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SCATTERING THEORY FOR RADIAL NONLINEAR SCHRÖDINGER EQUATIONS ON HYPERBOLIC SPACE

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Abstract. We study the long time behavior of radial solutions to nonlinear Schrödinger equations on hyperbolic space. We show that the usual distinction between short range and long range nonlinearity is modified: the geometry of the hyperbolic space makes every power-like nonlinearity short range. The proofs rely on weighted Strichartz estimates, which imply Strichartz estimates for a broader family of admissible pairs, and on Morawetz type inequalities. The latter are established without symmetry assumptions.

Contents

1. Introduction

This paper is devoted to the scattering theory for the nonlinear Schrödinger equation

$$i\partial_t u + \Delta_{\mathbb{H}^n} u = |u|^{2\sigma} u \quad ; \quad e^{-it\Delta_{\mathbb{H}^n}} u(t)\big|_{t=t_0} = \varphi\,, \tag{1.1}$$

on the hyperbolic space ($n \geqslant 2$):

$$\mathbb{H}^n = \{\mathbb{R}^{n+1} \ni \Omega = (x_0, \ldots, x_n) = (x_0, x') = (\cosh r, \sinh r\, \omega),\ r \geqslant 0,\ \omega \in \mathbb{S}^{n-1}\}.$$

We consider a defocusing power nonlinearity. One could also prove some results in the focusing case, but this case will not be discussed in this paper. When a

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function of time and space $u(t,\Omega)$ depends only on t and r, we say that it is radial. The reason is that r is the hyperbolic distance between Ω and the origin of the hyperboloid $O = (x_0 = 1, x' = 0)$. With the usual abuse of notation, we write $u(t,r)$. We prove that for any $\sigma > 0$, a short range scattering theory is available for radial solutions to (1.1). This is in sharp contrast with the Euclidean case, where the nonlinearity $|u|^{2\sigma}u$ cannot be short range as soon as $\sigma \leqslant 1/n$ (see Section 2.3). A crucial argument to prove this phenomenon is the existence of weighted Strichartz estimates for radial solutions to Schrödinger equations on $\mathbb{H}^n$, established in [2] and [29]. Note that if these weighted Strichartz estimates were available for general solutions to Schrödinger equations on $\mathbb{H}^n$ (and not only radial), then all the results of this paper could be adapted, with the same proofs. Also, similar results can be extended to the equation posed on Damek-Ricci spaces, thanks to the weighted Strichartz estimates obtained in [29].

With the above parameterization for the hyperbolic space, the Laplace-Beltrami operator reads:

$$\Delta_{\mathbb{H}^n} = \partial_r^2 + (n-1)\frac{\cosh r}{\sinh r}\,\partial_r + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}}\,. \tag{1.2}$$

In order to define wave operators, we introduce the free Schrödinger generalized initial value problem:

$$\left\{\begin{array}{c} i\partial_t u + \Delta_{\mathbb{H}^n} u = 0 \quad , \\ u_{|t=t_0} = \varphi. \end{array}\right. \tag{1.3}$$

We denote $U(t) = e^{it\Delta_{\mathbb{H}^n}}$, so that in (1.3), $u(t,\Omega) = U(t)\varphi(\Omega)$. When considering solutions to (1.1), we use the convention that if $t_0 = -\infty$ (resp. $t_0 = +\infty$), then we denote $\varphi = u_-$ (resp. $\varphi = u_+$), and solving (1.1) means that we construct wave operators. If $t_0 = 0$, then we denote $\varphi = u_0$, and (1.1) is the standard Cauchy problem. In all the cases, we seek mild solutions to (1.1), that is, we solve

$$u(t) = U(t)\varphi - i\int_{t_0}^{t} U(t)(t-s)\left(|u|^{2\sigma}u\right)(s)ds. \tag{1.4}$$

We can now state our main results. The first one deals with existence of wave operators and asymptotic completeness for small L^2 data:

Theorem 1.1. *Let $n \geqslant 2$, $0 < \sigma \leqslant 2/n$, and $t_0 \in \overline{\mathbb{R}}$. There exists $\epsilon = \epsilon(n,\sigma)$ such that if $\varphi \in L^2_{\rm rad}(\mathbb{H}^n)$ with $\|\varphi\|_{L^2} < \epsilon$, then (1.1) has a unique solution*

$$u \in C(\mathbb{R}; L^2) \cap L^{2+2\sigma}(\mathbb{R}\times\mathbb{H}^n).$$

Moreover, its L^2-norm is constant, $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ for all $t \in \mathbb{R}$.
There exist $u_\pm \in L^2_{\rm rad}(\mathbb{H}^n)$ such that

$$\|u(t) - U(t)u_\pm\|_{L^2} \to 0 \quad \text{as } t \to \pm\infty.$$

If $t_0 = -\infty$ (resp. $t_0 = +\infty$), then $u_- = \varphi$ (resp. $u_+ = \varphi$).

The existence of solutions in $C(\mathbb{R}; L^2)$ for data which are small in L^2 is analogous to the Euclidean case ([35], see also [12]). For $\sigma = 2/n$, our result is the exact analogue to its Euclidean counterpart recalled in Proposition 2.3. Note however that for $0 < \sigma < 2/n$, the space where the solutions belong, and the existence of a scattering theory, distinguish the hyperbolic space $\mathbb{H}^n$ from the Euclidean space $\mathbb{R}^n$. In particular, there is no long range effect in hyperbolic space, even if $0 < \sigma \leqslant 1/n$.

Our second result establishes the existence of the wave operator in the Sobolev space H^1, when the nonlinearity is H^1-subcritical (see Appendix A for the notion of criticality). Here again, the power σ can go down to 0, with no long range effect.

Theorem 1.2. *Let $n \geqslant 2$, $0 < \sigma < 2/(n-2)$, and $t_0 = -\infty$. For any $\varphi = u_- \in H^1_{\mathrm{rad}}(\mathbb{H}^n)$, there exists $T < \infty$ such that (1.4) has a unique solution in $C \cap L^\infty(]-\infty, -T]; H^1) \cap L^{2\sigma+2}(]-\infty, -T]; W^{1,2\sigma+2})$.*
Moreover, this solution u is defined globally in time: $u \in L^\infty(\mathbb{R}; H^1)$. That is, u is the only solution to (1.1) with

$$\|u(t) - U(t)u_-\|_{H^1} = \|U(-t)u(t) - u_-\|_{H^1} \to 0 \quad \text{as } t \to -\infty.$$

Of course, we could prove the existence of wave operators with data at time $t_0 = +\infty$. Since the proof is similar, we shall skip it.

The proofs of Theorems 1.1 and 1.2 rely on two remarks. First, the weighted Strichartz estimates proven in [2, 29] for radial solutions to Schrödinger equations on $\mathbb{H}^n$, $n \geqslant 3$, make it possible to state Strichartz estimates which are the same as on $\mathbb{R}^d$, for any $d \geqslant n$. We show in this paper that similar results are available when $n = 2$. Second, the classical proofs for the counterparts of Theorems 1.1 and 1.2 in the Euclidean space $\mathbb{R}^d$ rely only on functional analysis arguments, based on Strichartz estimates, Hölder inequality and Sobolev embeddings. This is why the proofs of Theorems 1.1 and 1.2, presented in Sections 4 and 5 respectively, are rather short.

The next natural step in scattering theory consists in proving the invertibility of the wave operators on their range, that is, asymptotic completeness. Note that the asymptotic completeness that we prove is for $n = 3$ only (see the discussion in Section 7).

Theorem 1.3. *Let $n = 3$, $0 < \sigma < 2$, and $t_0 = 0$. For any $\varphi = u_0 \in H^1_{\mathrm{rad}}(\mathbb{H}^3)$ (1.1) has a unique, global solution in $C(\mathbb{R}; H^1_{\mathrm{rad}}) \cap L^4(\mathbb{R} \times \mathbb{H}^3)$. Moreover, there exist u_- and u_+ in $H^1_{\mathrm{rad}}(\mathbb{H}^3)$ such that*

$$\|u(t) - U(t)u_\pm\|_{H^1(\mathbb{H}^3)} \to 0 \quad \text{as } t \to \pm\infty.$$

Moreover, if $2/3 < \sigma < 2$, then we need not assume that φ is radial: for any $\varphi = u_0 \in H^1(\mathbb{H}^3)$, (1.1) has a unique, global solution in $C(\mathbb{R}; H^1) \cap L^4(\mathbb{R} \times \mathbb{H}^3)$, and there exist u_- and u_+ in $H^1(\mathbb{H}^3)$ such that

$$\|u(t) - U(t)u_\pm\|_{H^1(\mathbb{H}^3)} \to 0 \quad \text{as } t \to \pm\infty.$$

Notation. In this paper we often use the notation $A \lesssim B$ to denote that there exists an absolute constant C such that $A \leq CB$. Another standard notation is to use for any $1 \leq p \leq \infty$ the symbol p' to denote the Hölder-conjugate exponent, that is $1/p + 1/p' = 1$. Finally we use the symbol $\Re z$ and $\Im z$ to denote respectively the the real and the imaginary part of a complex number z.

The rest of this paper is organized as follows. In Section 2, we review the scattering result for nonlinear Schrödinger equations on the Euclidean space: small L^2 data, existence of wave operators in $H^1(\mathbb{R}^d)$, non-existence of wave operators when $\sigma \leqslant 1/d$, and asymptotic completeness. In Section 3, we show that for radial solutions to (1.1), the same Strichartz estimates as in $\mathbb{R}^d$ are available in $\mathbb{H}^n$, for any $d \geqslant n \geqslant 2$. Theorems 1.1 and 1.2 are proven in Sections 4 and 5 respectively. We prove a general interaction Morawetz inequality in Section 6, and infer Theorem 1.3 in Section 7. In Appendix A , we prove that the notion of criticality, as far as the Cauchy problem (1.1) is concerned, is the same on $\mathbb{H}^n$ as on $\mathbb{R}^n$. We study the large time behavior of radial solutions to the linear Schrödinger equation (1.3) on $\mathbb{H}^3$ in Appendix B. Finally, we discuss the existence of an analogue to the Galilean operator in the radial framework on $\mathbb{H}^3$ in Appendix C.

2. A review of scattering theory in $\mathbb{R}^d$

In this paragraph, we consider, in the Euclidean space, the equation

$$i\partial_t u + \Delta_{\mathbb{R}^d} u = |u|^{2\sigma} u \quad ; \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d. \tag{2.1}$$

We recall some results concerning scattering theory, in order to compare them with their counterpart in hyperbolic space. We also sketch some proofs that we mimic in the hyperbolic setting.

First, the Schrödinger operator in the Euclidean space satisfies the following Strichartz estimates.

Definition 2.1. Let $d \geqslant 2$. A pair (p,q) is *d-admissible* if $2 \leqslant q \leqslant \frac{2d}{d-2}$ and

$$\frac{2}{p} = \delta(q) := d\left(\frac{1}{2} - \frac{1}{q}\right), \quad (p,q) \neq (2,\infty).$$

Proposition 2.2. *Let $d \geqslant 2$. Denote $S_d(t) = e^{it\Delta_{\mathbb{R}^d}}$.*
1. *For any d-admissible pair (p,q), there exists C_q such that*

$$\|S_d(\cdot)\phi\|_{L^p(\mathbb{R};L^q)} \leq C_q \|\phi\|_{L^2}, \quad \forall \phi \in L^2(\mathbb{R}^d).$$

2. *For any d-admissible pairs (p_1,q_1) and (p_2,q_2) and any interval I, there exists C_{q_1,q_2} independent of I such that*

$$\left\| \int_{I\cap\{s\leq t\}} S_d(t-s)F(s)ds \right\|_{L^{p_1}(I;L^{q_1})} \leq C_{q_1,q_2} \|F\|_{L^{p_2'}\left(I;L^{q_2'}\right)},$$

for every $F \in L^{p_2'}\left(I;L^{q_2'}(\mathbb{R}^d)\right)$.

Let $t_0 \in \overline{\mathbb{R}}$, and consider (2.1) along with the initial data:

$$S_d(-t)u(t)\big|_{t=t_0} = \varphi. \tag{2.2}$$

We use the convention that if $t_0 = -\infty$ (resp. $t_0 = +\infty$), then we denote $\varphi = u_-$ (resp. $\varphi = u_+$), and solving (2.1)–(2.2) means that we construct wave operators. If $t_0 = 0$, then we denote $\varphi = u_0$, and (2.1)–(2.2) is the standard Cauchy problem. In all these cases, we seek mild solutions to (2.1)–(2.2), that is, we solve

$$u(t) = S_d(t)\varphi - i\int_{t_0}^t S_d(t-s)\left(|u|^{2\sigma}u\right)(s)ds =: \Phi(u)(t). \tag{2.3}$$

2.1. Small data in the L^2-critical case.

Recall the result of [13]. The L^2-critical case corresponds to the power $\sigma = 2/d$. In that case, the pair

$$(p,q) = \left(2+\frac{4}{d}, 2+\frac{4}{d}\right).$$

is d-admissible, and this is the main remark to prove:

Proposition 2.3. *Let $d \geqslant 2$, $\sigma = 2/d$, and $t_0 \in \overline{\mathbb{R}}$. There exists $\epsilon = \epsilon(d)$ such that if $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\|_{L^2} < \epsilon$, then (2.1)–(2.2) has a unique solution*

$$u \in C(\mathbb{R};L^2) \cap L^{2+\frac{4}{d}}(\mathbb{R}\times\mathbb{R}^d).$$

Moreover, its L^2-norm is constant, $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ for all $t \in \mathbb{R}$.
There exist $u_\pm \in L^2(\mathbb{R}^d)$ such that

$$\|u(t) - S_d(t)u_\pm\|_{L^2} \to 0 \quad \text{as } t \to \pm\infty.$$

If $t_0 = -\infty$ (resp. $t_0 = +\infty$), then $u_- = \varphi$ (resp. $u_+ = \varphi$).

