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Moment estimates for Lévy Processes

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Abstract

For real Lévy processes \((X_t)_{t \geq 0}\) having no Brownian component with Blumenthal-Getoor index \(\beta\), the estimate \(\mathbb{E} \sup_{s \leq t} |X_s - a_p s|^p \leq C_p t\) for every \(t \in [0,1]\) and suitable \(a_p \in \mathbb{R}\) has been established by Millar [6] for \(\beta < p \leq 2\) provided \(X_1 \in L^p\). We derive extensions of these estimates to the cases \(p > 2\) and \(p \leq \beta\).

Key words: Lévy process increment, Lévy measure, \(\alpha\)-stable process, Normal Inverse Gaussian process, tempered stable process, Meixner process.

2000 Mathematics Subject Classification: 60G51, 60G18.

1 Introduction and results

We investigate the \(L^p\)-norm (or quasi-norm) of the maximum process of real Lévy processes having no Brownian component. A (càdlàg) Lévy process \(X = (X_t)_{t \geq 0}\) is characterized by its so-called local characteristics in the Lévy-Khintchine formula. They depend on the way the "big" jumps are truncated. We will adopt in the following the convention that the truncation occurs at size 1. So that

\[
\mathbb{E} e^{iuX_t} = e^{-t\Psi(u)} \text{ with } \Psi(u) = -iu + \frac{1}{2} \sigma^2 u^2 - \int (e^{iux} - 1 - iux1_{\{|x| \leq 1\}})d\nu(x),
\]

(1.1)

where \(u, a \in \mathbb{R}, \sigma^2 \geq 0\) and \(\nu\) is a measure on \(\mathbb{R}\) such that \(\nu(\{0\}) = 0\) and \(\int x^2 \land 1d\nu(x) < +\infty\). The measure \(\nu\) is called the Lévy measure of \(X\) and the quantities \((a, \sigma^2, \nu)\) are referred to as the characteristics of \(X\). One shows that for \(p > 0\), \(\mathbb{E} |X_1|^p < +\infty\) if and only if \(\mathbb{E} |X_t|^p < +\infty\) for every \(t \geq 0\) and this in turn is equivalent to \(\mathbb{E} \sup_{s \leq t} |X_s|^p < +\infty\) for every \(t \geq 0\). Furthermore,

\[
\mathbb{E} |X_1|^p < +\infty \text{ if and only if } \int_{\{|x| > 1\}} |x|^p d\nu(x) < +\infty
\]

(1.2)

(see [7]). The index \(\beta\) of the process \(X\) introduced in [2] is defined by

\[
\beta = \inf\{p > 0 : \int_{\{|x| \leq 1\}} |x|^p d\nu(x) < +\infty\}.
\]

Necessarily, \(\beta \in [0, 2]\).

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In the sequel we will assume that \( \sigma^2 = 0 \), i.e. that \( X \) has no Brownian component. Then the Lévy-Itô decomposition of \( X \) reads

\[
X_t = \alpha t + \int_0^t \int_{\{|x| \leq 1\}} x(\mu - \lambda \otimes \nu)(ds, dx) + \int_0^t \int_{\{|x| > 1\}} x\mu(ds, dx)
\]

where \( \lambda \) denotes the Lebesgue measure and \( \mu \) is the Poisson random measure on \( \mathbb{R}_+ \times \mathbb{R} \) associated with the jumps of \( X \) by

\[
\mu = \sum_{t \geq 0} \mathcal{E}_{(t, \triangle X_t)} \mathbf{1}_{\{\triangle X_t \neq 0\}},
\]

\( \triangle X_t = X_t - X_{t-}, \triangle X_0 = 0 \) (see [4], [7]).

**Theorem 1** Let \((X_t)_{t \geq 0}\) be a Lévy process with characteristics \((a, 0, \nu)\) and index \( \beta \) such that \( \mathbb{E}|X_1|^p < +\infty \) for some \( p \in (\beta, \infty) \) or for \( p = \beta \) provided \( \int_{\{|x| \leq 1\}} |x|^\beta \nu(dx) < +\infty \) and \( \beta > 0 \). Then for every \( t \geq 0 \)

\[
\mathbb{E} \sup_{s \leq t} |Y_s|^p \leq C_p t \quad \text{if } p < 1,
\]

\[
\mathbb{E} \sup_{s \leq t} |X_s - s| \mathbb{E} |X_1|^p \leq C_p t \quad \text{if } 1 \leq p \leq 2
\]

where \( Y_t = X_t - t(a - \int_{\{|x| \leq 1\}} x \nu(dx)) \). Furthermore, for every \( p > 2 \)

\[
\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t) \quad \text{as } t \to 0
\]

for a finite real constant \( C_p \).

