# The Cabibbo angle : an algebraic calculation

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# THE CABIBBO ANGLE : AN ALGEBRAIC CALCULATION

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We show that the Cabibbo angle $\theta_c$ satisfies the relation $\tan(2\theta_c) = \pm 1/2$ when universality for diagonal neutral currents of mass eigenstates is satisfied at the same level of accuracy as the absence of their off-diagonal counterparts. The predicted value is $\cos\theta_c \approx 0.9732$, only $7/10000$ away from experimental results. No mass ratio appears in the calculation. $\theta_c$ occurs *a priori* for both quark species, and, showing that one recovers the standard dependence of leptonic and semi-leptonic decays of pseudoscalar mesons, we advocate, like for neutrinos and charged leptons, for a symmetrical treatment of $u$ and $d$-type quarks.

## INTRODUCTION

Why mixing angles are what they are is, in addition to their more and more precise experimental determination, one of the most active present domain of research, both in the leptonic and hadronic sectors. The attempts that have been proposed up to now in order to address this fundamental question, such as the linking of the sine of the Cabibbo angle [1] to the $d$ and $s$ quark mass ratio [2], are often based on empirical evidence, and so remain unsatisfying [1] . Indeed, there are actually too many free parameters in the most general Yukawa couplings of the standard model to infer mixing angles from mass ratios.

In [4], we have shown that maximal mixing angles naturally arise for non-degenerate coupled systems of physical particles with two generations, like neutrinos, when one requires universality for the diagonal neutral currents of *mass eigenstates* and the absence of their off-diagonal counterparts. This follows from an important feature that we also demonstrated there, *i.e.* that the mixing matrix of such systems should never be parametrized as unitary. Such a property had also been checked before [5] for neutral kaons.

In this letter, we would like to suggest one promising step on the path to a better understanding of the enigma raised above. In direct continuation of [4], still working in the simple case of two

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[1] The rigorous treatment [3] shows that only weaker "asymptotic" relations hold, which involve both $m_d/m_s$ and $m_u/m_c$.

generations, we investigate non-degenerate coupled systems that do not exist on-shell, like quarks. Those systems, as shown in [4], are more naturally endowed with a unitary mixing matrix, due to the fact that their two mixing angles finally satisfy $\theta_2 = \pm\theta_1[\pi]$. Staying in an infinitesimal, but still 2-dimensional, neighborhood of such a "Cabibbo-case" (as referred to in [4]), we impose the physical requirement that universality for weak diagonal neutral currents of mass eigenstates is realized with the same accuracy as the absence of the off-diagonal mass changing neutral currents (MCNC's). As a consequence the value of the unique mixing angle is shown to be very close to that of the measured Cabibbo angle.

## GENERALITIES, STATES AND (WEAK) CURRENTS

In quantum field theory, the states of definite mass for a system of particles are defined as the eigenstates, at its poles, of the full (renormalized) propagator, which is given by a matrix of dimension $2n_f$ in flavour space. For non-degenerate coupled systems, like neutral kaons, leptons or quarks, these mass eigenstates belong to different bases [4]. Indeed, one finds only one such eigenstate at each of the poles of the propagator ; it is, among the $2n_f$ eigenstates of the latter, the one corresponding to the vanishing eigenvalue. Hence, the mass eigenstates do not make up an orthonormal basis, and the mixing matrix, which by definition connects the set of mass eigenstates to that of flavour eigenstates, is non-unitary.

Nevertheless, for systems of coupled fields which, like quarks, are never on-shell, a natural basis appears : the one that occurs at any given $z = q^2$. It is then orthonormal as soon as the inverse propagator $L^{(2)}(z)$ is hermitian. Thus, the mixing matrix relating it to the flavour basis is unitary (accorded that the flavour basis is orthonormal, too). This has, in the simple case of two generations, the following consequences. While, in general, $(2 \times 2)$ mixing matrices for coupled systems are to be parametrized with two mixing angles $\theta_1$ and $\theta_2$, quark-like (Cabibbo-like) systems finally shrink to one dimension.

As stressed in [4], the physics of mixing angles is underlain by weak currents in mass space. These we shall consider again now. Neutral (left-handed) weak currents for mass eigenstates are determined by the combinations $K_1^\dagger K_1$ and $K_2^\dagger K_2$, where $K_1$ and $K_2$ are the mixing matrices for the two types of fermions involved (for example neutral and charged leptons and, later, for our purpose, quarks of the $u$ and $d$-type), whereas charged currents give rise to the combination matrix $K = K_1^\dagger K_2$. As soon as neither $K_1$ nor $K_2$ is unitary, the conditions for universality (equality of diagonal neutral currents) and

absence of off-diagonal neutral currents, which are built-in properties in flavour space, appear on the contrary no longer trivial in the space of mass eigenstates.

