A MULTISCALE CORRECTION METHOD FOR LOCAL SINGULAR PERTURBATIONS OF THE BOUNDARY

Marc Dambrine, Grégory Vial

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M. Dambrine[*] and G.Vial[†]

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Abstract

In this work, we consider singular perturbations of the boundary of a smooth domain. We describe the asymptotic behavior of the solution u_ε of a second order elliptic equation posed in the perturbed domain with respect to the size parameter ε of the deformation. We are also interested in the variations of the energy functional. We propose a numerical method for the approximation of u_ε based on a multiscale superposition of the unperturbed solution u_0 and a profile defined in a model domain. We conclude with numerical results.

1   Introduction.

Various physical situations involve materials with a two-scale structure. From the macroscopic point of view, the considered body can usually be modeled by a smooth domain of $\mathbb{R}^2$ or $\mathbb{R}^3$, but this does not take into account the microscopic design of the material. We are specially interested in small inhomogeneities or inclusions on the border of the body. If they are arranged within a periodical network, homogenization techniques (see [1], for example) apply and the macroscopic model is valid, provided the characteristic properties of the material are modified accordingly. Such methods do not hold for local inhomogeneities, which are in the applications usually either omitted (for the smallest ones) or integrated into the macroscopic domain. Naturally, the numerical approximation of such problems requires a severe mesh refinement near the inclusions, which sometimes prevents from taking them into account in the computations.

In this paper, we deal with an elliptic partial differential equation in a domain with a small local boundary perturbation. We give the complete asymptotic expansion of its solution with respect to the size of the perturbing pattern, derive the variation of the associated energy (topological derivative) and propose a numerical method for the approximation of its solution based on the theoretical study.

Let us describe the geometrical setting we shall work within: Ω_0 is an open bounded subset of $\mathbb{R}^2$ with smooth boundary containing the origin O. We assume that the boundary $\partial\Omega_0$ coincides with a straight line near the origin, precisely for $|x| < r^*$. On the other hand, $\mathbf{H}_\infty$ denotes an infinite domain of $\mathbb{R}^2$, which coincides with the upper half-plane at infinity, precisely for $|x| > R^*$. The perturbed domain Ω_ε is defined for small ε by (see Figure 1)

$$\Omega_\varepsilon = \{x \in \Omega_0 \; ; \; |x| > \varepsilon R^*\} \cup \{x \in \varepsilon\mathbf{H}_\infty \; ; \; |x| < r^*\}. \tag{1}$$

Let us mention that we make *no assumption of inclusion* of the perturbed domain into the original one (or conversely). We will extend this framework to some curved smooth situations.

We define u_ε as the solution in $\mathrm{H}^1(\Omega_\varepsilon)$ of the equation $-\Delta u_\varepsilon = f$ in Ω_ε, where f is some function in $\mathrm{L}^2(\Omega_0)$ vanishing in a neighborhood of the origin. We consider Dirichlet boundary conditions on $\Gamma_D \subset \partial\Omega_\varepsilon$ (which does not reach the origin) and Neumann boundary conditions elsewhere (other types of boundary conditions can also be treated). The asymptotic analysis of similar problems have been investigated by several authors in various special cases, see [9, 11, 6, 4], and more recently in [13]. It appears that the solution u_ε can be approximated at first order by a superposition of the unperturbed solution u_0 and a *profile*, via cut-off functions in slow and rapid variables:

$$u_\varepsilon = \zeta(\tfrac{x}{\varepsilon})u_0(x) + \chi(x)W^1(\tfrac{x}{\varepsilon}) + \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}(\varepsilon^2). \tag{2}$$

[*]LMAC, Univers`té de Technologie de Compiègne.
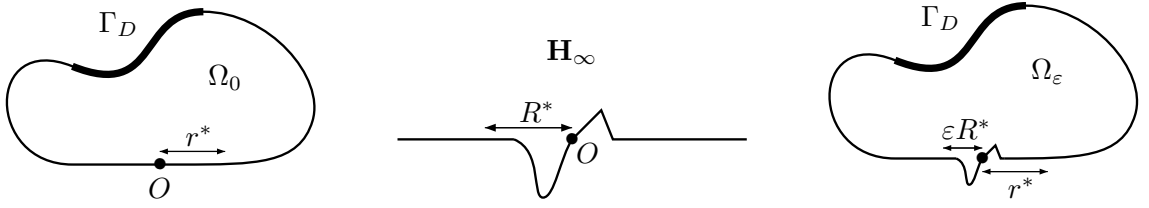[†]IRMAR, Antenne de Bretagne de l'ENS Cachan.

Figure 1: The original and pertubed domains.

The cut-off functions ζ and χ are chosen smooth, radial, and satisfying

- the function $\zeta(x)$ equals 1 for $|x| > R^*$, and vanishes for $|x| < R^*/2$;
- the function $\chi(x)$ equals 1 for $|x| < r^*/2$ and vanishes for $|x| > r^*$.

(3)

The profile W^1 is defined as the solution in the domain $\mathbf{H}_\infty$ of an homogeneous model problem. In the expansion (2), the term u_0 only contributes away from the origin and the information concerning the perturbing pattern is carried by the profile. These two contributions interact in the transition zone through the cut-off functions.

We can base a numerical approach for the approximation of u_ε on formula (2). Indeed, the computation of the term u_0 does not involve the perturbation and may therefore be done on a coarse mesh of Ω_0. If we have a suitable approximation of the profile W^1, the superposition formula (2) gives a numerical solution for u_ε. The cut-off functions are handled in the practical process by means of *patch of elements*.

Moreover, expression (2) allows to compute the *topological derivative* – see [7, 8, 12] – of the energy $j(\varepsilon)$:

$$j(\varepsilon) := -\frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 = j(0) + \varepsilon^2 |\nabla u_0(0)|^2 \mathcal{A}_{\mathbf{H}_\infty} + o(\varepsilon^2),$$

(4)

where the real number $\mathcal{A}_{\mathbf{H}_\infty}$ only depends on the geometry of $\mathbf{H}_\infty$.

The paper is divided as follows. In a first section, we give the full asymptotic expansion of the state function in the case of a straight boundary near the origin. This is based on a revisited multiscale asymptotic method. We extend then these results to a curved case. The third part is devoted to the numerical method using patch of elements near the perturbation. Last, we derive the leading terms in the asymptotical description of the energy functional. A numerical validation of our theoretical results is given.

2   Asymptotic expansion of the state function.

We consider u_ε solution of the following problem, posed in the geometry described by Figure 1:

$$\begin{cases} -\Delta u_\varepsilon &= f \quad \text{in } \Omega_\varepsilon, \\ u_\varepsilon &= 0 \quad \text{on } \Gamma_D, \\ \partial_{\mathbf{n}} u_\varepsilon &= 0 \quad \text{on } \partial\Omega_\varepsilon \backslash \Gamma_D. \end{cases}$$

(5)

The technique we use to build an asymptotic expansion of u_ε into powers of ε is adapted from the multiscale approach of [13].

