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## Multi-step limit cycle generation for Rabbit's walking based on a nonlinear low dimensional predictive control scheme

Ahmed Chemori \*, Mazen Alamir

*Laboratoire d'Automatique de Grenoble, UMR 5528, BP46, Domaine Universitaire, 38400 Saint Martin d'Hères, France*

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### 8 Abstract

9 In this paper, a new nonlinear predictive control scheme is proposed for a five-link planar under-actuated biped walking robot. The  
10 basic feature in the proposed strategy is to use on-line optimization to update the tracked trajectories in the completely controlled vari-  
11 ables (actuated coordinates) in order to enhance the behavior and the stability of the remaining indirectly controlled ones (unactuated  
12 coordinates). The stability issue is discussed using the Poincaré's section tool leading to a computable criterion that enables the stability  
13 of the overall scheme to be investigated as well as the computation of a candidate region of attraction. The whole framework is illustrated  
14 through simulation case-studies. To attest the efficiency of the proposed scheme, robustness against model uncertainties and ground  
15 irregularities are investigated by simulation studies.

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17 *Keywords:* Biped robots; Dynamic walking; Multi-step limit cycle; Nonlinear optimization; Nonlinear predictive control; Orbital stability

### 18 Résumé

19 Dans cet article une nouvelle approche de commande prédictive non linéaire est proposée pour un robot marcheur bipède à cinq seg-  
20 ments sous actionné. La caractéristique principale dans la stratégie proposée est d'utiliser l'optimisation en-ligne pour mettre à jour les  
21 trajectoires à poursuivre sur les variables complètement commandables (coordonnées actionnées) dans le but d'améliorer le comporte-  
22 ment et la stabilité des variables indirectement commandées (coordonnées non actionnées). La stabilité est analysée par un outil graphique  
23 basé sur la section de Poincaré. Ceci permet, en plus de l'analyse la stabilité du système en boucle fermée, d'estimer la région d'attraction.  
24 L'approche proposée est illustrée à travers différents scénarios de simulations. La robustesse, quant à elle, est analysée par rapport à des  
25 incertitudes dans le modèle du robot, et des irrégularités dans le sol.

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27 *Mots-clés:* Robots bipèdes; Marche dynamique; Cycle limite d'ordre multiple; Optimisation non linéaire; Commande prédictive non linéaire; Stabilité  
28 orbitale

### 30 1. Introduction

31 In recent years, the robotics community has shown  
32 increasing interest in the area of legged walking robots

[29,2]. An excellent database of climbing and walking 33  
robots built all over the world can be found in [2]. One 34  
of the serious reasons for exploring the use of legged robots 35  
is the mobility [23], there is a need for vehicles that can travel 36  
in difficult terrains, where existing wheeled vehicles cannot 37  
go, since wheels excel on prepared surfaces such as rails 38  
and roads, but they perform poorly on rough terrains. 39  
Moreover, walking robots could co-exist with their 40

\* Corresponding author.

*E-mail addresses:* [Ahmed.Chemori@inpg.fr](mailto:Ahmed.Chemori@inpg.fr) (A. Chemori), [Mazen.Alamir@inpg.fr](mailto:Mazen.Alamir@inpg.fr) (M. Alamir).

41 creators without any costly modification to the environ-  
42 ment created for humans.

43 In walking locomotion [23], two gaits could be under-  
44 lined. *Static walking* which refers to a system which stays  
45 balanced by always keeping the center of mass (c.o.m.) of  
46 the system vertically projected over the polygon of support  
47 formed by feet. On the contrary, *dynamic walking*  
48 [24,21,13,22] is not constrained in such a manner, therefore  
49 the c.o.m. may leave the support polygon for periods of  
50 time. Biped robots [23,34,24] have high mobility that  
51 allows them to achieve *dynamic walking*, consequently high  
52 speeds could be reached due to the horizontal acceleration.

53 Currently, many research groups in the world are work-  
54 ing on biped robots, either on optimization of leg and foot  
55 trajectory, stable walking control, or hardware design. The  
56 main thrust of current research on biped control includes  
57 many proposed control approaches, such as intuitive con-  
58 trol [28], intelligent learning control [19], neural network  
59 control [18], passivity based control [33], sliding control  
60 [6], impedance control [25], optimal control [14], computed  
61 torque control [5], and tracking control [31,30].

62 From a control viewpoint, the major academic interest  
63 of bipeds comes from (1) their hybrid nature resulting from  
64 the unavoidable impacts [3] with the ground which can pro-  
65 duce discontinuities (jumps) in the generalized velocities.  
66 (2) Another interesting point is under-actuation, indeed  
67 biped walking robots may be under-actuated (case of Rab-  
68 bit) that is the robot has fewer number of actuators than  
69 the number of degrees of freedom. The way this under-  
70 actuation is handled may be used to give a particularly  
71 clear insight into the set of solutions proposed so far within  
72 the academic control community.

73 1. One way to overcome the under-actuation related-diffi-  
74 culty is to define virtual controls [8]. Typically, reference  
75 trajectories are defined on the whole state including indi-  
76 rectly controlled sub-states. These trajectories are depen-  
77 dent on some parameter vector  $p$  that can be either a  
78 virtual time, a remaining free polynomial coefficient or  
79 both. Generally, the second time-derivative of the param-  
80 eter  $p$  becomes a virtual additional control enabling  
81 under-actuation to be conceptually overcome. Clearly,  
82 technical details are to be investigated when this second  
83 derivative is monitored by the tracking requirements (vir-  
84 tual time needs to be monotonic, coefficient excursions  
85 have to be compatible with geometric constraints, etc.)  
86 The reference trajectories to be tracked may be computed  
87 using classical constrained optimal control tools. Several  
88 optimization criterions have been proposed [31,26,30].  
89 2. A second way to handle under-actuation is to use the  
90 concept of virtual constraints and the associated zero-  
91 dynamics [12,35,36]. Namely, some regulated output is  
92 suitably defined that can be exactly tracked using the  
93 available control inputs. The constrained dynamics of  
94 the remaining sub-state on the zero-output manifold is  
95 then called the zero-dynamics [17]. This methodology  
96 is therefore based on the analytical study of the resulting

zero-dynamics that corresponds to each particular 97  
choice of the regulated output. If the latter is taken in 98  
a parameterized closed-loop form, off-line optimization 99  
can then be used to enhance the asymptotic stability of 100  
the zero-dynamics [12,35,36]. A particular feature when 101  
dealing with the zero-dynamics associated to bipeds is 102  
their hybrid nature [36]. 103

3. A third way to handle under-actuation in nonlinear 104  
dynamical systems is predictive control schemes 105  
[4,20,11]. Indeed, these schemes ensure stability by con- 106  
trolling the behavior of the whole state at some future 107  
time, say  $N$ -sampling times ahead. This naturally sup- 108  
presses under-actuation since the number of control 109  
d.o.f. is  $r \times N$  where  $r$  is the number of actuators while 110  
the controlled state is still  $n$ -dimensional. Therefore, 111  
under-actuation generically *disappears* as soon as 112  
 $Nr \geq n$ . 113  
114

The work proposed in this paper might be viewed as a 115  
*mixture* of the last two categories. Namely a nonlinear pre- 116  
dictive control scheme is proposed for the control of a five- 117  
link 7 d.o.f. under-actuated biped robot while the stability 118  
of the resulting zero-dynamics is explicitly studied. Con- 119  
trary to the approach adopted in [1], where a somehow 120  
*black-box* formulation is used to define the auxiliary 121  
open-loop optimization problem, our approach leads to 122  
low dimensional decision variables. This may be crucial 123  
in a real-time implementation context. In particular, it is 124  
shown that with a scalar such open-loop optimization 125  
problem, provably stable and quasi-cyclic motions can be 126  
generated. 127

The basic differences between the approach proposed in 128  
this paper and existing provably stable limit-cycle genera- 129  
tion can be summarized as follows: 130

- The limit cycle so-obtained may include several steps. 131  
Namely, the robot configuration just after the impact 132  
is not necessarily the same as the one just after the pre- 133  
ceding impact. This happens especially with very low 134  
dimensional (scalar) predictive control. By increasing 135  
the d.o.f. of the control parameterization, classical 136  
one-step limit cycles may be recovered, but the underly- 137  
ing predictive control may not be real-time 138  
implementable. 139
- The resulting closed-loop trajectories do not necessarily 140  
correspond to a periodic motion of the torso. The latter 141  
converges to a neighborhood of a stable limit cycle. This 142  
is a crucial point since it has been pointed out in [27,9] 143  
that such periodic motion is hard to achieve with at least 144  
polynomial trajectories of the actuated variables. 145  
146

Sufficient conditions for the stability of the feedback 147  
scheme are derived together with a concrete computation 148  
procedure to compute the corresponding region of attrac- 149  
tion related to the zero-dynamics of the closed-loop sys- 150  
tem. 151

152 This paper is organized as follows. First, the biped robot  
 153 prototype is described in Section 2. Then the proposed pre-  
 154 dictive control approach is presented in a rather general  
 155 setting (Section 3). Computable sufficient conditions for  
 156 stability are derived and implementation related topics  
 157 are discussed in Section 4. Finally simulation results are  
 158 given in Section 5, illustrating the potentiality of the pro-  
 159 posed solution. These simulations include scenarios where  
 160 a stable walk is obtained from the rest position as well as  
 161 transitions between different desired mean walking speeds.  
 162 Robustness against model parameters uncertainties and  
 163 ground irregularities are also verified. The paper ends by  
 164 some concluding remarks.

## 165 2. The RABBIT prototype description

166 The academic prototype RABBIT [10] is a biped walking  
 167 robot with five links and seven d.o.f. (see Fig. 1), which  
 168 results from the joint effort of several French laboratories  
 169 (Mechanical engineering, Automatic control, and Robot-  
 170 ics) working on a project on control of walking robots.<sup>1</sup>  
 171 By means of guidance device, RABBIT walks in a circular  
 172 path (see Fig. 2) while looking like a planar biped. The  
 173 counter-balance should be used to offset the weight of the  
 174 lateral stabilization bar in the guidance device. More tech-  
 175 nical details about the testbed can be found in [10].

### 176 2.1. Dynamic model

177 Using Lagrange formulation [32], the mathematical  
 178 model describing the biped moving in the sagittal plane is  
 179 as follows:  
 180

$$182 \quad M(q)\ddot{q} + N(q, \dot{q})\dot{q} + G(q) = Su + F_{\text{ext}} \quad (1)$$

183 where  $M(q) \in \mathbb{R}^{7 \times 7}$  is the inertia matrix,  $N(q, \dot{q}) \in \mathbb{R}^{7 \times 7}$  con-  
 184 tains the centrifugal and Coriolis forces terms,  $G(q) \in \mathbb{R}^7$  is  
 185 the vector of gravitational forces,  $u = [u_1 \ u_2 \ u_3 \ u_4]^T \in \mathbb{R}^4$   
 186 is the vector of control inputs,  $S$  is a torque distribution ma-  
 187 trix,  $q = [q_{31} \ q_{41} \ q_{32} \ q_{42} \ q_1 \ x \ y]^T \in \mathbb{R}^7$  is the vector of gen-  
 188 eralized coordinates (see Fig. 3). Finally,  $F_{\text{ext}}$  represents the  
 189 external forces acting on the robot (contact forces with the  
 190 ground). Following the proposed decomposition of the  
 191 walking cycle proposed in [12], the walking cycle can be di-  
 192 vided into two consecutive phases of motion (see Fig. 4). In  
 193 the first one, the biped remains in contact with the ground  
 194 through one foot (single support (SS) phase). The other  
 195 one is the impact phase [3] that is often considered as instan-  
 196 taneous and characterized by a collision between the swing  
 197 leg and the ground. Since the assumption that the robot is  
 198 walking on horizontal surface without obstacles is made,  
 199 the switching from one walking phase to another is closely  
 200 related to the vertical position of the robot free leg tip. Let  
 201 this position be denoted by  $\sigma(q)$ , the stance leg is denoted by  
 202  $(q_{31}, q_{41})$ , and the swing leg by  $(q_{32}, q_{42})$ , therefore



Fig. 1. RABBIT prototype testbed.



Fig. 2. The guidance device.

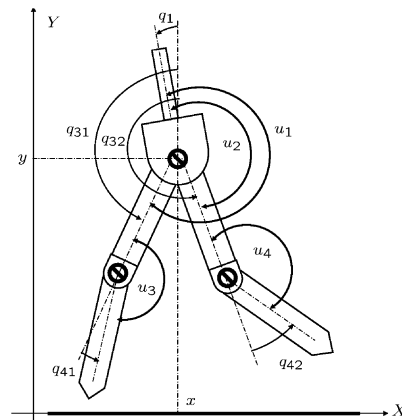


Fig. 3. Schematic view of RABBIT's mechanical structure.

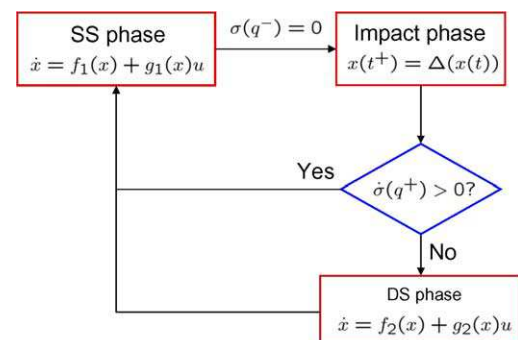


Fig. 4. The walking cycle decomposition.

<sup>1</sup> For a detailed information, see <http://robot-rabbit.lag.ensieg.inpg.fr/>.

$$\sigma(q) = l_3(\cos(q_{32}) - \cos(q_{31})) + l_4(\cos(q_{32} + q_{42}) - \cos(q_{31} + q_{41})) \quad (2)$$

Indeed the impact between the swing leg and the ground<sup>2</sup> occurs when the foot hits the ground, which can be expressed as

$$\sigma(q) = 0; \quad \dot{\sigma}(q) < 0$$

and it characterizes the switch from the single support phase to the impact phase (cf. Fig. 4). On the other hand, the lift-off from ground occurs just after the impact [3] and may be expressed as (after re-labelling the variables)

$$\sigma(q^+) = 0; \quad \dot{\sigma}(q^+) > 0$$

and it characterizes the switch from the impact phase to the single support phase (cf. Fig. 4). Note that  $q^+$  denotes the vector of the generalized coordinates just after the impact (cf. Section 2.1.2). In the following sections, the dynamic equations for these two phases are presented.