If \( X_1 \) is symmetric one observes that \( Y = X \) since the symmetry of \( X_1 \) implies \( a = 0 \) and the symmetry of \( \nu \) (see [7]). We emphasize that in view of the Kolmogorov criterion for continuous modifications the above bounds are best possible as concerns powers of \( t \). In case \( p > \beta \) and \( p \leq 2 \), these estimates are due to Millar [6]. However, the Laplace-transform approach in [6] does not work for \( p > 2 \). Our proof is based on the Burkholder-Davis-Gundy inequality.

For the case \( p < \beta \) we need some assumptions on \( X \). Recall that a measurable function \( \varphi: (0, c) \to (0, \infty) \) \((c > 0)\) is said to be regularly varying at zero with index \( b \in \mathbb{R} \) if, for every \( t > 0 \),

\[
\lim_{x \to 0} \frac{\varphi(tx)}{\varphi(x)} = t^b.
\]

This means that \( \varphi(1/x) \) is regularly varying at infinity with index \( -b \). Slow variation corresponds to \( b = 0 \).

**Theorem 2** Let \((X_t)_{t \geq 0}\) be a Lévy process with characteristics \((a, 0, \nu)\) and index \( \beta \) such that \( \beta > 0 \) and \( \mathbb{E}|X_1|^p < +\infty \) for some \( p \in (0, \beta) \). Assume that the Lévy measure satisfies

\[
\exists c \in (0, 1], 1_{\{0 < |x| \leq c\}} \nu(dx) \leq \varphi(|x|) 1_{\{0 < |x| \leq c\}} dx
\]

where \( \varphi: (0, c) \to (0, \infty) \) is a regularly varying function at zero of index \( -(\beta + 1) \). Let \( l(x) = x^{\beta+1} \varphi(x) \) and assume that \( l(1/x), x \geq c \) is locally bounded. Let \( l(x) = L_{\beta}(x) = l(x^{1/\beta}) \).

(a) Assume \( \beta > 1 \). Then as \( t \to 0 \), for every \( r \in (\beta, 2], q \in [p \vee 1, \beta) \),

\[
\mathbb{E} \sup_{s \leq t} |X_s|^p = O\left(t^{p/\beta} [l(t)^{p/r} + l(t)^{p/q}]\right) \quad \text{if } \beta < 2,
\]

where

\[
\varphi(x) = \frac{x^{\beta+1} \varphi(x)}{l(x)} \quad \text{for } 0 < x \leq c
\]

and

\[
\varphi(x) = \frac{x^{\beta+1} \varphi(x)}{l(x)} \quad \text{for } x > c
\]

and

\[
\varphi(x) = \frac{x^{\beta+1} \varphi(x)}{l(x)} \quad \text{for } x \geq c
\]

and

\[
\varphi(x) = \frac{x^{\beta+1} \varphi(x)}{l(x)} \quad \text{for } x \leq c
\]
If $\nu$ is symmetric then this holds for every $q \in [p, \beta]$.

(b) Assume $\beta < 1$. Then as $t \to 0$, for every $r \in (\beta, 1], q \in [p, \beta)\,$
$\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t^{p/\beta}[1 + \mathbb{I}(t)^{p/q}])$ if $\beta = 2$.

Observe that condition (1.5) is satisfied for a broad class of Lévy processes. It implies that the slowly varying part $l$ of $\varphi$ is decreasing and unbounded on $(0, c]$ (e.g. $-(\log x)^a, a > 0$). Then as $t \to 0$, for every $\varepsilon \in (0, \beta)$,
$\mathbb{E} \sup_{s \leq t} |U_s|^p = O(t^{p/\beta}[1 + \mathbb{I}(t)^{p/(\beta-\varepsilon)}]).$

(b) Assume that $l$ is increasing on $(0, c]$ satisfying $l(0+) = 0$ (e.g. $-(\log x)^{-a}, a > 0, c < 1$) and $\beta \in (0, 2)$. Then as $t \to 0$, for every $\varepsilon > 0$,
$\mathbb{E} \sup_{s \leq t} |U_s|^p = O(t^{p/\beta}[1 + \mathbb{I}(t)^{p/(\beta+\varepsilon)}]).$

The remaining cases $p = \beta \in (0, 2)$ if $\beta \neq 1$ and $p \leq 1$ if $\beta = 1$ are solved under the assumption that the slowly varying part of the function $\varphi$ in (1.5) is constant.

**Theorem 3** Let $(X_t)_{t \geq 0}$ be a Lévy process with characteristics $(a, 0, \nu)$ and index $\beta$ such that $\beta \in (0, 2)$ and $\mathbb{E}|X_1|^{\beta} < +\infty$ if $\beta \neq 1$ and $\mathbb{E}|X_1|^p < +\infty$ for some $p \leq 1$ if $\beta = 1$. Assume that the Lévy measure satisfies
$\exists c \in (0, 1), \exists C \in (0, \infty), \mathbb{1}_{\{0 < |x| \leq c\}} \nu(dx) \leq C \frac{1}{|x|^{\beta+1}} \mathbb{1}_{\{0 < |x| \leq c\}} dx.$

Then as $t \to 0$
$\mathbb{E} \sup_{s \leq t} |X_s|^\beta = O(t(-\log t))$ if $\beta \gg 1$,
$\mathbb{E} \sup_{s \leq t} |Y_s|^\beta = O(t(-\log t))$ if $\beta < 1$
and
$\mathbb{E} \sup_{s \leq t} |X_s|^p = O((t(-\log t))^p)$ if $\beta = 1, p \leq 1$
where the process $Y$ is defined as in Theorem 2.

