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# A remark about factorizing GCD-type Hyperdeterminants

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## Abstract

We show that a very elementary trick allows to generalize a theorem of Lindström to higher-dimensional determinants. As a special case of our main theorem (Theorem 3.3), one recovers some results due to Lehmer and Haukkanen on hyperdeterminants of higher-dimensional GCD matrices.

## 1 Introduction

Since the end of the nineteen century, it has been known that some determinants with entries depending only of the gcd of the indices factorize. Readers interested in the story of the problem can refer to [13] and [2]. The subject was born in 1876, in an article of Smith [15] which has evaluated the determinant of a GCD matrix whose entries belong to a factor closed set (*i.e.*, all the factors of an element belong to the set) as a product of Euler's totient. The interest of this kind of equalities lies in its links with arithmetic functions [1] and in particular multiplicative functions (see [7, 8] for interesting remarks about the last notion). During the last century, many generalizations of Smith's theorem have been investigated. One of the way to extend

this result consists on changing the set of the indices of the matrices. One of the most general result is due to Li [10] in 1990, which gives the value of GCD determinant for an arbitrary set of indices. The present paper is not devoted to such determinants which can not be easily factorized. Beslin and Ligh [3] gave a factorization of such a determinant when the indices run in a gcd-closed set (*i.e.*, the gcd of any two elements belongs in the set) as a product of certain functions evaluated in terms of Euler's totient. The fact that these determinants factorize can be seen as a particular case of a very elegant theorem due to Lindström [11] which evaluates the determinant of the GCD-matrix whose indices are taken in a meet semilattice (*i.e.*, a poset such that each pair admits a greatest lower bound). Another way to generalize Smith's result consists on computing multidimensional analogues. In 1930 Lehmer gave [9] the first multi-indexed version of the Smith's determinant. Other related computation are collected in [16, 17]. More recently, Haukkanen [12] gave a hyperdeterminantal generalization of the equality due to Beslin and Ligh (note that in the same paper he computed a multidimensional version of the equality of Li [10]).

We will see in Section 2 that the main trick for computing these multidimensional determinants is to consider it as a sum of (classical 2-way) determinants. In the aim to highlight this method, we apply it for a more general object  $\text{Det}_{\mathfrak{F}}$  which is defined in Section 2. In Section 3, we recall shortly a classical technic and give a slight generalization of Lindström's theorem. When we use our generalization in the expansion of  $\text{Det}_{\mathfrak{F}}$  we have a multidimensional analogue of Lindström's theorem.

## 2 Hyperdeterminants and $\mathfrak{F}$ -determinants

The question of extending the notion of determinant to higher dimensional arrays has been raised by Cayley [5, 6] few after he introduced the modern notation as square arrays [4]. The simplest generalization is defined for a  $k$ th order tensor on an  $n$ -dimensional space  $M = (M_{i_1, \dots, i_k})_{1 \leq i_1, \dots, i_k \leq n}$  by the alternated sum

$$\text{Det}M = \frac{1}{n!} \sum_{\sigma = (\sigma_1, \dots, \sigma_k) \in \mathfrak{S}_n^k} \text{sign}(\sigma) M^\sigma,$$

where  $\text{sign}(\sigma) = \text{sign}(\sigma_1) \cdots \text{sign}(\sigma_k)$ ,  $M^\sigma = M_{\sigma_1(1) \dots \sigma_k(1)} \cdots M_{\sigma_1(n) \dots \sigma_k(n)}$  and  $\mathfrak{S}_n$  is the symmetric group. A straightforward computation gives  $\text{Det}M = 0$

if  $k$  is odd.

For any  $k$  (even if  $k$  is odd), one defines the polynomial

$$\text{Det}_1 M = \sum_{\sigma=(\text{Id}, \sigma_2, \dots, \sigma_k) \in \mathfrak{S}_n^k} \text{sign}(\sigma) M^\sigma. \quad (1)$$

When  $k$  is even the two notions coincide but for  $k$  odd, only  $\text{Det}_1$  does not vanish. This is a special case of the "less-than-full-sign" determinant theory due to Rice [14].

Let  $\mathfrak{F}$  denote a map from  $\mathfrak{S}_n^{k-2}$  to a commutative ring. One defines a more general object, the  $\mathfrak{F}$ -determinant  $\text{Det}_{\mathfrak{F}}(M)$  of  $M$ , by

$$\text{Det}_{\mathfrak{F}}(M) = \sum_{\sigma=(\sigma_2, \dots, \sigma_k) \in \mathfrak{S}_n^{k-1}} \text{sign}(\sigma_2) \mathfrak{F}(\sigma_3, \dots, \sigma_k) \prod_i M_{i\sigma_2(i)\dots\sigma_k(i)}.$$

There exists a very elementary method for computing  $\text{Det}_{\mathfrak{F}}$  which consists in considering the  $\mathfrak{F}$ -determinant as a sum of  $(n!)^{k-2}$  classical (2-way) determinants.

