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**RIGOROUS ASYMPTOTICS FOR STEADY STATE VOLTAGE  
 POTENTIALS IN A BIDIMENSIONAL HIGHLY CONTRASTED  
 MEDIUM**

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ABSTRACT. We study the behavior of steady state voltage potentials in a bidimensional highly contrasted medium composed of a conducting cytoplasm surrounded by an insulating membrane of thickness  $h$ . We provide a rigorous derivation of the first two terms of the asymptotic expansion of steady state voltage potentials as  $h$  tends to zero. The first two terms of the potential in the membrane are given explicitly in local coordinates in terms of the boundary data, while the first terms of the cytoplasmic potential are the solutions of the so-called dielectric formulation with appropriate boundary conditions given in terms of the boundary data. The error estimates are given in terms of the conductivities and of the boundary data.

INTRODUCTION

We study in this paper the behavior of the solution of steady state voltage potentials in a bidimensional highly contrasted medium. This work is the generalization to a domain of class  $\mathcal{C}^2$  of arbitrary shape of the asymptotic expansion performed by the author in the case of a circular domain [12] for the so-called dielectric formulation with Neumann boundary condition (see Propositions III.1, III.2 and Corollary III.1 of [12]). The motivation of the present work and of [12] comes from numerical problems raised by the researchers in computational electromagnetics of CEGELY<sup>1</sup>, who want to compute the electric field in the biological cell. Because of its unusual dielectric parameters, the computation of the vector wave equation (see [12]) leads to matrices with very small coefficients, which are not easily invertible with the presently available numerical methods. To avoid these numerical difficulties, they used to neglect the curl part of the electric field. They compute the solution of the so called dielectric formulation (see [12]), which gives the steady state potentials. In the present paper, we give rigorous asymptotics of these potentials. In Fig. 1 we give the dielectric and geometric parameters of the biological cell. The permeability  $\mu_0$  and the permittivity  $\epsilon_0$  are constant, equal to:

$$\mu_0 = 4\pi 10^{-7}, \quad \epsilon_0 = 8.85 10^{-11}.$$

In [12], we denoted by  $q_h$  the following piecewise constant function, giving the non

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<sup>1</sup>The CEGELY is the center of electrical engineering of Lyon (France).

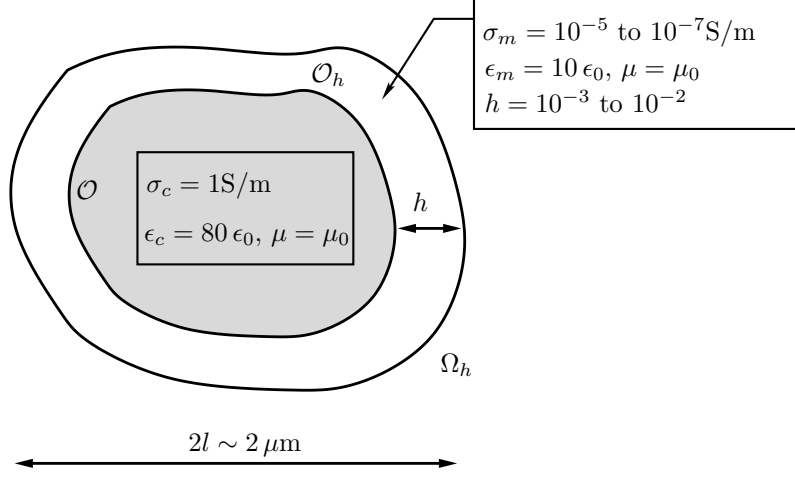


FIGURE 1. Typical parameters of the biological cell.

dimensionalized complex permittivity:

$$q_h = \begin{cases} q_c = l^2 \omega^2 \mu (\epsilon_c - i \sigma_c / \omega), & \text{in } \mathcal{O}, \\ q_m = l^2 \omega^2 \mu (\epsilon_m - i \sigma_m / \omega), & \text{in } \mathcal{O}_h. \end{cases}$$

As has been proved in [12] in the case of a circular biological cell, up to 100GHz the dielectric formulation gives an approximation of the solution of the vector wave equation; in this frequency range,  $|q_c| + |q_m|$  is at most  $10^{-2}$  and the relative error is of order  $|q_c| + |q_m|$ . This is the reason why, in this paper, we focus on the dielectric formulation. In upcoming papers, the author deals with the vector wave equation in a smooth domain of arbitrary shape; the circular case for the vector wave equation is treated in Section IV of [12].