The above estimates are optimal (see Section 3).

The paper is organized as follows. Section 2 is devoted to the proofs of Theorems 1, 2 and 3. Section 3 contains a collection of examples.
2 Proofs

We will extensively use the following compensation formula (see e.g. [4])

$$
\mathbb{E} \int_0^t \int f(s, x) \mu(ds, dx) = \mathbb{E} \sum_{s \leq t} f(s, \Delta X_s) 1_{\Delta X_s \neq 0} = \int_0^t \int f(s, x) d\nu(x) ds
$$

where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is a Borel function.

**Proof of Theorem 1.** Since $\mathbb{E} |X_1|^p < +\infty$ and $p > \beta$ (or $p = \beta$ provided $\int_{|x| \leq 1} |x|^\beta d\nu(x) < +\infty$ and $\beta > 0$), it follows from (1.2) that

$$
\int |x|^p d\nu(x) < +\infty.
$$

**CASE 1:** $0 < p < 1$. In this case we have $\beta < 1$ and hence $\int_{|x| \leq 1} |x|^\beta d\nu(x) < +\infty$. Consequently, $X$ a.s. has finite variation on finite intervals. By (1.4),

$$
Y_t = X_t - t \left( a - \int_{|x| \leq 1} x d\nu(x) \right) = \int_0^t \int x \mu(ds, dx) = \sum_{s \leq t} \Delta X_s
$$

so that, using the elementary inequality $(u + v)^p \leq u^p + v^p$,

$$
\sup_{s \leq t} |Y_s|^p \leq \left( \sum_{s \leq t} |\Delta X_s| \right)^p \leq \sum_{s \leq t} |\Delta X_s|^p = \int_0^t \int |x|^p \mu(ds, dx).
$$

Consequently,

$$
\mathbb{E} \sup_{s \leq t} |Y_s|^p \leq t \int |x|^p d\nu(x) \text{ for every } t \geq 0.
$$

**CASE 2:** $1 \leq p \leq 2$. Introduce the martingale

$$
M_t := X_t - t\mathbb{E} X_1 = X_t - t \left( a + \int_{|x| > 1} x d\nu(x) \right) = \int_0^t \int (x - \lambda \otimes \nu)(ds, dx).
$$

It follows from the Burkholder-Davis-Gundy inequality (see [5]) that

$$
\mathbb{E} \sup_{s \leq t} |M_s|^p \leq C \mathbb{E} [M]^p
$$

for some finite constant $C$. Since $p/2 \leq 1$, the quadratic variation $[M]$ of $M$ satisfies

$$
[M]_{t}^{p/2} = \left( \sum_{s \leq t} |\Delta X_s|^2 \right)^{p/2} \leq \sum_{s \leq t} |\Delta X_s|^p
$$

so that

$$
\mathbb{E} \sup_{s \leq t} |M_s|^p \leq Ct \int |x|^p d\nu(x) \text{ for every } t \geq 0.
$$

**CASE 3:** $p > 2$. One considers again the martingale Lévy process $M_t = X_t - t\mathbb{E} X_1$. For $k \geq 1$ such that $2^k \leq p$, introduce the martingales

$$
N^{(k)}_t := \int_0^t \int |x|^{2^k} (\mu - \lambda \otimes \nu)(ds, dx) = \sum_{s \leq t} |\Delta X_s|^{2^k} - t \int |x|^{2^k} d\nu(x).
$$
Set $m := \max\{k \geq 1 : 2^k < p\}$. Again by the Burkholder-Davis-Gundy inequality
\[
\mathbb{E} \sup_{s \leq t} |M_s|^p \leq C \mathbb{E} [M]_{t}^{p/2}
\]
\[
= C \mathbb{E} \left( t \int x 2d\nu(x) + N_t^{(1)} \right)^{p/2}
\]
\[
\leq C \left( t^{p/2} \left( \int x 2d\nu(x) \right)^{p/2} + \mathbb{E} |N_t^{(1)}|^{p/2} \right)
\]
\[
\leq C (t + \mathbb{E} |N_t^{(1)}|^{p/2})
\]
for every $t \in [0, 1]$ where $C$ is a finite constant that may vary from line to line. Applying successively the Burkholder-Davis-Gundy inequality to the martingales $N^{(k)}$ and exponents $p/2^k > 1, 1 \leq k \leq m$, finally yields
\[
\mathbb{E} \sup_{s \leq t} |M_s|^p \leq C(t + \mathbb{E} |N^{(m)}_t|^{p/2^{m+1}}) \text{ for every } t \in [0, 1].
\]
Using $p \leq 2^{m+1}$, one gets
\[
[N^{(m)}_t]^{p/2^{m+1}} = \left( \sum_{s \leq t} |\Delta X_s|^{2^{m+1}} \right)^{p/2^{m+1}} \leq \sum_{s \leq t} |\Delta X_s|^p
\]
so that
\[
\mathbb{E} \sup_{s \leq t} |M_s|^p \leq C \left( t + t \int |x|^p d\nu(x) \right) \text{ for every } t \in [0, 1].
\]
This implies $\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t)$ as $t \to 0$. □

