \theta) \right), \end{aligned}$$

which implies (27). This ends the proof of Lemma 5.3  $\square$

Now, we prove Theorem 5.2.

Denote by  $W^e$ ,  $W^c$  and  $W^m$  the following functions:

$$(29a) \quad W^e = u^e - (u_0^e + h u_1^e), \text{ in } \mathcal{O}_{e,h},$$

$$(29b) \quad W^c = u^c - (u_0^c + h u_1^c), \text{ in } \mathcal{O}_c,$$

$$(29c) \quad W^m = u^m - (u_0^m + h u_1^m), \text{ in } [0, 1] \times \mathbb{T}.$$

In order to simplify the notations, we introduce  $\mathcal{L}_{\eta,\theta}$ , the Helmholtz operator written in the local coordinates  $(\eta, \theta)$  given by

$$\mathcal{L}_{\eta,\theta} = \partial_\eta \left( \frac{1 + h f \eta \kappa}{h f} \partial_\eta \right) + \partial_\theta \left( \frac{h f}{1 + h f \eta \kappa} \partial_\theta \right) + z_m h f (1 + h f \eta \kappa).$$

Let us write the problem satisfied by  $(W^e, W^c, W^m)$ . We use the expressions of  $u_0^e$ ,  $u_1^e$ ,  $u_0^c$ ,  $u_1^c$ ,  $u_0^m$  and  $u_1^m$  found in Section 3 to obtain, by a simple calculation:

$$(30a) \quad \Delta W^e + z_e W^e = 0, \text{ in } \mathcal{O}_{e,h},$$

$$(30b) \quad \Delta W^c + z_c W^c = 0, \text{ in } \mathcal{O}_c,$$

$$\forall (\eta, \theta) \in [0, 1] \times \mathbb{T},$$

$$\begin{aligned} (30c) \quad \mathcal{L}_{\eta,\theta} W^m &= - \left\{ h (\kappa \partial_\eta u_m^1 + f \partial_\theta^2 u_0^m + z_m f u_0^m + f' \partial_\theta u_0^m) \right. \\ &\quad + h^2 f \left( - \frac{\eta \kappa f}{1 + h f \eta \kappa} \partial_\theta^2 u_0^m - \frac{f' (\eta \kappa + h f \eta^2 \kappa^2)}{(1 + h f \eta \kappa)} \partial_\theta u_0^m \right. \\ &\quad - \frac{\eta \kappa' f}{(1 + h f \eta \kappa)^2} \partial_\theta u_0^m + \partial_\theta \left( \frac{f \partial_\theta u_1^m}{1 + h f \eta \kappa} \right) \\ &\quad \left. \left. + z_m \eta f \kappa u_0^m + z_m (1 + h f \eta \kappa) f u_1^m \right) \right\}. \end{aligned}$$

with the transmission conditions coming from (2)

$$(30d) \quad \frac{1}{\mu_c} \partial_n W^c \circ \Phi_0 = \frac{1}{hf\mu_m} \left( \partial_\eta W^m|_{\eta=0} - h^2 f \frac{1}{\mu_c} \partial_n u_1^c \circ \Phi_0 \right),$$

$$(30e) \quad W^c \circ \Phi_0 = W^m|_{\eta=0},$$

$$(30f) \quad \begin{aligned} \frac{1}{\mu_e} \partial_n W^e \circ \Phi_1 &= \frac{1+hf\kappa}{hf\mu_m} \left( \partial_\eta W^m|_{\eta=1} + h \partial_\eta u_1^m|_{\eta=1} \right) \\ &\quad - \frac{1}{\mu_e} \partial_n u_0^e \circ \Phi_1 - \frac{h}{\mu_e} \partial_n u_1^e \circ \Phi_1, \end{aligned}$$

$$(30g) \quad W^e \circ \Phi_1 = W^m|_{\eta=1} + u_0^m|_{\eta=1} + hu_1^m|_{\eta=1} - u_0^e \circ \Phi_1 - hu_1^e \circ \Phi_1,$$

and the boundary condition

$$(30h) \quad \partial_\eta W^e|_{\eta=1} = 0.$$

Using (22b) and (23), and applying Taylor formula, we rewrite (30f) as follows:

$$(31) \quad \begin{aligned} \frac{1}{\mu_e} \partial_n W^e \circ \Phi_1 &= \frac{1+hf\kappa}{hf\mu_m} \left\{ \partial_\eta W^m|_{\eta=1} - \frac{h^2 f}{1+hf\kappa} \frac{\mu_m}{\mu_e} \left( -\kappa f \partial_n u_0^e \circ \Phi_0 \right. \right. \\ &\quad \left. \left. + f \partial_n^2 u_0^e \circ \Phi_0 + \partial_n u_1^e \circ \Phi_0 \right. \right. \\ &\quad \left. \left. + hf \int_0^1 (1-t) \left( \frac{1-t}{2} f \partial_n^3 u_0^e \circ \Phi(t, \theta) \right. \right. \right. \\ &\quad \left. \left. \left. + \partial_n^2 u_1^e \circ \Phi(t, \theta) \right) dt \right) \right\}, \end{aligned}$$

moreover, according to Lemma 5.3, we rewrite (30g) as:

$$(32) \quad W^e \circ \Phi_1 = W^m|_{\eta=1} - (v \circ \Phi - \tilde{v})|_{\eta=1}.$$

The trick of the proof consists in building a function  $B^m$  defined on  $[0, 1] \times \mathbb{T}$  and in introducing the function  $A^m$  defined on  $[0, 1] \times \mathbb{T}$  by:

$$A^m = W^m - h^2 B^m,$$

such that the term in  $h$  of the right-hand side of (30c) disappears. Let us define in  $[0, 1] \times \mathbb{T}$  the function  $b^m$  as follows:

$$(33a) \quad b^m = -\frac{\eta^2 f}{2} (\kappa \partial_\eta u_1^m + f \partial_\theta^2 u_0^m + z_m f u_0^m + f' \partial_\theta u_0^m) + \frac{\eta f \mu_m}{\mu_c} \partial_n u_1^c \circ \Phi_0,$$

and let  $B^m$  be such that:

$$(33b) \quad B^m(\eta, \theta) = b^m(\eta, \theta) - b^m(1, \theta) + (v \circ \Phi_1 - \tilde{v}|_{\eta=1}) / h^2.$$

According to Lemma (5.3), there exists a constant  $C$  depending on  $\mathcal{O}_c$ , on the function  $f$  and on  $\mu_e, \mu_m, \mu_c, q_e, q_m$  and  $q_c$  such that the following estimate holds:

$$(34) \quad \forall \eta \in [0, 1], \quad \|B^m(\eta, \cdot)\|_{H^2(\mathbb{T})} \leq C \|\phi\|_{H^{5/2}(\partial\Omega)}.$$



A simple calculation gives the following equalities in  $[0, 1] \times \mathbb{T}$ :

$$\begin{aligned}\partial_\eta B^m &= -\eta f \left( \kappa \partial_\eta u_1^m + f \partial_\theta^2 u_0^m + z_m f u_0^m + f' \partial_\theta u_0^m \right) + \frac{f \mu_m}{\mu_c} \partial_n u_1^c \circ \Phi_0, \\ \partial_\eta^2 B^m &= -f \left( \kappa \partial_\eta u_1^m + f \partial_\theta^2 u_0^m + z_m f u_0^m + f' \partial_\theta u_0^m \right).\end{aligned}$$

In particular, according to (22b), we have in  $\eta = 1$ :

$$\partial_\eta B^m|_{\eta=1} = \frac{f \mu_m}{\mu_e} \left( \partial_n u_1^e \circ \Phi_0 + f \partial_n^2 u_0^e \circ \Phi_0 - \kappa \partial_n u_0^e \circ \Phi_0 \right).$$

We define  $A^m$  in  $[0, 1] \times \mathbb{T}$  as follows:

$$A^m = W^m - h^2 B^m.$$

It is obvious that  $A^m$  satisfies the following equalities:

(35a)

$$\partial_\eta A^m|_{\eta=0} = \partial_\eta W^m|_{\eta=0} - \frac{h^2 f \mu_m}{\mu_c} \partial_n u_1^c \circ \Phi_0,$$

(35b)

$$\partial_\eta A^m|_{\eta=1} = \partial_\eta W^m|_{\eta=1} - \frac{h^2 f \mu_m}{\mu_e} \left( \partial_n u_1^e \circ \Phi_0 + f \partial_n^2 u_0^e \circ \Phi_0 - \kappa f \partial_n u_0^e \circ \Phi_0 \right),$$