Let  $\Omega_h$  be a smooth bounded bidimensional domain, composed of a smooth domain  $\mathcal{O}$  surrounded by a thin membrane  $\mathcal{O}_h$  with a small constant thickness  $h$  (see Fig. 1):

$$\Omega_h = \mathcal{O} \cup \mathcal{O}_h.$$

Since we impose a Neumann boundary condition on  $\partial\Omega_h$  the data  $\phi$  must satisfy the compatibility condition:

$$\int_{\partial\Omega_h} \phi \, d\sigma = 0.$$

Let  $q_c$  and  $q_m$  be two non vanishing complex numbers. We denote by  $q_h$  the following piecewise constant function:

$$\forall x \in \Omega_h, \quad q_h(x) = \begin{cases} q_c, & \text{if } x \in \mathcal{O}, \\ q_m & \text{if } x \in \mathcal{O}_h. \end{cases}$$

We would like to understand the behavior for  $h$  tending to zero of the solution  $V$  of Problem (1) with Neumann boundary condition:

$$(1a) \quad \operatorname{div}(q_h \operatorname{grad} V) = 0 \text{ in } \Omega_h, \text{ in the sense of distributions,}$$

$$(1b) \quad \frac{\partial V}{\partial n} = \phi \text{ on } \partial\Omega_h.$$

To determine  $V$ , we impose the following gauge condition on the boundary of the cytoplasm:

$$(1c) \quad \int_{\partial\mathcal{O}} V \, d\sigma = 0.$$

Several authors have worked on similar problems (see for instance Beretta *et al.* [4] and [5]). They compared the exact solution to the so-called background solution defined by replacing the material of the membrane by cytoplasmic material. The difference between these two solutions has then been given through an integral involving the polarization tensor defined for instance in [1], [2], [4], [5], [6], plus some remainder terms. The remainder terms are estimated in terms of the measure of the inhomogeneity. In this paper, we do not use this approach, for several reasons.

First of all, the Beretta *et al.* estimate of the remainder terms depends linearly on  $|q_m|$ ,  $|q_c|$  and  $|q_c/q_m|$ . Here the ratio  $|q_m/q_c|$  varies from  $10^{-5}$  to  $10^{-2}$  according to the frequency. Secondly,  $q_m$  and  $q_c$  are complex-valued, hence differential operators involved in our case are not self-adjoint, so that the  $\Gamma$ -convergence techniques of Beretta *et al.* do not apply. Thirdly, the potential in the membrane is not given explicitly in [4], [5] or [6], while we are definitely interested in this potential, in order to obtain the transmembranar potential (see Fear and Stuchly [9]). In this paper we work with bidimensional domain and we expect that the same analysis could be performed in higher dimensions.

The heuristics of this work consist in performing a change of coordinates in the membrane  $\mathcal{O}_h$ , so as to parameterize it by local coordinates  $(\eta, \theta)$ , which vary in a domain independently of  $h$ ; in particular, if we denote by  $L$  the length of  $\partial\mathcal{O}$ , the variables  $(\eta, \theta)$  should vary in  $[0, 1] \times \mathbb{R}/L\mathbb{Z}$ . This change of coordinates leads to an expression of the Laplacian in the membrane, which depends on  $h$ . Once the transmission conditions of the new problem are derived, we perform a formal asymptotic expansion of the solution of (1) in terms of  $h$ . It remains to validate this expansion.

This paper is structured as follows. In Section 1, we make precise our geometric conventions. We perform a change of variables in the membrane, and with the help of some differential geometry results, we write Problem (1) in the language of differential forms. We refer the reader to the book of Flanders [10] or those of Dubrovin *et al.* [8] (or [7] for the french version) for courses on differential geometry. We derive transmission conditions in the intrinsic language of differential forms, and we have to express these relations in local coordinates: euclidean coordinates in the cytoplasm, and local  $(\eta, \theta)$  coordinates in the membrane, and this is what we do in Section III. In Section 3, we derive formally the first two terms of the asymptotic expansion of the solution of our problem in terms of  $h$ . Section 4 is devoted to a proof of the estimate of the error. In the Appendix, we give some useful differential geometry formulae.