Sketch of the proof. The idea is to apply a fixed point argument to (2.3) in

$$X = \left\{ u \in C(\mathbb{R}; L^2) \cap L^{2+\frac{4}{d}}(\mathbb{R}\times\mathbb{R}^d) \quad ; \quad \|u\|_{L^{2+\frac{4}{d}}(\mathbb{R}\times\mathbb{R}^d)} \leqslant 2C_{2+\frac{4}{d}}\|\varphi\|_{L^2} \right\}.$$

Here, $C_{2+\frac{4}{d}}$ is the constant given in the first part of Proposition 2.2. Indeed, denoting $\gamma = 2 + 4/d$, Strichartz estimates and Hölder inequality yield:

$$\|\Phi(u)\|_{L^\gamma(\mathbb{R}\times\mathbb{R}^d)} \leqslant C_\gamma \|\varphi\|_{L^2(\mathbb{R}^d)} + C_{\gamma,\gamma}\|u\|^{1+\frac{4}{d}}_{L^\gamma(\mathbb{R}\times\mathbb{R}^d)}.$$

This shows that for $\|\varphi\|_{L^2}$ sufficiently small, X is invariant under the action of Φ. Similarly, Φ is a contraction on X if $\|\varphi\|_{L^2}$ is sufficiently small, thus providing a unique solution to (2.3) in X. The conservation of mass is classical, and holds without the smallness assumption.

Scattering then follows from the Cauchy criterion: for $t_1 \leqslant t_2$, we have

$$\|S_d(-t_2)u(t_2) - S_d(-t_1)u(t_1)\|_{L^2} \leqslant C_{2,\gamma}\|u\|^{1+\frac{4}{d}}_{L^\gamma([t_1,t_2]\times\mathbb{R}^d)}.$$

The right hand side goes to zero when $t_1, t_2 \to \pm\infty$. The proposition follows easily, since the group S_d is unitary on L^2. □

2.2. **Existence of wave operators in H^1.** We recall the existence of wave operators for negative time; for positive time, the proof is similar. This means that we solve (2.3) with $t_0 = -\infty$ (and $\varphi = u_-$). The strategy consists first in solving (2.3) in a neighborhood of $t = -\infty$, that is on $]-\infty, -T]$ for T possibly very large. Then the conservation of mass and energy makes it possible to extend the solution to $t \in \mathbb{R}$. We simply recall the first step. The proof of this result appears in [23]. The proof we give is a simplification, which may be found for instance in [19]. We shall not recall or use the results available in weighted Sobolev spaces (see e.g. [14, 21, 22]).

Proposition 2.4. *Let $t_0 = -\infty$, $d \geqslant 2$ and $2/d \leqslant \sigma < 2/(d-2)$. For any $\varphi = u_- \in H^1(\mathbb{R}^d)$, there exists $T < \infty$ such that (2.3) has a unique solution in $C \cap L^\infty(]-\infty, -T]; H^1) \cap L^p(]-\infty, -T]; W^{1,2\sigma+2})$, where p is such that $(p, 2\sigma+2)$ is d-admissible.*
Moreover, this solution u is defined globally in time: $u \in L^\infty(\mathbb{R}; H^1)$.

In other words, we construct the only solution u to (2.1) such that

$$\|u(t) - S_d(t)u_-\|_{H^1} = \|S_d(-t)u(t) - u_-\|_{H^1} \to 0 \quad \text{as } t \to -\infty.$$

The wave operator W_- is the map

$$W_- : \quad H^1 \ni u_- \mapsto u_{|t=0} \in H^1.$$

Proof. Recall that p is such that $(p, 2\sigma+2)$ is d-admissible:

$$p = \frac{4\sigma+4}{d\sigma}.$$

With the notation $L^\beta_T Y = L^\beta(]-\infty, -T]; Y)$, we introduce:

$$X_T := \Big\{ u \in C(]-\infty, -T]; H^1) \ ; \ \|u\|_{L^p_T W^{1,2\sigma+2}} \leqslant 2C_{2\sigma+2}\|u_-\|_{H^1},$$
$$\|u\|_{L^\infty_T H^1} \leqslant 2\|u_-\|_{H^1}, \quad \|u\|_{L^p_T L^{2\sigma+2}} \leqslant 2\,\|S_d(\cdot)u_-\|_{L^p_T L^{2\sigma+2}} \Big\},$$

where $C_{2\sigma+2}$ is given by Proposition 2.2. Set $q = s = 2\sigma+2$: we have

$$\frac{1}{q'} = \frac{1}{q} + \frac{2\sigma}{s},$$
$$\frac{1}{p'} = \frac{1}{p} + \frac{2\sigma}{k},$$

where (p,q) is d-admissible and $p \leqslant k < \infty$ since $2/d \leqslant \sigma < 2/(d-2)$. For $u \in X_T$, Strichartz estimates and Hölder inequality yield:

$$\begin{aligned}\|\Phi(u)\|_{L^p_T W^{1,2\sigma+2}} &\leqslant C_{2\sigma+2}\|u_-\|_{H^1} + C\left(\left\||u|^{2\sigma}u\right\|_{L^{p'}_T L^{q'}} + \left\||u|^{2\sigma}\nabla u\right\|_{L^{p'}_T L^{q'}}\right)\\ &\leqslant C_{2\sigma+2}\|u_-\|_{H^1} + C\|u\|^{2\sigma}_{L^k_T L^s}\left(\|u\|_{L^p_T L^q} + \|\nabla u\|_{L^p_T L^q}\right)\\ &\leqslant C_{2\sigma+2}\|u_-\|_{H^1} + C\|u\|^{2\sigma\theta}_{L^p_T L^q}\|u\|^{2\sigma(1-\theta)}_{L^\infty_T L^q}\|u\|_{L^p_T W^{1,2\sigma+2}}\ ,\end{aligned}$$

for some $0 < \theta \leqslant 1$, where we have used the property $q = s = 2\sigma+2$. Sobolev embedding and the definition of X_T then imply:

$$\|\Phi(u)\|_{L^p_T W^{1,2\sigma+2}} \leqslant C_{2\sigma+2}\|u_-\|_{H^1} + C\left\|S_d(\cdot)u_-\right\|^{2\sigma\theta}_{L^p_T L^q}\|u\|^{2\sigma(1-\theta)}_{L^\infty_T H^1}\|u\|_{L^p_T W^{1,2\sigma+2}}.$$

We have similarly

$$\|\Phi(u)\|_{L^\infty_T H^1} \leqslant \|u_-\|_{H^1} + C\left\|S_d(\cdot)u_-\right\|^{2\sigma\theta}_{L^p_T L^q}\|u\|^{2\sigma(1-\theta)}_{L^\infty_T H^1}\|u\|_{L^p_T W^{1,2\sigma+2}}$$

$$\|\Phi(u)\|_{L^p_T L^{2\sigma+2}} \leqslant \left\|S_d(\cdot)u_-\right\|_{L^p_T L^{2\sigma+2}} + C\left\|S_d(\cdot)u_-\right\|^{2\sigma\theta}_{L^p_T L^q}\|u\|^{2\sigma(1-\theta)}_{L^\infty_T H^1}\|u\|_{L^p_T W^{1,2\sigma+2}}.$$

From Strichartz estimates, $S_d(\cdot)u_- \in L^p(\mathbb{R};L^q)$, so

$$\left\|S_d(\cdot)u_-\right\|_{L^p_T L^q} \to 0 \quad \text{as } T \to +\infty.$$

Since $\theta > 0$, we infer that Φ sends X_T to itself, for T sufficiently large.

We have also, for $u_2, u_1 \in X_T$:

$$\begin{aligned}\|\Phi(u_2) - \Phi(u_1)\|_{L^p_T L^q} &\lesssim \max_{j=1,2}\|u_j\|^{2\sigma}_{L^k_T L^s}\|u_2 - u_1\|_{L^p_T L^q}\\ &\lesssim \left\|S_d(\cdot)u_-\right\|^{2\sigma\theta}_{L^p_T L^q}\|u_-\|^{2\sigma(1-\theta)}_{H^1}\|u_2 - u_1\|_{L^p_T L^q}.\end{aligned}$$

Up to choosing T larger, Φ is a contraction on X_T, and the proposition follows. □

2.3. **Non-existence of wave operators for $\sigma \leqslant 1/d$.** Even though the scattering result we recalled shows the existence of wave operators in $H^1(\mathbb{R}^d)$ for $\sigma \geqslant 2/d$, it is natural to expect the nonlinearity to be negligible for large time as soon as $\sigma > 1/d$. Many results exist, supporting this assertion; we shall not state them, but rather point out that it is not possible to go below $1/d$. The result recalled below was established in [32, 3] (see also [25]).

Proposition 2.5. *Let $d \geqslant 2$, $0 < \sigma \leqslant 1/d$ and $T > 0$. Let $u \in C(]-\infty,-T];L^2(\mathbb{R}^d))$ be a solution of (2.1) such that there exists $u_- \in L^2(\mathbb{R}^d)$ and*

$$\|u(t) - S_d(t)u_-\|_{L^2} = \|S_d(-t)u(t) - u_-\|_{L^2} \to 0 \quad \text{as } t \to -\infty.$$

Then $u \equiv 0$ and $u_- \equiv 0$.

Sketch of the proof. Let $\psi \in C^\infty_0(\mathbb{R}^d)$ and $t_1 \leqslant t_2 \leqslant -T$: by assumption,

$$\langle \psi, S_d(-t_2)u(t_2) - S_d(-t_1)u(t_1)\rangle = -i\int_{t_1}^{t_2}\left\langle S_d(\tau)\psi, \left(|u|^{2\sigma}u\right)(\tau)\right\rangle d\tau$$

goes to zero as $t_1, t_2 \to -\infty$. But for $\tau \to -\infty$, we have

$$S_d(\tau)\psi \sim c\frac{e^{i|x|^2/(4\tau)}}{|\tau|^{d/2}}\widehat{\psi}\left(\frac{x}{2\tau}\right) \quad ; \quad u(\tau) \sim S_d(\tau)u_- \sim c\frac{e^{i|x|^2/(4\tau)}}{|\tau|^{d/2}}\widehat{u_-}\left(\frac{x}{2\tau}\right).$$

Therefore,

$$\left\langle S_d(\tau)\psi, \left(|u|^{2\sigma}u\right)(\tau)\right\rangle \sim \frac{C}{|t|^{\sigma d}}\left\langle \widehat{\psi}, |\widehat{u_-}|^{2\sigma}\,\widehat{u_-}\right\rangle.$$

This function of τ is not integrable, unless

$$\left\langle \widehat{\psi}, |\widehat{u_-}|^{2\sigma}\,\widehat{u_-}\right\rangle = 0.$$

Since $\psi \in C_0^\infty(\mathbb{R}^d)$ is arbitrary, this means that $\widehat{u_-} \equiv 0 \equiv u_-$. The assumption and the conservation of mass then imply $u \equiv 0$. □

When $\sigma \leqslant 1/d$, long range effects must be taken into account, *even in a radial setting* (see e.g. [27, 8] for the case $d = 1$, [20] for $d \geqslant 2$).

2.4. **Asymptotic completeness in H^1.** To get a complete picture of large time behavior of solutions to (2.1), we proceed to the next step which consists in establishing asymptotic completeness, that is, proving that the wave operators $W_\pm$ are invertible on their range. Here we only recall some results in $H^1(\mathbb{R}^d)$, and we do not mention what can be done in weaker Sobolev spaces, or in weighted Sobolev spaces (see e.g. [14, 22, 26, 36]).

The original proof of the asymptotic completeness for (2.1) in $H^1(\mathbb{R}^d)$ is due to Ginibre and Velo [23]. Let $t_0 = 0$ and $\varphi \in H^1(\mathbb{R}^d)$: the local in time H^1 solution to the Cauchy problem (2.1)–(2.2) is actually global in time for $0 < \sigma < 2/(d-2)$, thanks to the conservations of mass and energy, since the nonlinearity is defocusing:

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2},$$
$$\|\nabla u(t)\|_{L^2}^2 + \frac{1}{\sigma+1}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} = \|\nabla\varphi\|_{L^2}^2 + \frac{1}{\sigma+1}\|\varphi\|_{L^{2\sigma+2}}^{2\sigma+2}.$$

Using Morawetz inequality and dispersive estimates for $S_d(t)$, they prove that

$$\|u(t)\|_{L^q(\mathbb{R}^d)} \to 0 \quad \text{as } t \to \pm\infty, \quad \forall q \in \left]2, \frac{2}{d-2}\right[.$$

This makes it possible to show that $u \in L^p(\mathbb{R}; L^q(\mathbb{R}^d))$ for all d-admissible pairs (p, q), as soon as $2/d < \sigma < 2/(d-2)$. Asymptotic completeness follows easily:

Proposition 2.6 ([23], see also [12])**.** *Let $d \geqslant 3$, $t_0 = 0$ and $\varphi \in H^1(\mathbb{R}^d)$. If $2/d < \sigma < 2/(d-2)$, then there exist $u_\pm \in H^1(\mathbb{R}^d)$ such that*

$$\|u(t) - S_d(t)u_\pm\|_{H^1(\mathbb{R}^d)} \to 0 \quad \textit{as } t \to \pm\infty.$$

More recently, a simplified proof was proposed by Tao, Visan and Zhang [34], relying on an *interaction* Morawetz inequality as introduced in [16]. We recall this approach for essentially two reasons:

- It is shorter than the original one [23] (or [12]).
- It does not use dispersive estimates for $S_d(t)$.

