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# A remark about factorizing GCD-type Hyperdeterminants

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## Abstract

We show that a very elementary trick allows to generalize a theorem of Linström to higher determinants. As a special case, one recovers some results due to Lehmer and Haukkannen on hyperdeterminants of GCD matrices.

## 1 Introduction

Since the end of the nineteen century, it is known that some determinants, with entries depending only of the gcd of the indices, factorize. Readers interested by the story of the problem can refer to [13] and [2]. The subject is born in 1876, in an article of Smith [15] which has evaluated the determinant of a GCD matrix whose entries belong in a factor closed set (*i.e.* all the factors of an element belong to the set) as a product of Euler's totient. The interest of this kind of equalities lies in its links with arithmetic functions [1] and in particular multiplicative functions (see [7, 8] for interesting remarks about the last notion). During the last century, many generalizations of the Smith theorem have been investigated. One of the way to extend this result

consists on changing the set of the indices of the matrices. The more general result is due to Li [10] in 1990, which gives the value of GCD determinant for an arbitrary set of indices. The present paper is not devoted to this more general determinants which can not be easily factorized. Beslin and Ligh [3] gave a factorization of such a determinant when the indices run in a gcd-closed set (*i.e.* the gcd of two elements belongs in the set) as a product of certain functions evaluated in terms of Euler's totient. The fact that these determinants factorize can be seen as a particular case of a very elegant theorem due to Lindström [11] which evaluates the determinant of GCD-matrices whose indices are taken in a meet semilattice (*i.e.* a poset such that each pair admits a greatest lower bound). An other way to generalize Smith identity consists on computing multidimensional analogous. Lehmer [9] has given in 1930 the first multi-indexed version of the Smith determinant. Other related computation are collected in [16, 17]. More recently, Haukkannen [12] gave a hyperdeterminantal generalization of the equality due to Beslin and Ligh (note that in the same paper he has computed a multidimensional version of the equality of Li [10]).

We will see in Section 2 that the main trick for computing these multidimensional determinants is to consider it as a sum of (classical) determinants. In the aim to highlight this method, i apply it for a more general object  $\text{Det}_F$  which is defined in Section 2. In Section 3, i recall shortly a classical technic and give a very slight generalization of the Lindström theorem. Combined with the expansion of  $\text{Det}_F$  in term of  $\det$ , this allows to give a multidimensional analogue of the Lindström theorem in Section 3.

## 2 Hyperdeterminants and $\mathfrak{F}$ -determinants

The question of extending the notion of determinant to higher dimensional arrays has been raised by Cayley [5, 6] few after he introduced the modern notation as square arrays [4]. The simplest generalization is defined for a  $k$ th order tensor on an  $n$ -dimensional space  $M = (M_{i_1, \dots, i_k})_{1 \leq i_1, \dots, i_k \leq n}$  by the alternated sum

$$\text{Det}M = \frac{1}{n!} \sum_{\sigma=(\sigma_1, \dots, \sigma_k) \in \mathfrak{S}_n^k} \text{sign}(\sigma) M^\sigma,$$

where  $\text{sign}(\sigma) = \text{sign}(\sigma_1) \cdots \text{sign}(\sigma_k)$  and  $M^\sigma = M_{\sigma_1(1) \dots \sigma_k(1)} \cdots M_{\sigma_1(n) \dots \sigma_k(n)}$ . A straightforward computation gives  $\text{Det}M = 0$  if  $k$  is odd.

For any  $k$  (even if  $k$  is odd), one defines the polynomial

$$\text{Det}_1 M = \sum_{\sigma=(\text{Id}, \sigma_2, \dots, \sigma_k) \in \mathfrak{S}_n^k} \text{sign}(\sigma) M^\sigma. \quad (1)$$

When  $k$  is even the two notions coincide but for  $k$  odd, only  $\text{Det}_1$  does not vanish. This is a special cases of the "less-than-full-sign" determinants theory due to Rice [14].