## 1. GEOMETRY STATEMENT

The boundary of the domain  $\mathcal{O}$  is assumed to be of class  $\mathcal{C}^2$ . The orientation of the boundary  $\partial\mathcal{O}$  is the trigonometric orientation and we denote by  $\partial_t$  the tangential derivative along  $\mathcal{O}$ . To simplify, we suppose that the length of  $\partial\mathcal{O}$  is equal to  $2\pi$ . We denote by  $\mathbb{T}$  the flat torus:

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$

Since  $\partial\mathcal{O}$  is of class  $\mathcal{C}^2$ , we can parameterize it by a function  $\Psi$  of class  $\mathcal{C}^2$  from  $\mathbb{T}$  to  $\mathbb{R}^2$  satisfying:

$$\forall \theta \in \mathbb{T}, \quad |\Psi'(\theta)| = 1.$$

Therefore the following identities holds:

$$\partial\mathcal{O} = \{\Psi(\theta), \theta \in \mathbb{T}\},$$

and

$$\partial\Omega_h = \{\Psi(\theta) + hn(\theta), \theta \in \mathbb{T}\}.$$

Here  $n(\theta)$  is the unitary exterior normal at  $\Psi(\theta)$  to  $\partial\mathcal{O}$ . The boundary  $\partial\Omega_h$  of the cell is parallel to the boundary  $\partial\mathcal{O}$  of the cytoplasm. We parameterize the membrane  $\mathcal{O}_h$  as follows:

$$\mathcal{O}_h = \{\Psi(\theta) + h\eta n(\theta), (\eta, \theta) \in ]0, 1[ \times \mathbb{T}\}.$$

We define now:

$$\Phi(\eta, \theta) = \Psi(\theta) + h\eta n(\theta).$$

Let us denote by  $\kappa$  the curvature of  $\partial\mathcal{O}$ . Let  $h_0$  belong to  $(0, 1)$  such that:

$$(2) \quad h_0 < \frac{1}{\|\kappa\|_\infty}.$$

Then, for all  $h$  in  $[0, h_0]$ , there exists an open interval  $I$  containing  $(0, 1)$  such that  $\Phi$  is a diffeomorphism of class  $\mathcal{C}^2$  from  $I \times \mathbb{R}/2\pi\mathbb{Z}$  to its image, which is a neighborhood of the membrane. The metric in  $\mathcal{O}_h$  is:

$$(3) \quad h^2 d\eta^2 + (1 + h\eta\kappa)^2 d\theta^2.$$

Thus, we use two systems of coordinates, depending on the domains  $\mathcal{O}$  and  $\mathcal{O}_h$ : in the interior domain  $\mathcal{O}$ , we use Euclidean coordinates  $(x, y)$  and in the membrane  $\mathcal{O}_h$ , we use local  $(\eta, \theta)$  coordinates with metric (3). Now, we translate into the language of differential forms Problem (1). We refer the reader to the book of Dubrovin, Fomenko and Novikov [8] or the book of Flanders [10] for the definition of the exterior derivative denoted by  $d$ , the exterior product denoted by  $\text{ext}$ , the interior derivative denoted by  $\delta$  and the interior product denoted by  $\text{int}$ . In the Appendix, we give the formulae describing these operators in the case of a general 2D metric. Our aim, while rewriting Problem (1) is to take into account nicely the change of coordinates in the thin membrane.

Let  $\mathbb{V}$  be the 0-form on  $\Omega_h$  such that, in the Euclidean coordinate  $(x, y)$ ,  $\mathbb{V}$  is equal to  $V$ , and let  $F$  be the 0-form, which is equal to  $\phi$  on  $\partial\Omega_h$ . We denote by  $\mathbb{N}$

the 1-form corresponding to the inward unit normal on the boundary  $\Omega_h$  (see for example the book Gilkey and *al.* [11] p.33):

$$\begin{aligned}\mathbb{N} &= N_x dx + N_y dy, \\ &= N_\eta d\eta.\end{aligned}$$

$\mathbb{N}^*$  is the inward unit normal 1-form. Problem (1) takes now the intrinsic form:

$$\begin{aligned}(4a) \quad & \delta(q_h d\mathbb{V}) = 0, \text{ in } \Omega_h, \\ (4b) \quad & \text{int}(\mathbb{N}^*) d\mathbb{V} = F, \text{ on } \partial\Omega_h.\end{aligned}$$

According to Green's formula (Lemma 1.5.1 of [11]), we obtain the following transmission conditions for  $\mathbb{V}$  along  $\partial\mathcal{O}$ :

$$(4c) \quad \begin{aligned}q_c \text{int}(\mathbb{N}^*) d\mathbb{V}|_{\partial\mathcal{O}} &= q_m \text{int}(\mathbb{N}^*) d\mathbb{V}|_{\partial\mathcal{O}_h \setminus \partial\Omega_h}, \\ \text{ext}(\mathbb{N}^*) \mathbb{V}|_{\partial\mathcal{O}} &= \text{ext}(\mathbb{N}^*) \mathbb{V}|_{\partial\mathcal{O}_h \setminus \partial\Omega_h}.\end{aligned}$$