**Proof of Theorems 2 and 3.** Let $p \leq \beta$ and fix $c \in (0, 1]$. Let $\nu_1 = 1_{\{|x| \leq c\}} \nu$ and $\nu_2 = 1_{\{|x| > c\}} \nu$. Construct Lévy processes $X^{(1)}$ and $X^{(2)}$ such that $X \overset{d}{=} X^{(1)} + X^{(2)}$ and $X^{(2)}$ is a compound Poisson process with Lévy measure $\nu_2$. Then $\beta = \beta(X) = \beta(X^{(1)}), \beta(X^{(2)}) = 0$, $\mathbb{E} |X^{(1)}|^q < +\infty$ for every $q > 0$ and $\mathbb{E} |X^{(2)}_t|^p < +\infty$. It follows e.g. from Theorem 1 that for every $t \geq 0$,
\[
\mathbb{E} \sup_{s \leq t} |X^{(2)}_s|^p \leq C_p t \quad \text{if } p < 1,
\]
\[
\mathbb{E} \sup_{s \leq t} |X^{(2)}_s - s E X^{(2)}_1|^p \leq C_p t \quad \text{if } 1 \leq p \leq 2
\]
where $\mathbb{E} X^{(2)}_1 = \int x d\nu_2(x) = \int_{|x| > c} x d\nu(x)$.

As concerns $X^{(1)}$, consider the martingale
\[
Z^{(1)}_t := X^{(1)}_t - t E X^{(1)}_1 = X^{(1)}_t - t \left( a - \int x 1_{|x| \leq 1 \leq T} d\nu(x) \right) = \int_0^t \int x (\mu_1 - \lambda \otimes \nu_1)(ds, dx)
\]
where $\mu_1$ denotes the Poisson random measure associated with the jumps of $X^{(1)}$. The starting idea is to part the 'small' and the 'big' jumps of $X^{(1)}$ in a non homogeneous way with respect to the function $s \mapsto s^{1/\beta}$. Indeed one may decompose $Z^{(1)}$ as follows
\[
Z^{(1)} = M + N
\]
where
\[
M_t := \int_0^t \int x 1_{|x| \leq s^{1/\beta}} (\mu_1 - \lambda \otimes \nu_1)(ds, dx)
\]
\[
N_t := \int_0^t \int x 1_{|x| > s^{1/\beta}} (\mu_1 - \lambda \otimes \nu_1)(ds, dx)
\]
and
\[ N_t := \int_0^t \int x^1_{\{|x| \geq s^{1/\beta}\}}(\mu_1 - \lambda \otimes \nu_1)(ds, dx) \]
are martingales. Observe that for every \( q > 0 \) and \( t \geq 0 \),
\[
\int_0^t \int |x|^q 1_{\{|x| > s^{1/\beta}\}}d\nu_1(x)ds = \int |x|^q (|x|^\beta + t)d\nu_1(x) \\
\leq \int_{\{|x| \leq c\}} |x|^{\beta+q}d\nu(x) < +\infty.
\]
Consequently,
\[ N_t = \int_0^t \int x^1_{\{|x| > s^{1/\beta}\}}d\mu_1(s, x) - \psi(t) \]
where \( \psi(t) := \int_0^t \int x^1_{\{|x| > s^{1/\beta}\}}d\nu_1(x)ds \).
Furthermore, for every \( r > \beta \) or \( r = 2 \) and \( t \geq 0 \)
\[
\int_0^t \int |x|^r 1_{\{|x| \leq s^{1/\beta}\}}d\nu_1(x)ds \leq t \int_{\{|x| \leq c\}} |x|^r d\nu(x) < +\infty. \tag{2.2}
\]
In the sequel let \( C \) denote a finite constant that may vary from line to line.
We first claim that for every \( t \geq 0, r \in (\beta, 2] \cap [1, 2] \) and for \( r = 2 \),
\[
\mathbb{E} \sup_{s \leq t} |M_s|^p \leq C \left( \int_0^t \int |x|^\beta 1_{\{|x| > s^{1/\beta}\}}d\nu_1(x)ds \right)^{p/r}. \tag{2.3}
\]
In fact, it follows from the Burkholder-Davis-Gundy inequality and from \( p/r \leq 1, r/2 \leq 1 \) that
\[
\mathbb{E} \sup_{s \leq t} |M_s|^p \leq \left( \mathbb{E} \sup_{s \leq t} |M_s|^r \right)^{p/r} \\
\leq C \left( \mathbb{E} |M|^r_{t/2} \right)^{p/r} \\
= C \left( \mathbb{E} \left( \sum_{s \leq t} |\triangle X^{(1)}_s|^2 1_{\{|\triangle X^{(1)}_s| \leq s^{1/\beta}\}} \right)^{r/2} \right)^{p/r} \\
\leq C \left( \mathbb{E} \sum_{s \leq t} |\triangle X^{(1)}_s|^r 1_{\{|\triangle X^{(1)}_s| \leq s^{1/\beta}\}} \right)^{p/r} \\
= C \left( \int_0^t \int |x|^\beta 1_{\{|x| \leq s^{1/\beta}\}}d\nu_1(x)ds \right)^{p/r}.
\]
Exactly as for \( M \), one gets for every \( t \geq 0 \) and every \( q \in [p, 2] \cap [1, 2] \) that
\[
\mathbb{E} \sup_{s \leq t} |N_s|^p \leq C \left( \int_0^t \int |x|^\beta 1_{\{|x| > s^{1/\beta}\}}d\nu_1(x)ds \right)^{p/q}. \tag{2.4}
\]
If \( \nu \) is symmetric then (2.4) holds for every \( q \in [p, 2] \) (which of course provides additional information in case \( p < 1 \) only). Indeed, \( \psi = 0 \) by the symmetry of \( \nu \) so that
\[
N_t = \int_0^t \int x^1_{\{|x| > s^{1/\beta}\}}d\mu_1(s, x)
\]
and for $q \in [p, 1]$