### 2.1.1. The single support phase model

In this phase only one foot is grounded, and the biped is modelled by the following differential equation [32]:

$$M(q)\ddot{q} + N(q, \dot{q})\dot{q} + G(q) = Su + J_1^T(q)\lambda \quad (3)$$

where  $J_1(q)$  represents the Jacobian matrix of the holonomic contact constraints, and  $\lambda$  the Lagrange multipliers of contact forces. Assuming that  $(q_{31}, q_{41})$  is the stance leg, the contact constraints may be expressed by

$$y_{p_1} = \dot{y}_{p_1} = \ddot{y}_{p_1} = 0; \quad x_{p_1} = \dot{x}_{p_1} = \ddot{x}_{p_1} = 0 \quad (4)$$

where  $(x_{p_1}, y_{p_1})$  denotes the cartesian coordinates of the stance leg's foot, given by

$$\begin{cases} y_{p_1}(q) = y + l_3 \cos(q_{31}) + l_4 \cos(q_{31} + q_{41}) \\ x_{p_1}(q) = x - l_3 \sin(q_{31}) - l_4 \sin(q_{31} + q_{41}) \end{cases} \quad (5)$$

Using (5) and (4) one obtains

$$J_1(q)\ddot{q} + \Pi_2(q, \dot{q}) = 0 \quad (6)$$

where  $\Pi_2(q, \dot{q})$  is defined by

$$\Pi_2(q, \dot{q}) := \begin{pmatrix} -l_3\dot{q}_{31}^2 \cos(q_{31}) - l_4(\dot{q}_{31} + \dot{q}_{41})^2 \cos(q_{31} + q_{41}) \\ l_3\dot{q}_{31}^2 \sin(q_{31}) + l_4(\dot{q}_{31} + \dot{q}_{41})^2 \sin(q_{31} + q_{41}) \end{pmatrix}$$

The constrained dynamic model in the single support phase is then given by

$$\begin{cases} M(q)\ddot{q} + N(q, \dot{q})\dot{q} + G(q) = Su + J_1^T(q)\lambda \\ J_1(q)\ddot{q} + \Pi_2(q, \dot{q}) = 0 \end{cases} \quad (7)$$

In simulation of the walking robot during the single support phase, a reduced order dynamic model, computed form (7), is used (cf. [7] and the references inside for more details).

### 2.1.2. The impact phase model

According to [15], the impact between the swing leg and the ground is considered as a rigid collision [3], it occurs when the swing leg hits the walking surface and it induces discontinuities (jumps) in the generalized velocities,<sup>3</sup> our objective is then to derive the post-impact velocities in terms of pre-impact positions and velocities. During the impact we have

$$M(q)\ddot{q} + N(q, \dot{q})\dot{q} + G(q) = Su + \delta F_{\text{ext}} \quad (8)$$

where  $F_{\text{ext}}$  represents the external contact forces.

Under suitable assumptions (see e.g. [15]) on the impact phenomenon, one can deduce the external acting forces by integration of (8) over the impact duration, so one obtains

$$M(q)(\dot{q}^+ - \dot{q}^-) = F_{\text{ext}} = J_2^T(q)\lambda \quad (9)$$

where  $\dot{q}^+$  (respectively,  $\dot{q}^-$ ) is the velocity just after (respectively before) impact, and  $F_{\text{ext}} = \int_{t^-}^{t^+} \delta F_{\text{ext}}$ .  $J_2(q)$  is the Jacobian matrix of the cartesian coordinates of the swing leg foot, given by

$$\begin{cases} y_{p_2}(q) = y + l_3 \cos(q_{32}) + l_4 \cos(q_{32} + q_{42}) \\ x_{p_2}(q) = x - l_3 \sin(q_{32}) - l_4 \sin(q_{32} + q_{42}) \end{cases} \quad (10)$$

Eq. (9) involves seven constraints and nine unknowns  $F_{\text{ext}}$  and  $\dot{q}^+$ . Two additional equations may be obtained from the condition that the impacted leg does not rebound nor slips at impact, that is

$$y_{p_2} = \dot{y}_{p_2}^+ = 0; \quad x_{p_2} = \dot{x}_{p_2}^+ = 0 \quad (11)$$

which gives using the expressions of  $x_{p_2}, y_{p_2}$

$$J_2(q)\dot{q}^+ = 0 \quad (12)$$

The solution of (9)–(12) leads to

$$\begin{cases} \dot{q}^+ = \left[ I - M^{-1}J_2^T(J_2M^{-1}J_2^T)^{-1}J_2 \right] \dot{q}^- = D(q)\dot{q}^- \\ \lambda = -\left[ (J_2M^{-1}J_2^T)^{-1}J_2 \right] \dot{q}^- \end{cases} \quad (13)$$

On the other hand, the impact model must account for the re-labelling of the robot coordinates (i.e. the swing leg becomes the new stance leg and vice versa), this can be expressed by

$$\begin{pmatrix} q^+ \\ \dot{q}^+ \end{pmatrix} = R(q) \begin{pmatrix} q^- \\ \dot{q}^- \end{pmatrix} \quad (14)$$

To summarize, the global impact model that includes both the jumps in velocities and the permutation of coordinates and velocities shortly writes

$$\begin{pmatrix} q^+ \\ \dot{q}^+ \end{pmatrix} = \Delta(q) \begin{pmatrix} q^- \\ \dot{q}^- \end{pmatrix} \quad (14)$$

<sup>2</sup> The system looks like a kinematic chain [15].

<sup>3</sup> But the generalized positions still unaltered i.e.  $q^+ = q^- = q$ .

304 where

$$\Delta(q) = \begin{pmatrix} R(q) & 0 \\ 0 & R(q)D(q) \end{pmatrix} R = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

306

### 307 3. The key idea: a predictive control scheme

308 Under single support assumption, the five independent  
309 degrees of freedom can be subdivided into two parts

$$311 z_1 := q_1 \in \mathbb{R}; \quad z_2 := (q_{31} \quad q_{41} \quad q_{32} \quad q_{42})^T \in \mathbb{R}^4$$

312 where  $z_2$  can be assumed to be completely controllable  
313 (provided that saturation constraints on actuators and con-  
314 tact conditions are fulfilled). In this section, the sequence of  
315 impact instants is denoted by  $(t_k)_{k \in \mathbb{N}}$  with  $t_k = kt_f$  where  $t_f$   
316 is the step duration.

317 **Remark 1.** Under nominal conditions, the step duration  $t_f$   
318 is fixed and it does not change from one step to another,  
319 nevertheless if the biped is required to change the walking  
320 speed, among others, a solution could be investigated to  
321 change this parameter, but for each speed and configura-  
322 tion corresponds a well specified value of  $t_f$ .

323 Let us choose some target configuration  $z_2^f \in \mathbb{R}^4$  (cf. Sec-  
324 tion 5.1) that is to be reached just before the impact  
325 instants  $t_k$  that is  $z_2(t_k^-) = z_2^f$ . This choice is fixed in all  
326 the forthcoming developments, in a way,  $z_2^f$  has to be con-  
327 sidered as a design parameter. The way  $z_2^f$  may be parame-  
328 terized is explained in Section 5.1.

329 Associated to this choice of  $z_2^f$ , the following choice  
330  $\dot{z}_2^f(z_2^f) \in \mathbb{R}^4$  is done for the desired  $\dot{z}_2(t_k^-)$ , this choice is  
331 defined given some desired foot impact velocity  $-v_{p_2}$   
332

$$334 \begin{aligned} \dot{z}_2^f(z_2^f, v_{p_2}) &:= \text{Arg min}_{\dot{z}_2} \|\dot{z}_2\|^2 \quad \text{under} \quad \frac{\partial y_{p_2}}{\partial z_2}(z_2^f) \dot{z}_2 = -v_{p_2} \\ &= - \left[ \frac{\partial y_{p_2}}{\partial z_2}(z_2^f) \right]^T v_{p_2} / \left\| \frac{\partial y_{p_2}}{\partial z_2}(z_2^f) \right\|^2 \end{aligned} \quad (15)$$

335 where  $y_{p_2}(z_2)$  is the  $y$ -coordinate of the swing foot. There-  
336 fore,  $\dot{z}_2^f$  is clearly the minimum norm velocity vector that  
337 corresponds to some impact velocity  $-v_{p_2}$ . Once this choice  
338 is done, a final desired “just before impact” sub-state  
339  $(z_2, \dot{z}_2) \in \mathbb{R}^8$  is completely defined by the choice of  $z_2^f \in \mathbb{R}^4$ .

340 In what follows, the following notations are used  
341

$$343 \begin{aligned} \mathcal{Z}_2 &:= \begin{pmatrix} z_2 \\ \dot{z}_2 \end{pmatrix} \in \mathbb{R}^8; \quad \mathcal{Z}_2^f := \begin{pmatrix} z_2^f \\ \dot{z}_2^f(z_2^f, v_{p_2}) \end{pmatrix} \in \mathbb{R}^8; \\ \mathcal{Z}_1 &:= \begin{pmatrix} q_1 \\ \dot{q}_1 \end{pmatrix} \in \mathbb{R}^2 \end{aligned} \quad (16)$$

Now, during the step, let us denote by  $\eta > 0$  the remaining  
time before impact. One has the following dynamic for  $\eta$

$$\dot{\eta} = -1 + \delta(\eta) \cdot t_f \quad (17)$$

where  $\delta(\cdot)$  is the generalized impulse function. Consider a  
control sampling period  $\tau_c > 0$  such that  $t_f/\tau_c = N_c \in \mathbb{N}$   
( $N_c$ : is also a design parameter).

Basically, a problem of synchronizing the sampling  
times to the impact times could appear when impact is  
either detected prematurely (i.e. before the expected  
instant) or detected with a delay (i.e. after the expected  
instant). Since RABBIT PROTOTYPE feet are equipped with  
switches, the impact instant could easily be detected. This  
situation is managed as indicated in Section 4.3 concerning  
implementation issues.

Let us use the following notation to refer to decision  
instants [4] on the interval  $[t_k, t_{k+1}]$

$$\tau_k^i = t_k + i\tau_c; \quad i \in \{0, \dots, N_c - 1\}; \quad k \in \mathbb{N}$$

During the step, at each decision instant  $\tau_k^i$ , a  $p$ -parameter-  
ized reference trajectory<sup>4</sup>

$$\mathcal{Z}_2^{\text{ref}}(\tau', \mathcal{Z}_2(\tau_k^i), \mathcal{Z}_2^f, \eta(\tau_k^i), p); \quad \tau' \in [\tau_k^i, t_{k+1}]; \quad p \in \mathcal{P} \quad (18)$$

is defined that satisfies for all parameter value  $p \in \mathcal{P}$  the  
following boundary (initial and final) conditions

$$\mathcal{Z}_2^{\text{ref}}(\tau_k^i, \mathcal{Z}_2(\tau_k^i), \mathcal{Z}_2^f, \eta(\tau_k^i), p) = \mathcal{Z}_2(\tau_k^i) \quad (19)$$

$$\mathcal{Z}_2^{\text{ref}}(t_{k+1}, \mathcal{Z}_2(\tau_k^i), \mathcal{Z}_2^f, \eta(\tau_k^i), p) = \mathcal{Z}_2^f \quad (20)$$

namely, the reference trajectory  $\mathcal{Z}_2^{\text{ref}}(\cdot, \mathcal{Z}_2(\tau_k^i), \mathcal{Z}_2^f, \eta(\tau_k^i), p)$   
is updated at each decision instant  $\tau_k^i$  to start at the present  
value  $\mathcal{Z}_2(\tau_k^i)$ , and to join the desired final value  $\mathcal{Z}_2^f$  just be-  
fore next impact.

It is worth noting that  $p \in \mathcal{P}$  is the remaining free  
parameter, once the constraints (19) and (20) have been  
structurally imposed, on some initial parameterization.  
This is typically easy to realize with polynomial parameter-  
ization [9] of trajectories since (19) and (20) are linear con-  
straints in the polynomial coefficients.

A relevant question is: how to choose  $p \in \mathcal{P}$ ?

The role of  $p$  is clearly to optimize the behavior of the  
indirectly controlled sub-state  $\mathcal{Z}_1$ . Indeed, imagine that a  
perfect tracking of the reference trajectory  $\mathcal{Z}_2^{\text{ref}}(\cdot, \mathcal{Z}_2(\tau_k^i),$   
 $\mathcal{Z}_2^f, \eta(\tau_k^i), p)$  is performed over  $[\tau_k^i, t_{k+1}]$ . What are the con-  
sequences of such tracking on the value of both  $\mathcal{Z}_1$  and  $\mathcal{Z}_2$   
just before the  $(k+1)$  impact?

- For  $\mathcal{Z}_2$ , one would clearly have, because of the perfect tracking [see (20)]

$$\mathcal{Z}_2(t_{k+1}^-) = \mathcal{Z}_2^f \quad (21)$$

- For the  $\mathcal{Z}_1$  dynamic, let us consider the torso equation extracted from the dynamic model (1), and given by

<sup>4</sup> In [16] for instance such trajectories are generated using Van der Pol oscillators.

$$\left(\frac{1}{4}m_1 l_1^2 + I_1\right)\ddot{q}_1 = \frac{1}{2}m_1 l_1 \cos(q_1)\ddot{x} + \frac{1}{2}m_1 l_1 \sin(q_1)(\ddot{y} + g) - u_1 - u_2 \quad (22)$$

where  $m_1$  is the mass of the torso,  $l_1$  its length,  $u_1$  and  $u_2$  are the torques of the femurs.