(35c)  $A^m|_{\eta=0} = W^m|_{\eta=0} - h^2 B^m|_{\eta=0},$

(35d)  $A^m|_{\eta=1} = W^m|_{\eta=1} - (v \circ \Phi - \tilde{v})|_{\eta=1}.$

Therefore, according to (30)–(31) and to (35),  $(W^e, A^m, W^c)$  satisfies the following transmission conditions respectively in  $\eta = 0$  and in  $\eta = 1$ :

$$(36a) \quad \frac{1}{\mu_c} \partial_n W^c \circ \Phi_0 = \frac{1}{h f \mu_m} \partial_\eta A^m|_{\eta=0},$$

$$(36b) \quad W^c \circ \Phi_0 = A^m|_{\eta=0} + h^2 B^m|_{\eta=0},$$

$$\begin{aligned}(36c) \quad \frac{1}{\mu_e} \partial_n W^e \circ \Phi_1 &= \frac{1 + h f \kappa}{h f \mu_m} \left( \partial_\eta A^m|_{\eta=1} - \frac{h^3 f^2}{1 + h f \kappa} \frac{\mu_m}{\mu_e} \left( f \kappa (\kappa f \partial_n u_0^e \circ \Phi_0 \right. \right. \\ &\quad \left. \left. - \partial_n u_1^e \circ \Phi_0 - f \partial_n^2 u_0^e \circ \Phi_0) + \int_0^1 (1-t) \left( \partial_n^2 u_1^e \circ \Phi(t, \theta) \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{(1-t)f}{2} \partial_n^3 u_0^e \circ \Phi(t, \theta) \right) dt \right) \right),\end{aligned}$$

$$(36d) \quad W^e \circ \Phi_1 = A^m|_{\eta=1},$$

We denote by  $b$  the right-hand side of (36c):

$$\begin{aligned}b &= \frac{f}{1 + h f \kappa} \frac{\mu_m}{\mu_e} \left( f \kappa (\kappa f \partial_n u_0^e \circ \Phi_0 - \partial_n u_1^e \circ \Phi_0 - f \partial_n^2 u_0^e \circ \Phi_0) \right. \\ &\quad \left. + \int_0^1 (1-t) \left( \partial_n^2 u_1^e \circ \Phi(t, \theta) + \frac{(1-t)f}{2} \partial_n^3 u_0^e \circ \Phi(t, \theta) \right) dt \right).\end{aligned}$$

Let us now introduce  $B^c$  defined in  $\mathcal{O}_c$  by:

$$(37) \quad \begin{cases} \Delta B^c + z_c B^c = 0, & \text{in } \mathcal{O}_c, \\ B^c|_{\Gamma_0} = B^m|_{\eta=0}, & \text{on } \Gamma_0. \end{cases}$$

We define in  $\mathcal{O}_c$  the function  $A^c$  as follows:

$$(38) \quad A^c = W^c - h^2 B^c.$$

According to equalities (30)–(31), (36) and (37), we see easily that the three functions  $A^c$ ,  $A^m$  and  $W^e$  satisfy the following problem

$$(39a) \quad \Delta W^e + z_e W^e = 0, \text{ in } \mathcal{O}_{e,h},$$

$$(39b) \quad \Delta A^c + z_c A^c = 0, \text{ in } \mathcal{O}_c,$$

$$\forall (\eta, \theta) \in [0, 1] \times \mathbb{T},$$

$$(39c) \quad \begin{aligned} \mathcal{L}_{\eta,\theta} A^m = & -h^2 f \left\{ -\frac{\eta f \kappa}{1 + h f \eta \kappa} \partial_\theta^2 u_0^m - \frac{f'(\eta \kappa + h f \eta^2 \kappa^2)}{1 + h f \eta \kappa} \partial_\theta u_0^m \right. \\ & - \frac{\eta f \kappa'}{(1 + h f \eta \kappa)^2} \partial_\theta u_0^m + \partial_\theta \left( \frac{f(\partial_\theta u_1^m + h B^m)}{1 + h f \eta \kappa} \right) \\ & + z_m f \eta \kappa u_0^m + z_m f(1 + h f \eta \kappa) u_1^m \\ & \left. + z_m f \left( \eta \kappa u_0^m + (1 + h f \eta \kappa)(u_1^m + h B^m) \right) \right\}. \end{aligned}$$

with the following transmission conditions

$$(39d) \quad \frac{1}{\mu_c} \partial_n A^c \circ \Phi_0 = \frac{1}{h f \mu_m} \partial_\eta A^m|_{\eta=0} - \frac{h^2}{\mu_c} \partial_n B^c \circ \Phi_0,$$