$$
\mathbb{E} \sup_{s \leq t} \left| \int_0^s \int x \chi_{\{|x| > u^{1/\beta}\}} \mu_1(du, dx) \right|^p \leq \left( \mathbb{E} \sup_{s \leq t} \left( \int_0^t \int x \chi_{\{|x| > u^{1/\beta}\}} \mu_1(du, dx) \right)^q \right)^{p/q} \tag{2.5}
$$

$$
\leq \left( \mathbb{E} \sum_{s \leq t} \left| \Delta X_s^{(1)} \right|^q \mathbb{1}_{\{|\Delta X_s^{(1)}| > s^{1/\beta}\}} \right)^{p/q}
$$

$$
= \left( \int_0^t \int |x|^q \mathbb{1}_{\{|x| > s^{1/\beta}\}} \mu_1(x) dx \right)^{p/q}.
$$

In the case $\beta < 1$ we consider the process

$$
Y_t^{(1)} := Z_t^{(1)} + t \int x \nu_1(x) = X_t^{(1)} - t \left( a - \int \chi_{\{|x| \leq 1\}} x \nu(x) \right)
$$

$$
= M_t + N_t + t \int x \nu_1(x)
$$

$$
= \int_0^t \int x \chi_{\{|x| \leq s^{1/\beta}\}} \mu_1(ds, dx) + \int_0^1 \int x \chi_{\{|x| > s^{1/\beta}\}} \mu_1(ds, dx).
$$

Exactly as in (2.5) one shows that for $t \geq 0$ and $r \in (\beta, 1]$

$$
\mathbb{E} \sup_{s \leq t} \left| \int_0^s \int x \chi_{\{|x| \leq u^{1/\beta}\}} \mu_1(du, dx) \right|^p \leq \left( \int_0^t \int |x|^r \chi_{\{|x| \leq u^{1/\beta}\}} \nu_1(x) dx \right)^{p/r}. \tag{2.6}
$$

Combining (2.1) and (2.3) - (2.6) we obtain the following estimates. Let

$$
Z_t = X_t - t \left( a - \int \chi_{\{|x| \leq 1\}} x \nu(x) \right).
$$

CASE 1: $\beta \geq 1$ and $p < 1$. Then for every $t \geq 0$, $r \in (\beta, 2] \cup \{2\}$, $q \in [1, 2]$,

$$
\mathbb{E} \sup_{s \leq t} |Z_s|^p \leq C \left( t + \left( \int_0^t \int |x|^r \chi_{\{|x| \leq s^{1/\beta}\}} \nu_1(x) dx \right)^{p/r} + \left( \int_0^t \int |x|^q \chi_{\{|x| > s^{1/\beta}\}} \nu_1(x) dx \right)^{p/q} \right). \tag{2.7}
$$

If $\nu$ is symmetric (2.7) is even valid for every $q \in [p, 2]$.