If  $\mathfrak{F}$  denotes a map from  $\mathfrak{S}_n^{k-2}$  to a commutative ring. One defines a more general object, the  $\mathfrak{F}$ -determinant  $\text{Det}_{\mathfrak{F}}(M)$  of  $M$ , by

$$\text{Det}_{\mathfrak{F}}(M) = \sum_{\sigma=(\sigma_2, \dots, \sigma_k) \in \mathfrak{S}_n^{k-1}} \text{sign}(\sigma_2) \mathfrak{F}(\sigma_3, \dots, \sigma_k) \prod_i M_{i\sigma_2(i) \dots \sigma_k(i)}.$$

It exists a very elementary method for computing  $\text{Det}_{\mathfrak{F}}$  which consists on considering the  $\mathfrak{F}$ -determinant as a sum of  $(n!)^{k-2}$  determinants

$$\text{Det}_{\mathfrak{F}} M = \sum_{\sigma_3, \dots, \sigma_k} \mathfrak{F}(\sigma_3, \dots, \sigma_k) \det(M^{\sigma_3, \dots, \sigma_k}) \quad (2)$$

where  $M^{\sigma_3, \dots, \sigma_k}$  denotes the  $n \times n$  matrix such that  $M_{i,j}^{\sigma_3, \dots, \sigma_k} = M_{i, j, \sigma_3(i), \dots, \sigma_k(i)}$ . The more interesting property of  $\text{Det}$  is its invariance under the action of  $k$  copies of the special linear group. It is a very classical result, which can be recover as a straightforward consequence of (2) via the following result.

**Proposition 2.1** *The polynomial  $\text{Det}_{\mathfrak{F}} M$  is invariant under the action of linear group on  $M$  in the following sense*

$$\text{Det}_{\mathfrak{F}} g.M = \det g \text{Det}_{\mathfrak{F}} M \quad (3)$$

where

$$g.M = \left( \sum_{1 \leq j_2 \leq n} g_{i_2 j_2} M_{i_1, j_2, i_3, \dots, i_k} \right)_{1 \leq i_1, \dots, i_k \leq n}.$$

**Proof** It suffices to apply (2) to  $g.M$

$$\begin{aligned} \text{Det}_{\mathfrak{F}} g.M &= \sum_{\sigma_3, \dots, \sigma_k} \mathfrak{F}(\sigma_3, \dots, \sigma_k) \det(g.M^{\sigma_3, \dots, \sigma_k}) \\ &= \sum_{\sigma_3, \dots, \sigma_k} \mathfrak{F}(\sigma_3, \dots, \sigma_k) \det g \det(M^{\sigma_3, \dots, \sigma_k}) \\ &= \det g \text{Det}_{\mathfrak{F}} M. \end{aligned}$$

□

### 3 Hyperdeterminant on a semilattice

The factorization properties of the GCD determinants are the consequence of the semilattice structure of the integers and can be stated in a more general way. Consider a partially ordered finite set  $L$  so that every pairs  $(x, y) \in L^2$  has a greatest lower bound denoted by  $x \wedge y$ . Such a poset is called a meet semilattice. One defines classically its  $\zeta$  function

$$\zeta_L(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

and its Mobius function

$$\mu_L(x, y) = \begin{cases} 1 & \text{if } x = y \\ -\sum_{x \leq z < y} \mu_L(z, y) & \text{if } x < y \\ 0 & \text{in the other cases} \end{cases} \quad (5)$$

The two functions are reciprocal. If  $F$  and  $f$  are two functions verifying

$$F(x) = \sum_{y \leq x} f(y) = \sum_{y \in L} \zeta_L(y, x) f(y) \quad (6)$$

one has

$$f(x) = \sum_{y \in L} \mu_L(y, x) F(y). \quad (7)$$

The manipulations of the identities (6) and (7) are the keys of the proof of the Lindström Theorem [11]. I recall its proof in a very slightly more general version.

I will denote by  $z_x$  an element of  $L$  lower than  $x$  and by  $F_x$  a function from  $L$  to a  $\mathbb{C}$  (or more generally in a commutative ring). Let  $M$  be the matrix defined by

$$M = (F_x(z_x \wedge y))_{x, y \in L}. \quad (8)$$

Remark that it suffices to define  $F_x(y)$  only when  $y \leq x$ . In particular, one can suppose that  $F_x(y) = F(y, x)$  is an incidence function (*i.e.*  $F(x, y) \neq 0$  unless  $y \leq x$ ). One has the equality

$$F_x(z_x \wedge y) = \sum_{z \in L} \zeta_L(z, z_x) \zeta(z, y) f_x(z). \quad (9)$$

where  $f_x(z) = \sum_{y \in L} \mu_L(y, z) F_x(y)$ . Hence, it follows  $\det M = \det \Phi \cdot \det Z$  where  $\Phi = (\zeta_L(y, z_x) f_x(y))_{x, y \in L}$  and  $Z = (\zeta_L(x, y))_{x, y \in L}$ . As  $\Phi$  and  $Z$  are

triangular  $\det Z = 1$  and

$$\det \Phi = \prod_x \zeta_L(x, z_x) f_x(x) = \begin{cases} \prod_x f_x(y) & \text{if } z_x = x \text{ for each } x \\ 0 & \text{otherwise,} \end{cases} \quad (10)$$

one obtains the Linström Theorem.