**Proposition 2.1** *One has*

$$\text{Det}_{\mathfrak{F}} M = \sum_{\sigma_3, \dots, \sigma_k} \mathfrak{F}(\sigma_3, \dots, \sigma_k) \det(M^{\sigma_3, \dots, \sigma_k}), \quad (2)$$

where  $M^{\sigma_3, \dots, \sigma_k}$  denotes the  $n \times n$  matrix such that  $M_{i,j}^{\sigma_3, \dots, \sigma_k} = M_{i, j, \sigma_3(i), \dots, \sigma_k(i)}$ .

**Proof** It suffices to remark that

$$\text{Det}_{\mathfrak{F}} M = \sum_{\sigma_3, \dots, \sigma_k} \mathfrak{F}(\sigma_3, \dots, \sigma_k) \sum_{\sigma_2} \text{sign}(\sigma_2) \prod_i M_{i\sigma_2(i)\sigma_3(i)\dots\sigma_k(i)} \quad (3)$$

□

The more interesting property of the hyperdeterminant is its invariance under the action of  $k$  copies of the special linear group. It is a very classical result which can be recover as a straightforward consequence of proposition 2.1 via the invariance of  $\text{Det}_{\mathfrak{F}}$  under the action of (one copy) of the special linear group.

**Proposition 2.2** *The polynomial  $\text{Det}_{\mathfrak{F}} M$  is invariant under the action of linear group on  $M$  in the following sense*

$$\text{Det}_{\mathfrak{F}} g.M = \det g \text{Det}_{\mathfrak{F}} M, \quad (4)$$

where

$$g.M = \left( \sum_{1 \leq j_2 \leq n} g_{i_2 j_2} M_{i_1, j_2, i_3 \dots, i_k} \right)_{1 \leq i_1, \dots, i_k \leq n}.$$

**Proof** It suffices to apply proposition 2.1 to  $g.M$

$$\begin{aligned} \text{Det}_{\mathfrak{F}} g.M &= \sum_{\sigma_3, \dots, \sigma_k} \mathfrak{F}(\sigma_3, \dots, \sigma_k) \det(g.M^{\sigma_3, \dots, \sigma_k}) \\ &= \sum_{\sigma_3, \dots, \sigma_k} \mathfrak{F}(\sigma_3, \dots, \sigma_k) \det g \det(M^{\sigma_3, \dots, \sigma_k}) \\ &= \det g \text{Det}_{\mathfrak{F}} M. \end{aligned}$$

□

### 3 Hyperdeterminant on a semilattice

The factorization properties of the GCD determinants are the consequence of the semilattice structure of the integers with respect to divisibility and can be stated in a more general way. Consider a partially ordered finite set  $L$  so that every pairs  $(x, y) \in L^2$  has a greatest lower bound denoted by  $x \wedge y$ . Such a poset is called a meet semilattice. One defines classically its  $\zeta$  function

$$\zeta_L(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise,} \end{cases} \quad (5)$$

and its Möbius function, which is the inverse of the zeta function of  $L$ ,

$$\mu_L(x, y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu_L(z, y) & \text{if } x < y, \\ 0 & \text{in the other cases.} \end{cases} \quad (6)$$

If  $F$  and  $f$  verify the equality

$$F(x) = \sum_{y \leq x} f(y) = \sum_{y \in L} \zeta_L(y, x) f(y), \quad (7)$$

then, one has

$$f(x) = \sum_{y \in L} \mu_L(y, x) F(y). \quad (8)$$

The manipulations of the identities (7) and (8) are the keys of the proof of Lindström's Theorem [11]. We recall its proof in a very slightly more general version.