Using Eqs. (4) and (5) and notations (16), this dynamic should be written

$$\dot{\mathcal{Z}}_1 = f(\mathcal{Z}_1, \mathcal{Z}_2, u) \quad (23)$$

The closed-loop system is obtained by state feedback, that is  $u = K(\mathcal{Z}, \mathcal{Z}_2^{\text{ref}})$ , therefore equation (23) could be rewritten as

$$\dot{\mathcal{Z}}_1 = f(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_2^{\text{ref}}) \quad (24)$$

Under the assumption of perfect tracking, by replacing  $\mathcal{Z}_2$  in (24) by the reference trajectory  $\mathcal{Z}_2^{\text{ref}}$ <sup>5</sup> one obtains:

$$\dot{\mathcal{Z}}_1 = f(\mathcal{Z}_1, \mathcal{Z}_2^{\text{ref}}) = f(\mathcal{Z}_1, \mathcal{Z}_2(\tau_k^i), \mathcal{Z}_2^f, p) \quad (25)$$

and integrating (25) starting from the initial condition  $(\tau_k^i, \mathcal{Z}_1(\tau_k^i))$  gives the predicted value of  $\mathcal{Z}_1(t_{k+1}^-)$  just before next impact. This can be rewritten formally as follows  $(\eta(\tau_k^i) = t_{k+1} - \tau_k^i)$

$$\widehat{\mathcal{Z}}_1(t_{k+1}^- | \tau_k^i) = \Psi(\mathcal{Z}_1(\tau_k^i), \mathcal{Z}_2(\tau_k^i), \mathcal{Z}_2^f, \eta(\tau_k^i), p) \quad (26)$$

and using the impact equation (cf. Eq. (14)) together with the predicted values (21) and (26) one can derive an expression of the predicted value of  $\mathcal{Z}_1$  just after impact

$$\widehat{\mathcal{Z}}_1(t_{k+1}^+ | \tau_k^i) = \Psi^+(\mathcal{Z}_1(\tau_k^i), \mathcal{Z}_2(\tau_k^i), \mathcal{Z}_2^f, \eta(\tau_k^i), p) \quad (27)$$

The value of the reference trajectory's parameter  $p(\tau_k^i)$  is then given by the optimal solution of the following quadratic optimization problem

$$\hat{p}(\tau_k^i) = \min_{p \in \mathcal{P}} \|\widehat{\mathcal{Z}}_1(t_{k+1}^+ | \tau_k^i) - \mathcal{Z}_1^f\|_Q^2 \text{ subject to} \\ C(\mathcal{Z}_2(\tau_k^i), \mathcal{Z}_2^f, p) > 0; \quad Q \in \mathbb{R}^{2 \times 2} \quad Q > 0 \quad (28)$$

where

- $\mathcal{Z}_1^f \in \mathbb{R}^2$  is some desired value just after the impact. This value (together with  $\mathcal{Z}_2^f$ ) defines the limit cycle one aims to establish.
- $C(\mathcal{Z}_2(\tau_k^i), \mathcal{Z}_2^f, p) > 0$  is a constraint expressing non penetration condition. This can be for instance

$$C(\mathcal{Z}_2(\tau_k^i), \mathcal{Z}_2^f, p) := \min_{\tau' \in [\tau_k^i, \tau_{k+1}^i - \epsilon]} y_{p_2}(\tau', \mathcal{Z}_2(\tau_k^i), \mathcal{Z}_2^f, p) \quad (29)$$

for some small  $\epsilon > 0$ .

To summarize,<sup>6</sup> during the step, at each decision instant  $\tau_k^i$  with  $i < N_c - 1$  the reference trajectory

$$\mathcal{Z}_2^{\text{ref}}(\tau', \mathcal{Z}_2(\tau_k^i), \mathcal{Z}_2^f, \eta(\tau_k^i), \hat{p}(\tau_k^i)); \quad \tau' \in [\tau_k^i, t_{k+1}] \quad 449$$

is defined on the completely controlled variables (actuated joints) and tracked using a nonlinear time varying feedback during the time interval  $[\tau_k^i, \tau_{k+1}^i]$ . At the next decision instant  $\tau_{k+1}^i$  a new reference trajectory

$$\mathcal{Z}_2^{\text{ref}}(\tau', \mathcal{Z}_2(\tau_{k+1}^i), \mathcal{Z}_2^f, \eta(\tau_{k+1}^i), \hat{p}(\tau_{k+1}^i)); \quad \tau' \in [\tau_k^i, t_{k+1}] \quad 455$$

is defined, based on the new measurements and is tracked during the time interval  $[\tau_{k+1}^i, \tau_{k+2}^i]$  and the scheme is repeated until the impact instant. This defines a predictive control scheme in which the open-loop auxiliary optimization problem is given by (28). The solution of such optimization problem is performed using the DBCPOL function from the IMSL math library of Digital Fortran 5.0.

#### 4. Stability and implementation issues

The stability can be investigated using the Poincaré's section [17] just before the impact, namely at instants  $t_k^-$ . Indeed, if this discrete-time map converges, then a cyclic trajectory results (see Figs. 5 and 6). To study the stability of the Poincaré's map, note that, by definition of the predictive control strategy depicted in the previous section, one clearly has

$$\mathcal{Z}_2(t_k^-) = \mathcal{Z}_2^f \quad (30)$$

where  $\mathcal{Z}_2^f$  is the desired final "just before impact" configuration defined by (15) and (16) and depending only on the desired final position  $z_2^f$ . Consequently, the overall stability depends on the stability of the sequence

$$(\mathcal{Z}_1(t_k^-))_{k \in \mathbb{N}} \quad 479$$

under the constraint (30).

##### 4.1. Stability definition

As it can be easily understood from Figs. 5 and 6, an asymptotically stable  $k_0$ -cyclic trajectory on the whole state results whenever the following property holds for the closed-loop system's behavior:

$$\|\mathcal{Z}_1(t_{(j+1)k_0}^-) - \mathcal{Z}_1^f\|_Q^2 \leq \mu \|\mathcal{Z}_1(t_{jk_0}^-) - \mathcal{Z}_1^f\|_Q^2; \quad \mu < 1 \quad (31)$$

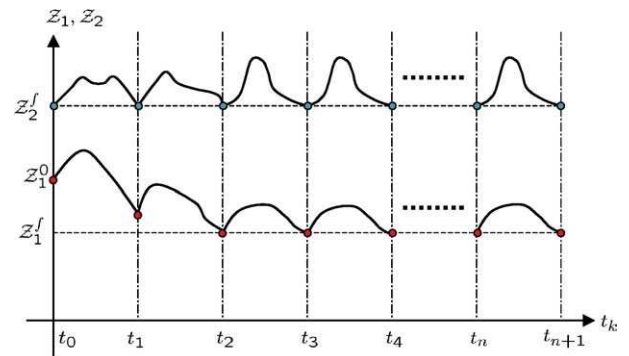


Fig. 5. Stability illustration ( $k_0 = 1$ ).

<sup>5</sup> Recall that  $\mathcal{Z}_2^{\text{ref}}$  depends on  $\mathcal{Z}_2(\tau_k^i)$ ,  $\mathcal{Z}_2^f$ , and  $p$ .

<sup>6</sup> A chart flow better illustrating the principle of the approach is given in Section 4.3.

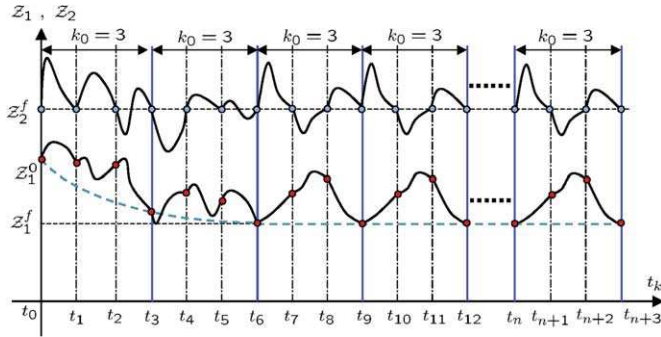


Fig. 6. Stability illustration ( $k_0 = 3$ ).

488 similarly, a neighborhood of a  $k_0$  cyclic trajectory on the  
 489 whole is asymptotically stabilized whenever the following  
 490 property holds for some small  $\varepsilon > 0$ ,

492 
$$\lim_{j \rightarrow \infty} \|\mathcal{Z}_1(t_{jk_0}^-) - \mathcal{Z}_1^f\|_Q^2 \leq \varepsilon \quad (32)$$

493 The aim of the following section is to give sufficient condi-  
 494 tions under which one of the above conditions is satisfied  
 495 for some  $k_0 \in \mathbb{N}$  with a graphical tools enabling a concrete  
 496 evaluation of the associated conditions (see Proposition 1  
 497 hereafter).

498 4.2. Stability result

499 Now let  $\mathcal{Z}_1(t_k^-)$  be given. Using (30) and the impact map  
 500 (14), the value of the whole  $\mathcal{Z}(t_k^+)$  just after the impact can  
 501 be computed and the predictive control closed-loop trajec-  
 502 tories may be predicted over  $[t_k^+, t_{k+1}^-]$ . Therefore, the pre-  
 503 dicted value of  $\mathcal{Z}_1(t_{k+1}^-)$  just before the next impact is  
 504 only function of  $\mathcal{Z}_1(t_k^-)$ ,  $N_c$  and  $\mathcal{Z}^f := (\mathcal{Z}_1^f, \mathcal{Z}_2^f)$ , namely

506 
$$\mathcal{Z}_1(t_{k+1}^-) =: \Gamma(\mathcal{Z}_1(t_k^-), \mathcal{Z}^f, N_c) \quad (33)$$

507 which is a discrete-time autonomous system (for fixed  $\mathcal{Z}^f$   
 508 and  $N_c$ ) in the sub-state  $\mathcal{Z}_1$  for which stability is to be  
 509 investigated. More generally, the following multi-step  
 510 map is particularly relevant to assess the stability of the  
 511 above predictive control scheme, namely,

514 
$$\mathcal{Z}_1(t_{k+k_0}^-) =: \Gamma^{k_0}(\mathcal{Z}_1(t_k^-), \mathcal{Z}^f, N_c) \quad (34)$$

515 where  $\Gamma^{k_0}$  is obtained by repetitive application of  $\Gamma(\cdot)$ . Note  
 516 that this map is easily computable by simulating  $k_0$  steps  
 517 under the closed-loop feedback law explained in the previ-  
 518 ous section. It is worth noting that such computations are  
 519 to be done off-line for stability investigations. The on-line  
 520 feedback however is still based on one-step scalar optimiza-  
 521 tion as explained in the preceding section. The whole  
 522 closed-loop system stability analysis is based on the follow-  
 523 ing proposition

524 **Proposition 1**

525 1. If for some  $(k_0, N_c) \in \mathbb{N} \times \mathbb{N}$ , there is  $\varrho > 0$  such that

$$\sup_{\|\mathcal{Z}_1 - \mathcal{Z}_1^f\|_Q^2 \leq \varrho} \|\Gamma^{k_0}(\mathcal{Z}_1, \mathcal{Z}^f, N_c) - \mathcal{Z}_1^f\|_Q^2 \leq \varrho \quad (35)$$

528 then the predictive control closed-loop leads to a stable  
 529 walk for all initial conditions belonging to the set  
 530

$$\mathcal{C}_0 := \left\{ \mathcal{Z} = \begin{pmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_2^f \end{pmatrix} \text{ s.t. } \mathcal{Z}_1 \in \mathbb{M}_\varrho \right\} \quad (36)$$

533 where for all  $\varrho \geq 0$ ,  $\mathbb{M}_\varrho := \left\{ \mathcal{Z}_1 \mid \|\mathcal{Z}_1 - \mathcal{Z}_1^f\|_Q^2 \leq \varrho \right\}$ .  
 534

535 2. If in addition, the following condition holds for some  
 536  $\mu \in [0, 1]$   
 537

For all  $0 < r < \varrho$   $\psi(r, N_c, k_0)$   

$$:= \sup_{\|\mathcal{Z}_1 - \mathcal{Z}_1^f\|_Q^2 = r} \|\Gamma^{k_0}(\mathcal{Z}_1, \mathcal{Z}^f, N_c) - \mathcal{Z}_1^f\|_Q^2 \leq \mu \cdot r \quad (37)$$

540 then the closed-loop trajectories asymptotically converges  
 541 to a stable limit cycle of length  $k_0$  defined by the pair  
 542  $(\mathcal{Z}_1^f, \mathcal{Z}_2^f)$  for all initial conditions in  $\mathcal{C}_0$ .  
 543

544 3. If (35) holds, and if (37) holds for all  $r \in [\varepsilon, \varrho]$  and  
 545 furthermore,

$$\sup_{\|\mathcal{Z}_1 - \mathcal{Z}_1^f\|_Q^2 \leq \varepsilon} \|\Gamma^{k_0}(\mathcal{Z}_1, \mathcal{Z}^f, N_c) - \mathcal{Z}_1^f\|_Q^2 \leq \varepsilon; \quad \varepsilon < \varrho \quad (38)$$

547 (see Fig. 7 for a typical situation) then the set  
 548

$$\mathcal{C}_1 := \left\{ \mathcal{Z} = \begin{pmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_2^f \end{pmatrix} \text{ s.t. } \mathcal{Z}_1 \in \mathbb{M}_\varepsilon \right\} \quad (39)$$

551 is invariant and attractive for all initial conditions in  $\mathcal{C}_0$   
 552 (an  $\varepsilon$ -neighborhood of the limit cycle is reached).  
 553  
 554

555 **Proof**

556 1. Straightforward since condition (35) implies that the set  
 557  $\mathbb{M}_\varrho$  is invariant under the composed map  $\Gamma^{k_0}(\cdot, \mathcal{Z}^f, N_c)$ ,  
 558 more precisely

$$\{\mathcal{Z}_1 \in \mathbb{M}_\varrho\} \Rightarrow \{\Gamma^{k_0}(\mathcal{Z}_1, \mathcal{Z}^f, N_c) \in \mathbb{M}_\varrho\} \quad (40)$$

560 Therefore, starting from some initial value  $\mathcal{Z}_1^0 \in \mathbb{M}_\varrho$ ,  
 561 the sequence  
 562

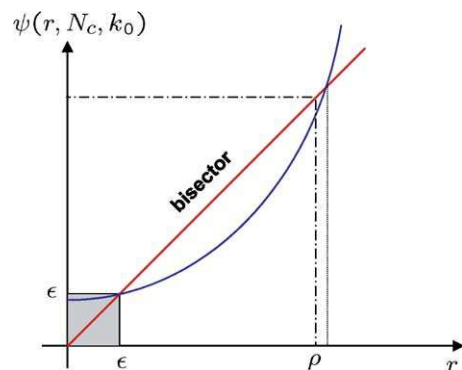


Fig. 7. Typical situation where (35), (37) and (38) hold (point 3 of Proposition 1).



$$564 \quad \left( \mathcal{Z}_1(t_{jk_0}^-) \right)_{j=1}^{\infty}$$

565 belongs to the compact set  $\mathbb{M}_\varrho$ .

566 2. Condition (37) implies that for all  $j \in \mathbb{N}$ , one has

$$569 \quad \|\mathcal{Z}_1(t_{(j+1)k_0}^-) - \mathcal{Z}_1^f\|_Q^2 \leq \mu \|\mathcal{Z}_1(t_{jk_0}^-) - \mathcal{Z}_1^f\|_Q^2 \quad (41)$$

570 (where  $\mu < 1$ ). Accordingly, by recurrence, one obtains  
571 (for  $m \in \mathbb{N}$ )

$$573 \quad \|\mathcal{Z}_1(t_{(j+m)k_0}^-) - \mathcal{Z}_1^f\|_Q^2 \leq \mu^m \|\mathcal{Z}_1(t_{jk_0}^-) - \mathcal{Z}_1^f\|_Q^2 \quad (42)$$

574 which implies that

$$576 \quad \lim_{j \rightarrow \infty} \mathcal{Z}_1(t_{jk_0}^-) = \mathcal{Z}_1^f$$

577 This shows that the closed-loop trajectories tends to a  
578 limit cycle of length  $k_0$  defined by the pair of desired val-  
579 ues  $(\mathcal{Z}_1^f, \mathcal{Z}_2^f)$ .