## 2. STATEMENT OF THE PROBLEM

In this section, we write Problem (4) in local coordinates. It is convenient to write:

$$\forall \theta \in \mathbb{T}, \quad \Phi_0(\theta) = \Phi(0, \theta), \quad \Phi_1(\theta) = \Phi(1, \theta).$$

We denote by  $f$  the function, defined on the torus  $\mathbb{T}$  by

$$\forall \theta \in \mathbb{T}, \quad f(\theta) = \phi \circ \Phi_1(\theta),$$

and we denote by  $f$  the function defined on  $\partial\mathcal{O}$  by

$$(5) \quad \forall x \in \partial\mathcal{O}, \quad f = f \circ \Phi_0^{-1}(x).$$

Let us denote by  $V^c$  the potential  $V$  in  $\mathcal{O}$ , written in Euclidean coordinates, and by  $V^m$  the potential  $V$  in  $\mathcal{O}_h$  in the local coordinates:

$$\begin{aligned}V^c &= V, \text{ in } \mathcal{O}, \\ V^m &= V \circ \Phi, \text{ in } [0, 1] \times \mathbb{T}.\end{aligned}$$

Using the expressions of the differential operators  $d$  and  $\delta$ , which are respectively the exterior and the interior derivatives (see the Appendix), applied to the metric (3), we can see that the Laplacian applied to  $V$  in the membrane is given in the local coordinates  $(\eta, \theta)$  by:

$$(6) \quad \begin{aligned}\forall (\eta, \theta) \in [0, 1] \times \mathbb{T}, \\ (\Delta V)|_{\Phi(\eta, \theta)} &= \frac{1}{h(1+h\eta\kappa)} \partial_\eta \left( \frac{1+h\eta\kappa}{h} \partial_\eta V^m \right) + \frac{1}{1+h\eta\kappa} \partial_\theta \left( \frac{1}{1+h\eta\kappa} \partial_\theta V^m \right).\end{aligned}$$

Therefore, we rewrite Problem (4) as follows:

$$(7a) \quad \begin{aligned}\Delta V^c &= 0, \text{ in } \mathcal{O}, \\ \forall (\eta, \theta) \in [0, 1] \times \mathbb{T},\end{aligned}$$

$$(7b) \quad \frac{1}{h^2} \partial_\eta \left( (1+h\eta\kappa) \partial_\eta V^m \right) + \partial_\theta \left( \frac{1}{1+h\eta\kappa} \partial_\theta V^m \right) = 0,$$

with the following transmission conditions according to (4c) translated into local coordinates

$$(7c) \quad q_c \partial_n V^c \circ \Phi_0 = \frac{q_m}{h} \partial_\eta V^m \Big|_{\eta=0},$$

$$(7d) \quad V^c \circ \Phi_0 = V^m \Big|_{\eta=0},$$

with the boundary condition according to (4b) translated into local coordinates

$$(7e) \quad (1 + h\kappa) \partial_\eta V^m \Big|_{\eta=1} = hf.$$

To determine completely  $(V^c, V^m)$ , we impose the same gauge condition as in (1c):

$$\int_{\partial\mathcal{O}} V \, d\sigma = 0.$$

### 3. FORMAL ASYMPTOTIC EXPANSION

In this section, we derive asymptotic expansions of the potentials  $(V^c, V^m)$  solution of (7) in terms of the thickness  $h$ . We write the following ansatz:

$$(8a) \quad V^c = V_0^c + hV_1^c + \dots,$$

$$(8b) \quad V^m = V_0^m + hV_1^m + \dots.$$

We multiply (7b) by  $h^2(1 + h\eta\kappa)^2$  and we order the powers of  $h$  to obtain:

$$(9) \quad \begin{aligned} & \forall(\eta, \theta) \in [0, 1] \times \mathbb{T}, \\ & \partial_\eta^2 V^m + h\kappa \{3\eta \partial_\eta^2 V^m + \partial_\eta V^m\} + h^2 \{3\eta^2 \kappa^2 \partial_\eta^2 V^m + 2\eta \kappa^2 \partial_\eta V^m + \partial_\theta^2 V^m\} \\ & + h^3 \{\eta^3 \kappa^3 \partial_\eta^2 V^m + \eta^2 \kappa^3 \partial_\eta V^m + \eta \kappa \partial_\theta^2 V^m - \eta \kappa' \partial_\theta V^m\} = 0 \end{aligned}$$

We are now ready to derive the first two terms of the asymptotic expansions of  $V^c$  and  $V^m$  by identifying the terms of the same power in  $h$ .