The second point seems to be crucial to prove Theorem 1.3 as a consequence of the proof in [34] and of the interaction Morawetz inequality that we establish in Section 6. The interaction Morawetz inequality presented in [34] reads as follows:

Proposition 2.7 ([34])**.** *Let $d \geqslant 3$, $t_0 = 0$ and $\varphi \in H^1(\mathbb{R}^d)$. Let I be a compact time interval. There exists C independent of I such that the following holds.*

- *If $d = 3$, then the solution to (2.1)–(2.2) satisfies:*

$$\int_I \int_{\mathbb{R}^3} |u(t,x)|^4 dxdt \leqslant C\|u\|_{L^\infty(I;H^1)}^4. \tag{2.4}$$

- *If $d \geqslant 4$, then the solution to (2.1)–(2.2) satisfies:*

$$\int_I \int_{\mathbb{R}^3} \frac{|u(t,y)|^2|u(t,x)|^2}{|x-y|^3} dxdydt \leqslant C\|u\|_{L^\infty(I;H^1)}^4. \tag{2.5}$$

These inequalities imply that there exists $\widetilde{C}$ independent of I such that:

$$\|u\|_{L^{d+1}\left(I;L^{\frac{2(d+1)}{d-1}}(\mathbb{R}^d)\right)} \leqslant \widetilde{C}\|u\|_{L^\infty(I;H^1)}. \tag{2.6}$$

In Section 6, we establish the analogue of (2.4)–(2.5) on the hyperbolic space $\mathbb{H}^n$, so we do not recall how (2.4) and (2.5) are proven here: the method on $\mathbb{H}^n$ is similar, with an additional drop of geometry.

If $d = 3$, then (2.6) is exactly (2.4). On the other hand, when $d \geqslant 4$, (2.6) follows from (2.5) by interpreting the convolution with $\frac{1}{|x|^3}$ as differentiation, and thanks to the inequality ([34, Lemma 5.6])

$$\left\| |\nabla|^{-\frac{d-3}{4}} f \right\|_{L^4}^2 \lesssim \left\| |\nabla|^{-\frac{d-3}{2}} |f|^2 \right\|_{L^2}, \tag{2.7}$$

which can be established by using paradifferential calculus.

Remark 2.8. The pair $\left(d+1, \frac{2(d+1)}{d-1}\right)$ present in (2.6) is 2-admissible.

Using the *a priori* estimate provided by the conservations of mass and energy, one infers the *a priori* bound

$$\|u\|_{L^{d+1}\left(\mathbb{R}; L^{\frac{2(d+1)}{d-1}}(\mathbb{R}^d)\right)} \leqslant \widetilde{C} \|u\|_{L^\infty(\mathbb{R}; H^1)} \lesssim \|\varphi\|_{H^1}.$$

Let $\eta > 0$ be a small constant to be fixed later. The line $\mathbb{R}$ can be divided into J (for some *finite* J from the above estimate) subintervals $I_j = [\tau_j, \tau_{j+1}]$ such that

$$\|u\|_{L^{d+1}\left(I_j; L^{\frac{2(d+1)}{d-1}}(\mathbb{R}^d)\right)} \leqslant \eta.$$

As a consequence of [34, Lemma 2.7], if $d \geqslant 3$ and $2/d < \sigma < 2/(d-2)$, there exist $C > 0$, $\delta \in]0, 1[$ and a d-admissible pair (p_0, q_0) such that for any time interval I:

$$\left\| |u|^{2\sigma} u \right\|_{L^2\left(I; W^{1, \frac{2d}{d+2}}\right)} \leqslant C \|u\|_{L^{d+1}\left(I; L^{\frac{2(d+1)}{d-1}}\right)}^{2\sigma\delta} \|u\|_{L^\infty(I; H^1)}^{2\sigma(1-\delta)} \|u\|_{L^{p_0}(I; W^{1,q_0})}. \tag{2.8}$$

This estimate follows from Hölder inequality (see [34]), and algebraic computations on d-admissible pairs. Using Strichartz estimates on (2.3) (with t_0 replaced by τ_j and φ replaced by $u(\tau_j)$), and (2.8), we get, for $1 \leqslant j \leqslant J$, and any d-admissible pair (p, q),

$$\begin{aligned}
\|u\|_{L^p(I_j; W^{1,q})} &\lesssim \|u(\tau_j)\|_{H^1} + \left\| |u|^{2\sigma} u \right\|_{L^2\left(I; W^{1, \frac{2d}{d+2}}\right)} \\
&\lesssim \|u\|_{L^\infty(\mathbb{R}; H^1)} + \|u\|_{L^{d+1}\left(I_j; L^{\frac{2(d+1)}{d-1}}\right)}^{2\sigma\delta} \|u\|_{L^\infty(I_j; H^1)}^{2\sigma(1-\delta)} \|u\|_{L^{p_0}(I_j; W^{1,q_0})} \\
&\lesssim \|u\|_{L^\infty(\mathbb{R}; H^1)} + \eta^{2\sigma\delta} \|u\|_{L^\infty(\mathbb{R}; H^1)}^{2\sigma(1-\delta)} \|u\|_{L^{p_0}(I_j; W^{1,q_0})}.
\end{aligned}$$

Fix $(p, q) = (p_0, q_0)$: taking $\eta > 0$ sufficiently small, we find

$$\|u\|_{L^{p_0}(I_j; W^{1,q_0})} \leqslant C \|u\|_{L^\infty(\mathbb{R}; H^1)}, \quad \forall 1 \leqslant j \leqslant J,$$

hence $u, \nabla u \in L^{p_0}(\mathbb{R}; W^{1,q_0})$. We deduce $u, \nabla u \in L^p(\mathbb{R}; W^{1,q})$ for all d-admissible pairs (p, q). Asymptotic completeness is then straightforward: let $t_2 \geqslant t_1 \geqslant 0$. From inhomogeneous Strichartz estimates with $(p_2, q_2) = (2, \frac{2n}{n-2})$ and (2.8), we have:

$$\begin{aligned}
\|S_d(-t_2)u(t_2) - S_d(-t_1)u(t_1)\|_{H^1} &\lesssim \left\| |u|^{2\sigma} u \right\|_{L^2\left([t_1, +\infty[; W^{1, \frac{2d}{d+2}}\right)} \\
&\lesssim \|u\|_{L^{d+1}\left([t_1, +\infty[; L^{\frac{2(d+1)}{d-1}}\right)}^{2\sigma\delta}.
\end{aligned}$$

Since the last term goes to zero as $t_1 \to +\infty$, this proves Proposition 2.6 for positive time. The proof for negative time is similar.

Remark 2.9. In Proposition 2.6, we assume that $\sigma < 2/(d-2)$. Scattering for the H^1-critical case $\sigma = 2/(d-2)$ was recently proved for $d = 3$ in [17], for $d = 4$ in [30] and finally for $d \geq 5$ in [33]. These results are not yet available in hyperbolic spaces since their proof, among other tools, uses very subtle arguments in Fourier analysis, arguments that are not at hand yet in $\mathbb{H}^n$.

3. Weighted Strichartz inequalities and consequences

The general idea is that weighted Strichartz estimates are available on hyperbolic space $\mathbb{H}^n$, provided that we restrict our study to *radial* functions. The weight has exponential decay in space. This decay gives us the "usual" Strichartz estimates recalled in Proposition 2.2, for d-admissible pairs, for *any* $d \geqslant n$. As usual for Strichartz estimates, we distinguish the case $n \geqslant 3$ from the case $n = 2$. The former is easier to present, and we start with it. In $\mathbb{H}^n$, we denote

$$\mathtt{w}_n(r) := \left(\frac{\sinh r}{r}\right)^{\frac{n-1}{2}} \quad ; \quad U(t) = e^{it\Delta_{\mathbb{H}^n}}.$$

3.1. Case $n \geqslant 3$.

The following global result was established in [2] for $n = 3$, and in [29] for $n \geqslant 4$:

Proposition 3.1 (Weighted Strichartz estimates in $\mathbb{H}^n$, $n \geqslant 3$)**.** *Let $n \geqslant 3$.*
1. *For any n-admissible pair (p, q), there exists C_q such that*

$$\left\| \mathtt{w}_n^{1-\frac{2}{q}} U(\cdot)\phi \right\|_{L^p(\mathbb{R};L^q)} \leqslant C_q \|\phi\|_{L^2}$$

for every radial *function $\phi \in L^2_{\mathrm{rad}}(\mathbb{H}^n)$.*
2. *For any n-admissible pairs (p_1, q_1) and (p_2, q_2) and any interval I, there exists C_{q_1,q_2} independent of I such that*

$$\left\| \mathtt{w}_n^{1-\frac{2}{q_1}} \int_{I\cap\{s\leqslant t\}} U(t-s)F(s)ds \right\|_{L^{p_1}(I;L^{q_1})} \leqslant C_{q_1,q_2} \left\| \mathtt{w}_n^{1-\frac{2}{q_2'}} F \right\|_{L^{p_2'}\left(I;L^{q_2'}\right)}$$

for every radial *function $F \in L^{p_2'}\left(I; L^{q_2'}_{\mathrm{rad}}(\mathbb{H}^n)\right)$.*

Corollary 3.2. *Let $d \geqslant n \geqslant 3$. Then Strichartz estimates hold for d-admissible pairs and radial functions on $\mathbb{H}^n$:*
1. *For any d-admissible pair (p, q), there exists $C_q = C_q(n, d)$ such that*

$$\|U(\cdot)\phi\|_{L^p(\mathbb{R};L^q)} \leqslant C_q\|\phi\|_{L^2}, \quad \forall \phi \in L^2_{\mathrm{rad}}(\mathbb{H}^n).$$

2. *For any d-admissible pairs (p_1, q_1) and (p_2, q_2) and any interval I, there exists $C_{q_1,q_2} = C_{q_1,q_2}(n, d)$ independent of I such that*

$$\left\| \int_{I\cap\{s\leqslant t\}} U(t-s)F(s)ds \right\|_{L^{p_1}(I;L^{q_1})} \leqslant C_{q_1,q_2} \|F\|_{L^{p_2'}\left(I;L^{q_2'}\right)},$$

for every $F \in L^{p_2'}\left(I; L^{q_2'}_{\mathrm{rad}}(\mathbb{H}^n)\right)$.

Proof. To prove the first estimate, it is enough to prove it for the endpoint estimate, $(p, q) = (2, \frac{2d}{d-2})$. Define s by

$$\frac{1}{n} = \frac{1}{d} + \frac{1}{s}.$$

We have $s \geqslant 0$, since $d \geqslant n$. Let $\phi \in L^2_{\mathrm{rad}}(\mathbb{H}^n)$. Hölder inequality and the first part of Proposition 3.1 yield:

$$\|U(\cdot)\phi\|_{L^2\left(\mathbb{R};L^{\frac{2d}{d-2}}\right)} \leqslant \left\| \mathtt{w}_n^{2/n} U(\cdot)\phi \right\|_{L^2\left(\mathbb{R};L^{\frac{2n}{n-2}}\right)} \left\| \mathtt{w}_n^{-2/n} \right\|_{L^s} \tag{3.1}$$

$$\leqslant C_{\frac{2n}{n-2}} \|\phi\|_{L^2(\mathbb{H}^n)} \left\| \mathtt{w}_n^{-2/n} \right\|_{L^s}.$$

If $d = n$, then $s = \infty$, and we have obviously $\mathtt{w}_n^{-2/n} \in L^\infty$. If $d > n$, then $\mathtt{w}_n^{-2/n} \in L^s$ if and only if

$$\int_0^\infty \left(\frac{r}{\sinh r}\right)^{s\frac{n-1}{n}} (\sinh r)^{n-1}\, dr < \infty.$$

This integral is convergent, since $s > n$ (d is finite). The first estimate of the corollary follows by interpolation, by conservation of the L^2 norm.