$$(39e) \quad A^c \circ \Phi_0 = A^m|_{\eta=0},$$

$$(39f) \quad \frac{1}{\mu_e} \partial_n W^e \circ \Phi_1 = \frac{1 + h f \kappa}{h f \mu_m} \left( \partial_\eta A^m|_{\eta=1} - h^3 f b \right),$$

$$(39g) \quad W^e \circ \Phi_1 = A^m|_{\eta=1},$$

and the boundary condition

$$(39h) \quad \partial_\eta W^e|_{\eta=1} = 0.$$

To simplify the notations, we denote by  $g$  the right-hand side of (39c):

$$(40) \quad \begin{aligned} \forall (\eta, \theta) \in [0, 1] \times \mathbb{T}, \\ g(\eta, \theta) = & -\frac{f \eta \kappa}{1 + h f \eta \kappa} \partial_\theta^2 u_0^m - \frac{f'(\eta \kappa + h \eta^2 \kappa^2)}{1 + h f \eta \kappa} \partial_\theta u_0^m \\ & - \frac{f \eta \kappa'}{(1 + h f \eta \kappa)^2} \partial_\theta u_0^m + \partial_\theta \left( \frac{f(\partial_\theta u_1^m + h f B^m)}{1 + h f \eta \kappa} \right) \\ & + z_m f \eta \kappa u_0^m + z_m f(1 + h f \eta \kappa) u_1^m. \end{aligned}$$

Now we are ready to perform  $L^2$  estimates as it has been performed in [10].

In  $\mathcal{O}_c$  parameterized by Euclidean coordinates, the  $L^2$  norm of a 0-form  $\beta$ , denoted by  $\|\beta\|_{L^2(\mathcal{O}_c)}$ , is equal to:

$$\|\beta\|_{\Lambda^0 L^2(\mathcal{O}_c)} = \|\beta\|_{L^2(\mathcal{O}_c)},$$

and the  $L^2$  norm of its exterior derivative  $d\beta$ , denoted by  $\|d\beta\|_{\Lambda^1 L^2(\mathcal{O}_c)}$  is equal to

$$\|d\beta\|_{\Lambda^1 L^2(\mathcal{O}_c)} = \|\text{grad } \beta\|_{L^2(\mathcal{O}_c)}.$$

In  $\mathcal{O}_{e,h}$  parameterized by Euclidean coordinates, the  $L^2$  norm of a 0-form  $\gamma$ , denoted by  $\|\gamma\|_{L^2(\mathcal{O}_{e,h})}$ , is equal to:

$$\|\gamma\|_{\Lambda^0 L^2(\mathcal{O}_{e,h})} = \|\gamma\|_{L^2(\mathcal{O}_{e,h})},$$

and the  $L^2$  norm of its exterior derivative  $dv$ , denoted by  $\|dv\|_{\Lambda^1 L^2(\mathcal{O}_{e,h})}$  is equal to

$$\|d\gamma\|_{\Lambda^1 L^2(\mathcal{O}_{e,h})} = \|\text{grad } \gamma\|_{L^2(\mathcal{O}_{e,h})}.$$

We multiply equalities (39a)–(39b)–(39c) respectively by the conjugates of  $W^e$ ,  $A^c$  and  $A^m$ . Using the transmission conditions (39), we integrate by part and we take the imaginary part of the result. To simplify the notations, we define  $\|A\|_{\Lambda^0 L^2(\Omega)}^2$  and  $\|dA\|_{\Lambda^1 L^2(\Omega)}^2$  as follows:

$$\begin{aligned} \|A\|_{\Lambda^0 L^2(\Omega)}^2 &= \|W^e\|_{\Lambda^0 L^2(\mathcal{O}_{e,h})}^2 + \|A^m\|_{\Lambda^0 L_m^2(D)}^2 + \|A^c\|_{\Lambda^0 L^2(\mathcal{O}_c)}^2, \\ \|dA\|_{\Lambda^1 L^2(\Omega)}^2 &= \|dW^e\|_{\Lambda^1 L^2(\mathcal{O}_{e,h})}^2 + \|dA^m\|_{\Lambda^1 L_m^2(D)}^2 + \|dA^c\|_{\Lambda^1 L^2(\mathcal{O}_c)}^2, \end{aligned}$$

we obtain:

$$\begin{aligned} \min(\Im(z_e), \Im(z_m), \Im(z_c)) \|A\|_{\Lambda^0 L^2(\Omega)}^2 &\leq \frac{h^2}{\mu_c} \left| \int_{\Gamma_0} \partial_n B^c \overline{A^c} \, d\text{vol}_{\Gamma_0} \right| \\ &\quad + \frac{h^2}{\mu_m} \left| \int_0^{2\pi} b \overline{A^e} \circ \Phi_1 \, d\theta \right| \\ &\quad + h^2 f \left| \int_0^1 \int_0^{2\pi} g \overline{A^m} \, d\eta \, d\theta \right|. \end{aligned}$$