580 3. Using the same argumentation as in the last point, Eq.  
581 (41) may be rewritten for all  $\mathcal{Z}_1(t_{jk_0}^-)$  that lies in  
582  $\mathbb{M}_\varrho \setminus \mathbb{M}_\varepsilon$ . This proves that  $\mathbb{M}_\varepsilon$  is attractive. Further-  
583 more,  $\mathbb{M}_\varepsilon$  is invariant.  $\square$   
584

585 Note again that the investigation of (35)–(38) may be  
586 done off-line simultaneously and in a deterministic way  
587 by solving the following two-dimensional optimization  
588 problem

$$590 \quad \psi(r, N_c, k_0) := \sup_{\|\mathcal{Z}_1 - \mathcal{Z}_1^f\|_Q^2 = r} \|I^{k_0}(\mathcal{Z}_1, \mathcal{Z}_1^f, N_c) - \mathcal{Z}_1^f\|_Q^2$$

591 for increasing values of  $r$  and check whether the curve so ob-  
592 tained [see Fig. 7] satisfies (35)–(38) for some  $\varrho > 0$  and  
593  $\varepsilon > 0$ . The whole procedure may be repeated for different  
594 values of  $k_0$ . Concrete examples of such plots are given in  
595 Section 5 for a specific choice of the design parameters  
596  $\mathcal{Z}_1^f$ ,  $Q$  and  $N_c$ . It is then shown that the condition of point  
597 3. of Proposition 1 are satisfied for  $k_0 = 3$  (see Fig. 11)  
598 while it is not satisfied for  $k_0 = 1, 2$ . This shows the need  
599 for non trivial multi-step map (34) in establishing the sta-  
600 bility of the underlying closed-loop behaviour.

### 601 4.3. Implementation issues

602 The reference trajectories (18) are implemented using  
603 Matlab cubic spline interpolation functions with various  
604 end-conditions. They are parameterized with a free param-  
605 eter  $p$  which should be computed by solution of the optimi-  
606 zation problem (28). The use of the cubic spline functions  
607 requires the definition of the end-conditions, in our case  
608 they are given by

- 609 • initial-time conditions, given by Eq. (19),
- 610 • intermediate-time conditions, given by the parameter  $p$   
611 to be computed,
- 612 • final-time conditions, given by Eq. (20).  
613

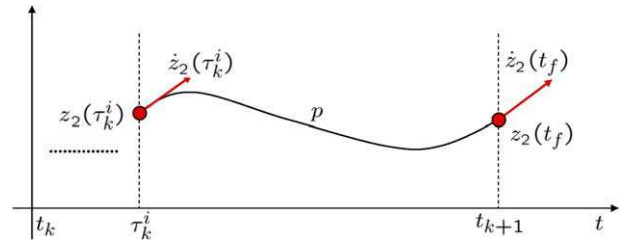


Fig. 8. The directly controlled variables trajectories.

The obtained trajectory, that satisfy these constraints, 614  
may be illustrated in Fig. 8. 615

The used subroutines provide the cubic spline interpolant, which should be used to evaluate the trajectory and its derivatives, at each instant  $\tau'$ . The switching to a new step is closely related to the impact occurrence. The implemented simulator (using visual Fortran 5.0 and Matlab 6.5 softwares) switches to a new step once it detects an impact, therefore three possible cases could be underlined 623

- 624 1. The biped walks without external disturbances, the 624  
dynamic model is perfect, as well as the tracking of 625  
the optimal reference trajectories. 626
- 627 2. During walking, because of external disturbances, 627  
model imperfections, or obstacles, the impact is detected 628  
prematurely. 629
- 630 3. The biped, during walking is subject to external distur- 630  
bances, model imperfections, or environment changes, 631  
as a consequence the impact is not detected at the 632  
expected time (instant). 633  
634

How the control system would react? 635

In the first case there is no problem, the whole closed-loop system behavior looks like the predicted one. In the second case, when the impact is detected, the reached configuration just after the impact is considered as an initial configuration, the final desired configuration is then computed, and a new step starts (illustrated in simulation 4). While in the third case, the impact is not detected at the expected instant, so the control system proceed to an extrapolation of the computed trajectories (using the Matlab PPVAL function) until the occurrence of the forthcoming impact. The whole control approach is summarized in the diagram depicted in Fig. 9 that illustrates how it works. 648

## 649 5. Illustrative simulations

650 Consider the biped robot model (7) and (14) with the 650  
parameters summarized in Table 1. The control parameter 651  
 $N_c = 1$  is used in the following simulations, enabling a large 652  
admissible on-line computation time. Indeed, with this 653  
choice,  $\tau_c = t_f$  and the trajectories being tracked during 654  
the step are updated just after each impact. The following 655

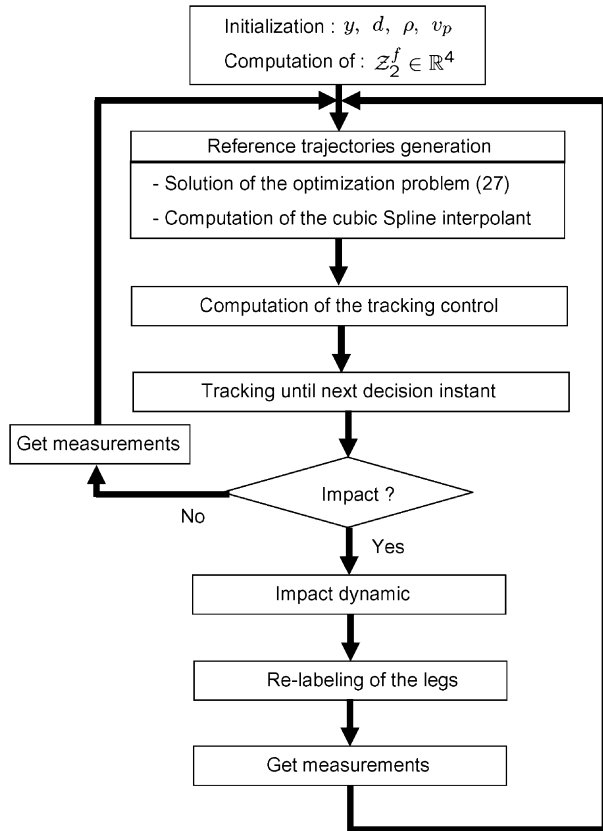


Fig. 9. Algorithm of the approach.

Table 1  
The model parameters

Parameter	Mass (kg)	Length (m)	Inertia (kg m <sup>2</sup> )
Torso	20	0.625	2.22
Femur	6.8	0.4	1.08
Tibia	3.2	0.4	0.93

- The second one illustrates the transition between several walks with different mean walking velocities.

For robustness evaluation of the proposed controller, two scenarios are investigated, namely

- Robustness against uncertainties in the robot model parameters.
- Robustness against ground irregularities.

Let us first illustrate how  $z_2^f$  is chosen by means of a reduced dimensional parameterization. The way such choice of  $z_2^f$  may be made optimal in some sense is beyond the scope of the present paper and will be investigated in later works.

5.1. Reduced dimensional parameterization of the position vector  $z_2$

Consider the instantaneous double support configuration. The position vector  $z_2 := (q_{31} \ q_{41} \ q_{32} \ q_{42})^T$  is defined by three simple parameters, namely  $y, d$  and  $\rho$  that are illustrated in Fig. 10.

Indeed, simple computations give

$$\begin{cases} q_{31} = \pi - \arctan\left(\frac{\rho d}{y}\right) - \varphi_{31} \\ q_{32} = \pi + \arctan\left(\frac{(1-\rho)d}{y}\right) - \varphi_{32} \\ q_{41} = \pi - \varphi_{41} = \arccos\left(-\frac{l_3^2 + l_4^2 - \rho^2 d^2 - y^2}{2l_3 l_4}\right) \\ q_{42} = \pi - \varphi_{42} = \arccos\left(-\frac{l_3^2 + l_4^2 - (1-\rho)^2 d^2 - y^2}{2l_3 l_4}\right) \end{cases} \quad (43)$$

where

choice of the parameter  $p$  is used in the definition of the predictive control law (see Section 3):

$$p(t_k) := \pi - q_{31}(t_k + t_f/2)$$

**Remark 2.** The proposed choice of the optimization parameter represents the angular position of the femur of the swing leg at median instant between two impacts. This is a particular choice among many others, for instance one can imagine any free parameter on the trajectories of the actuated coordinates or their derivatives, it can also be one of the configuration parameters ( $\rho$  for instance).

Two simulation scenarios are proposed:

- The first one shows how the biped reaches a stable walk with constant mean velocity starting from rest.

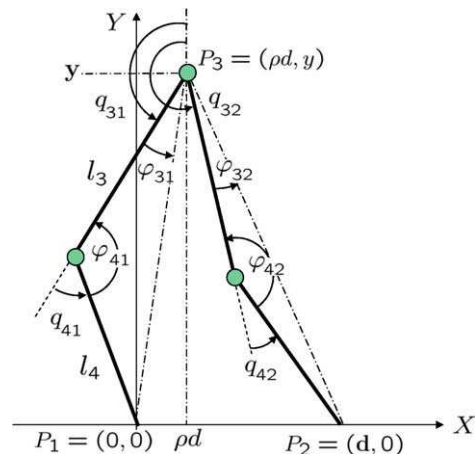


Fig. 10. Computation scheme for the position's reduced parameterization.

$$\left\{ \begin{array}{l} \varphi_{31} = \arccos \left( \frac{l_3^2 - l_4^2 + \rho^2 d^2 + y^2}{2l_3 \sqrt{\rho^2 d^2 + y^2}} \right) \\ \varphi_{32} = \arccos \left( \frac{l_3^2 - l_4^2 + (1 - \rho)^2 d^2 + y^2}{2l_3 \sqrt{(1 - \rho)^2 d^2 + y^2}} \right) \\ \varphi_{41} = \arccos \left( \frac{l_3^2 + l_4^2 - \rho^2 d^2 - y^2}{2l_3 l_4} \right) \\ \varphi_{42} = \arccos \left( \frac{l_3^2 + l_4^2 - (1 - \rho)^2 d^2 - y^2}{2l_3 l_4} \right) \end{array} \right. \quad (44)$$

697  
698 This enables a simple choice of the desired final configura-  
699 tion just before the impact  $z_2^f$  using parameters that are di-  
700 rectly linked to the mean velocity and the geometric  
701 configuration [32].

## 702 5.2. Simulation 1: cyclic forward walking starting 703 from rest (standing position)

704 The aim of this simulation is to take the robot from a  
705 rest position to a constant speed periodic walking. The con-  
706 figuration  $z_2^f$  and the other control design parameters are  
707 summarized in Table 2.

### 708 5.2.1. Stability analysis according to Proposition 1

709 In this section, it is shown that under the feedback  
710 defined above, the sufficient conditions invoked in point  
711 3. of Proposition 1 are satisfied. This can be verified on  
712 Fig. 11, that shows the multi-step map

$$\psi(r, N_c, k_0) = \sup_{\|\mathcal{Z}_1 - \mathcal{Z}_1^f\|_Q^2 = r} \|I^{k_0}(\mathcal{Z}_1, \mathcal{Z}_1^f, N_c) - \mathcal{Z}_1^f\|_Q^2$$

713 invoked in Proposition 1, for the two cases corresponding  
714 to  $k_0 = 1, 2$ . Note that:

- 717 • For  $k_0 = 1$  the conditions of Proposition 1 are not satis-  
718 fied. Higher values of  $k_0$  must be investigated in order to  
719 prove stability of the closed-loop system. Recall that  $k_0$   
720 is only an analysis tool and not a design tool.
- 721 • For  $k_0 = 2$ , the conditions of point 3 of Proposition 1  
722 are satisfied with  $\varrho \approx 0.56$  and  $\varepsilon \approx 0.08$ , therefore for  
723 all initial conditions in the set  $\mathcal{C}_0$  given by (36) with

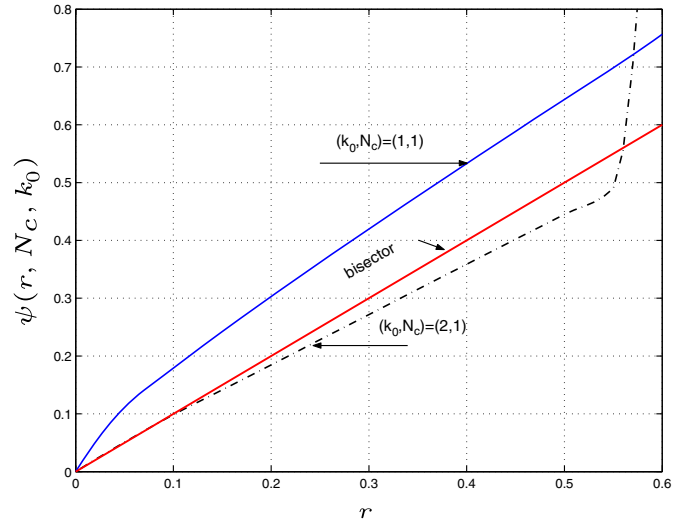


Fig. 11. (Sim 1) A stability analysis tool: the curve  $\psi(r, N_c, k_0)$  for different values of  $k_0$ .