We turn to the inhomogeneous estimates. Let (p_1, q_1) and (p_2, q_2) be d-admissible pairs. Let (p_1, r_1) and (p_2, r_2) be the corresponding n-admissible pairs:

$$\frac{2}{p_j} = d\left(\frac{1}{2} - \frac{1}{q_j}\right) = n\left(\frac{1}{2} - \frac{1}{r_j}\right).$$

Note that since $d \geqslant n$, $q_j \leqslant r_j$. Therefore, s_j, given by

$$\frac{1}{q_j} = \frac{1}{r_j} + \frac{1}{s_j},,$$

is non-negative. Using Hölder inequality and the second part of Proposition 3.1, we find:

$$\begin{aligned}
\left\| \int_{I\cap\{s\leqslant t\}} U(t-s)F(s)ds \right\|_{L^{p_1}(I;L^{q_1})} &\leqslant \\
&\leqslant \left\| \mathtt{w}_n^{1-\frac{2}{r_1}} \int_{I\cap\{s\leqslant t\}} U(t-s)F(s)ds \right\|_{L^{p_1}(I;L^{r_1})} \left\| \mathtt{w}_n^{-1+\frac{2}{r_1}} \right\|_{L^{s_1}} \\
&\lesssim \left\| \mathtt{w}_n^{1-\frac{2}{r_2'}} F \right\|_{L^{p_2'}(I;L^{r_2'})} \left\| \mathtt{w}_n^{-1+\frac{2}{r_1}} \right\|_{L^{s_1}} \\
&\lesssim \|F\|_{L^{p_2'}(I;L^{q_2'})} \left\| \mathtt{w}_n^{1-\frac{2}{r_2'}} \right\|_{L^{s_2}} \left\| \mathtt{w}_n^{-1+\frac{2}{r_1}} \right\|_{L^{s_1}}.
\end{aligned}$$

Therefore, we have to check that $\mathtt{w}_n^{-1+\frac{2}{r_j}} \in L^{s_j}(\mathbb{H}^n)$. If $q_j = 2$, then $r_j = 2$ and $s_j = \infty$. If $q_j > 2$, then the above integrability condition is equivalent to:

$$s_j\left(\frac{1}{2} - \frac{1}{r_j}\right) > 1 \Leftrightarrow \frac{1}{2} - \frac{1}{r_j} > \frac{1}{s_j} = \frac{1}{q_j} - \frac{1}{r_j}.$$

Since $q_j > 2$, this is satisfied, and the corollary follows. □

3.2. **Case $n = 2$.** When $n = 2$, the analogue of Proposition 3.1 is not proven, but we have from [2]:

$$e^{it\Delta_{\mathbb{H}^2}}\phi(\Omega) = \frac{c}{|t|^{3/2}} e^{-it/2} \int_{\mathbb{H}^2} \phi(\Omega') \int_\rho^\infty \frac{s e^{is^2/4t}}{\sqrt{\cosh s - \cosh \rho}} ds d\Omega', \tag{3.2}$$

where $\rho = d(\Omega, \Omega')$. The following weighted dispersion estimate holds for *radial* functions in $\mathbb{H}^2$. Denote

$$\widetilde{\mathtt{w}}_2(r) = \left(\frac{\sinh r}{r(1+r)}\right)^{1/2}.$$

Proposition 3.3. *Let $\varepsilon \in]0,1[$. There exists $C_\varepsilon > 0$ such that*

$$\widetilde{\mathtt{w}}_2^{1-\varepsilon}(r) e^{it\Delta_{\mathbb{H}^2}}\phi(\Omega) \leqslant \frac{C_\varepsilon}{|t|^{3/2}} \int_{\mathbb{H}^2} |\phi(\Omega')| \frac{d\Omega'}{\widetilde{\mathtt{w}}_2^{1-\varepsilon}(r')}, \quad \forall t \neq 0, \quad \forall \phi \in L^1_{\mathrm{rad}}(\mathbb{H}^2),$$

where $r = d(0, \Omega)$ and $r' = d(0, \Omega')$.

Remark 3.4. For small time, this weighted estimate is worse than the one in $\mathbb{R}^2$ in terms of powers of t: $|t|^{-3/2}$ instead of $|t|^{-1}$. Formally, we could get a rate in $|t|^{-1-\varepsilon}$ by integration by parts in the s integral in (3.2), by considering derivatives of order $1/2-\varepsilon$. We shall not pursue this approach here, and content ourselves with Proposition 3.3.

Proof. We first prove

$$\left|\int_\rho^\infty \frac{se^{is^2/4t}}{\sqrt{\cosh s-\cosh\rho}}ds\right| \leqslant C\left(\frac{\rho}{\sinh\rho}\right)^{1/2}\sqrt{1+\rho}, \tag{3.3}$$

where C is independent of $\rho\geqslant 0$. Note that this estimate is analogous to the one given in [18]: for the heat operator, an additional Gaussian decay is available (replace $e^{is^2/4t}$ with $e^{-s^2/4t}$). The computation below shows that this extra decay is not necessary in order for (3.3) to be true. We have obviously

$$\left|\int_\rho^\infty \frac{se^{is^2/4t}}{\sqrt{\cosh s-\cosh\rho}}ds\right| \leqslant \int_\rho^\infty \frac{s}{\sqrt{\cosh s-\cosh\rho}}ds.$$

Using "trigonometry", we find:

$$\int_\rho^\infty \frac{s}{\sqrt{\cosh s-\cosh\rho}}ds = \int_\rho^\infty \frac{s}{\sqrt{2\sinh\left(\frac{s+\rho}{2}\right)\sinh\left(\frac{s-\rho}{2}\right)}}ds.$$

With the change of variable $y=s-\rho$, we estimate:

$$\int_0^\infty \frac{y+\rho}{\sqrt{\sinh\left(\rho+\frac{y}{2}\right)\sinh\left(\frac{y}{2}\right)}}dy = \int_0^\infty \frac{(y+2\rho)-\rho}{\sqrt{\sinh\left(\rho+\frac{y}{2}\right)\sinh\left(\frac{y}{2}\right)}}dy.$$

For the first term, we use the fact that

$$s\mapsto \frac{s}{\sinh s} \text{ is non-increasing,}$$

to have the estimate:

$$\begin{aligned}\int_0^\infty \frac{y+2\rho}{\sqrt{\sinh\left(\rho+\frac{y}{2}\right)\sinh\left(\frac{y}{2}\right)}}dy &\leqslant \sqrt{2\frac{\rho}{\sinh\rho}}\int_0^\infty \frac{\sqrt{y+2\rho}}{\sqrt{\sinh\left(\frac{y}{2}\right)}}dy\\ &\lesssim \sqrt{\frac{\rho}{\sinh\rho}}\left(\int_0^\infty \sqrt{\frac{y}{\sinh\left(\frac{y}{2}\right)}}dy+\sqrt{\rho}\int_0^\infty \frac{dy}{\sqrt{\sinh\left(\frac{y}{2}\right)}}\right)\\ &\lesssim \sqrt{\frac{\rho}{\sinh\rho}}\sqrt{1+\rho}.\end{aligned}$$

For the second term, we have:

$$\rho\int_0^\infty \frac{1}{\sqrt{\sinh\left(\rho+\frac{y}{2}\right)\sinh\left(\frac{y}{2}\right)}}dy \leqslant \frac{\rho}{\sqrt{\sinh\rho}}\int_0^\infty \frac{dy}{\sqrt{\sinh\left(\frac{y}{2}\right)}},$$

and (3.3) follows. To infer the proposition, we mimic the computations of [2], §5. From (3.2) and (3.3), we have:

$$\left|e^{it\Delta_{\mathbb{H}^2}}\phi(\Omega)\right| \lesssim \frac{1}{|t|^{3/2}}\int_{\mathbb{H}^2}|\phi(\Omega')|\left(\frac{\rho}{\sinh\rho}\right)^{1/2}\sqrt{1+\rho}\,d\Omega'.$$

Recall that ϕ is radial: with the usual abuse of notations,

$$\phi(\Omega')=\phi(\cosh r',\omega'\sinh r')=\phi(r').$$

Using hyperbolic coordinates,

$$\rho=d(\Omega,\Omega')=\cosh^{-1}\left(\cosh r\cosh r'-\sinh r\sinh r'\omega\cdot\omega'\right),$$

and we can write:

$$\left|e^{it\Delta_{\mathbb{H}^2}}\phi(\Omega)\right| \lesssim \frac{1}{|t|^{3/2}}\int_{\mathbb{H}^2}|\phi(\Omega')|\,K(r,r')d\Omega',$$

where the kernel K is given by:

$$K(r,r') = \int_{\mathbb{S}^1}\sqrt{f\left(\cosh^{-1}\left(\cosh r\cosh r' - \sinh r\sinh r'\omega\cdot\omega'\right)\right)}d\omega',$$

with

$$f(y) = \frac{y}{\sinh y}(1+y).$$

With $x = \omega\cdot\omega'$, we have:

$$K(r,r') = \int_{-1}^{1}\sqrt{f\left(\cosh^{-1}\left(\cosh r\cosh r' - x\sinh r\sinh r'\right)\right)}\frac{dx}{\sqrt{1-x^2}}.$$

The lemma follows from Hölder inequality. For $\lambda > 0$, we have

$$K(r,r') \leqslant C_\lambda\left(\int_{-1}^{1} f\left(\cosh^{-1}\left(\cosh r\cosh r' - x\sinh r\sinh r'\right)\right)^{1+\lambda}dx\right)^{1/(2+2\lambda)}.$$

With the change of variable

$$\cosh y = \cosh r\cosh r' - x\sinh r\sinh r',$$

this yields

$$\begin{aligned}
K(r,r') &\leqslant C_\lambda\left(\frac{1}{\sinh r\sinh r'}\int_{|r-r'|}^{r+r'} f(y)^{1+\lambda}\sinh y dy\right)^{1/(2+2\lambda)}\\
&\leqslant C'_\lambda\left(\frac{1}{\sinh r\sinh r'}\int_{|r-r'|}^{r+r'} y(1+y)dy\right)^{1/(2+2\lambda)}\\
&\leqslant C'_\lambda\left(\frac{(r+r')^3 - |r-r'|^3}{\sinh r\sinh r'}\right)^{1/(2+2\lambda)}\\
&\leqslant C''_\lambda\left(\frac{r'(1+r')r(1+r)}{\sinh r\sinh r'}\right)^{1/(2+2\lambda)}.
\end{aligned}$$

This completes the proof of the proposition, with $\frac{1}{1+\lambda} = 1-\varepsilon$. □

We find a weighted dispersive estimate and weighted Strichartz estimates which are similar to the ones for radial functions in $\mathbb{H}^3$. The difference is that we must replace $\mathtt{w}_3$ with $\widetilde{\mathtt{w}}_2^{1-\varepsilon}$. Even though the value $\varepsilon = 0$ is excluded, we can consider $d > 3$ arbitrarily close to 3 and repeat the argument in (3.1). On the other hand, since $\widetilde{\mathtt{w}}_2$ is bounded, we have the Strichartz estimates as in $\mathbb{R}^3$ for free: this yields the Corollary 3.2 with $n = 2$ and $d \geqslant 3$.

¿From Proposition 3.3 we have the dispersion

$$\left\|e^{it\Delta_{\mathbb{H}^2}}\phi\right\|_{L^\infty_{\mathrm{rad}}(\mathbb{H}^2)} \lesssim \frac{1}{|t|^{3/2}}\|\phi\|_{L^1_{\mathrm{rad}}(\mathbb{H}^2)},\quad \forall t\neq 0.$$

For small time, [2, Theorem 1.2] yields:

$$\left\|e^{it\Delta_{\mathbb{H}^2}}\phi\right\|_{L^\infty(\mathbb{H}^2)} \lesssim \frac{1}{|t|}\|\phi\|_{L^1(\mathbb{H}^2)},\quad \forall t\in[-1,1]\setminus\{0\}.$$

We infer the global dispersive estimate, for $2 \leqslant d \leqslant 3$ (not necessarily an integer):

$$\left\|e^{it\Delta_{\mathbb{H}^2}}\phi\right\|_{L^\infty_{\mathrm{rad}}(\mathbb{H}^2)} \leqslant \frac{C_d}{|t|^{d/2}}\|\phi\|_{L^1_{\mathrm{rad}}(\mathbb{H}^2)},\quad \forall t\neq 0.$$

We conclude:

Corollary 3.5. *Let $d \geqslant 2$. Then Strichartz estimates hold for d-admissible pairs and radial functions on $\mathbb{H}^2$:*
1. *For any d-admissible pair (p,q), there exists $C_q = C_q(d)$ such that*

$$\|U(\cdot)\phi\|_{L^p(\mathbb{R};L^q)} \leqslant C_q \|\phi\|_{L^2}, \quad \forall \phi \in L^2_{\mathrm{rad}}(\mathbb{H}^2).$$

2. *For any d-admissible pairs (p_1,q_1) and (p_2,q_2) and any interval I, there exists $C_{q_1,q_2} = C_{q_1,q_2}(d)$ independent of I such that*

$$\left\| \int_{I\cap\{s\leqslant t\}} U(t-s)F(s)ds \right\|_{L^{p_1}(I;L^{q_1})} \leqslant C_{q_1,q_2} \left\|F\right\|_{L^{p_2'}\left(I;L^{q_2'}\right)},$$

for every $F \in L^{p_2'}\left(I; L^{q_2'}_{\mathrm{rad}}(\mathbb{H}^2)\right)$.

4. Scattering for small data in $L^2_{\mathrm{rad}}(\mathbb{H}^n)$

The proof of Theorem 1.1 is straightforward in view of Section 2.1 and Corollaries 3.2 and 3.5.

Let $n \geqslant 2$ and $0 < \sigma \leqslant 2/n$. Set $d = 2/\sigma$: Corollaries 3.2 and 3.5 yield the same Strichartz estimates as in $\mathbb{R}^d$, provided that we work with radial functions. Simply notice that the proof of Proposition 2.3 relies only on functional analysis: Hölder inequality and Strichartz estimates. Theorem 1.1 follows: the statement is the analogue of Proposition 2.3, with $d = 2/\sigma$.