CASE 2: $\beta \geq 1$ and $p \geq 1$. Then for every $t \geq 0$, $r \in (\beta, 2] \cup \{2\}$, $q \in [p, 2]$,

$$
\mathbb{E} \sup_{s \leq t} |X_s - s \mathbb{E} X_1|^p \leq C \left( t + \left( \int_0^t \int |x|^r \chi_{\{|x| \leq s^{1/\beta}\}} \nu_1(x) dx \right)^{p/r} + \left( \int_0^t \int |x|^q \chi_{\{|x| > s^{1/\beta}\}} \nu_1(x) dx \right)^{p/q} \right). \tag{2.8}
$$

CASE 3: $\beta < 1$. Then for every $t \geq 0$, $r \in (\beta, 1], q \in [p, 1]$

$$
\mathbb{E} \sup_{s \leq t} |Y_s|^p \leq C \left( t + \left( \int_0^t \int |x|^r \chi_{\{|x| \leq s^{1/\beta}\}} \nu_1(x) dx \right)^{p/r} + \left( \int_0^t \int |x|^q \chi_{\{|x| > s^{1/\beta}\}} \nu_1(x) dx \right)^{p/q} \right). \tag{2.9}
$$
If \( \nu \) is symmetric then \( Y = Z = (X_t - at)_{t \geq 0} \) and (2.9) is valid for every \( r \in (\beta, 2], q \in [p, 2] \).

Now we deduce Theorem 2. Assume \( p \in (0, \beta) \) and (1.5). The constant \( c \) in the above decomposition of \( X \) is specified by the constant from (1.5). Then one just needs to investigate the integrals appearing in the right hand side of the inequalities (2.7) - (2.10). One observes that Theorem 1.5.11 in [1] yields for \( r > \beta \),

\[
\int |x|^r \varphi(|x|) 1_{\{|x| \leq s^{1/\beta}\}} \, dx \sim \frac{2}{r - \beta} s^{r-1} l(s^{1/\beta}) \text{ as } s \to 0
\]

which in turn implies that for small \( t \),

\[
\int_0^t \int |x|^r 1_{\{|x| \leq s^{1/\beta}\}} \, d\nu_1(x) \, ds \leq \int_0^t \int |x|^r \varphi(|x|) 1_{\{|x| \leq s^{1/\beta}\}} \, dx \, ds \sim \frac{2\beta}{(r - \beta)r} t^{r/\beta} l(t^{1/\beta}) \text{ as } t \to 0.
\]

Similarly, for \( 0 < q < \beta \),

\[
\int_0^t \int |x|^q 1_{\{|x| > s^{1/\beta}\}} \, d\nu_1(x) \, ds \leq \int_0^t \int |x|^q \varphi(|x|) 1_{\{|x| > s^{1/\beta}\}} \, dx \, ds \sim \frac{2\beta}{(\beta - q)q} t^{q/\beta} l(t^{1/\beta}) \text{ as } t \to 0.
\]

Using (2.2) for the case \( \beta = 2 \) and \( t + tp = o(t^{p/\beta}(t^\alpha)) \) as \( t \to 0, \alpha > 0 \), for the case \( \beta > 1 \) one derives Theorem 2.

As for Theorem 3, one just needs a suitable choice of \( q \) in (2.7) - (2.9). Note that by (1.6) for every \( \beta \in (0, 2) \) and \( t \leq c^\beta \),

\[
\int_0^t \int |x|^\beta 1_{\{|x| > s^{1/\beta}\}} \, d\nu_1(x) \, ds = \int_0^t \int |x|^\beta 1_{\{c \geq |x| > s^{1/\beta}\}} \, d\nu(x) \, ds \leq C \int_0^t \int |x|^{-1} 1_{\{c \geq |x| > s^{1/\beta}\}} \, dx \, ds = C t (-\log t)
\]

so that \( q = \beta \) is the right choice. (This choice of \( q \) is optimal.) Since by (2.10), for \( r \in (\beta, 2] \neq \emptyset \),

\[
\int_0^t \int |x|^r 1_{\{|x| \leq s^{1/\beta}\}} \, d\nu_1(x) \, ds = O(t^{r/\beta})
\]

the assertions follow from (2.7) - (2.9).

3 Examples

Let \( K_\nu \) denote the modified Bessel function of the third kind and index \( \nu \) given by

\[
K_\nu(z) = \frac{1}{2} \int_0^\infty u^{\nu-1} \exp(-\frac{z}{2}(u + \frac{1}{u})) \, du, \quad z > 0.
\]

- The \( \Gamma \)-process is a subordinator (increasing Lévy process) whose distribution \( \mathbb{P}_{X_t} \) at time \( t > 0 \) is a \( \Gamma(1, t) \)-distribution

\[
\mathbb{P}_{X_t}(dx) = \frac{1}{\Gamma(t)} x^{t-1} e^{-x} 1_{(0, \infty)}(x) \, dx.
\]
The characteristics are given by
\[ \nu(dx) = \frac{1}{x}e^{-x}1_{(0,\infty)}(x)ds \]
and \( a = \int_0^1 x d\nu(x) = 1 - e^{-1} \) so that \( \beta = 0 \) and \( Y = X \). It follows from Theorem 1 that
\[ \mathbb{E} \sup_{s \leq t} X_s^p = \mathbb{E} X_t^p = O(t) \]
for every \( p > 0 \). This is clearly the true rate since
\[ \mathbb{E} X_t^p = \frac{\Gamma(p+1)}{\Gamma(t+1)}t^p \sim \Gamma(p)t \text{ as } t \to 0. \]