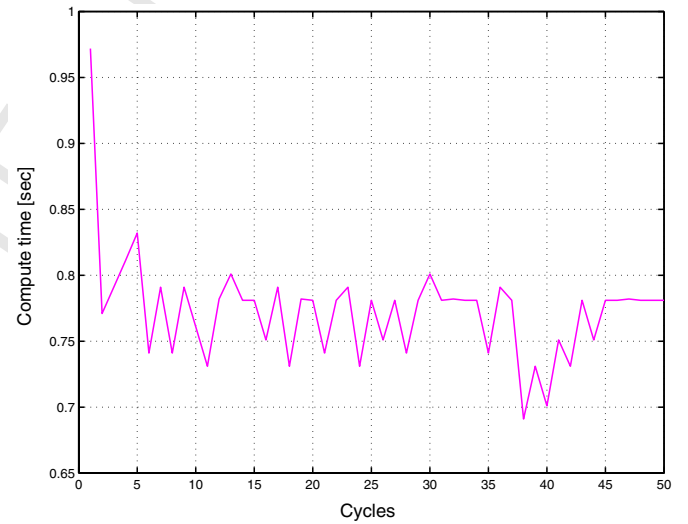


Fig. 12. (Sim 1) Evaluation of computation time.

$$\mathbb{M}_\varrho := \left\{ \mathcal{Z}_1 \mid \|\mathcal{Z}_1 - \mathcal{Z}_1^f\|_Q^2 \leq 0.56 \right\} \quad 725$$

726 the closed-loop trajectories impacts on the Poincaré sec-  
727 tion converge to the invariant and attractive set  $\mathcal{C}_1$  given  
728 by (39) with

$$\mathbb{M}_\varepsilon := \left\{ \mathcal{Z}_1 \mid \|\mathcal{Z}_1 - \mathcal{Z}_1^f\|_Q^2 \leq 0.08 \right\} \quad 730$$

731 which is a neighborhood of the desired limit cycle of  
732 length 2 defined by  $\mathcal{Z}_1^f = 0$ .

733 This example shows clearly the need to the multi-step  
734 stability analysis tool developed in Proposition 1, since  
735 for  $k_0 < 2$ , stability cannot be claimed.

736 **Remark 3.** To give some concrete idea about how large is  
737 the region of attraction, note that the set of initial  
738 conditions  $\mathbb{M}_\varrho$  leading to convergence to the neighborhood

Table 2  
The approach's parameters description

	Significance	Value
$t_f$	Step duration	0.75 s
$y$	Hips height	$y = 0.775$
$d$	Step length	0.3 m
$\rho$	Hip's $x$ position w.r.t. step length	0.5
$v_p$	Foot impact velocity	-0.25 m/s
$\mathcal{Z}_{10}$	Initial conditions on the torso	(0, 0)
$Q$	Weighting matrix in optimization	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$v_{\text{mean}}$	Mean walking velocity	-0.4 m/s

739 of the limit cycle corresponds, among others to the  
 740 following two initial conditions:

742  $(q_1, \dot{q}_1)_0 = (\pm 42.87^\circ, 0^\circ/\text{s})$

743 5.2.2. More simulation results

744 The behavior of the closed-loop system is to be illus-  
 745 trated through the following simulation results. In Fig. 13  
 746 the phase portrait  $(q_1 - \dot{q}_1)$  [16] of the unactuated coordi-  
 747 nate (torso) is displayed, where we note the convergence  
 748 to a neighborhood of a limit cycle of length 2, which con-  
 749 firm the stability result discussed above. The mean walking  
 750 velocity evolution is shown in Fig. 14 where the transition  
 751 from a rest to the desired mean velocity stable walk can be  
 752 observed. Note that the mean velocity is computed as the  
 753 ratio between  $\Delta x(k)$  and  $t_f$  where  $\Delta x(k) = x(t_k) - x(t_{k-1})$ .  
 754 The position and velocity of the torso coordinate is shown  
 755 in Fig. 15 where we note through its trajectory that it

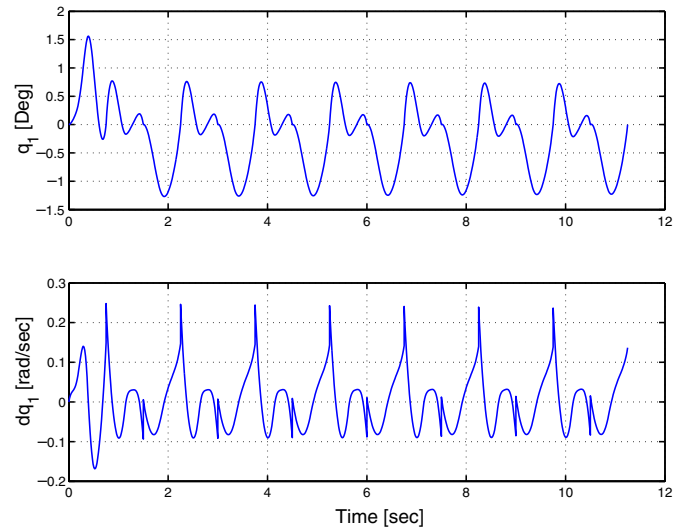


Fig. 15. (Sim 1) The torso position and velocity versus time.

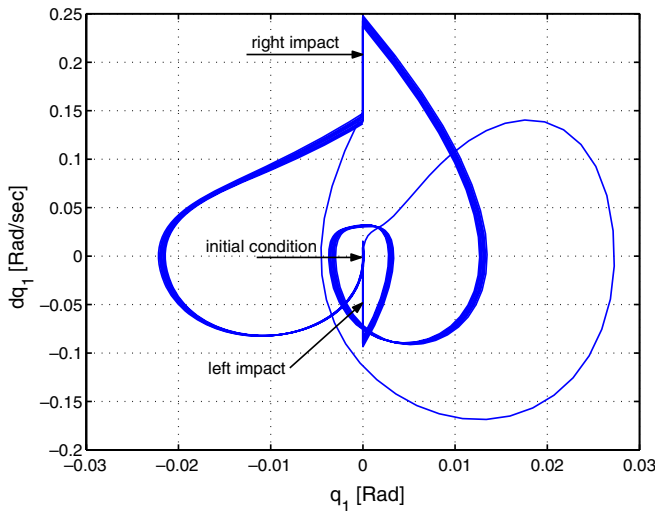


Fig. 13. (Sim 1) The phase portrait of the non actuated coordinate (torso).

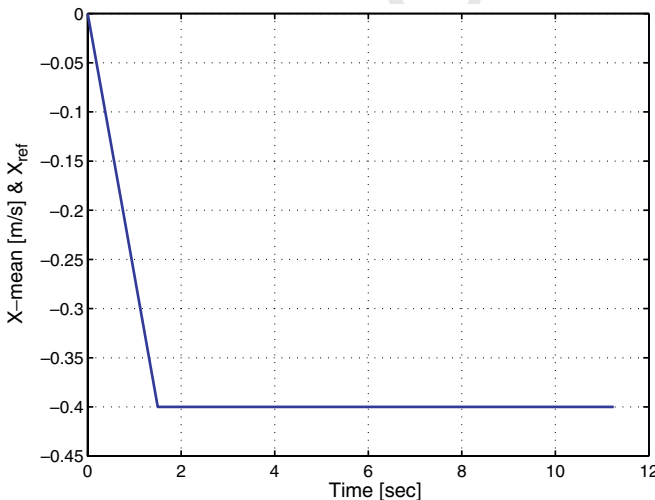


Fig. 14. (Sim 1) The mean walking velocity.

remains close to the vertical. The cartesian coordinates  
 (and their corresponding velocities) of the hips are depicted  
 in Fig. 17 (for the  $x$  coordinate) and in Fig. 18 (for the  $y$   
 coordinate), furthermore the resulting trajectory of the hips  
 in the plane  $x - y$  is illustrated in Fig. 16.

The system control inputs (i.e. joint torques) to be  
 applied to the actuated joints are depicted in Fig. 19 for  
 both femurs, and in Fig. 20 for both tibias. We note that  
 RABBIT is equipped with dc motors of a maximum torque  
 of 150 N m, therefore according to the figures of the gener-  
 ated torques we conclude that this bound is largely satis-  
 fied, but it should also be checked that the power  
 requirement remain within the admissible limit.

To check the admissibility of the actuators required  
 power, the idea is to plot the angular velocity of the actu-  
 ators versus their absolute torques, and check if the  
 obtained curves remain within the admissible region given  
 by the manufacturer of the actuators (DC motors). If it is

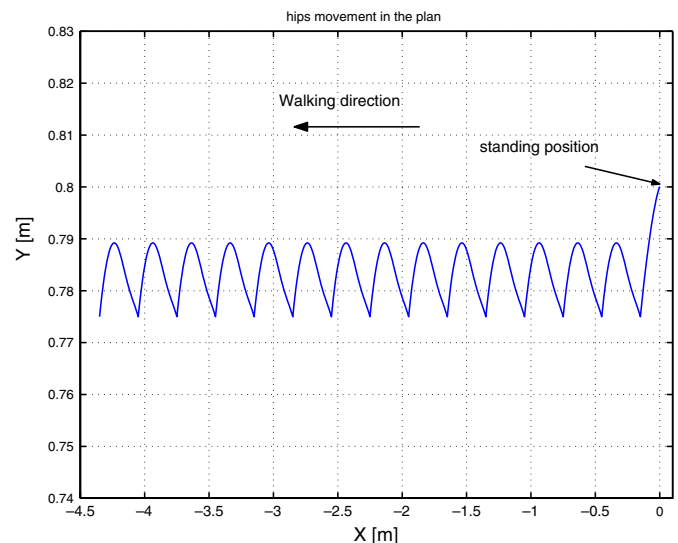


Fig. 16. (Sim 1) The hips movement trajectory.

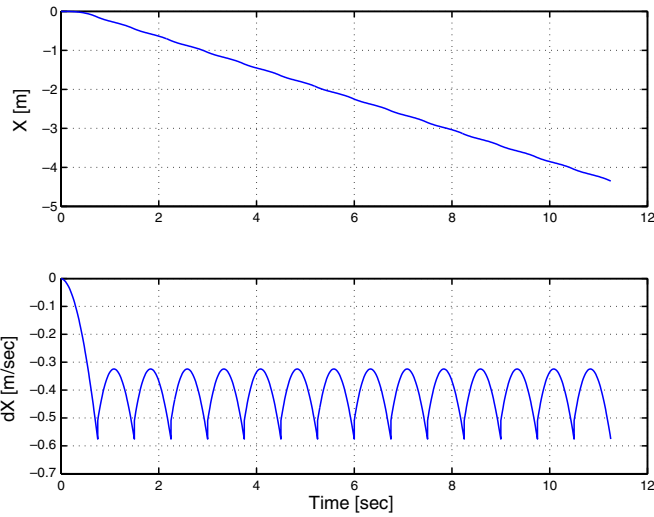


Fig. 17. (Sim 1) The x position and velocity versus time.

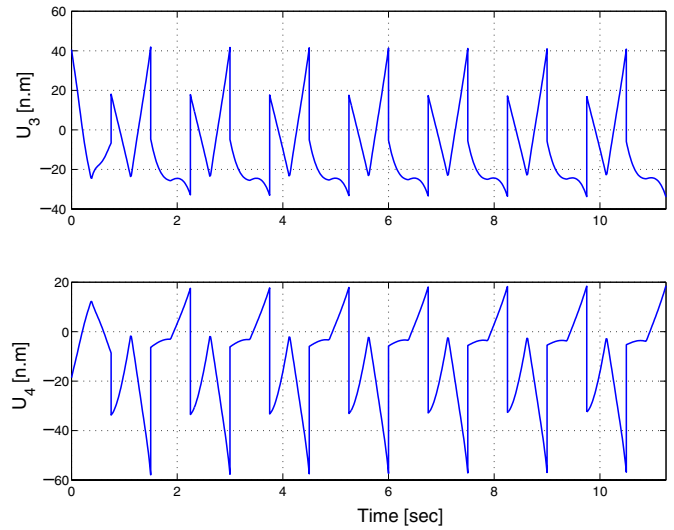


Fig. 20. (Sim 1) The torques of the tibias versus time.

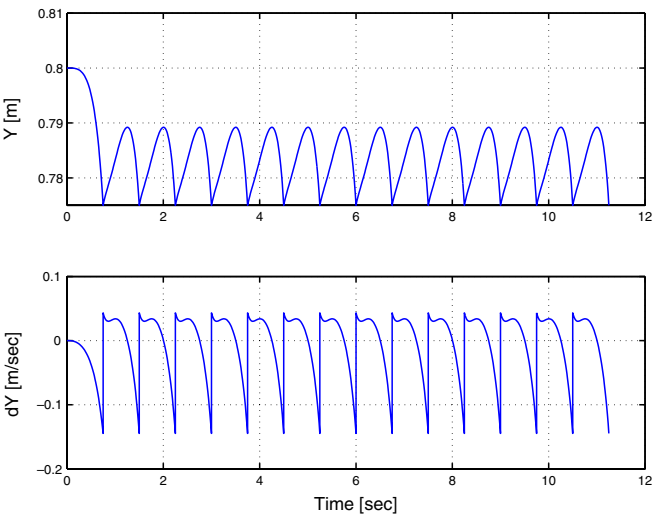


Fig. 18. (Sim 1) The y position and velocity versus time.

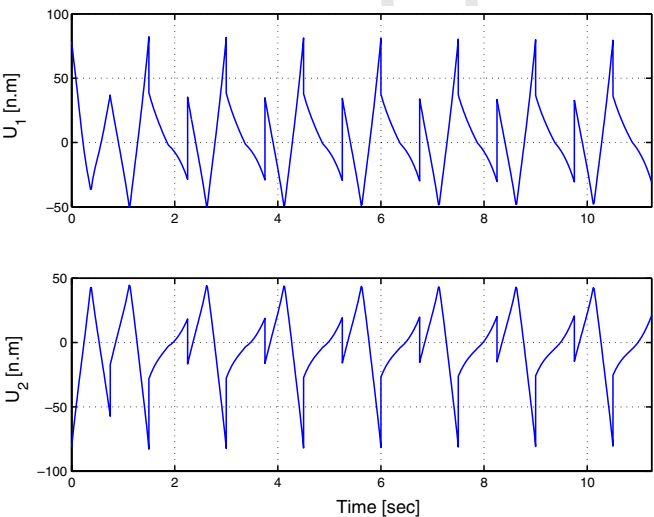


Fig. 19. (Sim 1) The torques of the femurs versus time.

the case it could be concluded that required actuators power is admissible. To compute the velocities of the motors and their absolute torques, based on articular velocities and the motors gear ratio which is of 50, the following formulas are used:

$$\begin{cases} v_{mot} \text{ [rpm]} = \frac{v_{art} \times 50 \times 60}{2 \times \pi} \\ \tau_{mot} \text{ [N m]} = \frac{\tau_{art}}{50} \end{cases} \quad (45)$$

where  $v_{mot}$  [rpm] is the velocity of the motor shaft,  $v_{art}$  [rad/s] is the relative velocity between the two adjacent links of the concerned articulation,  $\tau_{mot}$  [N m] is the motor torque, and  $\tau_{art}$  [N m] is the torque applied on the links.