Let us parametrize like in [4] ($c_1$ stands for example for $\cos\theta_1$)

$$K_1 = \begin{pmatrix} e^{i\alpha}c_1 & e^{i\delta}s_1 \\ -e^{i\beta}s_2 & e^{i\gamma}c_2 \end{pmatrix}. \tag{1}$$

It is then straightforward to show that universality in mass space for weak neutral currents imposes the vanishing of

$$R = c_1^2 + s_2^2 - c_2^2 - s_1^2, \tag{2}$$

while the absence of MCNC's requires that of

$$S = c_1 s_1 - c_2 s_2 \quad \text{or} \quad T = c_1 s_1 + c_2 s_2. \tag{3}$$

**GETTING THE CABIBBO ANGLE FROM AN EXTRA TOUCH OF PHYSICS**

Let us now demand that the two conditions for universality and for the absence of MCNC's are realized with the same accuracy. We consider accordingly the equations

$$|R| = |S| \Leftrightarrow R \pm S = 0 \quad \text{or} \quad |R| = |T| \Leftrightarrow R \pm T = 0, \tag{4}$$

*i.e.*

$$\cos(2\theta_1) - \cos(2\theta_2) \pm \frac{1}{2}\big(\sin(2\theta_1) - \sin(2\theta_2)\big) = 0, \tag{5a}$$

$$\cos(2\theta_1) - \cos(2\theta_2) \pm \frac{1}{2}\big(\sin(2\theta_1) + \sin(2\theta_2)\big) = 0, \tag{5b}$$

which directly lead to the following conditions :

$$\sin(\theta_1 - \theta_2)\big(\cos(\theta_1 + \theta_2) \mp 2\sin(\theta_1 + \theta_2)\big) = 0, \tag{6a}$$

$$\sin(\theta_1 + \theta_2)\big(\cos(\theta_1 - \theta_2) \mp 2\sin(\theta_1 - \theta_2)\big) = 0. \tag{6b}$$

First, one immediately sees that a trivial solution of the equations (6a) and (6b) is, resp., $\theta_2 = \theta_1 \mod \pi$ and $\theta_2 = -\theta_1 \mod \pi$. This corresponds to Cabibbo-like solutions, for which the two

conditions $R = 0$ and $T$ or $S = 0$ are separately satisfied, ensuring universality for diagonal neutral currents in mass space, and the absence of MCNC's.

Let us now expand (6a) and (6b) in the vicinity of these Cabibbo-like solutions by writing respectively in there $\theta_2 = \pm\theta_1 + \epsilon$ :

$$\epsilon\Big(\big(\cos(2\theta_1) \mp 2\sin(2\theta_1)\big) - 2\epsilon\big(\sin(2\theta_1) \pm 2\cos(2\theta_1)\big)\Big) = 0, \tag{7a}$$

$$\epsilon\Big(\big(\cos(2\theta_1) \mp 2\sin(2\theta_1)\big) + 2\epsilon\big(\sin(2\theta_1) \pm 2\cos(2\theta_1)\big)\Big) = 0, \tag{7b}$$

and impose that the two conditions (7a,7b) are satisfied with the same accuracy $o(\epsilon^2)$. This leads to the following equation (we replace hereafter $\theta_1$ with $\theta_c$) :

$$\tan(2\theta_c) = \pm\frac{1}{2}, \tag{8}$$

that can be rewritten

$$\big(\tan(\theta_c)\big)^2 \pm 4\tan(\theta_c) - 1 = 0. \tag{9}$$

Eqs. (9) admit the solutions : $\tan(\theta_c) = \pm(-2 \pm \sqrt{5})$, the smaller of which (they have opposite signs) have the following value for their cosine : $\cos\big(\pm\arctan(2 - \sqrt{5})\big) \simeq 0.9732$. This result lies remarkably close to the present experimental range [6] for the cosine of the Cabibbo angle : $[0.9739, 0.9751]$. The other solution corresponds to $\theta_c - \pi/2$, *i.e.* to a simple relabeling of the two corresponding mass states (for example $u_m \leftrightarrow c_m$). It is noticeable that this approach yields a constant value for the Cabibbo angle, which should be confronted with experiment. The most natural, *a priori $q^2$-dependent*, orthonormal basis related to Cabibbo-like systems mentioned in section 2, should then also exhibit special properties with respect to its $q^2$ dependence. This will be investigated in a forthcoming work.

A similar demonstration holds for $K_2$.

### PLEA FOR A SYMMETRICAL TREATMENT OF BOTH QUARK SPECIES

We already advocated in [4] for a symmetrical treatment of charged and neutral leptons, with maximal mixing for both species. The PMNS matrix then reduces to the diagonal unit matrix, without putting in jeopardy the interpretation of the solar neutrino deficit on earth in terms of neutrino oscillations.