Therefore, there exists a constant  $C$  depending on  $\mathcal{O}_c$ , on  $f$  and on the dielectric constant  $\mu_c$ ,  $\mu_e$ ,  $q_c$ ,  $q_m$  and  $q_e$  such that:

$$\begin{aligned} \|A\|_{\Lambda^0 L^2(\Omega)}^2 &\leq \frac{Ch^{3/2}}{\Im_{e,m,c}} \left( \sqrt{h} \left( \|\partial_n B^c\|_{H^{-1/2}(\Gamma_0)} + \|b\|_{L^2(\mathbb{T})} \right) \right. \\ (41) \quad &\quad \left. + \left( \int_0^1 \int_0^{2\pi} |g|^2 \, d\eta \, d\theta \right)^{1/2} \right) \left( \|A\|_{\Lambda^0 L^2(\Omega)} + \|dA\|_{\Lambda^1 L^2(\Omega)} \right). \end{aligned}$$

One more time, we multiply equalities (39a)–(39b)–(39c) respectively by the conjugates of  $W^e$ ,  $A^c$  and  $A^m$ . Using the transmission conditions (39), we integrate by part and we take the real part of the result. We infer:

$$\begin{aligned} \|dA\|_{\Lambda^1 L^2(\Omega)}^2 &\leq \max\{\Re(z_e), \Re(z_m), \Re(z_c)\} \|A\|_{\Lambda^0 L^2(\Omega)}^2 + \frac{h^2}{\mu_c} \left| \int_{\Gamma_0} \partial_n B^c \overline{A^c} \, d\text{vol}_{\Gamma_0} \right| \\ &\quad + \frac{h^2}{\mu_m} \left| \int_0^{2\pi} b \overline{A^e} \circ \Phi_1 \, d\theta \right| + h^2 f \left| \int_0^1 \int_0^{2\pi} g \overline{A^m} \, d\eta \, d\theta \right|. \end{aligned}$$

Using (41) we infer easily that there exists a constant  $C$  depending on the domain  $\mathcal{O}_c$ , on  $f$  and on the dielectric parameters  $\mu_c$ ,  $\mu_e$ ,  $q_c$ ,  $q_m$  and  $q_e$  such that:

$$\begin{aligned} \|A\|_{\Lambda^0 L^2(\Omega)} + \|dA\|_{\Lambda^1 L^2(\Omega)} &\leq Ch^{3/2} \left( \sqrt{h} \left( \|\partial_n B^c\|_{H^{-1/2}(\Gamma_0)} + \|b\|_{L^2(\mathbb{T})} \right) \right. \\ &\quad \left. + \left( \int_0^1 \int_0^{2\pi} |g|^2 \, d\eta \, d\theta \right)^{1/2} \right). \end{aligned}$$

According to the estimate of  $B^m$  given in (34) and according to the definition of  $B^c$  given in (37), it is obvious that there exists a constant  $C$  depending on  $\mathcal{O}_c$  and the dielectric constants such that the following estimates hold:

$$\left( \int_0^1 \int_0^{2\pi} |g|^2 d\eta d\theta \right)^{1/2} \leq C \|\phi\|_{H^{5/2}(\partial\Omega)},$$

$$\|\partial_n B^c\|_{H^{-1/2}(\Gamma_0)} \leq C \|\phi\|_{H^{5/2}(\partial\Omega)}.$$

Therefore, there exists a constant  $C$  depending on the domain  $\mathcal{O}_c$ , on  $f$  and on the dielectric constants such that:

$$\|A\|_{\Lambda^0 H^1(\Omega)} \leq Ch^{3/2} \|\phi\|_{H^{5/2}(\partial\Omega)}.$$

□

**Remark 5.4.** Now we consider the domain  $\Omega_h$  defined by:

$$\Omega_h = \mathcal{O}_c \cup \mathcal{O}_h.$$

Let  $\gamma$  be in  $H^{5/2}(\partial\Omega_h)$ , and we denote by  $g$  the following function defined on the torus:

$$\forall \theta \in \mathbb{T}, \quad g(\theta) = \gamma \circ \Phi_1(\theta).$$

Let  $u$  be the solution of the following problem:

$$\operatorname{div} \left( \frac{1}{\mu} \operatorname{grad} u \right) + qu = 0, \text{ in } \Omega_h,$$

$$\partial_n u|_{\partial\Omega_h} = \gamma, \text{ in } \Omega_h,$$

Then, we have the following theorem:

**Theorem 5.5.** We remember that  $h_0$  and  $f$  are defined in Theorem 5.2. Let  $\gamma$  be in  $H^{5/2}(\partial\Omega_h)$ .