The application of this verification technique is illustrated on Fig. 21, which depicts the shaft speed versus tor-

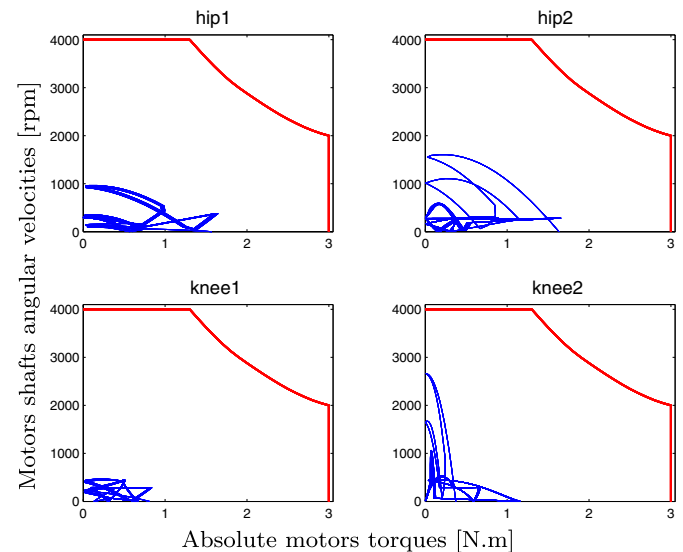


Fig. 21. (Sim 1) Absolute value of actuator angular velocities (revolutions per minute) versus absolute value of actuator torques [N m], and admissible region.

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787 que, for the four robot actuators, where it could be clearly  
 788 seen that the actuators power requirement is admissible.  
 789 The contact foot interaction forces with ground are plotted  
 790 versus time in Fig. 22, where we note that the condition of  
 791 the friction Coulomb's law is largely satisfied (this is clearly  
 792 seen through the amount of the ratio  $\lambda_t/\lambda_n$  with respect to  
 793 the friction coefficient which is  $\mu_0 = 0.7$ ). Fig. 23 illustrates  
 794 the movement of the robot by means of a set of walking  
 795 stick figures (for the three first steps).

### 796 5.2.3. Computation time evaluation

797 In order to evaluate the computation time of the pro-  
 798 posed control scheme, let us consider biped walking for  
 799 50 steps with a constant speed. The evaluation of the com-  
 800 puting time is displayed in Fig. 12, which represents the  
 801 evolution of the computing time versus cycles (steps). The  
 802 maximum value is given by  $t_{\max} = 0.97$  s. For real time

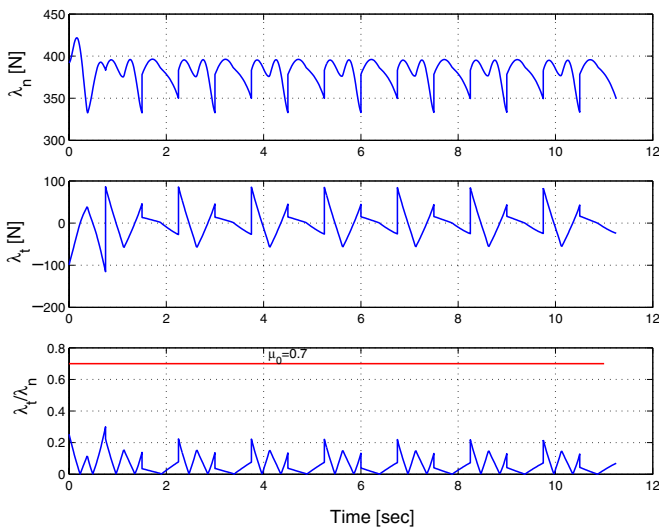


Fig. 22. (Sim 1) The stance foot interaction forces with ground and their ratio ( $\lambda_t/\lambda_n$ ).

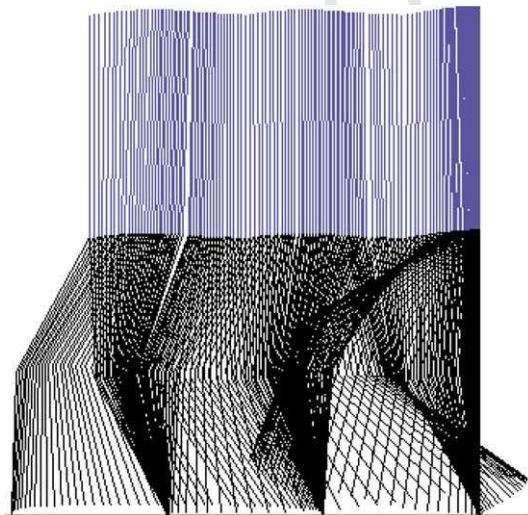


Fig. 23. (Sim 1) Stick figures of the walking robot.

803 implementation the on-line optimization is replaced by an  
 804 interpolation procedure. The basic idea is to define a grid  
 805 on the space  $(q_1, \dot{q}_1)$ , and for all the points the optimization  
 806 problem is resolved off-line to define the corresponding  
 807 optimization parameter  $p$ , so that at the end of the proce-  
 808 dure a look-up table is obtained. In the experiments this  
 809 look-up table is used to find, for the chosen configuration,  
 810 the optimization parameter at each sample time knowing  
 811 the initial condition (position and velocity) on the unactu-  
 812 ated coordinate.

### 5.3. Simulation 2: transition between different mean walking velocities

815 In this simulation, it is shown that the proposed feed-  
 816 back enables transitions between different mean walking  
 817 velocities to be easily obtained. To show this, 46 walking  
 818 cycles have been produced during which different desired  
 819 velocities of 0.24 m/s, 0.3 m/s and 0.40 m/s are successively  
 820 applied during 12, 14 and 20 cycles respectively. Because of  
 821 the proportional dependency between the duration of the  
 822 cycle  $t_f$  and the mean walking speed we have chosen to  
 823 change  $t_f$  under constant  $d = 0.3$  m in order to increase  
 824 (go faster), or to decrease (go slower) the walking speed.  
 825 Since the step length is 0.3 m, the choice of the cycle end-  
 826 time corresponding to the yet mentioned speeds (0.24,  
 827 0.3, and 0.40 m/s respectively) is (1.25, 1, and 0.75 s respec-  
 828 tively).

829 Fig. 24 shows the phase portrait of the unactuated coordi-  
 830 nate (torso), where it is well shown the transition between  
 831 the different stable limit cycles (each limit cycle is relative to  
 832 a walking speed). In Fig. 25 the mean walking speed is plot-  
 833 ted, showing thus the switching between the different pro-  
 834 posed walking speeds. The behavior of the unactuated  
 835 coordinate (torso) is illustrated in Fig. 26 giving its position  
 836 as well as its velocity versus time. The cartesian coordinates  
 837 (horizontal, respectively vertical) of the hips are depicted

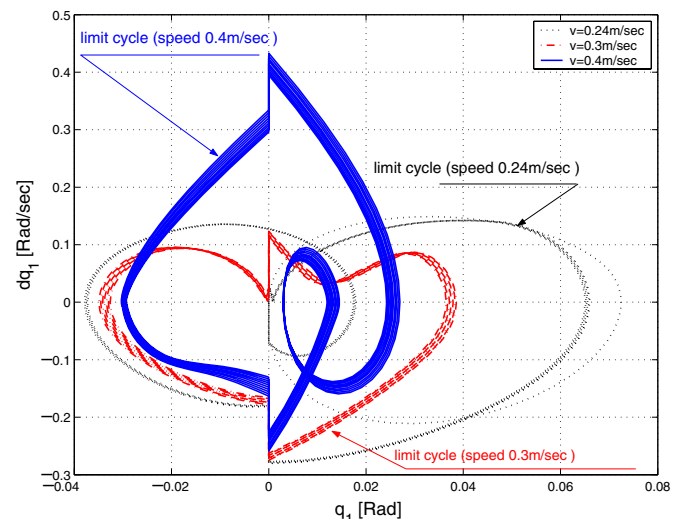


Fig. 24. (Sim 2) The phase portrait of the non actuated coordinate (torso).

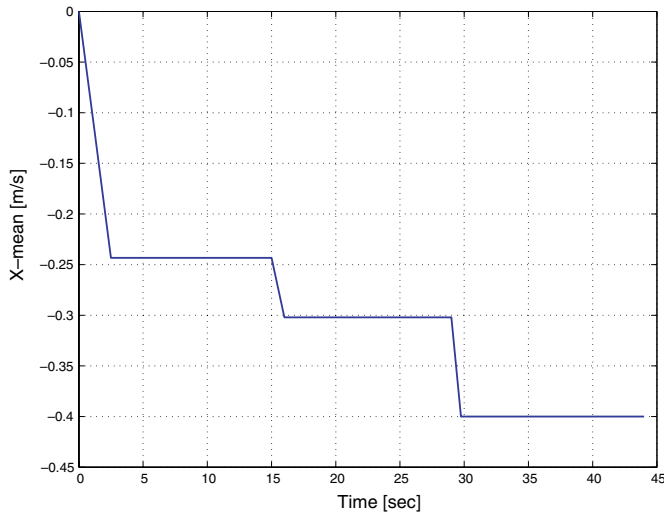


Fig. 25. (Sim 2) The mean walking velocity.

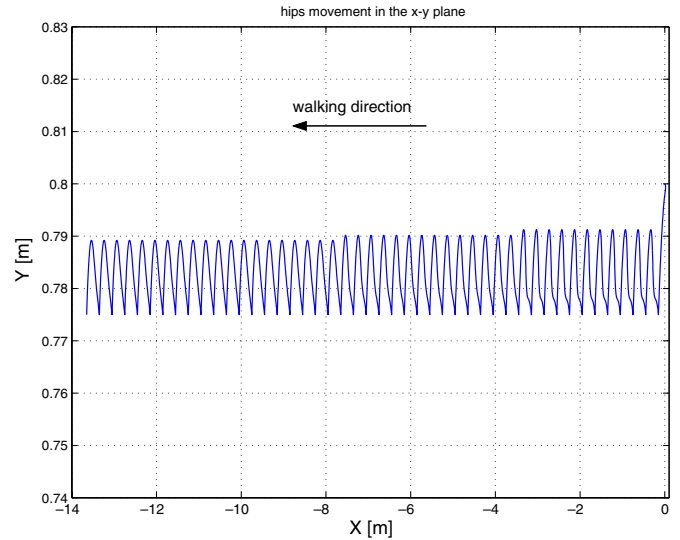


Fig. 27. (Sim 2) The hips movement trajectory.

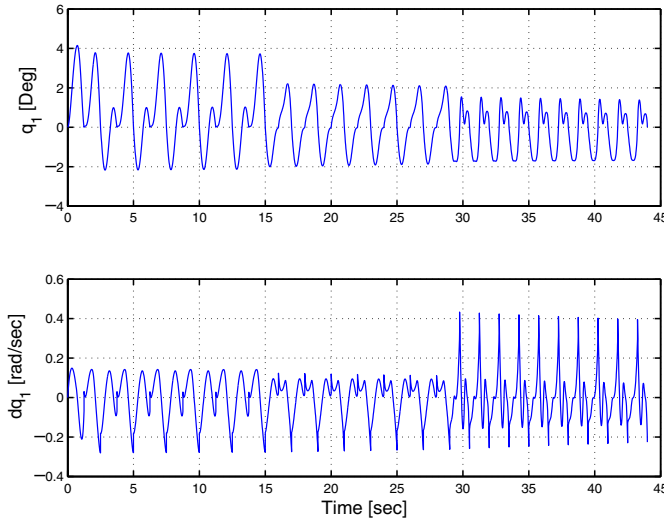


Fig. 26. (Sim 2) The torso position and velocity versus time.

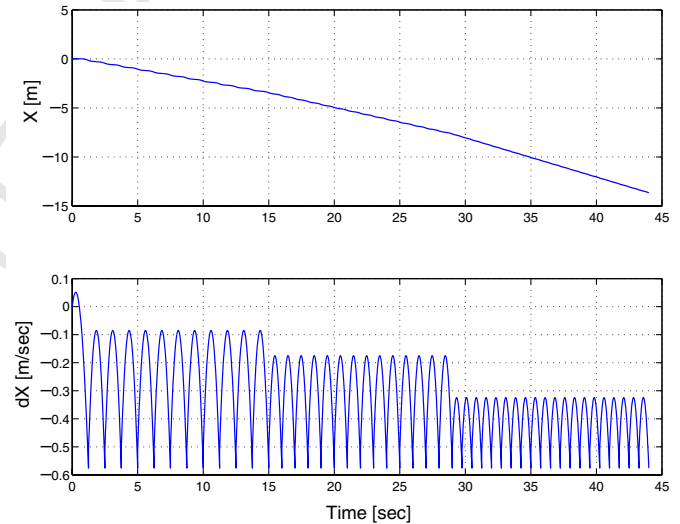


Fig. 28. (Sim 2) The x position and velocity versus time.

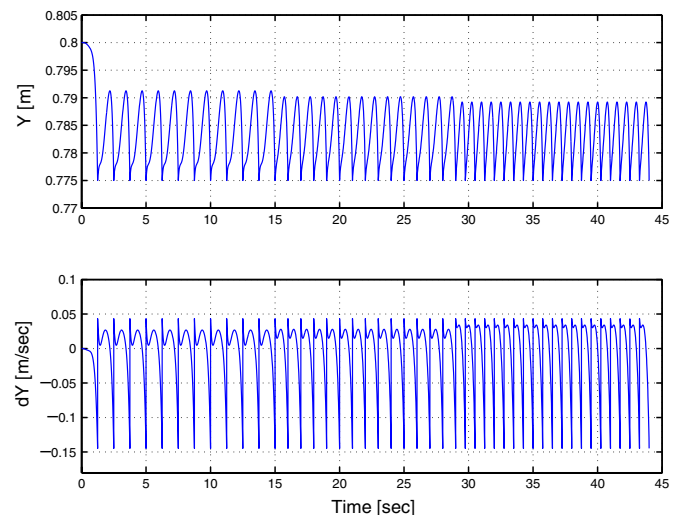


Fig. 29. (Sim 2) The y position and velocity versus time.

838 respectively in Figs. 28 and 29, and a more illustrative plot  
839 of the hip's trajectory in the sagittal plane is given in  
840 Fig. 27.

841 The generated torques are plotted in Figs. 32 and 33 for  
842 the femurs and tibias respectively, while in Fig. 30 it is  
843 checked that power requirement remain within the permitted  
844 limit. The contact forces with ground of the stance leg  
845 foot are depicted in Fig. 31, where we see clearly that the  
846 robot keeps contact with ground during walking.