- The \( \alpha \)-stable Lévy Processes indexed by \( \alpha \in (0, 2) \) have Lévy measure
\[ \nu(dx) = \left( \frac{C_1}{x^{\alpha+1}}1_{(0,\infty)}(x) + \frac{C_2}{|x|^{\alpha+1}}1_{(-\infty,0)}(x) \right)dx \]
with \( C_1 \geq 0, C_1 + C_2 > 0 \) so that \( \mathbb{E} |X|^p < +\infty \) for \( p \in (0, \alpha) \), \( \mathbb{E} |X|^\alpha = \infty \) and \( \beta = \alpha \). It follows from Theorems 2 and 3 that for \( p \in (0, \alpha) \),
\[ \mathbb{E} \sup_{s \leq t} |X_s|^p = O(t^{p/\alpha}) \text{ if } \alpha > 1, \]
\[ \mathbb{E} \sup_{s \leq t} |Y_s|^p = O(t^{p/\alpha}) \text{ if } \alpha < 1, \]
\[ \mathbb{E} \sup_{s \leq t} |X_s|^p = O((t(-\log t))^p) \text{ if } \alpha = 1. \]

Here Theorem 3 gives the true rate provided \( X \) is not strictly stable. In fact, if \( \alpha = 1 \) the scaling property in this case says that \( X_t \overset{d}{=} tX_1 + Ct \log t \) for some real constant \( C \neq 0 \) (see [7], p.87) so that for \( p < 1 \)
\[ \mathbb{E} |X|^p = t^p\mathbb{E} |X_1 + C\log t|^p \sim C^p t^p |\log t|^p \text{ as } t \to 0. \]

Now assume that \( X \) is strictly \( \alpha \)-stable. If \( \alpha < 1 \), then \( a = \int_{|x| \leq 1} x d\nu(x) \) and thus \( Y = X \) and if \( \alpha = 1 \), then \( \nu \) is symmetric (see [7]). Consequently, by Theorem 2, for every \( \alpha \in (0, 2), p \in (0, \alpha) \),
\[ \mathbb{E} \sup_{s \leq t} |X_s|^p = O(t^{p/\alpha}). \]

In this case Theorem 2 provides the true rate since the self-similarity property of strictly stable Lévy processes implies
\[ \mathbb{E} \sup_{s \leq t} |X_s|^p = t^{p/\alpha}\mathbb{E} \sup_{s \leq 1} |X_s|^p. \]

- Tempered stable processes are subordinators with Lévy measure
\[ \nu(dx) = \frac{2^\alpha \cdot \alpha}{\Gamma(1-\alpha)}x^{-(\alpha+1)}\exp\left(-\frac{1}{2}x^{1/\alpha}\right)1_{(0,\infty)}(x)dx \]
and first characteristic \( a = \int_0^1 x d\nu(x), \alpha \in (0, 1), \gamma > 0 \) (see [8]) so that \( \beta = \alpha, Y = X \) and \( \mathbb{E} X_t^p < +\infty \) for every \( p > 0 \). The distribution of \( X_t \) is not generally known. It follows from Theorems 1, 2 and 3 that
\[ \mathbb{E} X_t^p = O(t) \text{ if } p > \alpha, \]
\[ \mathbb{E} X_t^p = O(t^{p/\alpha}) \text{ if } p < \alpha, \]
\[ \mathbb{E} X_t^\alpha = O(t(-\log t)) \text{ if } p = \alpha. \]
For $\alpha = 1/2$, the process reduces to the inverse Gaussian process whose distribution $\mathbb{P}_{X_t}$ at time $t > 0$ is given by

$$\mathbb{P}_{X_t}(dx) = \frac{t}{\sqrt{2\pi}} x^{-3/2} \exp\left(-\frac{1}{2} \left(\frac{t}{\sqrt{x}} - \gamma \sqrt{x}\right)^2\right) 1_{(0,\infty)}(x)dx.$$  

In this case all rates are the true rates. In fact, for $p > 0$,

$$\mathbb{E} X_t^p = \frac{t}{\sqrt{2\pi}} \int_0^t y^{p-3/2} \exp\left(-\frac{1}{2} \left(\frac{1}{y} + \gamma^2 y\right)\right) dy$$

$$= \frac{2}{\sqrt{2\pi}} t^{p-3/2} \frac{p+1/2}{\beta} K_{p-1/2}(t\gamma)$$

and, as $z \to 0$,

$$K_{p-1/2}(z) \sim \frac{C_p}{z^{p-1/2}}$$ if $p > 1/2$,

$$K_{p-1/2}(z) \sim \frac{C_p}{z^{1/2-p}}$$ if $p < 1/2$,

$$K_0(z) \sim |\log z|$$

where $C_p = 2^{p-3/2} \Gamma(p-1/2)$ if $p > 1/2$ and $C_p = 2^{-p-1/2} \Gamma(1/2 - p)$ if $p < 1/2$.