We first write the Taylor expansion at a target precision K of the limit term u_0 at point $x = 0$ (thanks to standard elliptic regularity, u_0 is a smooth function up to the boundary):

$$u_0(x) = \chi(x) \sum_{k=0}^{K} u^k(x) + R_K(x) = \chi(x) T_K(x) + R_K(x),$$

(6)

the first terms of the Taylor polynomial T_K being given by $u^0(x) = u_0(0)$, $u^1(x) = |\nabla u_0(0)| x_1$ (more generally u^k is an homogeneous polynomial of total degree k). The limit term u_0 is not necessarily defined in the whole domain Ω_ε, but its Taylor part may be extended to Ω_ε. For this reason a better start is given by the truncated function

$$\tilde{u}_0(x) = \chi(x) T_K(x) + \zeta(\tfrac{x}{\varepsilon}) R_K(x) \in \mathrm{H}^1(\Omega_\varepsilon).$$

(7)

2

The difference between u_0 and $\tilde{u}_0$ is small since the remainder R_K is flat in the cut-off region. Let us denote by r_ε^0 the difference between u_ε and $\tilde{u}_0$, it naturally satisfies the following problem

$$\begin{cases} -\Delta r_\varepsilon^0 & = & \varphi_\varepsilon^0 & \text{in } \Omega_\varepsilon, \\ r_\varepsilon^0 & = & 0 & \text{on } \Gamma_D, \\ \partial_{\mathbf{n}} r_\varepsilon^0 & = & -\chi(x)\partial_{\mathbf{n}} T_K + \psi_\varepsilon^0 & \text{on } \partial\Omega_\varepsilon\backslash\Gamma_D, \end{cases} \tag{8}$$

where the data φ_ε^0 and ψ_ε^0 arise from the cut-off and are supported in the ring of size ε defined as $\{x \in \Omega_\varepsilon \; ; \; \varepsilon R^*/2 < |x| < \varepsilon R^*\}$, they will contribute to the remainder since they are essentially of order ε^K. Thus, the principal defect in equation (8) comes from the normal derivative of the Taylor expansion of u_0, whose leading term reads

$$-\chi(x)|\nabla u_0(0)|\partial_{\mathbf{n}} x_1 = -\chi(x)|\nabla u_0(0)|n_1, \tag{9}$$

which does not vanish only on the boundary part of Ω_ε which corresponds to the perturbing pattern. Following the ideas of [2, 3, 13], we introduce the profile V^1 as the solution of the problem in the infinite domain $\mathbf{H}_\infty$:

$$\begin{cases} -\Delta V^1 & = & 0 & \text{in } \mathbf{H}_\infty, \\ \partial_{\mathbf{n}} V^1 & = & -|\nabla u_0(0)|N_1 & \text{on } \partial\mathbf{H}_\infty, \\ V^1 & \to & 0 & \text{at infinity}, \end{cases} \tag{10}$$

where N_1 denote the first component of the unitary normal vector on $\partial\mathbf{H}_\infty$. The existence and uniqueness of such a profile follows from the lemma

Lemma 2.1 *Problem* (10) *admits a unique weak solution V^1 in the variational space*

$$\left\{V \; ; \; \nabla V \in \mathrm{L}^2(\mathbf{H}_\infty) \quad \text{and} \quad \frac{V}{(1+|X|)\log(2+|X|)} \in \mathrm{L}^2(\mathbf{H}_\infty)\right\}. \tag{11}$$

Furthermore, we have the following behaviors at infinity:

$$V^1(X) = \mathcal{O}(|X|^{-1}) \quad \text{and} \quad \nabla V^1(X) = \mathcal{O}(|X|^{-2}) \quad \text{when } |X| \to \infty. \tag{12}$$

The proof makes use of a weighted Poincaré-like inequality, for the first part, and the Mellin transform for the behavior at infinity, cf. [2].

Thanks to the profile V^1, we are able to write the beginning of the asymptotic expansion of u_ε: we set

$$r_\varepsilon^1 = u_\varepsilon - \left[\tilde{u}_0 + \chi(x)\varepsilon\, V^1(\tfrac{x}{\varepsilon})\right]. \tag{13}$$

By construction, this remainder satisfies

$$\begin{cases} -\Delta r_\varepsilon^1 & = & \varphi_\varepsilon^0 + \varphi_\varepsilon^1 & \text{in } \Omega_\varepsilon, \\ r_\varepsilon^1 & = & 0 & \text{on } \Gamma_D, \\ \partial_{\mathbf{n}} r_\varepsilon^1 & = & \psi_\varepsilon^0 + \psi_\varepsilon^1 & \text{on } \partial\Omega_\varepsilon\backslash\Gamma_D. \end{cases} \tag{14}$$

The function φ_ε^1 comes from the cut-off function χ:

$$\varphi_\varepsilon^1 = \Delta\left[\chi(\cdot)\varepsilon V^1(\tfrac{\cdot}{\varepsilon})\right]. \tag{15}$$

Note that in the Laplacian, only derivatives of χ are involved since V^1 is harmonic, and only $|x| > r^*/2$ has to be considered in (15). The function ψ_ε^1 has its support inside the ball $|x| < \varepsilon R^*$ and is given by

$$\psi_\varepsilon^1 = \chi(x)\partial_{\mathbf{n}} V^1(\tfrac{x}{\varepsilon}) - \chi(x)\partial_{\mathbf{n}} T_K = -\chi(x)\sum_{k=2}^{K} \partial_{\mathbf{n}} u^k(x) = \mathcal{O}_{\mathrm{L}^2(\Omega_\varepsilon)}(\varepsilon^2), \tag{16}$$

since V^1 stands for the term corresponding to $k = 1$ of the Taylor expansion (the constant term u^0 does not contribute to the normal derivative).

It is not straightforward to obtain a remainder estimate on r_ε^1 since the L^2-norm of φ_ε^1 is only $\mathcal{O}(1)$. We need to build further terms to get the (optimal) estimate

$$\|r_\varepsilon^1\|_{\mathrm{H}^1(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^2). \tag{17}$$

The proof will follow from Theorem 2.2 below.

To continue the construction of the expansion, we need to take into account the next terms in the Taylor expansion of u_0 by new profiles, and add correctors for the cut-off. The technology used in [2, 3, 13] can be extended, the main differences have been described just above for the first terms. Precisely, we get

Theorem 2.2 *We assume that f in an L^2-function, with compact support inside Ω_0. Then the solution u_ε of (5) admits the following asymptotic expansion for $N < K$*

$$u_\varepsilon(x) = \tilde{u}_0(x) + \chi(x)\sum_{i=1}^{N}\varepsilon^i V^i(\tfrac{x}{\varepsilon}) + \sum_{i=2}^{N}\varepsilon^i w_\varepsilon^i(x) + \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}(\varepsilon^{N+1}). \tag{18}$$

The term $\tilde{u}_0$ is defined by (7), the profiles V^i is a counterpart for the ith term u^i of the Taylor expansion of u_0 – see (20), and w_ε^i are cut-off correctors satisfying $\|w_\varepsilon^i\|_{\mathrm{H}^1(\Omega_\varepsilon)} = \mathcal{O}(1)$.