Remark 4.1. This result shows that there are no long range effects, at least in a neighborhood of the origin in $L^2_{\mathrm{rad}}(\mathbb{H}^n)$. This can be compared to [9, Proposition 1.1]. There, the following nonlinear Schrödinger equation is considered:

$$i\partial_t u + \frac{1}{2}\Delta_{\mathbb{R}^n} u = -\frac{x_1^2}{2}u + V(x_2,\dots,x_n)u + \kappa|u|^{2\sigma}u \quad ; \quad x \in \mathbb{R}^n,\ n \geqslant 1,$$

where V is any quadratic polynomial ($V \equiv 0$ if $n = 1$). It is proved that for $0 < \sigma \leqslant 2/n$, there is a small data scattering theory in L^2, just as in Theorem 1.1. This is because the repulsive potential $-x_1^2$ yields an exponential decay in time of the free solution. Here, this exponential decay in time is replaced by an exponential decay in space. The proof relies on the same idea though: we have Strichartz estimates that make it possible to pretend that we work in $\mathbb{R}^d$ with $d \geqslant n$.

5. Wave operators in $H^1_{\mathrm{rad}}(\mathbb{H}^n)$

The argument for the proof of Theorem 1.2 is similar. First, taking $d = n$, we cover the range $2/n \leqslant \sigma < 2/(n-2)$. To cover the range $0 < \sigma < 2/n$, we keep the value $d = 2/\sigma$: $d > n$. The proof of Proposition 2.4 uses the same arguments as the proof of Proposition 2.3, plus Sobolev embeddings. Therefore, we simply have to check that the step where Sobolev embeddings are used can be adapted.

Recall that we work in $\mathbb{H}^n$, and that we pretend that we work in $\mathbb{R}^d$, with $d = 2/\sigma \geqslant n$. In the proof of Proposition 2.4, we used the embedding:

$$H^1(\mathbb{R}^d) \subset L^{2\sigma+2}(\mathbb{R}^d).$$

Since we assume $\sigma < 2/(n-2)$, we have:

$$H^1(\mathbb{H}^n) \subset L^{2\sigma+2}(\mathbb{H}^n).$$

Therefore, we can argue as in Section 4: we can mimic the approach to prove Proposition 2.4, which is based on functional analysis.

6. Morawetz estimates in $\mathbb{H}^n$

In this section we prove some Morawetz type estimates for general solutions to the equation (1.1). Note that in this section the solutions are not necessarily radial. We start by stating these inequalities. Their proof will be a consequence of more general geometric set up already used by Hassell, Tao and Wunsch [24] while studying the same type of estimates for non-trapping asymptotically conic manifolds. This last paper in turn was based on the interaction Morawetz inequality introduced by Colliander et al. [16]. Define the operator $\tilde{H}$ acting on a function f defined on $\mathbb{H}^n \times \mathbb{H}^n$ as

$$\tilde{H} f(\Omega, \Omega') = -\Delta_{\mathbb{H}^n \times \mathbb{H}^n} f(\Omega, \Omega') = -\Delta_\Omega f(\Omega, \Omega') - \Delta_{\Omega'} f(\Omega, \Omega'),$$

and the distance function $d(\Omega, \Omega') = \mathrm{dist}_{\mathbb{H}^n}(\Omega, \Omega')$. Then we can state the following theorem

Theorem 6.1. *For any compact interval of time I and for any u solution to* (1.1)*,*

$$-\int_I \int_{\mathbb{H}^n \times \mathbb{H}^n} \tilde{H}^2(d(\Omega, \Omega'))|u(t,\Omega)|^2 |u(t,\Omega')|^2 \, d\Omega d\Omega' \, dt \leqslant C\|u\|^4_{L^\infty(I;H^1)}.$$

Since

$$-\tilde{H}^2\left(d(\Omega, \Omega')\right) = \begin{cases} \delta_{\Omega'}(\Omega) + \delta_\Omega(\Omega') & \text{if } n = 3, \\ 2\dfrac{\cosh(d(\Omega, \Omega'))}{\sinh^3(d(\Omega, \Omega'))} & \text{if } n > 3, \end{cases}$$

we also have the following corollary:

Corollary 6.2. *For any compact interval of time I and for any u solution to* (1.1) *we have:*

- *If $n = 3$:*

$$\int_I \int_{\mathbb{H}^3} |u(t,\Omega)|^4 \, d\Omega \, dt \leqslant C\|u\|^4_{L^\infty(I;H^1)}. \tag{6.1}$$

- *If $n > 3$:*

$$\int_I \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{\cosh(d(\Omega, \Omega'))}{\sinh^3(d(\Omega, \Omega'))} |u(t,\Omega')|^2 |u(t,\Omega)|^2 \, d\Omega \, d\Omega' \, dt \leqslant C\|u\|^4_{L^\infty(I;H^1)}.$$

Remark 6.3. Note that in the above results, we do not assume that u is radial.

Let M be a general Riemannian manifold with metric g. We denote by $< \cdot, \cdot >_g$ the product on the tangent space given by the metric g. We define the real inner product for functions on M

$$\langle u, v \rangle_M = \Re \int_M u(z)\overline{v}(z) \, dV_g(z). \tag{6.2}$$

We will often use the commutator $[A, B]$ among pseudo-differential operators A and B defined as $[A, B] = AB - BA$. We have the following lemma corresponding to [24, Lemma 2.1].

Lemma 6.4. *Let $a(x)$ be a real-valued tempered distribution on a manifold M, acting as a multiplier operator $(af)(x) = a(x)f(x)$ on Schwartz functions. Then we have the commutator identities, with $H = -\Delta_M$:*

$$i[H, a] = -i\langle \nabla a, \nabla \rangle_g + iHa = -i(\nabla^\alpha a)\nabla_\alpha + iH_a, \tag{6.3}$$

and the double commutator identity

$$-[H, [H, a]] = -\nabla_\beta \mathrm{Hess}(a)^{\alpha\beta}\nabla_\alpha - (H^2 a), \tag{6.4}$$

where $\mathrm{Hess}(a)^{\alpha\beta}$ *is the symmetric tensor*

$$\mathrm{Hess}(a)^{\alpha\beta} = (\nabla da)^{\alpha\beta} = g^{\alpha\gamma} g^{\beta\delta}(\partial_\gamma \partial_\delta + \Gamma^\rho_{\gamma\delta}\partial_\rho a).$$

We now assume that U is solution to

$$i\partial_t U + \Delta_M U = F. \tag{6.5}$$

Then it is easy to see that given a real pseudo-differential operator A on M we have

$$\begin{aligned} \partial_t \langle A\,U(t), U(t)\rangle_M &= \langle i[H,A]U(t), U(t)\rangle_M + \\ &\quad + \langle -iA\,F(t), U(t)\rangle_M + \langle iA\,U(t), F(t)\rangle_M. \end{aligned} \tag{6.6}$$

Next, given a real valued tempered distribution a and a function $U(t,x)$ we define

$$M_a(t) = \langle i[H,a]U(t), U(t)\rangle_M. \tag{6.7}$$

By using (6.3) and the definition (6.2) we recover the familiar form of the first order momentum

$$M_a(t) = \Im \int_M \langle \nabla a, \nabla U(t)\rangle_g \overline{U}(t) dV_g. \tag{6.8}$$

Now, by taking U solution to (6.5) and $A = i[H,a]$ in (6.6) and using (6.4), one gets

$$\begin{aligned} \frac{d}{dt} M_a(t) &= -\langle [H,[H,a]]U(t), U(t)\rangle_M + \\ &\quad + \langle [H,a]\,F(t), U(t)\rangle_M + \langle -[H,a]\,U(t), F(t)\rangle_M \\ &= -\langle H^2 a\,U(t), U(t)\rangle_M - \langle \nabla_\beta \mathrm{Hess}(a)^{\alpha\beta}\nabla_\alpha U(t), U(t)\rangle_M \\ &\quad + \langle [H,a]\,F(t), U(t)\rangle_M + \langle -[H,a]\,U(t), F(t)\rangle_M. \end{aligned} \tag{6.9}$$

Lemma 6.5. *If M is a Riemannian manifold with a nonpositive sectional curvature and if a is a distance function defined on M, that is $|\nabla a| = 1$, then for any smooth function ϕ,*

$$\langle \mathrm{Hess}(a)^{\alpha\beta}\nabla_\alpha \phi, \nabla_\beta \phi\rangle_g \geqslant 0. \tag{6.10}$$

Proof. This is a well-known result in Riemannian geometry. We refer the reader for example to Theorem 3.6 in [28]. □

Using (6.9) and (6.10) after an integration by parts in space variable, we obtain for all $T \geqslant 0$ the key inequality:

$$\begin{aligned} M_a(T) - M_a(0) \geqslant \int_0^T -\langle H^2 a\,U(t), U(t)\rangle_M + \\ + \int_0^T \langle [H,a]\,F(t), U(t)\rangle_M + \langle -[H,a]\,U(t), F(t)\rangle_M\, dt. \end{aligned} \tag{6.11}$$

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. Following again the argument in [24], we assume now that $M = \mathbb{H}^n \times \mathbb{H}^n$, with the usual metric $\tilde{g} = g \otimes g$. Assume also that u is a solution to the equation (1.1). It is easy to show that $U(t,\Omega,\Omega') := u(t,\Omega)\,u(t,\Omega')$ is solution to the equation

$$\begin{aligned} i\partial_t U(t,\Omega,\Omega') + \Delta_\Omega \oplus \Delta_{\Omega'} U(t,\Omega,\Omega') &= \left(|u|^{2\sigma}u\right)(t,\Omega)\,u(t,\Omega') + \\ &\quad + \left(|u|^{2\sigma}u\right)(t,\Omega')\,u(t,\Omega) \\ &=: F(t,\Omega,\Omega'). \end{aligned}$$

We now set $\tilde{H} = -\left(\Delta_\Omega \oplus \Delta_{\Omega'}\right)$ and $a(\Omega,\Omega') = \mathrm{dist}_{\mathbb{H}^n}(\Omega,\Omega')$. It is easy to see that this function a is a distance function with respect to the manifold $(M,\tilde{g})$. Also one

can check that this manifold has a nonpositive sectional curvature. Finally one can also prove, using (6.3) and the definition of the real inner product (6.2), that

$$\langle [\tilde{H}, a]\, F(t), U(t)\rangle_M + \langle -[\tilde{H}, a]\, U(t), F(t)\rangle_M = \tag{6.12}$$

$$= \langle -\langle \nabla a, \nabla F(t)\rangle_{\tilde{g}}, U(t)\rangle_M + \langle \langle \nabla a, \nabla U(t)\rangle_{\tilde{g}}, F(t)\rangle_M = \tag{6.13}$$

$$= 2\frac{\sigma}{\sigma+1}\int_M \Delta_\Omega a(\Omega,\Omega')\,|u|^{2\sigma+2}(t,\Omega)\,|u|^2(t,\Omega').$$

Since (§5.7 of [18])

$$\Delta_\Omega a(\Omega,\Omega') = (n-1)\operatorname{cotanh} d(\Omega,\Omega'),$$

we infer by (6.11) that

$$M_a(T) - M_a(0) \geqslant \int_0^T -\langle \tilde{H}^2 a U(t), U(t)\rangle_M\, dt. \tag{6.14}$$

Using (6.8) and the fact that a is a distance function we also have for any $t \in I = [0,T]$

$$|M_a(t)| \leqslant C\|u\|^4_{L^\infty(I;H^1)}. \tag{6.15}$$

The proof of the theorem now follows by combining (6.14) and (6.15). □

7. Asymptotic completeness in $H^1(\mathbb{H}^3)$

First, we note that the proof recalled in Section 2.4 can be mimicked on $\mathbb{H}^3$: the Morawetz estimate (2.4) was adapted to the hyperbolic case, (6.1). This proves Theorem 1.3 for $2/3 < \sigma < 2$. The reason why we do not have to assume that u is radial at this stage is that global in time Strichartz estimates are available on $\mathbb{H}^3$, from [2].

To decrease σ, we proceed with the same idea as before, and pretend that we work on $\mathbb{R}^d$, for $d > 3$. Fix $d > 3$: we first claim that (6.1) implies

$$\big\|u\big\|_{L^{d+1}\left(\mathbb{R};L^{\frac{2(d+1)}{d-1}}(\mathbb{H}^3)\right)} \lesssim \|u\|_{L^\infty(\mathbb{R};H^1)}. \tag{7.1}$$

Indeed, we noticed in Section 2.4 that for any $d \geqslant 3$, the pair $\left(d+1, \frac{2(d+1)}{d-1}\right)$ is 2-admissible. Interpolating between the pairs $(4,4)$ and $(\infty,2)$ yields the pair $\left(d+1, \frac{2(d+1)}{d-1}\right)$, hence (7.1). Having the analogue of (2.6), we can go on with the proof presented in Section 2.4, thanks to Corollary 3.2 for *radial* solutions. This completes the proof of Theorem 1.3 for any $0 < \sigma < 2$. Note also that if Corollary 3.2 holds for solutions that are not necessarily radial, then Theorem 1.3 will be true without the symmetry hypothesis.