We denote by  $u_0^c$ ,  $u_0^m$ ,  $u_1^c$ , and  $u_1^m$  the functions defined as follows:

$$\begin{cases} \Delta u_0^c + z_c u_0^c = 0, \text{ in } \mathcal{O}_c, \\ \partial_n u_0^c|_{\Gamma_0} = (\mu_c/\mu_m) g \circ \Phi_0^{-1}, \text{ on } \Gamma_0. \end{cases}$$

In the membrane, the field  $u_0^m$  is equal to:

$$\forall (\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad u_0^m = u_0^c \circ \Phi_0(\theta).$$

The field  $u_1^c$  is the solution of the following problem in  $\mathcal{O}_c$ :

$$\begin{cases} \Delta u_1^c + z_c u_1^c = 0, \text{ in } \mathcal{O}_c, \\ (\mu_m/\mu_c) \partial_n u_1^c|_{\Gamma_0} = f \partial_t^2 u_0^c|_{\Gamma_0} + z_m f u_0^c|_{\Gamma_0} + f' \partial_t u_0^c|_{\Gamma_0}, \text{ on } \Gamma_0. \end{cases}$$

In the membrane, we have:

$$\forall (\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad u_1^m = f \eta \gamma + u_1^c \circ \Phi_0.$$

Let  $W$  be the function defined on  $\Omega_h$  by:

$$W = \begin{cases} u - (u_0^c + h u_1^c), \text{ in } \mathcal{O}_c, \\ u - (u_0^m \circ \Phi^{-1} + h u_1^m \circ \Phi^{-1}), \text{ in } \mathcal{O}_h. \end{cases}$$

Then, there exists a constant  $C > 0$  independant of  $h$  but depending on the domain  $\mathcal{O}_c$ , on the function  $f$  and on the dielectric parameters  $z_m$  and  $z_c$  of the material such that

$$\|W\|_{H^1(\Omega_h)} \leq Ch^{3/2} \|g \circ \Phi_0^{-1}\|_{H^{5/2}(\partial\mathcal{O}_c)}.$$

**Remark 5.6.** In Theorem 5.2, we can replace  $z_m$  by zero. The proof is then very similar, except that we need the following inequality.

**Proposition 5.7.** Let  $h$  and  $f$  be as in Theorem 5.2. Let  $u$  be a function of class  $C^1([0, 1] \times \mathbb{T})$ . On the domain  $[0, 1] \times \mathbb{T}$ , we use the metric  $h^2 f^2 d\eta + (1 + hf\eta\kappa)d\theta$ .

Then, there exists a constant  $C$  depending on  $\mathcal{O}$  and on  $f$  and independant of  $h$  such that

$$(42) \quad \|u\|_{\Lambda^0 L_m^2(D)}^2 \leq C \left( \|du\|_{\Lambda^1 L_m^2(D)}^2 + \int_0^{2\pi} |u(0, \theta)|^2 d\theta \right).$$

*Proof.* Actually, according to the definition of  $h_0$  in (3) there exists two constants  $C_1$  and  $C_2$  depending on the domain  $\mathcal{O}$  and on  $f$  such that the following inequalities hold:

$$(43a) \quad \|u\|_{\Lambda^0 L_m^2(D)}^2 \leq C_1 h \int_0^1 \int_0^{2\pi} |u(\eta, \theta)|^2 d\theta d\eta,$$

$$(43b) \quad \|du\|_{\Lambda^1 L_m^2(D)}^2 \geq C_2 \left( \int_0^1 \int_0^{2\pi} \frac{|\partial_\eta u(\eta, \theta)|^2}{h} + h |\partial_\theta u|^2 d\theta d\eta \right).$$