Proof of Theorem 2.2:   We give a sketch of the proof for the complete asymptotic expansion. Supposing the expansion built until rank $N-1$, we set

$$r_\varepsilon^N(x) = u_\varepsilon(x) - \tilde{u}_0(x) - \chi(x)\sum_{i=1}^{N-1}\varepsilon^i V^i(\tfrac{x}{\varepsilon}) - \zeta(\tfrac{x}{\varepsilon})\sum_{i=2}^{N-1}\varepsilon^i w^i(x), \tag{19}$$

the remainder of order $N-1$. By definition, the profiles V^i satisfies

$$\begin{cases} -\Delta V^i & = & 0 & \text{in } \mathbf{H}_\infty, \\ \partial_{\mathbf{n}} V^i & = & -\partial_{\mathbf{n}} u^i & \text{on } \partial\mathbf{H}_\infty, \\ V^i & \to & 0 & \text{at infinity.} \end{cases} \tag{20}$$

(again, the datum is compactly supported and Lemma 2.1 ensures[1] existence and uniqueness of V^1).
Laplacian. By construction, the residual in Δr_ε^N is corrected up to order $N-1$ by the w^i. But the term $\Delta[\chi(x)\varepsilon^{N-1}V^{n_1}(\tfrac{x}{\varepsilon})]$ is of order ε^N (in $\mathrm{L}^\infty(\Omega_\varepsilon)$) thanks to an estimate similar to (12). We define hence $w^N\varepsilon$ as the solution in $\mathrm{H}^1(\Omega_\varepsilon)$ of

$$-\Delta w^N = -\Delta[\chi(x)\varepsilon^{N-1}V^{N+1}(\tfrac{x}{\varepsilon})] \quad \text{with same boundary conditions as } u_0. \tag{21}$$

Boundary conditions. The Dirichlet boundary condition on Γ_D is fully satisfied by r_ε^N, but the Neumann boundary condition is not. Indeed, only the $N-1$ first Neumann-traces have been taken into account so far by the profiles V^i: the leading term in $\partial_{\mathbf{n}} r_\varepsilon^N$ on $\partial\Omega_\varepsilon\backslash\Gamma_D$ is given by $-\partial_{\mathbf{n}} u^N(x) = -\varepsilon^N\partial_{\mathbf{n}} u^N(\tfrac{x}{\varepsilon})$. This naturally leads to the definition of V^N, according to (20).

Conclusion. The introduction of the terms w^N and V^N allows to define the remainder r_ε^N of order N, which satisfies

- the laplacian $-\Delta r_\varepsilon^N$ is small: precisely, its leading term is $\varepsilon^{N+1}\Delta[\chi(x)V^{N+1}(\tfrac{x}{\varepsilon})]$, whose $\mathrm{L}^2(\Omega_\varepsilon)$-norm is of order ε^N;

- the Neumann boundary condition is satisfied up to a term in $\mathcal{O}_{\mathrm{L}^2(\partial\Omega_\varepsilon)}$.

Using an *a priori* estimate on Problem (5) (independent on ε) we immediately get the estimate $r_\varepsilon^N = \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}(\varepsilon^N)$. To obtain ε^{N+1}, we simply write

$$r_\varepsilon^N = r_\varepsilon^{N+1} + \chi(x)\varepsilon^{N+2}V^{N+2}(\tfrac{x}{\varepsilon}) + \varepsilon^{N+2}w_\varepsilon^{N+2}(x), \tag{22}$$

yielding the result from the estimates

$$\chi(x)V^{N+2}(\tfrac{x}{\varepsilon}) = \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}(\varepsilon^{N+1}) \quad \text{and} \quad w_\varepsilon^{N+2} = \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}(1). \tag{23}$$

$\blacksquare$

---

[1]Since Neumann conditions are considered, we have to make sure that the right hand-side of (20) meets the compatibility requirement. This is the case here: since u_0 is harmonic, it is also the case of the terms in its Taylor expansion.

Remark 2.3 *By a mere rearrangement of the terms, the expansion of u_ε can read as follows*

$$u_\varepsilon = \zeta(\tfrac{x}{\varepsilon})u_0(x) + \chi(x)\sum_{i=1}^{N}\varepsilon^i W^i(\tfrac{x}{\varepsilon}) + \sum_{i=2}^{N}\varepsilon^i \tilde{w}_\varepsilon^i(x) + \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}(\varepsilon^N). \tag{24}$$

The new profiles W^i are defined by $W^i(X) = V^i(X) + \big(1 - \zeta(X)\big)u^i(X)$ and the $\tilde{w}_\varepsilon^i$ are new correctors. The advantage of this formulation is to involve u_0 itself, instead of $\tilde{u}_0$.

In the case of an inclusion, i.e. $\Omega_\varepsilon \subset \Omega_0$, the function ζ can be chosen identically equal to 1, and $W^i = V^i$.

Remark 2.4 *We can deplore that the correcting terms w_ε^i do depend on ε, though weakly since they are of order $\mathcal{O}(1)$ in the $\mathrm{H}^1(\Omega_\varepsilon)$-norm. It is possible to remove this feature from the asymptotic expansion by introducing correctors z^i defined in the limit domain Ω_0 (with same right-hand side), and using the cut-off function ζ. Of course, the normal trace does no more vanish on the perturbed boundary and we have to take this into account in the definition of the profiles. The resulting expansion reads*

$$u_\varepsilon(x) = \tilde{u}_0(x) + \chi(x)\sum_{i=1}^{N}\varepsilon^i \tilde{V}^i(\tfrac{x}{\varepsilon}) + \zeta(\tfrac{x}{\varepsilon})\sum_{i=2}^{N}\varepsilon^i z^i(x) + \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}(\varepsilon^{N+1}). \tag{25}$$

or, with the previous remark,

$$u_\varepsilon(x) = \zeta(\tfrac{x}{\varepsilon})u_0(x) + \chi(x)\sum_{i=1}^{N}\varepsilon^i \tilde{W}^i(\tfrac{x}{\varepsilon}) + \zeta(\tfrac{x}{\varepsilon})\sum_{i=2}^{N}\varepsilon^i \tilde{z}^i(x) + \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}(\varepsilon^{N+1}). \tag{26}$$

3 Extension to some curved boundaries.

In this section, for the lightness of the presentation, we consider the case of Dirichlet boundary conditions. Let u_ε solve $-\Delta u = f$ in $\mathrm{H}_0^1(\Omega_\varepsilon)$ while u_0 solves the same equation in $\mathrm{H}_0^1(\Omega_0)$. We also restrict ourself to the inclusion case to avoid the need of $\tilde{u}_0$, and we make the assumption that the initial domain is convex in the neighborhood of O. The geometrical situation is illustrated in Figure 2.
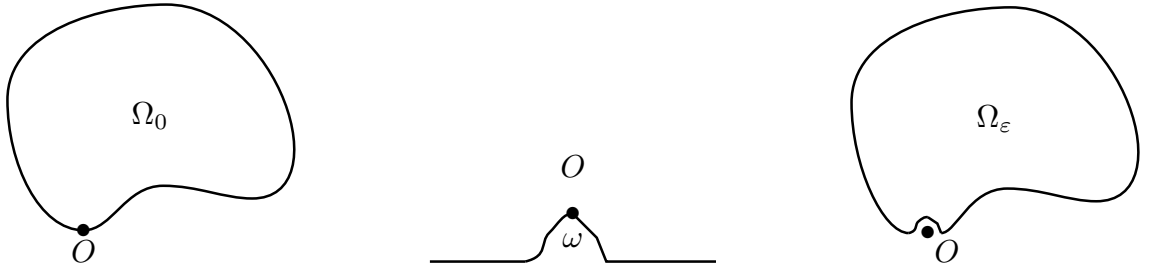

Figure 2: Domains in the case of locally convex curved boundary .

This situation in not a mere extension of the flat one, considered previously. Indeed, if we rectify the boundary locally near O, the perturbation is not selfsimilar anymore!

Following the analysis performed in [3], we introduce the profile V_d^1 as the solution of the problem in the infinite domain $\mathbf{H}_\infty$:

$$\begin{cases} -\Delta V_d^1 &=& 0 & \text{in } \mathbf{H}_\infty, \\ V_d^1 &=& -|\nabla u_0(0)|x_2 & \text{on } \partial\mathbf{H}_\infty, \\ V_d^1 &\to& 0 & \text{at infinity,} \end{cases} \tag{27}$$

where x_2 denote the second component of the position on $\partial\mathbf{H}_\infty$. As for the Neumann case, existence and uniqueness of such a profile follows from next lemma, similar to lemma 2.1.