#### 847 5.4. Simulation 3: robustness against parameters 848 uncertainty

849 In order to investigate the robustness of the proposed  
850 controller, let us introduce parameter uncertainties. The  
851 inertias of the robot links, namely  $I_1$  (the torso),  $I_3$  (the  
852 femur) and  $I_4$  (the tibia) cf. Table 1, are considered with  
853 an uncertainty of 10% of their nominal values, that is

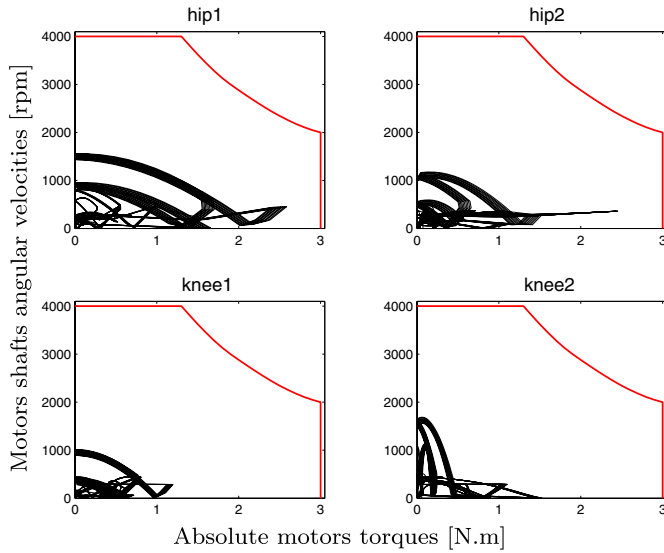


Fig. 30. (Sim 2) Absolute value of actuator angular velocities (revolutions per minute) versus absolute value of actuator torques [N.m], and admissible region.

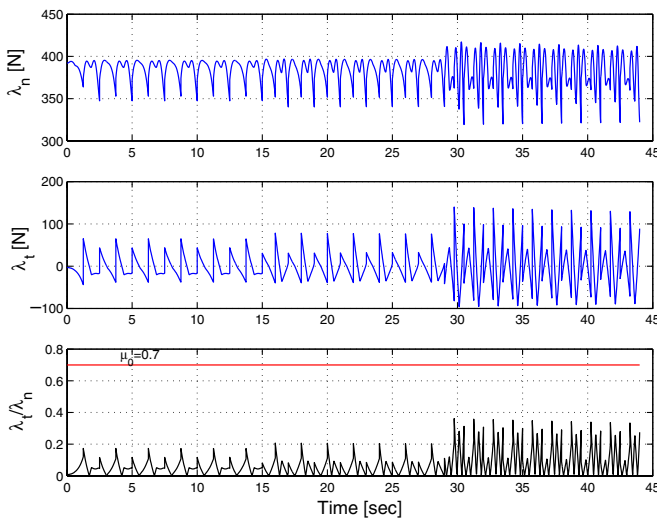


Fig. 31. (Sim 2) The stance foot interaction forces with ground and their ratio  $\lambda_t/\lambda_n$ .

$$855 \quad I_{1u} = I_1 + \Delta I_1; \quad I_{3u} = I_3 + \Delta I_3; \quad I_{4u} = I_4 + \Delta I_4$$

856 where the uncertainties  $\Delta I_i = 0.1I_i$ , for  $i \in \{1, 3, 4\}$ . Figs.  
857 34–39 present the corresponding simulation results over  
858 12 walking steps.

859 In Fig. 34 the positions and velocities of the first leg  
860 femur are plotted for the nominal system (solid line), as  
861 well as for the uncertain system (dashed line). Whereas in  
862 Fig. 35, the positions and velocities of the tibia are plotted.  
863 It can be seen clearly that the introduced uncertainty affects  
864 more the femur coordinates.

865 The behavior of the unactuated coordinate (torso) is  
866 represented in Fig. 36 which plots the evolution of its posi-  
867 tion and velocity versus time. A convergence to a new sta-  
868 ble cyclic trajectory is observed. This fact can be seen also

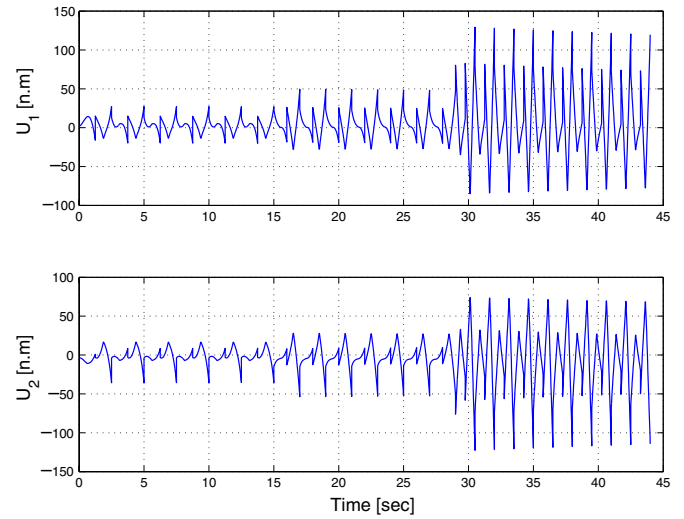


Fig. 32. (Sim 2) The torques of the femurs versus time.

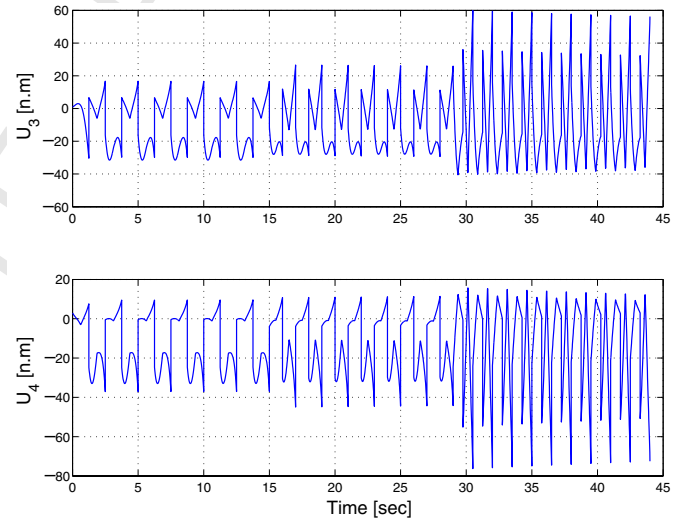


Fig. 33. (Sim 2) The torques of the tibias versus time.

869 on the phase portrait of Fig. 37, where a convergence to a  
870 neighborhood of a new limit cycle (different from that of  
871 the nominal system) of length 2 is observed for the uncertain  
872 system.

873 In Figs. 38 and 39 the control inputs of the robot are  
874 plotted, they represent the torques generated by the pro-  
875 posed controller for the femurs articulations (Fig. 38) and  
876 for the tibias articulations (Fig. 39). For both figures the  
877 uncertain system torques are slightly different from those  
878 of the nominal system.

879 Let us now consider an other test of the robustness of  
880 the proposed control approach. This time consider an  
881 uncertainty of 15% on the mass of the unactuated coordi-  
882 nate (torso), that is

$$884 \quad m_{1u} = m_1 + \Delta m_1; \quad \Delta m_1 = 15\%$$

885 To see the effect of the introduced uncertainty on the  
886 closed-loop system two figures are given. On Fig. 40 the



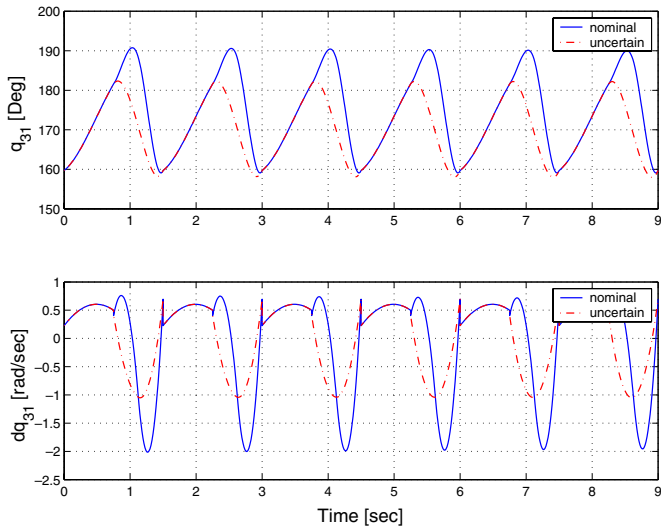


Fig. 34. (Sim 3) Position and velocity of the femur of the first leg.

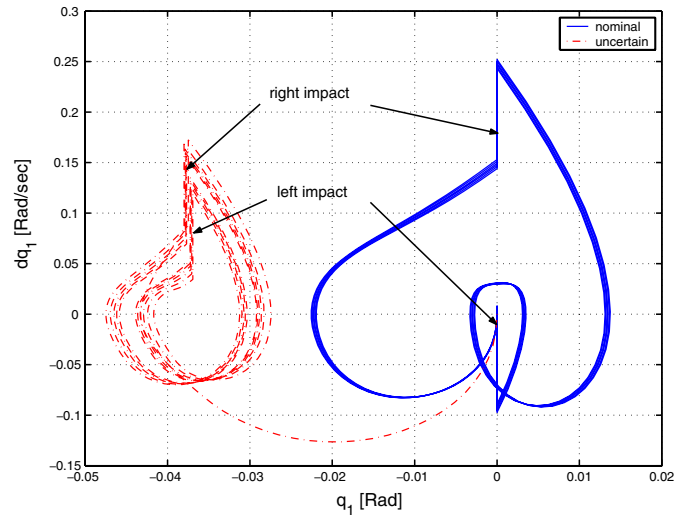


Fig. 37. (Sim 3) Phase portrait of the torso.

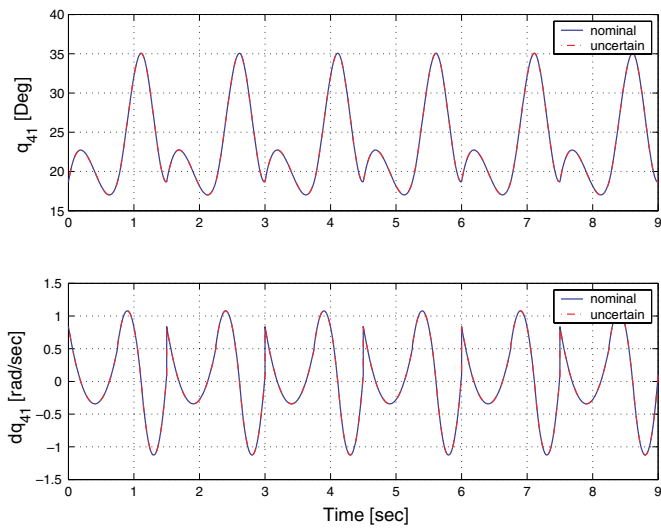


Fig. 35. (Sim 3) Position and velocity of the tibia of the first leg.

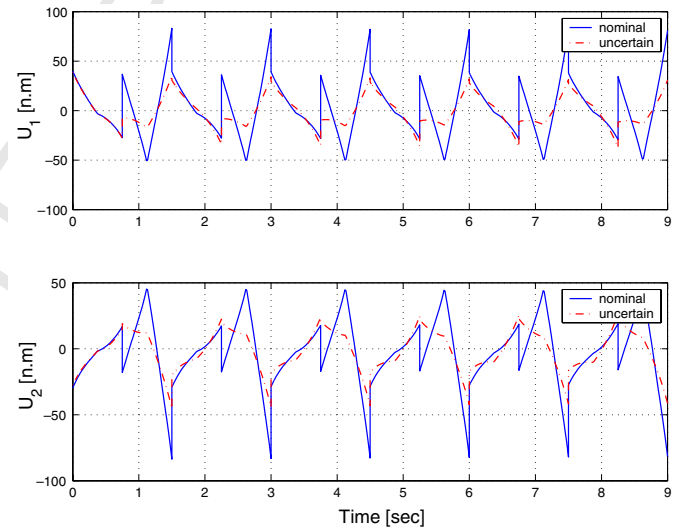


Fig. 38. (Sim 3) Torques of the femurs.

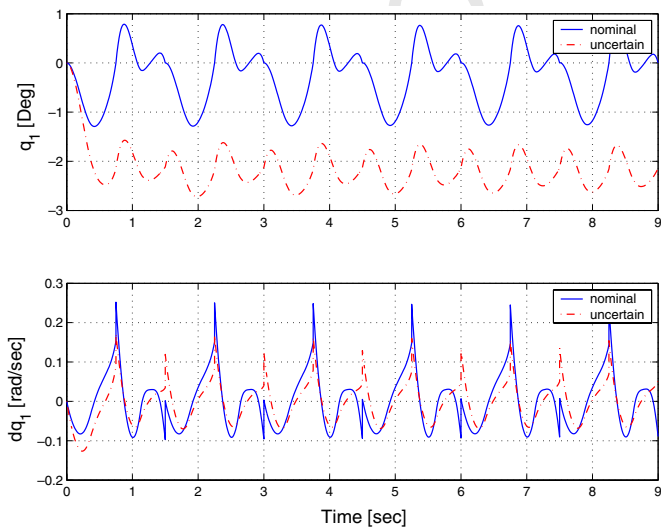


Fig. 36. (Sim 3) Position and velocity of the torso.

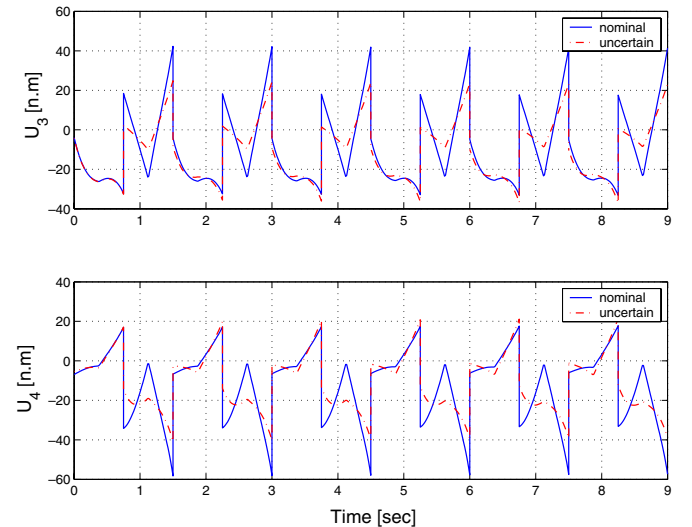


Fig. 39. (Sim 3) Torques of the tibias.