Let us denote by  $(\widehat{u})_k$  for  $k \in \mathbb{Z}$  the  $k^{\text{th}}$ -Fourier coefficient (with respect to  $\theta$ ) of  $u$ :

$$\widehat{u}_k = \int_0^\pi u(\theta) e^{-2i\pi k/L} d\theta.$$

Since  $\left(\widehat{\partial_\theta u}\right)_k = 2i\pi k \widehat{u}_k$ , it is easy to see that:

$$\forall k \neq 0, \quad \int_0^1 |\widehat{u}_k(\eta)|^2 d\eta \leq 4\pi^2 \int_0^1 \left| \left(\widehat{\partial_\theta u}\right)_k(\eta) \right|^2 d\eta.$$

Using the following equality

$$\widehat{u}_0(\eta) = \int_0^\eta \left(\widehat{\partial_\eta u}\right)_0(s) ds + \widehat{u}_0(0),$$

we infer

$$\int_0^1 |\widehat{u}_0(\eta)|^2 d\eta \leq 2 \int_0^1 \left| \left(\widehat{\partial_\eta A^m}\right)_0(\eta) \right|^2 + 2|\widehat{u}_0(0)|^2 d\eta.$$

We deduce easily inequality (42).  $\square$

## 6. APPLICATION TO THE BIOLOGICAL CELL

In the biological cells, the membrane is insulating (see Fear and Stuchly [8]). This means that at mid frequencies, the ratio  $|z_m|/|z_c|$  and  $|z_m|/|z_e|$  are small compared to  $h$ . Actually, the thickness is of order  $10^{-3}$ , while  $|z_m|/|z_c|$  is about  $10^{-5}$  (see [10]). We say that we work at mid frequency since we suppose that  $z_c$  and  $z_e$  are of order 1. The following results show that the asymptotics obtained by replacing  $z_m$  by zero in the expansions of Theorem 5.2 give a good approximation of the electric field in the biological cell. We have the following proposition.

**Proposition 6.1.** *Let  $z_c$  and  $z_e$  be complex constants with strictly negative imaginary part.*

*We suppose that  $|z_m|$  tends to zero and that there exists a constant  $c > 0$  such that:*

$$(44) \quad 0 < -\frac{|z_m|}{\Im(z_m)} < c.$$

*Let  $z$  and  $\tilde{z}$  be such that:*

$$z = \begin{cases} z_e, & \text{in } \mathcal{O}_{e,h}, \\ z_m, & \text{in } \mathcal{O}_h, \\ z_c, & \text{in } \mathcal{O}_c, \end{cases} \quad \tilde{z} = \begin{cases} \tilde{z}_e, & \text{in } \mathcal{O}_{e,h}, \\ 0, & \text{in } \mathcal{O}_h, \\ \tilde{z}_c, & \text{in } \mathcal{O}_c. \end{cases}$$

*Let  $\phi$  in  $H^{1/2}(\partial\Omega)$ . Let  $u$  the solution of the following problem:*

$$(45a) \quad \Delta u + zu = 0, \text{ in } \Omega,$$

$$(45b) \quad \partial_n u|_{\partial\Omega} = \phi, \text{ in } \Omega,$$

*and let  $v$  be such that*

$$(45c) \quad \Delta v + \tilde{z}v = 0, \text{ in } \Omega,$$

$$(45d) \quad \partial_n v|_{\partial\Omega} = \phi, \text{ in } \Omega.$$

*Then, there exists a constant  $C$  depending on  $\mathcal{O}_c$ ,  $c$  and on the function  $f$  such that:*

$$\|u - v\|_{H^1(\Omega)} \leq C|z_m|\|\phi\|_{H^{1/2}(\partial\Omega)}.$$

*Proof.* To prove this theorem, first, using hypothesis (44), we prove that there exists a constant  $C$  depending on  $\mathcal{O}_c$ , and on the constants  $c$ ,  $z_c$  and  $z_e$  such that:

$$\|u\|_{H^1(\Omega)} \leq C\|\phi\|_{H^{1/2}(\partial\Omega)}.$$

Then, we just have to write the problem satisfies by  $u - v$  in local coordinates in the membrane. As usual, we multiply in by  $\overline{u - v}$  and we integrate by parts. Then, using inequality (42), we show easily the following inequality:

$$\|u - v\|_{H^1(\Omega)} \leq C|z_m|\|u\|_{H^1(\Omega)},$$

which ends the proof of the theorem.  $\square$

Using Proposition 6.1 and Remark 5.6, we obtain easily the following theorem.