Lemma 3.1 *Problem* (10) *admits a unique weak solution V_d^1 in the variational space*

$$\left\{ V \; ; \; \nabla V \in \mathrm{L}^2(\mathbf{H}_\infty) \quad and \quad \frac{V}{1+|X|} \in \mathrm{L}^2(\mathbf{H}_\infty) \right\}. \tag{28}$$

Furthermore, there is a constant C depending only $\mathbf{H}_\infty$ such that

$$|V_d^1(X)| \le \frac{C}{|X|} \quad and \quad |\nabla V_d^1(X)| = \frac{C}{|X|^2} \quad when \; |X| \to \infty. \tag{29}$$

As in [3], we approximate u_ε by $u_0 + \chi V_d^1(\frac{\cdot}{\varepsilon})$ and we set

$$r_\varepsilon^d(x) = u_\varepsilon(x) - \left[u_0(x) + \chi(x) V_d^1(\tfrac{x}{\varepsilon}) \right]. \tag{30}$$

This remainder solves

$$\begin{cases} -\Delta r_\varepsilon^d(x) &= \Delta \left[\chi(x)\varepsilon V_d^1(\tfrac{x}{\varepsilon}) \right], & \text{in } \Omega_\varepsilon, \\ r_\varepsilon^d(x) &= u_0(x) - \chi(x)\varepsilon V_d^1(\tfrac{x}{\varepsilon}) & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{31}$$

The difference with the flat case treated is the presence of boundary condition on $\partial\Omega_0 \cap \partial\Omega_\varepsilon$. The expansions obtained in [3] and in Section 2 were justified without taking into account the short range interaction between the profiles and the geometry of the initial domain Ω_0. The flatness assumption of Ω_0 around O cancels the interaction between slow and rapid variable. Let us emphasize the fact that the approximation $u_0 + \chi V_d^1(\frac{\cdot}{\varepsilon})$ does not satisfy the homogeneous Dirichlet boundary conditions on $\partial\Omega_0 \cap \partial\Omega_\varepsilon$. However, its trace almost vanishes.

Like in the previous section, the laplacian part is easy to handle and it holds:

$$\left\| \Delta \left[\chi(x)\varepsilon V_d^1(\tfrac{x}{\varepsilon}) \right] \right\|_{\mathrm{L}^2(\Omega_\varepsilon)} \le C\varepsilon^2.$$

We need to consider the boundary conditions on $\partial\Omega_\varepsilon$ in the two natural parts: on $\partial\Omega_\varepsilon \cap \Omega_0$, we immediately get $r_\varepsilon^d = u^2$, which is naturally of order ε^2 as a reminder of order 2 in a Taylor expansion. To prove that this estimate extends to $\partial\Omega_0 \cap \partial\Omega_\varepsilon$ requires some precautions and turns out to be the most difficult part of the extension of to curved boundaries. The leading idea of the analysis is a decomposition of profiles in terms of homogeneous functions, usually obtained from the Mellin transform, see [5, 2]. Here, we only need the weak following statement.

Lemma 3.2 *The profil V_d^1 can be written as the sum $\mathcal{V}_d^1 + \mathcal{R}$ where $\mathcal{V}_d^1$ is an homogeneous function of degree -1 and the remainder $\mathcal{R}$ has a precised behavior at infinity: there is a constant C depending only $\mathbf{H}_\infty$ such that*

$$|\mathcal{R}(X)| \le \frac{C}{|X|} \quad and \quad |\nabla\mathcal{R}(X)| = \frac{C}{|X|^2} \quad when \; |X| \to \infty. \tag{32}$$

Proof of Lemma 3.2:   Fix $R > 0$ large enough so that $\omega \subset B(O, R)$. Then, the trace of V_d^1 on the curve $\partial B(O, R) \cap \mathbf{H}_\infty$ is smooth and can be written as the sum of its Fourier series. Thanks to the boundary conditions, only the sinus appear and one gets

$$V_d^1(R, \theta) = a_0 + \sum_{n \ge 1} a_n \sin n\theta.$$

Using Poisson's kernel, we then get that

$$V_d^1(r, \theta) = a_0 + \sum_{n \ge 1} a_n \frac{R^n}{r^n} \sin n\theta.$$

The behavior at infinity of V_d^1 imposes $a_0 = 0$ and we set $\mathcal{V}_d^1(r, \theta) = a_1 a_n \dfrac{R}{r} \sin\theta$. Note that the dependency of expression of $\mathcal{V}_d^1$ with respect to R is fictious thanks to the homogeneity of its expression. Setting $\mathcal{R} = V_d^1 - \mathcal{V}_d^1$, leads to the stated result. ∎

Let us specify the geometry of $\partial\Omega_0$ around O. We assume $\partial\Omega_0$ to be $\mathcal{C}^2$ and fix the coordinate axis such that $\partial\Omega_0$ is the graph $x_2 = h(x_1)$ of a function h in the neighborhood of O with $h(0) = h'(0) = 0$. Then, there exists a number $C > 0$ and a radius $r > 0$ such that for $x = (x_1, x_2) \in \partial\Omega_0$, it holds

$$|x| \leq r \Rightarrow 0 \leq h(x_1) \leq C|x_1|^2 \text{ and } |h'(x_1)| \leq C|x_1|;$$

this property is connected to the $\mathcal{C}^2$ regularity od $\partial\Omega_0$. We fix $r^* = r$ and choose $\varepsilon \ll r^*$. This assumption the characteristic size of the inclusion is small with respect of the radius of curvature of $\partial\Omega_0$ at O is a natural limitation of the method. The geometrical context is summed up in Figure 3.
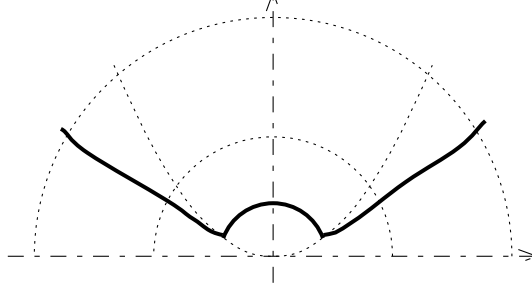

Figure 3: The geometrical setting of the inclusion in the convex case.

We can now state the estimates on the boundary conditions. The homogeneous part $\mathcal{V}_d^1$ of V_d^1 is homogeneous of order -1, therefore it is easy to check that $\|\mathcal{V}_d^1\|_{\mathrm{H}^{1/2}(\partial\Omega_\varepsilon)}$ is of order two in ε. We focus on the remainder

$$\tilde{r}(x) = r_\varepsilon^d + \varepsilon\chi(x_1, h(x_1))\, \mathcal{V}_d^1\left(\frac{[x_1, h(x_1)]}{\varepsilon}\right).$$

Proposition 3.3 *One has*

$$\|\tilde{r}_\varepsilon^d\|_{\mathrm{H}^{1/2}(\partial\Omega_\varepsilon)} \leq C\varepsilon^2. \tag{33}$$

Proof of Proposition 3.3: In order to split the norm on the differents parts of $\partial\Omega_\varepsilon$, we first study the L^2 and H^1 norms of the trace. In a first step, we show:

$$\|\tilde{r}_\varepsilon^d\|_{\mathrm{L}^2(\partial\Omega_\varepsilon)} \leq C\varepsilon^{5/2}, \tag{34}$$

$$\|\tilde{r}_\varepsilon^d\|_{\mathrm{H}^1(\partial\Omega_\varepsilon)} \leq C\varepsilon^{3/2}. \tag{35}$$