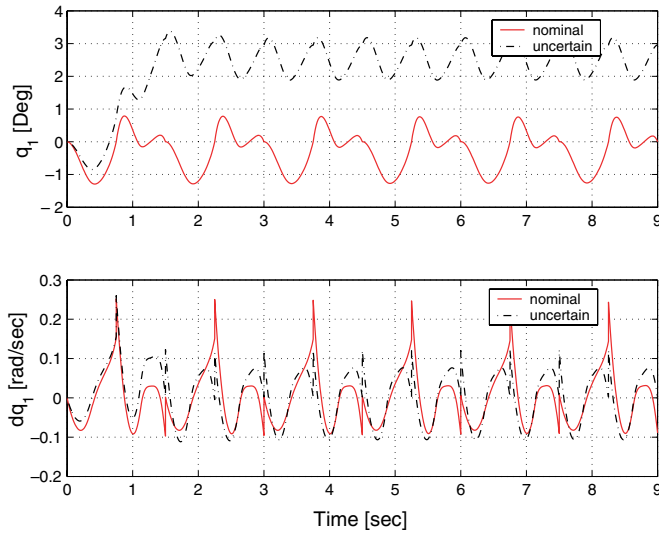


Fig. 40. (Sim 3) Position and velocity of the torso.

887 evolution of the position and the velocity of the torso are  
 888 displayed. Where it could be seen clearly the convergence  
 889 to a new cyclic trajectory for the uncertain system. An  
 890 other interesting point result in the periodicity of the velocity  
 891 trajectory which is of 1 cycle. This last fact could be better  
 892 observed on Fig. 41 of the phase portrait, where one  
 893 notices a convergence to an other stable limit cycle. However  
 894 the new limit cycle is of length 1, consequently the introduced  
 895 uncertainty on the torso masse has induced a deformation of  
 896 the limit cycle.

897 5.5. Simulation 4: robustness against ground irregularities

898 The aim is to investigate the robustness of the proposed  
 899 controller, against ground irregularities. Let us make the  
 900 robot walking on a horizontal surface with a stair at a distance  
 901 of 1.4 m from the robot, the stair is 1 cm height.

902 According to the robot configuration, namely the step  
 903 length  $d = 0.3$  m, the robot hits the stair during the fifth  
 904 walking step.

905 The approach parameters are the same as previous simulations  
 906 except the weighting matrix in the optimization criterion which  
 907 is chosen  $Q = \text{Diag}\{1, 0.1\}$ , and the step duration which is  
 908 of  $t_f = 1$  s now. This change in these two parameters  
 909 have been adopted because it gives a better results, namely  
 910 a configuration with these parameters choice is more robust  
 911 than the configuration with the old values.

912 The corresponding simulation results are depicted in  
 913 Figs. 42–47. It is worth to note that the impact is detected  
 914 at the instants  $t = 1$  s for all steps, except for the step  
 915 during which it hits the stair (fifth step), where the impact  
 916 instant corresponding to this last one is  $t = 0.935$  s, which  
 917 is before the expected time  $t = 1$  s.  
 918

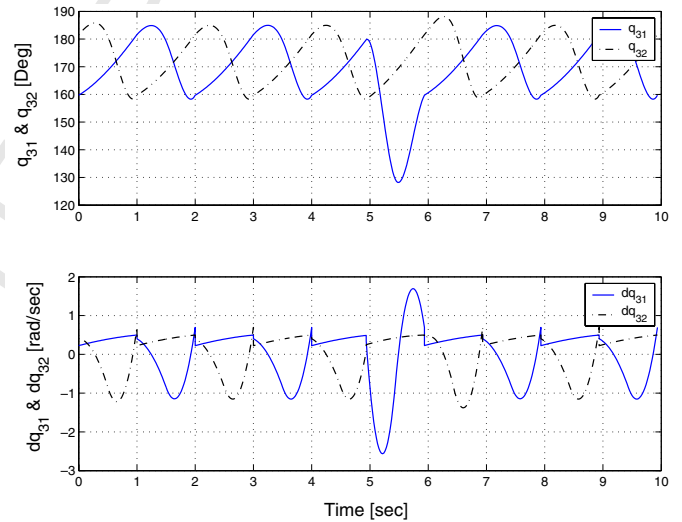


Fig. 42. (Sim 4) Position and velocity of the robot femurs.

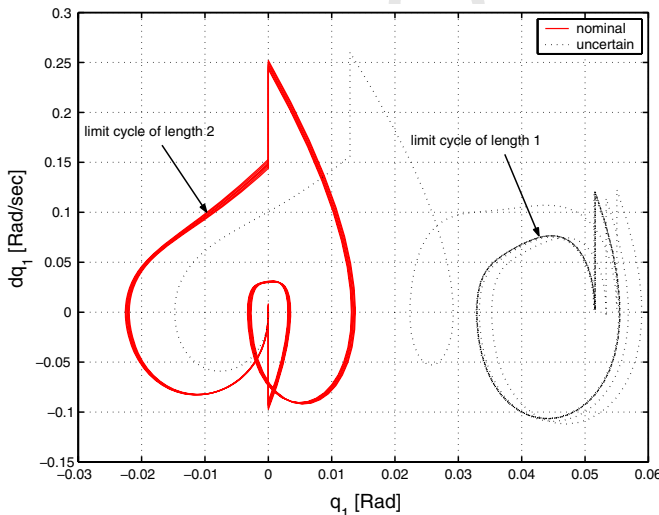


Fig. 41. (Sim 3) Phase portrait of the torso.

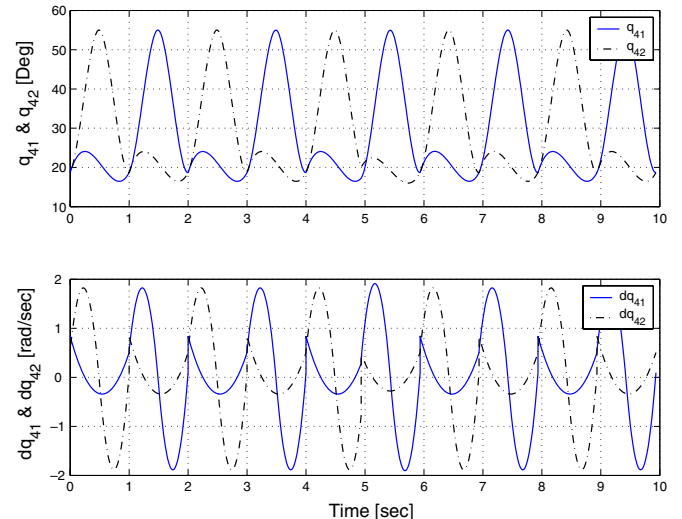


Fig. 43. (Sim 4) Position and velocity of the robot tibias.

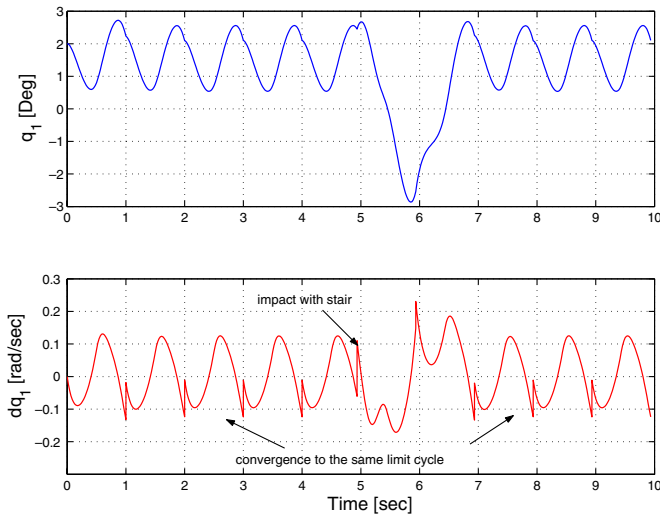


Fig. 44. (Sim 4) Position and velocity of the torso.

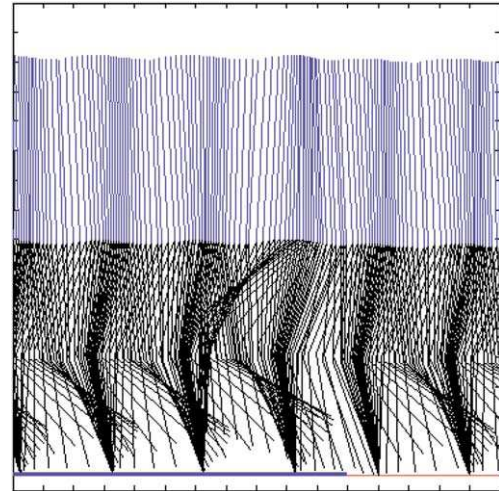


Fig. 47. (Sim 4) Stick figures of stair climbing.

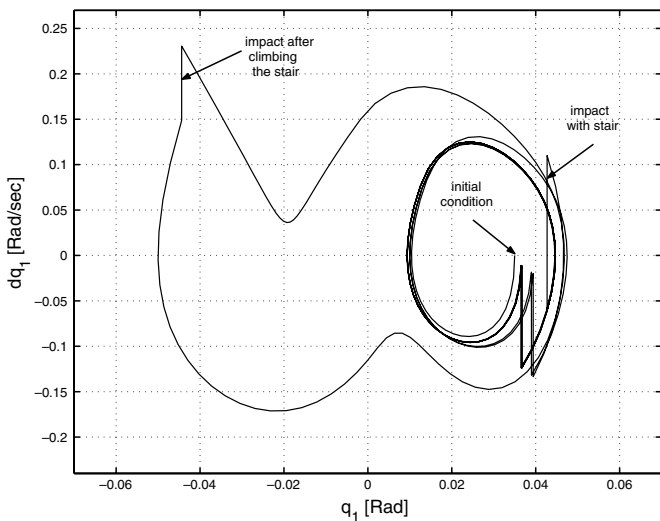


Fig. 45. (Sim 4) The phase portrait of the torso.

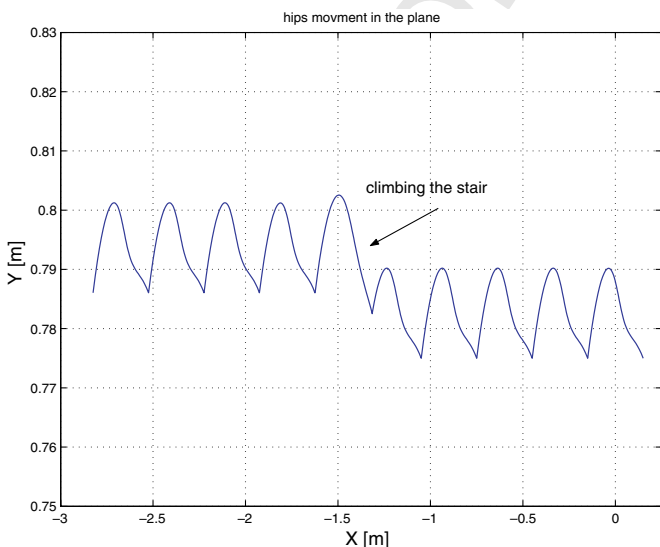


Fig. 46. (Sim 4) The hips trajectory in the sagittal plane.

Figs. 42 and 43 display the positions and velocities of the 919  
 actuated robot's limbs versus time. The effect of the unex- 920  
 pected impact could be seen on Fig. 42 as a removal from 921  
 the cyclic trajectory followed by a convergence to the same 922  
 trajectory. This effect can also be observed on the behavior 923  
 of the unactuated coordinate (torso). Especially Fig. 44 924  
 which shows the evolution of the angular position and 925  
 velocity of the torso, and Fig. 45 which displays its phase 926  
 portrait. On both figures a removal from the cyclic trajec- 927  
 tory is observed, followed by a convergence to the same 928  
 trajectory. It is worth to note that the limit cycle is of 929  
 length 1 for this simulation. Fig. 46 depicts the hips trajec- 930  
 tory in the sagittal plane, whereas Fig. 47 illustrates 931  
 through a stick figures of the robot postures the climbing 932  
 of the stair. 933

The limit of stair height beyond it the robot falls is closely 934  
 related to the chosen configuration. For the actual confi- 935  
 guration the limit is of 5 cm. Nevertheless this limit could 936  
 be increased by changing the configuration, or the 937  
 approach parameters. 938

## 6. Conclusion and future work 939

In this paper, a nonlinear low dimensional predictive 940  
 control approach is proposed for the control of RABBIT, a 941  
 walking five-link, seven d.o.f. under-actuated biped robot. 942  
 The basic idea of the approach is to split up the vector of 943  
 coordinates into actuated and unactuated variables. Then 944  
 on-line optimization is used to update reference trajectories 945  
 on the actuated coordinates, to aim to enhance the behavior 946  
 as well as the stability of the unactuated variables. 947

The stability analysis of the resulting closed-loop system 948  
 is carried out using a graphical tool based on the Poincaré 949  
 section. Sufficient conditions for the stability of the motion 950  
 are proposed and a concrete computation procedure is 951  
 given to estimate the corresponding region of attraction 952  
 related to the zero-dynamics of the closed-loop system. 953  
 The particular case of scalar predictive control is success- 954

955 fully investigated by simulation and a reasonable regions of  
956 attraction are obtained.

957 The resulting feedback seems to be real-time implement-  
958 able thanks to the low dimension of the optimization  
959 problem.

960 The whole framework is illustrated through simulation  
961 case studies. Indeed four simulations are proposed. In the  
962 first one walking at constant average speed starting from  
963 rest is investigated, while the second scenario concerns  
964 switching between different walking speeds. Robustness  
965 of the proposed nonlinear predictive based-upon controller  
966 is verified through the two last applications. In the first one  
967 a robot model including parameters uncertainties (namely  
968 uncertainties on inertias and masses) is considered, while  
969 in the second, ground irregularities are considered. In spite  
970 of these both significant disturbances the controller is able  
971 to guide the robot suitably for walking while keeping it sta-  
972 ble (no slipping, no falling).

973 In simulations, after each impact the reference trajecto-  
974 ries on actuated coordinates are computed based on a pre-  
975 defined step frequency. These trajectories are tracked while  
976 checking at each decision instant if there is impact (RABBIT  
977 prototype is equipped with switches at feet used to detect  
978 impacts). If an impact is detected, the two legs are re-  
979 labelled, the configuration of the robot is measured and a  
980 new step starts up.

981 Future works may include other features, that should be  
982 deeply investigated. In particular, one may be able to  
983 choose the design parameters  $t_f$ ,  $z_2^f$  and  $Q$  in order to opti-  
984 mize some desired feature (mean energy, torque, robust-  
985 ness). However, the key future work is naturally the  
986 experimentations. This is currently in progress.

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