The reason why we stated Theorem 1.3 in the case $n = 3$ only is the following. In Section 6, we have proved the analogue of (2.4) and (2.5) for the solutions to (1.1) (not necessarily radial). To go on with the proof of [34], we would need the analogue of (2.7) on hyperbolic space $\mathbb{H}^n$, $n \geqslant 4$. A paradifferential calculus on $\mathbb{H}^n$ would be welcome then, which we do not have at hand.

If the recent proof of Tao, Visan and Zhang [34] cannot be used on $\mathbb{H}^n$ for $n \geqslant 4$, one might want to use the original proof of Ginibre and Velo [23] (or [12]). As we recalled in Section 2.4, this proof uses dispersive estimates for the free Schrödinger group. Unfortunately, the dispersive estimates of the free Schrödinger group on $\mathbb{H}^n$ are only *local in time* as soon as $n \geqslant 4$ [2]. For this reason, even proving the same scattering results on $\mathbb{H}^n$ as on $\mathbb{R}^n$, $n \geqslant 4$, does not seem obvious at all.

Appendix A. On the notion of criticality

Consider the equation (1.1). The critical scaling in $\mathbb{R}^n$ is

$$s_c = \frac{n}{2} - \frac{1}{\sigma}\,.$$

At first glance, it is not clear whether the critical indices for (1.1) are the same as in the Euclidean case, since the linear part of the equation is not scale invariant.

Note that in the case of a positive curvature, the geometry changes the notion of criticality (see e.g. [6, 5], [1]), but always in the "same order": the positive curvature "creates" more instabilities.

However, it is established in [2] that there is local well-posedness in $H^s(\mathbb{H}^n)$ for (1.1) if $s > s_c$:

$$\text{subcritical in the Euclidean case} \Rightarrow \text{subcritical in the hyperbolic case.}$$

This stems from the fact that we have the same local Strichartz inequalities as in the Euclidean space. Moreover, in the focusing setting, blow-up may occur "as in the Euclidean case", thanks to a new virial identity, where the negative curvature of the hyperbolic space shows up.

On the other hand, some proofs of ill-posedness rely on highly concentrated initial data and solutions, so that the geometry is not relevant. To prove that the notion of criticality is the same in the Euclidean and in the hyperbolic case, it suffices to prove the analogue of the results of [15]. We resume the semi-classical approach of these results, as in [7, Appendix] and [10, Appendix B].

We give the proof for a nonlinearity which may be focusing: let $\kappa \in \{-1, +1\}$. Assume $s < s_c$, and for $0 < \lambda \leqslant 1$, consider:

$$i\partial_t u + \Delta_{\mathbb{H}^n} u = \kappa |u|^{2\sigma} u \quad ; \quad u_{|t=0} = \lambda^{-\frac{n}{2}+s} a_0\left(\frac{r}{\lambda}\right)\,, \tag{A.1}$$

where $a_0 \in C_0^\infty(\mathbb{R}_+;\mathbb{C})$ is such that

$$\operatorname{supp} a_0 \subset \{1 \leqslant r \leqslant 2\}\,.$$

Then $u(0,\cdot)$ is bounded in $H^s(\mathbb{H}^n)$, uniformly for $\lambda \in]0,1]$. Since $u(0,\cdot)$ is radially symmetric, so is $u(t,\cdot)$, and we write $u(t,r)$. Define ψ by:

$$\psi(t,r) = \lambda^{\frac{n}{2}-s} u\left(\lambda^{\frac{n\sigma}{2}+1-s\sigma} t, \lambda r\right)\,, \text{ or } u(t,r) = \lambda^{-\frac{n}{2}+s}\psi\left(\frac{t}{\lambda^{\frac{n\sigma}{2}+1-s\sigma}}, \frac{r}{\lambda}\right).$$

Denote $\varepsilon = \lambda^{\frac{n\sigma}{2}-1-s\sigma}$. Because we assumed $s < s_c$, λ and ε go to zero simultaneously. We have:

$$i\varepsilon\partial_t\psi^\varepsilon + \varepsilon^2\partial_r^2\psi^\varepsilon + (n-1)\varepsilon^2\lambda\operatorname{cotanh}(\lambda r)\partial_r\psi^\varepsilon = \kappa|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon \quad ; \quad \psi^\varepsilon(0,r) = a_0(r)\,.$$

The idea is that for very small times, the Laplacian is negligible. Introduce the approximate solution

$$i\varepsilon\partial_t\varphi^\varepsilon = \kappa|\varphi^\varepsilon|^{2\sigma}\varphi^\varepsilon \quad ; \quad \varphi^\varepsilon(0,r) = a_0(r)\,.$$

We have explicitly:

$$\varphi^\varepsilon(t,r) = a_0(r)e^{-i\kappa\frac{t}{\varepsilon}|a_0(r)|^2}.$$

In particular, the support of $\varphi^\varepsilon(t,\cdot)$ is the same as that of a_0.

Proposition A.1. *Fix $k > n/2$. Then we can find $c_0, c_1, \theta, C > 0$ independent of $\varepsilon \in]0,1]$ such that ψ^ε and φ^ε satisfy:*

$$\|\psi^\varepsilon - \varphi^\varepsilon\|_{L^\infty([0,c_0\varepsilon|\ln\varepsilon|^\theta];H^k)} \leqslant C\varepsilon|\ln\varepsilon|^{c_1}\,.$$

Proof. Denote $w^\varepsilon = \psi^\varepsilon - \varphi^\varepsilon$. It solves:

$$\begin{aligned} i\varepsilon\partial_t w^\varepsilon + \varepsilon^2\partial_r^2 w^\varepsilon + (n-1)\varepsilon^2\lambda\operatorname{cotanh}(\lambda r)\partial_r w^\varepsilon =& \kappa\left(F(\psi^\varepsilon) - F(\varphi^\varepsilon)\right) - \varepsilon^2\partial_r^2\varphi^\varepsilon \\ & - (n-1)\varepsilon^2\lambda\operatorname{cotanh}(\lambda r)\partial_r\varphi^\varepsilon, \end{aligned}$$

with $w^\varepsilon_{|t=0} = 0$, where we have set $F(z) = |z|^{2\sigma}z$. Introduce the vector-fields

$$H_j = \omega_j\partial_r\,.$$

They commute with the Laplacian $\Delta_{\mathbb{H}^n}$. Moreover, since w is radially symmetric,

$$\begin{aligned} H_j\big(\partial_r^2 + (n-1)\lambda\operatorname{cotanh}(\lambda r)\partial_r\big)w^\varepsilon =& \\ &= H_j\left(\partial_r^2 + (n-1)\lambda\operatorname{cotanh}(\lambda r)\partial_r + \frac{\lambda^2}{\sinh^2(\lambda r)}\Delta_{\mathbb{S}^{n-1}}\right)w^\varepsilon \\ &= \left(\partial_r^2 + (n-1)\lambda\operatorname{cotanh}(\lambda r)\partial_r + \frac{\lambda^2}{\sinh^2(\lambda r)}\Delta_{\mathbb{S}^{n-1}}\right)H_j w^\varepsilon\,. \end{aligned}$$

For $k \geqslant 0$, we apply $H_{j_1}\circ\ldots\circ H_{j_k}$ to the equation solved by w^ε. The usual L^2 estimate yields:

$$\begin{aligned} \|w^\varepsilon\|_{L^\infty([0,t];H^k)} \lesssim& \frac{1}{\varepsilon}\left\|F(w^\varepsilon+\varphi^\varepsilon) - F(\varphi^\varepsilon)\right\|_{L^1([0,t];H^k)} \\ &+ \varepsilon\|\partial_r^2\varphi^\varepsilon\|_{L^1([0,t];H^k)} + \varepsilon\|\lambda\operatorname{cotanh}(\lambda r)\partial_r\varphi^\varepsilon\|_{L^1([0,t];H^k)}. \end{aligned}$$

Since $\operatorname{supp}\varphi^\varepsilon(t,\cdot) \subset \{1 \leqslant r \leqslant 2\}$,

$$\|\lambda\operatorname{cotanh}(\lambda r)\partial_r\varphi^\varepsilon\|_{L^1([0,t];H^k)} \lesssim \|\partial_r\varphi^\varepsilon\|_{L^1([0,t];H^k)}\,,$$

and we infer:

$$\begin{aligned} \|w^\varepsilon\|_{L^\infty([0,t];H^k)} &\lesssim \frac{1}{\varepsilon}\left\|F(w^\varepsilon+\varphi^\varepsilon) - F(\varphi^\varepsilon)\right\|_{L^1([0,t];H^k)} + \varepsilon\|\varphi^\varepsilon\|_{L^1([0,t];H^{k+2})} \\ &\lesssim \frac{1}{\varepsilon}\left\|F(w^\varepsilon+\varphi^\varepsilon) - F(\varphi^\varepsilon)\right\|_{L^1([0,t];H^k)} + \varepsilon\int_0^t\left\langle\frac{s}{\varepsilon}\right\rangle^{k+2}ds. \end{aligned}$$

Since σ is an integer, the fundamental theorem of calculus yields, when $k > n/2$:

$$\begin{aligned} \left\|F(w^\varepsilon(t)+\varphi^\varepsilon(t)) - F(\varphi^\varepsilon(t))\right\|_{H^k} &\lesssim \left(\|w^\varepsilon(t)\|_{H^k}^{2\sigma} + \|\varphi^\varepsilon(t)\|_{H^k}^{2\sigma}\right)\|w^\varepsilon(t)\|_{H^k} \\ &\lesssim \left(\|w^\varepsilon(t)\|_{H^k}^{2\sigma} + \left\langle\frac{t}{\varepsilon}\right\rangle^{2\sigma k}\right)\|w^\varepsilon(t)\|_{H^k}. \end{aligned}$$

On any time interval where we have, say, $\|w^\varepsilon\|_{H^k} \leqslant 1$, we infer:

$$\|w^\varepsilon\|_{L^\infty([0,t];H^k)} \leqslant \frac{C_0}{\varepsilon}\int_0^t\left\langle\frac{s}{\varepsilon}\right\rangle^{2\sigma k}\|w^\varepsilon(s)\|_{H^k}ds + C_1\varepsilon\int_0^t\left\langle\frac{s}{\varepsilon}\right\rangle^{k+2}ds.$$

Gronwall lemma yields:

$$\begin{aligned} \|w^\varepsilon\|_{L^\infty([0,t];H^k)} &\lesssim \int_0^t \varepsilon\left\langle\frac{s}{\varepsilon}\right\rangle^{k+2}\exp\left(C_0\int_s^t\frac{1}{\varepsilon}\left\langle\frac{\tau}{\varepsilon}\right\rangle^{2\sigma k}d\tau\right)ds \\ &\lesssim \int_0^t \varepsilon\left\langle\frac{s}{\varepsilon}\right\rangle^{k+2}\exp\left(C'\frac{t}{\varepsilon}\left\langle\frac{t}{\varepsilon}\right\rangle^{2\sigma k}\right)ds \\ &\lesssim \varepsilon t\left\langle\frac{t}{\varepsilon}\right\rangle^{k+2}\exp\left(C'\frac{t}{\varepsilon}\left\langle\frac{t}{\varepsilon}\right\rangle^{2\sigma k}\right). \end{aligned}$$

For $t = c_0\varepsilon|\ln\varepsilon|^\theta$, we have:

$$\|w^\varepsilon\|_{L^\infty([0,c_0\varepsilon|\ln\varepsilon|^\theta];H^k)} \lesssim \varepsilon^2|\ln\varepsilon|^{(k+3)\theta}\exp\left(C'c_0^{1+2\sigma k}|\ln\varepsilon|^{(1+2\sigma k)\theta}\right).$$

Now if we take $\theta = (1+2\sigma k)^{-1}$ and c_0 sufficiently small, we have the estimate of the proposition. A continuity argument completes the proof, for ε sufficiently small. □

Corollary A.2. *Let $n \geqslant 2$, $\kappa \in \mathbb{R}\setminus\{0\}$ and $\sigma > 0$. For $s < \frac{n}{2} - \frac{1}{\sigma}$, (1.1) is not locally well-posed in $H^s(\mathbb{H}^n)$: for any $\delta > 0$, we can find families $(u_{01}^\varepsilon)_{0<\varepsilon\leqslant 1}$ and $(u_{02}^\varepsilon)_{0<\varepsilon\leqslant 1}$ of radially symmetric functions with*

$$u_{0j}^\varepsilon(r) \in C_0^\infty(\mathbb{R}_+) \quad ; \quad \|u_{01}^\varepsilon\|_{H^s}, \|u_{02}^\varepsilon\|_{H^s} \leqslant \delta\,, \quad \|u_{01}^\varepsilon - u_{02}^\varepsilon\|_{H^s} \to 0 \text{ as } \varepsilon \to 0\,,$$

such that if u_1^ε and u_2^ε denote the solutions to (1.1) with these initial data, there exist $0 < t^\varepsilon \to 0$, and $c > 0$ independent of $\varepsilon \in]0,1]$, such that

$$\left\|u_1^\varepsilon(t^\varepsilon) - u_2^\varepsilon(t^\varepsilon)\right\|_{H^s} \geqslant c.$$

Remark A.3. We could also prove the *norm inflation* phenomenon, as called in [15]. It is rather this result which is proven in [7, Appendix].