Thanks to the assumption made on the truncation in slow variable, the only two areas are to be considered: $\varepsilon\partial\omega$ the boundary of the inclusion itself, and the part of $\partial\Omega_\varepsilon \setminus \varepsilon\partial\omega$ in the support of the truncation χ. On $\varepsilon\partial\omega$, r_ε^d is by construction the remainder of order two in the Taylor expansion of u_{Ω_0}. Therefore, it is smooth with a L^∞-norm of order ε^2 and there is a constant $C > 0$ such that

$$\int_{\varepsilon\partial\omega} (\tilde{r}_\varepsilon^d(s))^2 ds \leq C\varepsilon^5,$$

After one derivation, one looses one order and gets

$$\int_{\varepsilon\partial\omega} (\nabla_\tau \tilde{r}_\varepsilon^d(s))^2 ds \leq C\varepsilon^3.$$

For $x = (x_1, h(x_1)) \in \partial\Omega_\varepsilon \setminus \varepsilon\partial\omega$, the remainder r_ε^d is

$$\tilde{r}_\varepsilon^d(x_1, h(x_1)) = -\varepsilon\chi(x, h(x_1))\, \mathcal{R}\left(\frac{(x, h(x_1))}{\varepsilon}\right).$$

Now, for $x_1 \in (-r^*, r^*)$, we take advantage of the homogeneous Dirichlet boundary conditions and write the remainder as an integral and make the change of variable $y = \varepsilon s$:

$$\tilde{r}_\varepsilon^d(x_1, h(x_1)) \leq \varepsilon \mathcal{R}\left(\frac{(x, h(x_1))}{\varepsilon}\right) = \varepsilon \int_0^{h(x_1)/\varepsilon} \partial_2 \mathcal{R}\left(\frac{x}{\varepsilon}, s\right) ds = \int_0^{h(x_1)} \partial_2 \mathcal{R}\left(\frac{x_1}{\varepsilon}, \frac{y}{\varepsilon}\right) dy$$

Using the upper bound (29) on the profile, we get the pointwise estimate

$$\tilde{r}_\varepsilon^d(x_1, h(x_1)) \leq \int_0^{h(x_1)} \frac{C}{1 + \left|\frac{x_1}{\varepsilon}\right|^3} dy \leq \frac{C|x|^4 \varepsilon^3}{\varepsilon^3 + |x_1|^3}$$

that leads to

$$\int_\varepsilon^{r^*} \left(\tilde{r}_\varepsilon^d(x_1, h(x_1))\right)^2 dx_1 \leq C\varepsilon^6 \int_\varepsilon^{r^*} \frac{|x_1|^4}{(\varepsilon^3 + |x_1|^3)^2} dx_1$$

After the change of variables $x_1 = \varepsilon y$, we finally get

$$\int_\varepsilon^{r^*} \left(\tilde{r}_\varepsilon^d(x_1, h(x_1))\right)^2 dx_1 \leq C\varepsilon^5 \int_1^{r^*/\varepsilon} \frac{|y|^4}{(1 + |y|^3)^2} dy \leq C\varepsilon^5.$$

Let us turn ourselves to the derivative. For $x = (x_1, h(x_1)) \in \partial\Omega_\varepsilon \setminus \varepsilon\partial\omega$, one has

$$\nabla_\tau \tilde{r}_\varepsilon^d(x) = \chi(x)\left[\partial_1 \mathcal{R}\left(\frac{x_1}{\varepsilon}, \frac{h(x_1)}{\varepsilon}\right) + h'(x_1)\partial_2 \mathcal{R}\left(\frac{x_1}{\varepsilon}, \frac{h(x_1)}{\varepsilon}\right)\right] + \nabla_\tau \chi(x)\mathcal{R}\left(\frac{x_1}{\varepsilon}, \frac{h(x_1)}{\varepsilon}\right).$$

We decompose this sum into

$$
\begin{aligned}
T_1(x) &= \chi(x)\partial_1 \mathcal{R}\left(\frac{x_1}{\varepsilon}, \frac{h(x_1)}{\varepsilon}\right), \\
T_2(x) &= \chi(x)h'(x_1)\partial_2 \mathcal{R}\left(\frac{x_1}{\varepsilon}, \frac{h(x_1)}{\varepsilon}\right), \\
T_3(x) &= \nabla_\tau \chi(x)\mathcal{R}\left(\frac{x_1}{\varepsilon}, \frac{h(x_1)}{\varepsilon}\right).
\end{aligned}
$$

The study of T_3 is a corollary of (34) and the Cauchy-Schwarz inequality leads to

$$\int_\varepsilon^{r^*} |T_3(x)|^2 dx_1 \leq C\varepsilon^5.$$

The other terms involve derivation in the fast variable and hence loss of order. More precisely, we have:

$$T_1(x) = \int_0^{h(x_1)/\varepsilon} \partial_{2,1}^2 \mathcal{R}\left(\frac{x_1}{\varepsilon}, s\right) ds \leq \frac{C|x_1|^2 \varepsilon^3}{\varepsilon^4 + |x_1|^4}.$$

Once we integrate over x_1, we obtain

$$\int_\varepsilon^{r^*} |T_1(x_1)|^2 dx_1 \leq C\varepsilon^3 \int_1^{r^*/\varepsilon} \frac{|y|^4}{(1 + |y|^3)^2} dy.$$

Finally, we write

$$T_2(x) \leq C|x_1| \frac{C}{1 + \left|\frac{x_1}{\varepsilon}\right|^3} \leq \frac{C|x|\varepsilon^3}{\varepsilon^3 + |x|^3}.$$

$$\int_\varepsilon^{r^*} |T_1(x_1)|^2 dx_1 \leq C\varepsilon^3 \int_1^{r^*/\varepsilon} \frac{|y|^2}{(1 + |y|^3)^2} dy.$$

We treat the double products thanks to the Cauchy-Schwarz inequality to get (35). The estimate (33) is easily deduced from (34) and (35) by interpolation. ∎

All the ingredients needed to prove the main result of this section are now at our disposal.

Theorem 3.4 *In the curved situation described previously , it holds:*

$$u_\varepsilon(x) = u_0(x) + \chi(x)V_d^1(\tfrac{x}{\varepsilon}) + \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}(\varepsilon^2). \tag{36}$$

while the boundary condition satisfies

$$u_0(x) + \chi(x)V_d^1(\tfrac{x}{\varepsilon}) = \mathcal{O}_{\mathrm{H}^{1/2}(\partial\Omega_\varepsilon)}(\varepsilon^2) \tag{37}$$

4 Variations of energy integrals for singular domain deformations.

In this section, we investigate the behavior of the Dirichlet energy with respect to singular deformations of the boundary, our presentation is similar to [10]. We recall that for a given function $f \in \mathcal{D}(\mathbb{R}^d)$ the Dirichlet energy of Ω_0 an bounded open subset of $\mathbb{R}^d$ with $\mathrm{supp}(f) \subset\subset \Omega_0$ is

$$J(\Omega_0) = -\frac{1}{2}\int_{\Omega_0} |\nabla u_{\Omega_0}(x)|^2\, dx,$$

where u_{Ω_0} is the solution of $-\Delta u = f$ in $\mathrm{H}_0^1(\Omega_0)$. We consider the same class of singular deformations than in the previous section. The notations are recalled in Figure 4. The first result is the following.
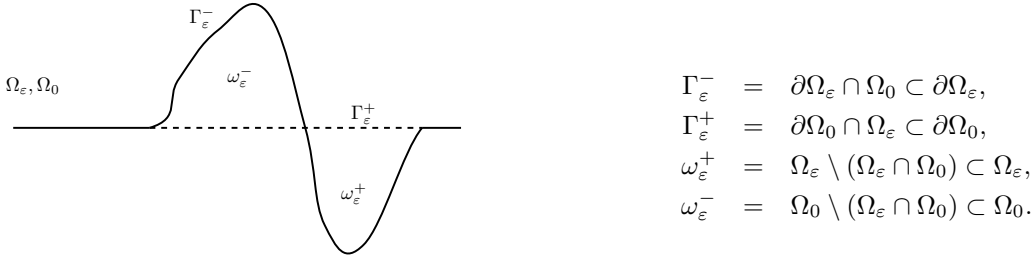