**Theorem 3.1** (*Linström*)

$$\det (F_x(z_x \wedge y))_{x,y \in L} = \begin{cases} \prod_x f_x(x) & \text{if } z_x = x \text{ for each } x \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

In the original Linström Theorem  $z_x = x$  for each  $x$ , furthermore equality (10) generalizes a Cesaro lemma

**Lemma 3.2** (*Cesaro*)

Denote by  $\gcd_m(n) = \gcd(m, n)$ . One has,

$$(\mu * (f \circ \gcd_m))(n) = \begin{cases} (f * \mu)(n) & \text{if } m = n \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

where  $*$  is the Dirichlet convolution.

The Linström Theorem can be extended for  $\mathfrak{F}$ -determinants.

**Theorem 3.3** (*Linström Theorem for  $\mathfrak{F}$ -determinants*)

Let  $L$  be a meet semilattice and set  $L = \{x_1, \dots, x_n\}$ . One has,

$$\text{Det}_{\mathfrak{F}} (F_{x_{i_1}}(x_{i_1} \wedge \dots \wedge x_{i_k}))_{1 \leq i_1, \dots, i_k \leq n} = \mathfrak{F}(\text{Id}, \dots, \text{Id}) \prod_x f_x(x) \quad (13)$$

**Proof** Equality (2) allows to write

$$\text{Det}_{\mathfrak{F}} (F_{x_1}(x_1 \wedge \dots \wedge x_k))_{x_1, \dots, x_k \in L} = \sum_{\sigma_3, \dots, \sigma_k} \mathfrak{F}(\sigma_3, \dots, \sigma_k) \det (F_{x_i}(x_i \wedge x_j \wedge x_{\sigma_3(i)} \wedge \dots \wedge x_{\sigma_k(i)}))_{i,j} \quad (14)$$

and the result follows from the Linström Theorem (Theorem 3.1).  $\square$

Now, consider a sublattice  $S \subset L$  which is meet closed (*ie* closed for the operation  $\wedge$ ) and fix a linear extension  $l = y_1 \dots y_n$  of  $S$ . As in [2], i denote by  $x \trianglelefteq y_i$  the relation  $x \leq y_i$  and  $x \not\leq y_j$  for each  $j < i$ . Consider a pair of function  $f$  and  $F$  verifying

$$F(y_i) = \sum_{x \leq y_i} f(x) \quad (15)$$

and set  $\hat{f}(y_i) = \sum_{d \trianglelefteq y_i} f(x)$ . One has

**Lemma 3.4**

$$F(y_i) = \sum_{y_k \leq y_i} \hat{f}(y_k) \quad (16)$$

**Proof** This identities appears in [2] Theorem 4.1 p 7 and its proof is based on a one-to-one correspondance between the terms in (16) and (15). I do not repeat it here.  $\square$

Hence, using Theorem 3.3 and Lemma 15, one generalizes a result by Altinisk, Sagan and Tuglu ([2] Theorem 4.1 p 7).

**Corollary 3.5**

$$\text{Det}_{\mathfrak{F}}(F_{y_{i_1}}(y_{i_1} \wedge \cdots \wedge y_{i_k})) = \mathfrak{F}(\text{Id}, \cdots, \text{Id}) \prod_{i=1}^n \left( \sum_{x_1 \triangleleft y_i} \sum_{x_2 \in L} \mu_L(x_1, x_2) F_{y_i}(x_2) \right).$$

If we set  $\mathfrak{F}(\sigma_3, \dots, \sigma_k) = \text{sign}(\sigma_3) \dots \text{sign}(\sigma_k)$ , one recovers the computation of Lehmer [9] as a special case of Theorem 3.3 and the result of Haukannen ([12] Theorem 1. p 56) from Corollary 3.5.

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