We advocate for the same attitude in the quark sector. First, there is not a single good reason to assume that mass and flavour eigenstates coincide for one species and not for the other one. Secondly, the Cabibbo angle naturally arises by the mechanism explained above for both quark types. Last, it is easy to demonstrate that, by a suitable analysis, we recover standard results. We give below the example of semi-leptonic and leptonic decays of the $K^+$ meson.

From the most common example of neutral kaons [7], two species of mesons are known to occur in physics : flavor and mass eigenstates. The first are produced by strong interactions (which commute with flavour) while the second also identify with electroweak eigenstates [2] . When studying its semi-leptonic decays, the $K^+$ is produced by strong interactions while electroweak eigenstates are detected ; the following subscripts should be accordingly attributed to this reaction : $K_f^+ \to \pi_m^0 \ell_m^+ \nu_{em}$. In a quark-spectator model, quark flavors are by definition weakly coupled with $diag(1,1)$ ; the decay channels of $K_f^+$ can only be, accordingly, $(d_f \bar{s}_f) \ell^+ \nu_\ell$ and $(u_f \bar{c}_f) \ell^+ \nu_\ell$, with coefficient 1 since no mixing angle occurs. When projected on mass states, these final flavour states undergo the Cabibbo rotations which take place for both quark species. This gives $(d_f \bar{s}_f) \equiv K_f^0 = c_{\theta_c}^2 K_m^0 + c_{\theta_c} s_{\theta_c} \big( (s_m \bar{s}_m) - (d_m \bar{d}_m) \big) - s_{\theta_c}^2 \overline{K_m^0}$, $(u_f \bar{c}_f) \equiv \overline{D_f^0} = c_{\theta_c}^2 \overline{D_m^0} + c_{\theta_c} s_{\theta_c} \big( (c_m \bar{c}_m) - (u_m \bar{u}_m) \big) - s_{\theta_c}^2 D_m^0$. All semi-leptonic decays of the $K_f^+$ mesons are then described with their observed dependence on the Cabibbo angle (the ones into $\pi^0$, with their $\cos \theta_c \sin \theta_c$ dependence, are built up from the $u \bar{u}$ and $d \bar{d}$ final states). We also find double-Cabibbo-suppressed $|\Delta S| = 2$ transitions.

Leptonic decays are likewise easily explained by the decomposition $K_f^+ \equiv (u_s \bar{s}_f) = c_{\theta_c}^2 K_m^+ + c_{\theta_c} s_{\theta_c} (D_{sm}^+ - \pi_m^+) - s_{\theta_c}^2 D_m^+$ and with the leptonic decay of the $\pi_m^+$ that takes place with a coefficient 1 since, now, no mixing angle occurs in charged weak currents.

This supports our point of view according to which the Cabibbo angle appears at the level of both quark species but not in the charged weak couplings of the Lagrangian. While the usual "Cabibbo matrix", which connects mass eigenstates for $u$ and $d$-type quarks, reduces then to the unit matrix, the distinction between flavour and mass eigenstates for mesonic bound states must be carefully performed [3] .

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[2] This is not to be confused with the convention that flavor states of quarks and leptons are the ones which diagonally couple, by definition, to weak gauge bosons.

[3] Since the Cabibbo-angle keeps acting between flavour eigenstates of one type and mass eigenstates of the other type, the evaluation of "box-diagram" controlling, for example, the transitions between the flavour $K^0$ and $\overline{K^0}$ mesons stays unchanged since internal quark lines are *propagating*, that is *mass* eigenstates, while external lines are the $d_f$ and $s_f$ *flavour* states.

**CONCLUSION AND PROSPECTS**

Realizing, as was first done in [5], that mixing matrices of non-degenerate coupled systems should not be parametrized as unitary, led in [4] to uncover maximal mixing of leptons as a class of solutions of very simple physical conditions for their mass eigenstates.

Now, we have shown that the measured value of the Cabibbo angle $\theta_c$ is such that, in its vicinity (allowing the second mixing angle $\theta_2$ to be close to $\theta_1 = \theta_c$), universality of diagonal neutral currents for mass eigenstates is realized with the same $o\big((\theta_2 - \theta_1)^2\big)$ accuracy as the absence of MCNC's.

In this elementary algebraic calculation, no mass ratio appears. It may thus help to provide independent information on the latter. On another side, this feature is welcome for quark-like systems which cannot be defined on-shell and for which, accordingly, the notion of physical mass is ill-defined.

Like for leptons, we advocate that the two species of quarks should be treated on equal footing. Indeed, the Cabibbo angle, as a mixing angle, must occur for both, while it should not appear inside charged weak couplings of mass eigenstates.

This work, together with [4], strongly suggests that the observed values of mixing angles for quarks and leptons follow from well defined physical requirements. The generalization to three generations is currently under investigation.

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