$$
\begin{aligned}
\Gamma_\varepsilon^- &= \partial\Omega_\varepsilon \cap \Omega_0 \subset \partial\Omega_\varepsilon, \\
\Gamma_\varepsilon^+ &= \partial\Omega_0 \cap \Omega_\varepsilon \subset \partial\Omega_0, \\
\omega_\varepsilon^+ &= \Omega_\varepsilon \setminus (\Omega_\varepsilon \cap \Omega_0) \subset \Omega_\varepsilon, \\
\omega_\varepsilon^- &= \Omega_0 \setminus (\Omega_\varepsilon \cap \Omega_0) \subset \Omega_0.
\end{aligned}
$$

Figure 4: The notations.

Proposition 4.1 *Let $\varepsilon > 0$ be such that $\mathrm{supp}(f) \subset\subset \Omega_\varepsilon$ and u_ε (resp. u_0) denotes the solution of $-\Delta u = f$ in $\mathrm{H}_0^1(\Omega_\varepsilon)$ (resp. $\mathrm{H}_0^1(\Omega_0)$). Then, one has :*

$$J(\Omega_\varepsilon) = J(\Omega_0) - \frac{1}{2}\int_{\Gamma_\varepsilon^-} u_0 \frac{\partial u_\varepsilon}{\partial n} d\sigma + \frac{1}{2}\int_{\Gamma_\varepsilon^+} u_\varepsilon \frac{\partial u_0}{\partial n} d\sigma \tag{38}$$

Proof of Proposition 4.1.: The proof is elementary and based on the Gauss formula. Hence, it cannot be extended to more general shape functional. Since $\Omega_\varepsilon = (\Omega_\varepsilon \cap \Omega_0) \cup \omega_\varepsilon^+$, we write

$$J(\Omega_\varepsilon) = -\frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\, dx = -\frac{1}{2}\int_{\Omega_\varepsilon \cap \Omega_0} |\nabla u_0 + \nabla(u_\varepsilon - u_0)|^2\, dx - \frac{1}{2}\int_{\omega_\varepsilon^+} |\nabla u_\varepsilon|^2\, dx.$$

The second integral can be rewritten by the Gauss formula. One has to be careful with the outer normal vector field. By $\mathbf{n}$, we denote the outer normal vector field of $\partial\Omega_\varepsilon$ or $\partial\Omega_0$ depending on the context. It may be the opposite to the outer normal field to ω_ε^-, ω_ε^+ denoted by n :

$$\int_{\omega_\varepsilon^+} |\nabla u_\varepsilon|^2\, dx = \int_{\omega_\varepsilon^+} u_\varepsilon(-\Delta u_\varepsilon)dx + \int_{\Gamma_\varepsilon^+} u_\varepsilon \frac{\partial u_\varepsilon}{\partial n} d\sigma = -\int_{\Gamma_\varepsilon^+} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \mathbf{n}} d\sigma.$$

We expend the first integral and get :

$$
\int_{\Omega_\varepsilon \cap \Omega_0} |\nabla u_0 + \nabla(u_\varepsilon - u_0)|^2\, dx = \int_{\Omega_\varepsilon \cap \Omega_0} |\nabla u_0|^2\, dx + 2\int_{\Omega_\varepsilon \cap \Omega_0} \langle \nabla u_0, \nabla(u_\varepsilon - u_0)\rangle dx
$$
$$
+ \int_{\Omega_\varepsilon \cap \Omega_0} |\nabla(u_\varepsilon - u_0)|^2\, dx.
$$

9

Applying Green's formula and using the homogeneous Dirichlet boundary conditions on $\partial\Omega_\varepsilon$ and $\partial\Omega_0$, we have :

$$\int_{\Omega_\varepsilon \cap \Omega_0} |\nabla u_0|^2 \, dx = \int_\Omega |\nabla u_0|^2 \, dx + \int_{\Gamma_\varepsilon^-} u_0 \frac{\partial u_0}{\partial \mathbf{n}} d\sigma;$$

$$\int_{\Omega_\varepsilon \cap \Omega_0} |\nabla(u_\varepsilon - u_0)|^2 \, dx = \int_{\Gamma_\varepsilon^-} u_0 \left(\frac{\partial u_0}{\partial \mathbf{n}} - \frac{\partial u_\varepsilon}{\partial \mathbf{n}} \right) d\sigma + \int_{\Gamma_\varepsilon^+} u_\varepsilon \left(\frac{\partial u_\varepsilon}{\partial \mathbf{n}} - \frac{\partial u_0}{\partial \mathbf{n}} \right) d\sigma;$$

$$\int_{\Omega_\varepsilon \cap \Omega_0} \langle \nabla u_0, \nabla(u_\varepsilon - u_0) \rangle dx = \int_{\Gamma_\varepsilon^-} u_0 \frac{\partial u_\varepsilon}{\partial \mathbf{n}} d\sigma - \int_{\Gamma_\varepsilon^-} u_0 \frac{\partial u_0}{\partial \mathbf{n}} d\sigma.$$

We now sum up all these intermediary computations, and we get :

$$\int_{\Omega_\varepsilon \cap \Omega_0} |\nabla u_\varepsilon|^2 \, dx = \int_{\Omega_0} |\nabla u_0|^2 \, dx + \int_{\Gamma_\varepsilon^-} u_0 \frac{\partial u_\varepsilon}{\partial \mathbf{n}} d\sigma + \int_{\Gamma_\varepsilon^+} u_\varepsilon \left(\frac{\partial u_\varepsilon}{\partial \mathbf{n}} - \frac{\partial u_0}{\partial \mathbf{n}} \right) d\sigma;$$

and

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx = \int_{\Omega_0} |\nabla u_0|^2 \, dx + \int_{\Gamma_\varepsilon^-} u_0 \frac{\partial u_\varepsilon}{\partial \mathbf{n}} d\sigma - \int_{\Gamma_\varepsilon^+} u_\varepsilon \frac{\partial u_0}{\partial \mathbf{n}} d\sigma.$$

This concludes the proof. ∎

Change of boundary conditions. The same method allows to treat the change of boundary conditions imposed on the perturbed part of the boundary. Assume that the state function u_{Ω_ε} solves now the mixed problem

$$\begin{cases} -\Delta u &= f \in \Omega_\varepsilon, \\ u &= 0 \in \partial\Omega_\varepsilon \cap \partial\Omega_0, \\ \partial_n u &= 0 \in \partial\Omega_\varepsilon \setminus (\partial\Omega_\varepsilon \cap \partial\Omega_0). \end{cases} \tag{39}$$

We can state the counterpart of Theorem 4.1.

Proposition 4.2 *Let $\varepsilon > 0$ be such that* $\operatorname{supp}(f) \subset\subset \Omega_\varepsilon$ *and u_ε denotes the solution of (39) in* $\mathrm{H}^1(\Omega_\varepsilon)$. *Let u_0 be the solution of $-\Delta u = f$ in* $\mathrm{H}_0^1(\Omega_0)$*). Then, one has :*

$$J(\Omega_\varepsilon) = J(\Omega_0) + \frac{1}{2} \int_{\Gamma_\varepsilon^-} u_0 \frac{\partial u_0}{\partial n} d\sigma - \frac{1}{2} \int_{\Gamma_\varepsilon^+} u_\varepsilon \frac{\partial u_\varepsilon}{\partial n} d\sigma. \tag{40}$$

The proof is very similar to the proof of Theroem 4.1. The changes appear in the Green formula .