We will denote by  $z_x$  an element of  $L$  lower than  $x$  and by  $F_x$  a function from  $L$  to  $\mathbb{C}$  (or more generally in a commutative ring). Let  $M$  be the matrix defined by

$$M = (F_x(z_x \wedge y))_{x,y \in L}. \quad (9)$$

Remark that it suffices to define  $F_x(y)$  only when  $y \leq x$ . In particular, one can suppose that  $F_x(y) = F(y, x)$  is an incidence function (*i.e.*,  $F(x, y) = 0$  unless  $x \leq y$ ). One has

$$F_x(z_x \wedge y) = \sum_{z \in L} \zeta_L(z, z_x) \zeta(z, y) f_x(z), \quad (10)$$

where  $f_x(z) = \sum_{y \in L} \mu_L(y, z) F_x(y)$ . Hence, it follows

$$\det M = \det \Phi \cdot \det Z,$$

where  $\Phi = (\zeta_L(y, z_x) f_x(y))_{x,y \in L}$  and  $Z = (\zeta_L(x, y))_{x,y \in L}$ . As  $\Phi$  and  $Z$  are triangular,  $\det Z = 1$  and

$$\det \Phi = \prod_x \zeta_L(x, z_x) f_x(x) = \begin{cases} \prod_x f_x(x) & \text{if } z_x = x \text{ for each } x \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

Then one can obtain Lindström's theorem.

**Theorem 3.1** (*Lindström*)

$$\det (F_x(z_x \wedge y))_{x,y \in L} = \begin{cases} \prod_x f_x(x) & \text{if } z_x = x \text{ for each } x, \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

Note that, in the original Lindström's Theorem,  $z_x = x$  for each  $x$ . Furthermore, equality generalizes a lemma of Cesaro (11).

**Lemma 3.2** (*Cesaro*)

Denote by  $\text{gcd}_m(n) = \text{gcd}(m, n)$ . One has,

$$(\mu * (f \circ \text{gcd}_m))(n) = \begin{cases} (f * \mu)(n) & \text{if } m = n, \\ 0 & \text{otherwise,} \end{cases} \quad (13)$$

where  $*$  is the Dirichlet convolution and  $\circ$  is the composition of functions.

Lindström's Theorem can be extended to  $\mathfrak{F}$ -determinants.

**Theorem 3.3** (*Lindström's theorem for  $\mathfrak{F}$ -determinants*)  
If  $L = \{x_1, \dots, x_n\}$  denotes a meet semilattice, one has

$$\text{Det}_{\mathfrak{F}}(F_{x_{i_1}}(x_{i_1} \wedge \dots \wedge x_{i_k}))_{1 \leq i_1, \dots, i_k \leq n} = \mathfrak{F}(\text{Id}, \dots, \text{Id}) \prod_x f(x) \quad (14)$$

**Proof** Proposition 2.1 allows to write

$$\text{Det}_{\mathfrak{F}}(F_{x_1}(x_1 \wedge \dots \wedge x_k))_{x_1, \dots, x_k \in L} = \sum_{\sigma_3, \dots, \sigma_k} \mathfrak{F}(\sigma_3, \dots, \sigma_k) \det(F_{x_i}(x_i \wedge x_j \wedge x_{\sigma_3(i)} \wedge \dots \wedge x_{\sigma_k(i)}))_{i,j} \quad (15)$$

and the result follows from Lindström's Theorem (Theorem 3.1).  $\square$

Now, consider a subset  $S \subset L$  which is meet closed (*i.e.*, closed under the operation  $\wedge$ ) and fix a linear extension  $l = y_1 \dots y_n$  of  $S$ . As in [2], we denote by  $x \trianglelefteq y_i$  the relation  $x \leq y_i$  and  $x \not\leq y_j$  for each  $j < i$ . Consider a pair of function  $f$  and  $F$  verifying

$$F(y_i) = \sum_{x \leq y_i} f(x) \quad (16)$$

and set  $\hat{f}(y_i) = \sum_{d \trianglelefteq y_i} f(x)$ . One has the following lemma.

**Lemma 3.4**

$$F(y_i) = \sum_{y_k \leq y_i} \hat{f}(y_k) \quad (17)$$

**Proof** These identities appear in [2] (Theorem 4.1, p 7) and its proof is based on a one-to-one correspondance between the terms in (17) and (16). We do not repeat it here.  $\square$

Hence, using Theorem 3.3 and Lemma 3.4, one generalizes a result by Altınışık, Sagan and Tuglu ([2], Theorem 4.1 p 7).

**Corollary 3.5**

$$\text{Det}_{\mathfrak{F}}(F_{y_{i_1}}(y_{i_1} \wedge \dots \wedge y_{i_k})) = \mathfrak{F}(\text{Id}, \dots, \text{Id}) \prod_{i=1}^n \left( \sum_{x_1 \trianglelefteq y_i} \sum_{x_2 \in L} \mu_L(x_1, x_2) F_{y_i}(x_2) \right).$$

By the specialization  $\mathfrak{F}(\sigma_3, \dots, \sigma_k) = \text{sign}(\sigma_3) \dots \text{sign}(\sigma_k)$ , one recovers the computation of Lehmer [9] as a special case of Theorem 3.3 and the result of Haukkanen ([12] Theorem 1. p 56) from Corollary 3.5.

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