Proof. Let a_0 as above, and u_1 the solution to (A.1). Let

$$u_{02}^\varepsilon(r) = (1+\delta^\varepsilon)u_{01}^\varepsilon(r),$$

where $\delta^\varepsilon \to 0$ as $\varepsilon \to 0$. Denote

$$\psi_j^\varepsilon(t,r) = \lambda^{\frac{n}{2}-s} u_j\left(\lambda^{\frac{n\sigma}{2}+1-s\sigma}t, \lambda r\right).$$

From Proposition A.1, for $k > n/2$,

$$\left\|\psi_j^\varepsilon - \varphi_j^\varepsilon\right\|_{L^\infty([0,c_0\varepsilon|\ln\varepsilon|^\theta];H^k)} \to 0 \text{ as } \varepsilon \to 0\,.$$

Note that the constants c_0 and θ for ψ_2 can be taken uniform with respect to ε, since δ^ε is bounded. Now we have

$$\begin{aligned}
\|\varphi_1^\varepsilon(t) - \varphi_2^\varepsilon(t)\|_{\dot H^s} &= \left\|a_0 e^{i\kappa\frac{t}{\varepsilon}|a_0|^2} - (1+\delta^\varepsilon)\, a_0 e^{i\kappa\frac{t}{\varepsilon}(1+\delta^\varepsilon)^2|a_0|^2}\right\|_{\dot H^s} \\
&\underset{\varepsilon\to 0}{\sim} \left\|a_0\left(e^{i\kappa\frac{t}{\varepsilon}|a_0|^2} - e^{i\kappa\frac{t}{\varepsilon}(1+\delta^\varepsilon)^2|a_0|^2}\right)\right\|_{\dot H^s} \\
&\underset{\varepsilon\to 0}{\sim} \left\|a_0 e^{i\kappa\frac{t}{\varepsilon}|a_0|^2}\left(1 - e^{i\kappa\frac{t}{\varepsilon}\left(2\delta^\varepsilon + (\delta^\varepsilon)^2\right)|a_0|^2}\right)\right\|_{\dot H^s}.
\end{aligned}$$

With $t^\varepsilon = c_0\varepsilon|\ln\varepsilon|^\theta$, we find:

$$\|\varphi_1^\varepsilon(t^\varepsilon) - \varphi_2^\varepsilon(t^\varepsilon)\|_{\dot H^s} \underset{\varepsilon\to 0}{\sim} \left\|a_0\left(1 - e^{i\kappa 2\delta^\varepsilon|\ln\varepsilon|^\theta|a_0|^2}\right)\right\|_{\dot H^s}.$$

If we take $\delta^\varepsilon = |\ln\varepsilon|^{-\theta}$, then the corollary follows, since ψ and u have the same $\dot H^s$ norms. □

Appendix B. Asymptotic behavior of free solutions in $L^2_{\mathrm{rad}}(\mathbb{H}^3)$

We stick to the case $n = 3$, because the Harish-Chandra coefficient is simpler. In this appendix, all irrelevant "physical"/geometrical constants are denoted by c.

The Fourier transform in the radially symmetric case is defined as:

$$\widehat f(\lambda) = \frac{c}{\lambda}\int_0^\infty \sin(\lambda r) f(r) \sinh r dr\,. \tag{B.1}$$

Plancherel formula reads:

$$\int_0^\infty |f(r)|^2 \sinh^2 r dr = c\int_0^\infty \left|\widehat f(\lambda)\right|^2 \lambda^2 d\lambda\,. \tag{B.2}$$

Lemma B.1. *For $u_0 \in L^2(\mathbb{H}^3)$, radially symmetric, denote:*

$$u_{\rm asym}(t,r) = c\frac{e^{-it+i\frac{r^2}{4t}}}{t^{3/2}}\frac{r}{\sinh r}\widehat{u_0}\left(\frac{r}{2t}\right).$$

Then we have:

$$\left\|e^{it\Delta_{\mathbb{H}^3}}u_0 - u_{\rm asym}(t)\right\|_{L^2(\mathbb{H}^3)} \underset{t\to+\infty}{\longrightarrow} 0\,.$$

Proof. First, we show an explicit representation for radial solutions:

$$e^{it\Delta_{\mathbb{H}^3}}u_0(r) = c\frac{e^{-it+i\frac{r^2}{4t}}}{t^{3/2}}\int_0^\infty \frac{e^{i\frac{\rho^2}{4t}}}{\sinh r\sinh\rho}t\sin\left(\frac{r\rho}{2t}\right)u_0(\rho)\sinh^2\rho d\rho\,. \tag{B.3}$$

To prove this we recall the representation of the free solution for $n=3$:

$$u(t,\Omega) = \frac{c}{|t|^{3/2}}e^{-it}\int_{\mathbb{H}^3} u_0(\Omega')\,e^{i\frac{d^2(\Omega,\Omega')}{4t}}\frac{d(\Omega,\Omega')}{\sinh d(\Omega,\Omega')}\,d\Omega'\,, \tag{B.4}$$

where $d(\Omega,\Omega')$ is the hyperbolic distance between Ω and Ω' (see [2]). From (B.4), one gets that for radial initial data, the free solution writes

$$e^{it\Delta_{\mathbb{H}^3}}u_0(\cosh r,\sinh r\omega) = c\frac{e^{-it}}{t^{3/2}}\int_0^\infty\int_{\mathbb{S}^2} K(t,r,\rho,\omega\cdot\omega')d\omega' u_0(\rho)\sinh^2\rho d\rho\,,$$

with

$$K(t,r,\rho,\omega\cdot\omega') = e^{i\frac{z^2}{4t}}\frac{z}{\sinh z}\Big|_{z=\cosh^{-1}(\cosh r\cosh\rho-\sinh r\sinh\rho\,\omega\cdot\omega')}.$$

Let us consider an isometry $T\in SO(3)$ such that $T(1,0,0)=\omega$. Then a given $\omega'\in\mathbb{R}^3\cap\mathbb{S}^2$ defines a unique pair $(\alpha,\theta)\in(0,\pi)\times\mathbb{R}^2\cap\mathbb{S}^1$, related by the formula:

$$\omega' = T(\cos\alpha,\sin\alpha\,\theta).$$

Moreover, $\omega\cdot\omega'=\cos\alpha$ and $d\omega'=\sin\alpha\,d\alpha\,d\theta$. With this change of variable,

$$\int_{\mathbb{S}^2} K(t,r,\rho,\omega\cdot\omega')d\omega' = c\int_0^\pi K(t,r,\rho,\cos\alpha)\sin\alpha\,d\alpha\,.$$

Next, we change $\cos\alpha$ into x, so

$$\int_{\mathbb{S}^2} K(t,r,\rho,\omega\cdot\omega')d\omega' = c\int_{-1}^1 K(t,r,\rho,x)\,dx\,.$$

Finally, we do a last change of variable,

$$\cosh r\cosh\rho-\sinh r\sinh\rho\,x = \cosh y\,,$$

and get

$$\int_{\mathbb{S}^2} K(t,r,\rho,\omega\cdot\omega')d\omega' = \frac{c}{\sinh r\,\sinh\rho}\int_{|r-\rho|}^{r+\rho} e^{i\frac{y^2}{2t}}y\,dy.$$

By simple integration formula (B.3) follows. We infer:

$$e^{it\Delta_{\mathbb{H}^3}}u_0(r) - u_{\rm asym}(t,r) = \\ = c\frac{e^{-it+i\frac{r^2}{4t}}}{t^{3/2}}\frac{r}{\sinh r}\int_0^\infty\left(e^{i\frac{\rho^2}{4t}}-1\right)\frac{2t}{r}\sin\left(\frac{r\rho}{2t}\right)u_0(\rho)\sinh\rho d\rho\,.$$

Therefore:

$$\left\|e^{it\Delta_{\mathbb{H}^3}}u_0 - u_{\rm asym}(t)\right\|_{L^2}^2 = \\ = \frac{c}{t^3}\int_0^\infty r^2\left|\int_0^\infty\left(e^{i\frac{\rho^2}{4t}}-1\right)\frac{t}{r}\sin\left(\frac{r\rho}{2t}\right)u_0(\rho)\sinh\rho d\rho\right|^2 dr \\ = c\int_0^\infty r^2\left|\int_0^\infty\left(e^{i\frac{\rho^2}{4t}}-1\right)\frac{1}{r}\sin\left(r\rho\right)u_0(\rho)\sinh\rho d\rho\right|^2 dr\,.$$

Use Plancherel formula (B.2):

$$\left\| e^{it\Delta_{\mathbb{H}^3}} u_0 - u_{\mathrm{asym}}(t) \right\|_{L^2}^2 = c \int_0^\infty \left| \left(e^{i\frac{\rho^2}{4t}} - 1 \right) u_0(\rho) \right|^2 \sinh^2 \rho d\rho \,.$$

Now we conclude with a density argument, thanks to the estimate (for instance)

$$|e^{i\theta} - 1| \lesssim |\theta| \,.$$

□

Remark B.2. This asymptotic behavior is essentially the same as in the Euclidean case, up to a new oscillation in time, and the weight $\frac{r}{\sinh r}$. This can also be seen as follows: in the proof, we have used the identity

$$e^{it\Delta_{\mathbb{H}^3}} u_0(r) = W \mathcal{M}_t D_t \mathcal{F} \mathcal{M}_t u_0 \,,$$

where $\mathcal{F}$ is the Fourier transform, $\mathcal{M}_t(r)$ is the multiplication by $e^{ir^2/(4t)}$, D_t is the dilation at scale $1/(2t)$ with L^2 scaling:

$$D_t \varphi(r) = \frac{1}{t^{3/2}} \varphi\left(\frac{r}{2t}\right),$$

and W is the weight $W = e^{-it}\frac{r}{\sinh r}$. In the Euclidean case, we have the same formula, with the only change $W = \mathrm{Id}$, and the usual asymptotics is:

$$e^{it\Delta_{\mathbb{R}^n}} = \mathcal{M}_t D_t \mathcal{F} \mathcal{M}_t \underset{t\to+\infty}{\sim} \mathcal{M}_t D_t \mathcal{F} \,.$$

The proof of the L^2 asymptotics is the same as above.

Appendix C. A Galilean operator?

In the Euclidean case, a nice object for scattering theory (and also blow-up, see e.g. [37]) is the Galilean operator

$$J_{\mathrm{eucl}}(t) = x + 2it\nabla_x = 2ite^{i\frac{|x|^2}{4t}} \nabla_x \left(e^{-i\frac{|x|^2}{4t}} \cdot \right) = e^{it\Delta_{\mathbb{R}^n}} x e^{-it\Delta_{\mathbb{R}^n}} \,.$$

Recall that for any *radial* function ϕ on $\mathbb{R}^3$, we have:

$$\Delta_{\mathbb{R}^3} \left(\frac{\phi(r)}{r} \right) = \frac{1}{r} \partial_r^2 \phi.$$

A similar identity is available for radial functions on $\mathbb{H}^3$:

$$\Delta_{\mathbb{H}^3} \left(\frac{\phi(r)}{\sinh r} \right) = \frac{1}{\sinh r} \partial_r^2 \phi.$$

Using the Galilean operator on the (half-)line, it is then natural to introduce the following operator, acting on radial functions on $\mathbb{H}^3$:

$$J(t) = \frac{2it}{\sinh r} e^{i\frac{r^2}{4t}} \partial_r \left(e^{-i\frac{r^2}{4t}} \sinh r \cdot \right) = r + 2it\partial_r + 2it \operatorname{cotanh} r \,. \tag{C.1}$$

Now we have

$$[J, i\partial_t + \Delta_{\mathbb{H}^3}] \, u(t,r) = 0 \,,$$

provided that u is radial. Note that as in the Euclidean case, J is an Heisenberg observable (see e.g. [31]):

$$J(t) = e^{it\Delta_{\mathbb{H}^3}} r e^{-it\Delta_{\mathbb{H}^3}} \,. \tag{C.2}$$

We already saw that in the radial framework, J commutes with the Schrödinger operator. Proceeding as in [11, Lemma 6.2], we find:

Lemma C.1. *Let $n = 3$. For every $q \in [2, 6[$, there exists c_q such that*

$$\left\| \mathtt{w}_3^{1-\frac{2}{q}} \phi \right\|_{L^q} \leqslant \frac{c_q}{|t|^{\delta(q)}} \|\phi\|_{L^2}^{1-\delta(q)} \|J(t)\phi\|_{L^2}^{\delta(q)}, \ \forall t \neq 0, \quad \textit{for every } \text{radial } \textit{function } \phi,$$

where we have denoted $\mathtt{w}_3(r) = \dfrac{\sinh r}{r}$ *and* $\delta(q) = 3\left(\dfrac{1}{2} - \dfrac{1}{q}\right) \in [0, 1]$.

It is important to understand how J acts on nonlinear terms. Let F be a C^1 function such that $F(z) = G(|z|^2)z$ (the usual gauge invariance). We compute:

$$J(t)F(u) = \partial_z F(u) J(t)u - \partial_{\bar z} F(u)\overline{J(t)u} + it \operatorname{cotanh} r F(u). \tag{C.3}$$

Forgetting the last term, we would have the same expression as in $\mathbb{R}^n$, and J would act on such nonlinearities like a derivative. Unfortunately, this last term cumulates two features: extra linear growth in time, and singularity as $r \to 0$.