Inserting the asymptotic expansion of u_ε into formulæ (38) and (40), we easily obtain

Proposition 4.3 *In the framework of Proposition 4.1, the Dirichlet energy admits the following asymptotic expansion:*
$$J(\Omega_\varepsilon) = J(\Omega_0) + \varepsilon^2 |\nabla u_0(0)|^2 \mathcal{A}_{\mathbf{H}_\infty} + o(\varepsilon^2), \tag{41}$$

where
$$\mathcal{A}_{\mathbf{H}_\infty} = -\frac{1}{2} \int_{\Gamma^-} K(y) \partial_N K(y) d\sigma_y + \frac{1}{2} \int_{\Gamma^+} K(y) N_2(y) d\sigma_y,$$

V is the normalized profile: $K = V_d^1 / |\nabla u_0(0)|$, *cf. (10).*

For 4.2, formula (41) hold with

$$\mathcal{A}_{\mathbf{H}_\infty} = -\frac{1}{2} \int_{\Gamma^-} N_2(y) d\sigma_y + \frac{1}{2} \int_{\Gamma^+} K(y) \partial_N K(y) d\sigma_y.$$

with the modified boundary conditions.

5 Numerics

5.1 Strategy

As already mentioned, the solution u_ε of the model problem (5) is difficult to approximate from a numerical point of view: the refinement needed near the perturbation for a reasonable precision prevents (a least for small values of ε) to compute u_ε directly. The asymptotic expansion, see Theorem 2.2, suggests a the following numerical strategy.

Writing the expansion (24) of u_ε at order 1, we get

$$u_\varepsilon(x) \simeq \zeta(\tfrac{x}{\varepsilon})u_0(x) + \varepsilon\chi(x)V^1(\tfrac{x}{\varepsilon}). \tag{42}$$

For simplicity, we consider here the case of an inclusion ($\Omega_\varepsilon \subset \Omega_0$) and thanks to Remark 2.3, the cut-off function ζ may be chosen identically equal to 1. A natural approximation of u_ε reads then $u_\varepsilon(x) \simeq u_0(x) + \varepsilon\chi(x)V^1(\tfrac{x}{\varepsilon})$.

- the limit term u_0 may be computed accurately in a pretty coarse mesh *independently of ε* ;

- the profile V^1 does not depend on ε, but only on the geometry of the pattern $\mathbf{H}_\infty$. Its approximation is not straightforward, since it is defined on an infinite domain, but a technique of artificial boundary is efficient in this case (with a suitable boundary condition, base for example on an integral representation).

The functions u_0 and V^1 being computed, it remains to perform the superposition of $u_0(x)$ with the correcting term $\varepsilon\chi(x)V^1(\tfrac{x}{\varepsilon})$. Since the mesh used for the approximations do not coincide, we need to transfer $V^1(\tfrac{\cdot}{\varepsilon})$ onto the mesh where u_0 has been computed. This step can be facilitated by using a regular mesh for V^1 (e.g. tensorial in polar coordinates, except near the perturbing pattern). The function χ is replaced in the computations by the use of a patch of elements: V^1 is not taken into account except in this patch.

The obtained approximation is close to u_ε up to order $\mathcal{O}(\varepsilon^2)$. For small values of ε, we expect the method to work fine; for larger ε, the results may be inaccurate, but in that case the perturbation can be incorporated directly to the initial mesh without harsh refinement. Of course, from a practical point of view *small* and *large* have to be adapted to the considered situation.

5.2 Numerical results

We conclude the paper with some numerical results which validate our approach. The considered problem is the following

$$u_\varepsilon \in \mathrm{H}^1(\Omega_\varepsilon), \; -\Delta u_\varepsilon = f \text{ in } \Omega_\varepsilon, \text{ and } \partial_n u = 0 \text{ on } \partial\Omega_\varepsilon, \tag{43}$$

where $f(x,y) = 2\pi^2\cos(\pi x)\sin(\pi y)$ and Ω_ε is the square $(-1/2, 1/2) \times (0,1)$ with a semicircular hole of radius ε, centered at $(0,0)$. Figure 5 shows, for $\varepsilon = 1/32$ the solution u_ε (top-left picture), the difference between u_ε and the limit term u_0 (top-right picture), the difference between u_ε and the corrected limit term $u_1 = u_0 + \varepsilon V^1(\tfrac{\cdot}{\varepsilon})$ (bottom-right picture[2]). The bottom-left graph represents, for various values of ε, the norm of the errors (L^2, H^1 and L^∞-norms).

The efficiency of the correction by the first profile clearly appears in these results: for example, with $\varepsilon = 1/128$, the L^∞-norm of $u_\varepsilon - u_1$ is about 40 times less than the L^∞-norm of $u_\varepsilon - u_0$.

In Figure 6, we present the same results in the Dirichlet case for a curved Ω_0:

$$u_\varepsilon \in \mathrm{H}^1(\Omega_\varepsilon), \; -\Delta u_\varepsilon = f \text{ in } \Omega_\varepsilon. \tag{44}$$

The same conclusions arise; the gain in L^∞-norm is here around 50.