The Normal Inverse Gaussian (NIG) process was introduced by Barndorff-Nielsen and has been used in financial modeling (see [8]). The NIG process is a Lévy process with characteristics $(\alpha, 0, \nu)$ where

$$\nu(dx) = \frac{\delta \alpha}{\pi} \exp(\gamma x) K_1(\alpha|x|) dx, $$

$$\alpha > 0, \gamma \in (-\alpha, \alpha), \delta > 0. $$

Since $K_1(|z|) \sim |z|^{-1}$ as $z \to 0$, the Lévy density behaves like $\delta \pi^{-1} |x|^{-2}$ as $x \to 0$ so that (1.6) is satisfied with $\beta = 1$. One also checks that $\mathbb{E} |X_1|^p < +\infty$ for every $p > 0$. It follows from Theorems 1 and 3 that, as $t \to 0$

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t)$$ if $p > 1$,

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O((t(-\log t))^p)$$ if $p \leq 1$.

If $\gamma = 0$, then $\nu$ is symmetric and by Theorem 2,

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t^p)$$ if $p < 1$.

The distribution $\mathbb{P}_{X_t}$ at time $t > 0$ is given by

$$\mathbb{P}_{X_t}(dx) = \frac{t \delta \alpha}{\pi} \exp(t \sqrt{\alpha^2 - \gamma^2 + \gamma x}) K_1(\alpha \sqrt{t^2 \delta^2 + x^2}) dx$$

so that Theorem 3 gives the true rate for $p = \beta = 1$ in the symmetric case. In fact, assuming $\gamma = 0$, we get as $t \to 0$

$$\mathbb{E} |X_t| = \frac{2t \delta \alpha}{\pi} e^{t \delta} \int_0^\infty x K_1(\alpha \sqrt{t^2 \delta^2 + x^2}) dx$$
\[
\frac{2t\delta\alpha}{\pi}e^{t\delta\alpha}\int_0^\infty K_1(\alpha y)dy
\sim \frac{2\delta}{\pi} t \int_0^1 \frac{1}{y} dy \\
\sim \frac{2\delta}{\pi} t(-\log(t)).
\]

• **Hyperbolic Lévy motions** have been applied to option pricing in finance (see [3]). These processes are Lévy processes whose distribution \(P_{X_1}\) at time \(t = 1\) is a symmetric (centered) hyperbolic distribution
\[
P_{X_1}(dx) = C \exp(-\delta \sqrt{1 + (x/\gamma)^2}) dx, \quad \gamma, \delta > 0.
\]

Hyperbolic Lévy processes have characteristics \((0, 0, \nu)\) and satisfy \(E|X_1|^p < +\infty\) for every \(p > 0\). In particular, they are martingales. There (rather involved) symmetric Lévy measure has a Lebesgue density that behaves like \(Cx^{-2}\) as \(x \to 0\) so that (1.6) is satisfied with \(\beta = 1\). Consequently, by Theorems 1, 2 and 3, as \(t \to 0\)
\[
E \sup_{s \leq t} |X_s|^p = O(t) \quad \text{if} \quad p > 1,
\]
\[
E \sup_{s \leq t} |X_s|^p = O(t^p) \quad \text{if} \quad p < 1,
\]
\[
E \sup_{s \leq t} |X_s| = O(t(-\log t)) \quad \text{if} \quad p = 1.
\]

• **Meixner processes** are Lévy processes without Brownian component and with Lévy measure given by
\[
\nu(dx) = \frac{\delta e^{\gamma x}}{x \sinh(\pi x)} dx, \quad \delta > 0, \quad \gamma \in (-\pi, \pi)
\]
(see [8]). The density behaves like \(\delta/\pi x^2\) as \(x \to 0\) so that (1.6) is satisfied with \(\beta = 1\). Using (1.2) one observes that \(E|X_1|^p < +\infty\) for every \(p > 0\). It follows from Theorems 1 and 3 that
\[
E \sup_{s \leq t} |X_s|^p = O(t) \quad \text{if} \quad p > 1,
\]
\[
E \sup_{s \leq t} |X_s|^p = O((t(-\log t))^p) \quad \text{if} \quad p \leq 1.
\]

If \(\gamma = 0\), then \(\nu\) is symmetric and hence Theorem 2 yields
\[
E \sup_{s \leq t} |X_s|^p = O(t^p) \quad \text{if} \quad p < 1.
\]

**References**