**Theorem 6.2.** *Let  $f$  be in  $C^\infty(\mathbb{T})$ , such that*

$$\forall \theta \in \mathbb{T}, \quad f(\theta) > 0,$$

*and let  $h_0$  be in  $(0, 1)$  such that*

$$h_0 < \frac{1}{\|f\kappa\|_\infty}.$$

*Let  $h$  be in  $(0, h_0)$ .*

*Let  $z_c$  and  $z_e$  be complex constants with strictly negative imaginary part.*

*We suppose that  $|z_m| = o(h)$  and that there exists a constant  $c > 0$  such that:*

$$0 < -\frac{|z_m|}{\Im(z_m)} < c.$$

Let  $z$  and  $\tilde{z}$  be such that:

$$z = \begin{cases} z_e, & \text{in } \mathcal{O}_{e,h}, \\ z_m, & \text{in } \mathcal{O}_h, \\ z_c, & \text{in } \mathcal{O}_c, \end{cases}$$

Let  $\phi$  be in  $H^{5/2}(\partial\Omega)$ .

We denote by  $u$  the solution of the following problem:

$$\begin{cases} \Delta u + zu = 0, & \text{in } \Omega, \\ \partial_n u = \phi, & \text{on } \partial\Omega. \end{cases}$$

We define  $(u_0^e, u_0^c)$ ,  $u_0^m$ ,  $(u_1^e, u_1^c)$ , and  $u_1^m$  are defined as follows.

- The 0<sup>th</sup> order terms. The electric fields  $u_0^e$  and  $u_0^c$  are solution of the following problem in  $\mathcal{O}_{e,h} \cup \mathcal{O}_c$ :

$$(46a) \quad \begin{cases} \Delta u_0^e + z_e u_0^e = 0, & \text{in } \mathcal{O}_{e,h}, \\ \Delta u_0^c + z_c u_0^c = 0, & \text{in } \mathcal{O}_c, \end{cases}$$

with the transmission conditions

$$(46b) \quad u_0^c|_{\Gamma_0} = u_0^e|_{\Gamma_0},$$

$$(46c) \quad \partial_n u_0^c|_{\Gamma_0} = \partial_n u_0^e|_{\Gamma_0},$$

with Neumann boundary condition:

$$(46d) \quad \partial_n u_0^e|_{\partial\Omega} = \phi.$$

In the membrane, the field  $u_0^m$  is equal to:

$$(47) \quad \forall(\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad u_0^m = u_0^c \circ \Phi_0(\theta).$$

- The first order terms. The fields  $u_1^e$  and  $u_1^c$  are solution of the following problem in  $\mathcal{O}_{e,h} \cup \mathcal{O}_c$ :

$$(48a) \quad \begin{cases} \Delta u_1^e + z_e u_1^e = 0, & \text{in } \mathcal{O}_{e,h}, \\ \Delta u_1^c + z_c u_1^c = 0, & \text{in } \mathcal{O}_c, \\ \partial_n u_1^e|_{\partial\Omega} = 0, \end{cases}$$

with the following transmission conditions

$$(48b) \quad \partial_n u_1^c|_{\Gamma_0} - \partial_n u_1^e|_{\Gamma_0} = f \partial_t^2 u_0^c|_{\Gamma_0} + f' \partial_t u_0^c|_{\Gamma_0} + f \partial_n^2 u_0^e|_{\Gamma_0},$$

$$(48c) \quad u_1^c|_{\Gamma_0} - u_1^e|_{\Gamma_0} = 0.$$

In the membrane, we have:

$$(49) \quad \forall(\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad u_1^m = \eta f \partial_n u_0^c \circ \Phi_0 + u_1^c \circ \Phi_0.$$

Let  $W$  be the function defined on  $\Omega$  by:

$$W = \begin{cases} u - (u_0^e + h u_1^e), & \text{in } \mathcal{O}_{e,h}, \\ u - (u_0^c + h u_1^c), & \text{in } \mathcal{O}_c, \\ u - (u_0^m \circ \Phi^{-1} + h u_1^m \circ \Phi^{-1}), & \text{in } \mathcal{O}_h. \end{cases}$$

Then, there exists a constant  $C > 0$  independent of  $h$  but depending on the domain  $\mathcal{O}_c$ , on the function  $f$  and on the dielectric parameters  $z_e$  and  $z_c$  of the material such that

$$\|W\|_{H^1(\Omega)} \leq C \left( h^{3/2} + |z_m| \right) \|\phi\|_{H^{5/2}(\partial\Omega)}.$$

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INSTITUT CAMILLE JORDAN AND CEGELY, UNIVERSITÉ CLAUDE BERNARD LYON1

POIGNARD@MATH.UNIV-LYON1.FR