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[2]The profile V^1 has been computed on a (quasi-)regular mesh, independently of the value of ε, and it has been projected onto the initial grid for the computation of u_1.
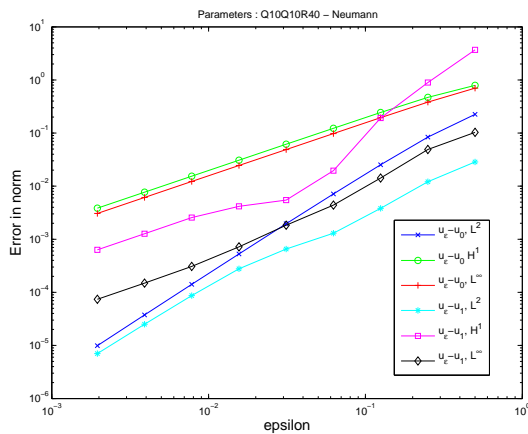
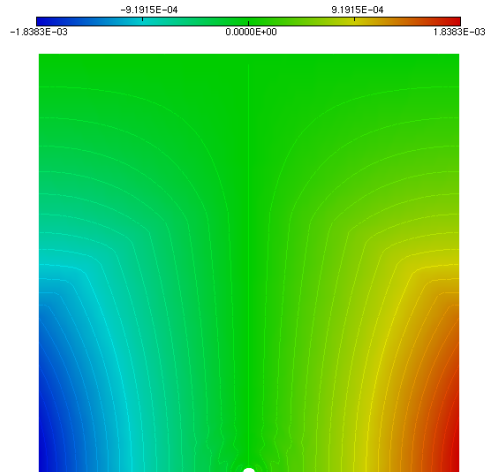| ε	$\|u_\varepsilon - u_0\|_{\mathrm{H}^1(\Omega_\varepsilon)}$	$\|u_\varepsilon - u_1\|_{\mathrm{H}^1(\Omega_\varepsilon)}$	Gain
$\varepsilon = 1/2$	7.9168e-01	3.6946e+00	0.2
$\varepsilon = 1/4$	4.6937e-01	8.9397e-01	0.5
$\varepsilon = 1/8$	2.4354e-01	1.9181e-01	1.3
$\varepsilon = 1/16$	1.2286e-01	1.9593e-02	6.3
$\varepsilon = 1/32$	6.1569e-02	5.4704e-03	11.3
$\varepsilon = 1/64$	3.0802e-02	4.1649e-03	7.4
$\varepsilon = 1/128$	1.5403e-02	2.5574e-03	6.0
$\varepsilon = 1/256$	7.7017e-03	1.2696e-03	6.1
$\varepsilon = 1/512$	3.8509e-03	6.3292e-04	6.1
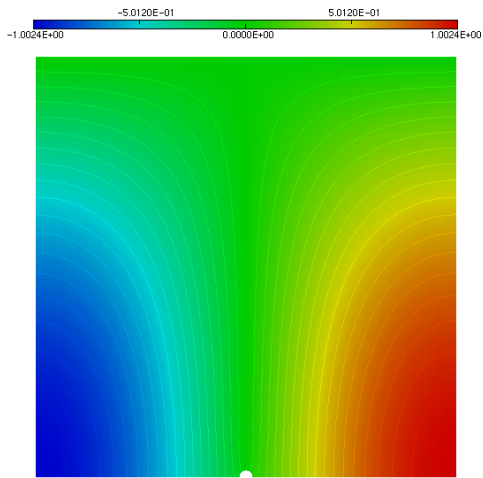
u_ε for $\varepsilon = 1/32$
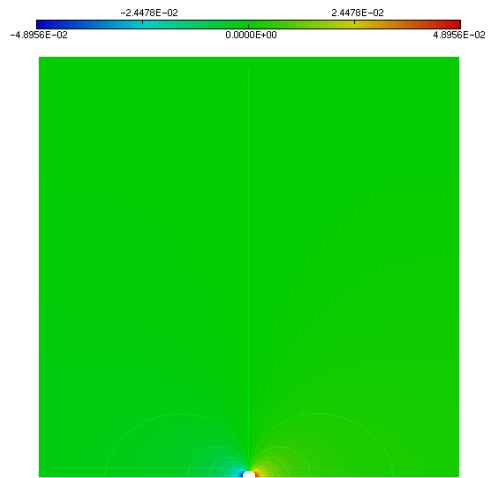
$u_\varepsilon - u_0$ for $\varepsilon = 1/32$

Norms of the errors

$u_\varepsilon - u_1$ for $\varepsilon = 1/32$

Figure 5: Computations for the Neumann problem (43).

ε	$\|u_\varepsilon - u_0\|_{\mathrm{H}^1(\Omega_\varepsilon)}$	$\|u_\varepsilon - u_1\|_{\mathrm{H}^1(\Omega_\varepsilon)}$	Gain
$\varepsilon = 1/2$	2.1645e-01	2.1029e+00	1.0
$\varepsilon = 1/4$	1.3264e+00	5.4458e-01	2.4
$\varepsilon = 1/8$	7.4406e-01	1.4416e-01	5.2
$\varepsilon = 1/16$	3.9208e-01	3.5187e-02	11.1
$\varepsilon = 1/32$	2.0097e-01	8.8438e-03	22.7
$\varepsilon = 1/64$	1.0170e-01	2.3011e-03	44.2
$\varepsilon = 1/128$	5.1153e-02	7.0180e-04	72.9
$\varepsilon = 1/256$	2.5652e-02	1.8811e-04	136.4
$\varepsilon = 1/512$	1.2845e-02	5.7410e-05	223.7

u_ε for $\varepsilon = 1/32$

$u_\varepsilon - u_0$ for $\varepsilon = 1/32$

Norms of the errors

$u_\varepsilon - u_1$ for $\varepsilon = 1/32$
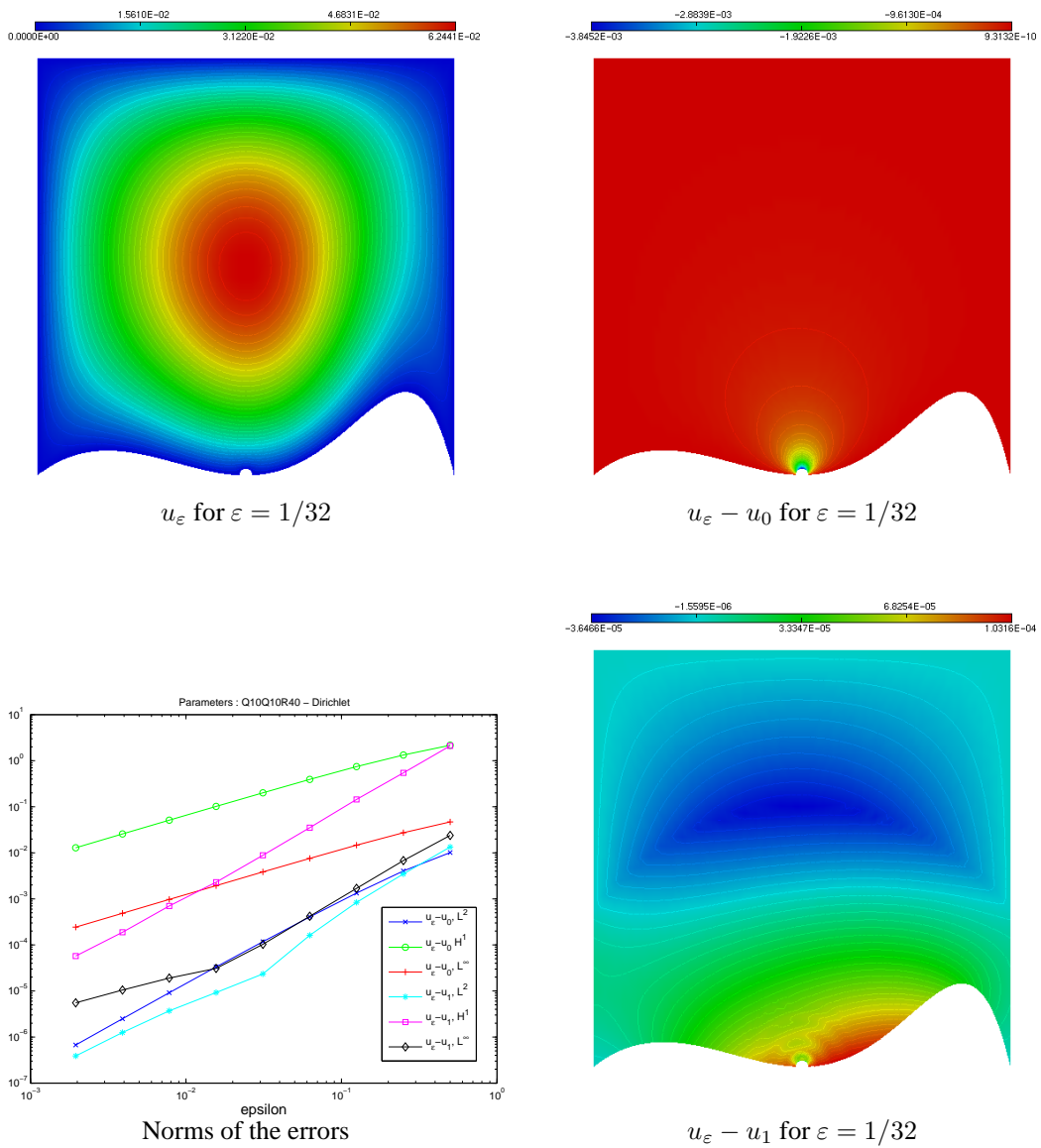
Figure 6: Computations for the Dirichlet problem (44).

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