As a matter of fact, the above drawback is also present in the radial Euclidean case. There, it can be removed by using J_{eucl}, even in a radial framework (see e.g. [4]). However, the analogue for J_{eucl} in hyperbolic space (not only in the radial case) may just not exist... Consider the Euclidean case in $\mathbb{R}^3$. We have

$$\Delta_{\mathbb{R}^3} = \partial_r^2 + \frac{2}{r}\,\partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^2}\,.$$

For a radial function $u(t, r)$, we have the commutation relation

$$[x_j + 2it\partial_j, i\partial_t + \Delta_{\mathbb{R}^3}]\, u(t, r) = 0,$$

where $(x_j + 2it\partial_j)u(t, r) = (r\omega_j + 2it\omega_j\partial_r)u(t, r)$. The factor ω_j is crucial: in general,

$$[r + 2it\partial_r, i\partial_t + \Delta_{\mathbb{R}^3}]\, u(t, r) \neq 0.$$

On the other hand,

$$\left[r + 2it\partial_r + 2i\frac{t}{r}, i\partial_t + \Delta_{\mathbb{R}^3}\right] = 0.$$

The above operator can also be written as

$$\mathbf{J}(t) = r + 2it\partial_r + 2i\frac{t}{r} = \frac{2it}{r} e^{i\frac{r^2}{4t}} \partial_r \left(e^{-i\frac{r^2}{4t}} r\,\cdot\right)\,.$$

When acting on gauge invariant nonlinearities, it is not like a derivative:

$$\mathbf{J}(t)F(u) = \partial_z F(u)\mathbf{J}(t)u - \partial_{\bar z} F(u)\overline{\mathbf{J}(t)u} + \frac{it}{r}F(u).$$

The last factor has the same drawback as above. Note that even in the radial case, one uses J_{eucl} and not $\mathbf{J}$ (see e.g. [4]). So we may try to extend J to a non-radial framework. Yet, seeking J_{hyper} of the form

$$J_{\text{hyper}} = r\omega_j + 2it\omega_j\partial_r + 2ith(r) = \frac{2it}{f(r)} e^{i\frac{r^2}{4t}} \omega_j \partial_r \left(e^{-i\frac{r^2}{4t}} f(r)\,\cdot\right) \quad \left(h = \frac{f'}{f}\right),$$

and writing

$$[J_{\text{hyper}}, i\partial_t + \Delta_{\mathbb{H}^3}]\, u(t, r) = 0,$$

yields incompatible conditions.

Using the operator J and Lemma C.1, one can prove the following scattering result though. For $T > 0$ possibly large, define:

$$\begin{aligned} Y_T = \{u, Ju \in C(]-\infty, -T]; L^2),\ &\text{with } \mathtt{w}_3^{2/3} u, \mathtt{w}_3^{2/3} Ju \in L^2(]-\infty, -T]; L^6)\ ; \\ &\|u\|_{L^\infty(]-\infty,-T;L^2)} \leqslant 2\|u_-\|_{L^2},\ \|\mathtt{w}_3^{2/3} u\|_{L^2(\mathbb{R};L^6)} \leqslant 2C_6\|u_-\|_{L^2}, \\ &\|Ju\|_{L^\infty(]-\infty,-T;L^2)} \leqslant 2\|ru_-\|_{L^2},\ \|\mathtt{w}_3^{2/3} Ju\|_{L^2(\mathbb{R};L^6)} \leqslant 2C_6\|ru_-\|_{L^2}\}, \end{aligned}$$

where C_6 is given by Proposition 3.1 when $n = 3$.

Proposition C.2. *Let $n = 3$, $t_0 = -\infty$ and $1/4 < \sigma < 1$. For every* radial *function $\varphi = u_- \in L^2(\mathbb{H}^3)$ with $ru_- \in L^2(\mathbb{H}^3)$, there exists $T = T(\sigma, \|u_-\|_{L^2}, \|ru_-\|_{L^2})$ such that* (1.4) *has a unique solution in Y_T.*

Remark C.3. The condition $1/4 < \sigma < 1$ looks rather strange at first glance. It appears because of the singular term that shows up when J acts on the nonlinearity, as discussed above. Without this term, we could virtually cover the range $0 < \sigma < 2$. Note however that we can go below $\sigma = 1/3$, thus showing the absence of (the usual) long range effects.

Proof. We want to show that the map

$$\Phi(u)(t) := U(t)u_- - i\int_{-\infty}^{t} U(t-s)|u|^{2\sigma}u(s)ds$$

has a fixed point in Y_T for T sufficiently large. Let (q,p) be an 3-admissible pair to be chosen later. Proposition 3.1 yields:

$$\left\|\mathtt{w}_3^{2/3}\Phi(u)\right\|_{L^2_T L^6} \leqslant C_6\|u_-\|_{L^2} + C_{6,p}\left\|\mathtt{w}_3^{1-2/p'}|u|^{2\sigma}u\right\|_{L^{q'}_T L^{p'}},$$

where we denote from now on: $L^a_T L^b := L^a(]-\infty,-T]; L^b(\mathbb{H}^3))$. Introduce indices such that:

$$\text{(C.4)}\qquad \begin{cases} \dfrac{1}{q'} = \dfrac{1}{q} + \dfrac{2\sigma}{s} + \dfrac{1}{\theta}, \\ \dfrac{1}{p'} = \dfrac{1}{p} + \dfrac{2\sigma}{k}, \end{cases} \qquad \text{with } s \in]2,6[.$$

Hölder's inequality then yields:

$$\left\|\mathtt{w}_3^{1-2/q'}|u|^{2\sigma}u\right\|_{L^{p'}_T L^{q'}} \leqslant \left\|\mathtt{w}_3^{1-2/q}u\right\|_{L^p_T L^q}\left\|\mathtt{w}_3^{1-2/s}u\right\|^{2\sigma}_{L^k_T L^s}\left\|\mathtt{w}_3^{4/q-2+2\sigma(2/s-1)}\right\|_{L^\theta}.$$

The last term is finite provided that:

$$\text{(C.5)}\qquad 2 - \frac{4}{q} + 2\sigma\left(1-\frac{2}{s}\right) > \frac{2}{\theta}.$$

In view of (C.4), this is equivalent to $\sigma > 0$.

Now we show that (C.4) can be achieved for a 3-admissible pair (p,q). Letting $(p,q) = (2,6)$, we find $k = \infty$ and (C.5) is satisfied for any $\sigma > 0$. Then $s \in]2,6[$ provided that $\sigma < 2$. Since these algebraic conditions are open, they still hold if we take $q = 6 - \varepsilon$ for $\varepsilon > 0$ sufficiently small, and p such that (p,q) is 3-admissible:

$$\left\|\mathtt{w}_3^{1-2/q'}|u|^{2\sigma}u\right\|_{L^{p'}_T L^{q'}} \leqslant C\left\|\mathtt{w}_3^{1-2/q}u\right\|_{L^p_T L^q}\left\|\mathtt{w}_3^{1-2/s}u\right\|^{2\sigma}_{L^k_T L^s}.$$

Note also that now, $p > 2$, hence $k < \infty$. By interpolation, the first factor of the right hand side is controlled by:

$$\left\|\mathtt{w}_3^{1-2/q}u\right\|_{L^p_T L^q} \leqslant \|u\|^{1-\delta(q)}_{L^\infty_T L^2}\left\|\mathtt{w}_3^{2/3}u\right\|^{\delta(q)}_{L^2_T L^6}.$$

¿From Lemma C.1, we have, for $t \leqslant -T$:

$$\left\|\mathtt{w}_3^{1-2/s}u(t)\right\|_{L^s} \lesssim \frac{1}{|t|^{\delta(s)}}\|u\|^{1-\delta(s)}_{L^\infty_T L^2}\|Ju\|^{\delta(s)}_{L^\infty_T L^2}.$$

We check that (C.4) implies

$$\begin{aligned} k\delta(s) &= 2 + \frac{k}{\sigma}\left(-1 + \frac{3\sigma}{2} + \frac{3}{2}\left(1-\frac{2}{q}\right) - \frac{3\sigma}{s}\right) \\ &> 2 + \frac{k}{\sigma}\left(\frac{3\sigma}{2} - \frac{3\sigma}{s}\right) > 2. \end{aligned}$$

Therefore,

$$\left\|\mathtt{w}_3^{1-2/s}u\right\|_{L^k_TL^s}^{2\sigma}\lesssim T^{-2\sigma/k}\left(\|u\|_{L^\infty_TL^2}+\|Ju\|_{L^\infty_TL^2}\right)^{2\sigma}.$$

Then choosing T sufficiently large, we see that for $u\in Y_T$,

$$\left\|\mathtt{w}_3^{2/3}\Phi(u)\right\|_{L^2_TL^6}\leqslant 2C_6\|u_-\|_{L^2}.$$

The similar estimate for $\|\Phi(u)\|_{L^\infty_TL^2}$ proceeds along the same lines.

To estimate $J\Phi(u)$, we find:

$$\begin{aligned}\left\|\mathtt{w}_3^{2/3}J\Phi(u)\right\|_{L^2_TL^6}&\leqslant C_6\|ru_-\|_{L^2}+C\left\|\mathtt{w}_3^{1-2/q'}|u|^{2\sigma}Ju\right\|_{L^{p'}_TL^{q'}}\\&\quad+C\left\|\mathtt{w}_3^{1-2/q_1'}t\operatorname{cotanh}r|u|^{2\sigma+1}\right\|_{L^{p_1'}_TL^{q_1'}},\end{aligned}$$

with the same admissible pair (p,q) as before, and where (p_1,q_1) is a possibly different admissible pair. For the second term of the right hand side, we proceed as before, to find:

$$\left\|\mathtt{w}_3^{1-2/q'}|u|^{2\sigma}Ju\right\|_{L^{p'}_TL^{q'}}\leqslant C\left\|\mathtt{w}_3^{1-2/q}Ju\right\|_{L^p_TL^q}\left\|\mathtt{w}_3^{1-2/s}u\right\|_{L^k_TL^s}^{2\sigma}.$$

We are left with the next term, involving $\mathtt{w}_3^{1-2/q'}t\operatorname{cotanh}r|u|^{2\sigma+1}$. Introduce the condition

$$\frac{1}{q_1'}=\frac{1}{q_1}+\frac{2\sigma}{s_1}+\frac{1}{\theta_1}\cdot\tag{C.6}$$

When it is satisfied for $s_1,\theta_1\geqslant 1$, we have:

$$\begin{aligned}\left\|\mathtt{w}_3^{1-2/q_1'}\operatorname{cotanh}r|u|^{2\sigma+1}\right\|_{L^{q_1'}}&\leqslant\left\|\mathtt{w}_3^{1-2/q_1}u\right\|_{L^{q_1}}\left\|\mathtt{w}_3^{1-2/s_1}u\right\|_{L^{s_1}}^{2\sigma}\times\\&\quad\times\left\|\mathtt{w}_3^{4/q_1-2+2\sigma(2/s_1-1)}\operatorname{cotanh}r\right\|_{L^{\theta_1}}.\end{aligned}$$

For the last term to be finite, a new condition appears, for the integral to converge near $r=0$ (for $r\to\infty$, nothing is changed):

$$1\leqslant\theta_1<3.\tag{C.7}$$

For $q_1,s_1\in]2,6[$, Lemma C.1 then yields:

$$\left\|\mathtt{w}_3^{1-2/q_1'}\operatorname{cotanh}r|u|^{2\sigma+1}\right\|_{L^{q_1'}}\lesssim\frac{1}{|t|^{\delta(q_1)+2\sigma\delta(s_1)}}\left(\|u\|_{L^\infty_TL^2}+\|Ju\|_{L^\infty_TL^2}\right)^{2\sigma+1}.$$

The right hand side multiplied by t is in $L^{p_1'}(]-\infty,-T])$ as soon as:

$$\delta(q_1)+2\sigma\delta(s_1)-1>\frac{1}{p_1'}\cdot\tag{C.8}$$

So we are left with the following situation: if we can meet (C.6), (C.7) and (C.8) with (p_1,q_1) admissible and $q_1,s_1\in]2,6[$, then choosing T sufficiently large, Φ maps Y_T to itself.

The line of reasoning is the same as above: if we pick $q_1=s_1=6$, then (C.6) and (C.7) imply $\sigma<1$, and (C.8) yields $\sigma>1/4$. Conversely, if $1/4<\sigma<1$, then taking $(p_1,q_1)=(2,6)$ and $s_1=6$, θ_1 given by (C.6) satisfies (C.7), and (C.8) holds. By continuity, all the conditions required are satisfied if we take $q_1=6-\varepsilon_1$, with $\varepsilon_1>0$ sufficiently small.

Up to increasing T, Φ is a contraction on Y_T, and the Proposition C.2 follows. □

We finally notice that an analogue to the pseudo-conformal conservation law [22] is available:

$$\begin{aligned} &\frac{d}{dt}\left(\|J(t)u\|_{L^2}^2+\frac{4t^2}{\sigma+1}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}\right)= \\ &=\frac{4t}{\sigma+1}\int_0^\infty (2-\sigma-2\sigma r\operatorname{cotanh} r)\,|u|^{2\sigma+2}(t,r)\sinh^2 r dr. \end{aligned} \tag{C.9}$$

This evolution law should make it possible to establish some asymptotic completeness results in weighted Sobolev spaces, but we leave out the discussion here.

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