**First step.** Substituting in (9) the potential  $V^m$  by its expansion (8b), and using the boundary condition (7e), we obtain:

$$\begin{cases} \partial_\eta^2 V_0^m = 0, \\ \partial_\eta V_0^m \Big|_{\eta=1} = 0. \end{cases}$$

Thus, we obtain:

$$(10) \quad \forall(\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad V_0^m(\eta, \theta) = V_0^m(\theta).$$

We will determine  $V_0^m$  in the following.

**Second step.** Substituting in (9) the potential  $V^m$  by its expansion (8b), and using boundary condition (7e) and equality (10), we obtain:

$$\begin{cases} \partial_\eta^2 V_1^m = 0, \\ \partial_\eta V_1^m \Big|_{\eta=1} = f. \end{cases}$$

Thus, we infer:

$$(11) \quad \forall(\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad \partial_\eta V_1^m(\eta, \theta) = f(\theta).$$

Substituting in (7a) the potential  $V^c$  by its expansion (8a), and substituting in the transmission conditions (7c) expression (11) of  $\partial_\eta V_1^m$  we obtain:

$$(12a) \quad \begin{cases} \Delta V_0^c = 0, \text{ in } \mathcal{O}, \\ \partial_n V_0^c|_{\partial\mathcal{O}} = (q_m/q_c)f, \end{cases}$$

with gauge condition:

$$(12b) \quad \int_{\partial\mathcal{O}} V_0^c \, d\sigma = 0.$$

According to the transmission condition (7d),  $V_0^m$  is equal to:

$$(13) \quad \forall (\eta, \theta) \in [0, L] \times \mathbb{T}, \quad V_0^m(\eta, \theta) = V_0^c \circ \Phi_0(\theta).$$

We have determined  $V_0^c$  and  $V_0^m$ .

**Third step.** As in the previous paragraph, substituting in (9) the potential  $V^m$  by its expansion (8b) and using equalities (10)–(11), we obtain:

$$(14a) \quad \partial_\eta^2 V_2^m + \kappa f + \partial_\theta^2 V_0^m = 0,$$

$$(14b) \quad \partial_\eta V_2^m|_{\eta=1} = -\kappa f,$$

and hence integrating (14) with respect to  $\eta$  the following equality holds

$$(15) \quad \partial_\eta V_2^m = -\eta\kappa f + (1-\eta)\partial_\theta^2 V_0^m.$$

By the transmission condition (7c) and equality (13),  $V_1^c$  is the solution of:

$$(16a) \quad \begin{cases} \Delta V_1^c = 0, \text{ in } \mathcal{O}, \\ \partial_n V_1^c|_{\partial\mathcal{O}} = (q_m/q_c)\partial_t^2 V_0^c, \end{cases}$$

with gauge condition

$$(16b) \quad \int_{\partial\mathcal{O}} V_1^c \, d\sigma = 0.$$

Integrating (11) with respect to  $\eta$  we obtain the value of  $V_1^m$ :

$$(17a) \quad \forall (\eta, \theta) \in [0, L] \times \mathbb{T}, \quad V_1^m(\eta, \theta) = \eta f + V_1^m(0, \theta),$$

with  $V_1^m$  determined by the transmission condition (7d)

$$(17b) \quad \forall \theta \in \mathbb{T}, \quad V_1^m(0, \theta) = V_1^c \circ \Phi_0(\theta).$$

We have given the first two terms of the asymptotic expansion of  $V^c$  and  $V^m$ . It remains to prove that the remainder terms are small.

#### 4. ERROR ESTIMATES

We give an error estimate, which proves that the first two terms found in Section 3 by a formal argument are indeed the first terms, in the sense that the remainder is smaller. We have the following theorem.

**Theorem 4.1.** *We remember that  $h_0$  is defined in (2). Let  $h$  be in  $(0, h_0)$ . Let  $q_c$  and  $q_m$  be two non vanishing complex numbers such that:*

$$(18) \quad \Re(q_c/q_m) > 0.$$

*Let  $\phi$  be in  $H^3(\partial\Omega_h)$ . We remember that  $f$  is defined in (5). We denote by  $V$  the solution of Problem (1), and  $V_0^c$ ,  $V_0^m$ ,  $V_1^c$ , and  $V_1^m$  are defined in Section 3 respectively by equalities (12)–(13)–(16)–(17). More precisely  $V_0^c$  is the solution of the following problem:*

$$\begin{cases} \Delta V_0^c = 0, \text{ in } \mathcal{O}, \\ \partial_n V_0^c|_{\partial\mathcal{O}} = (q_m/q_c)f, \end{cases}$$

$V_1^c$  satisfies

$$\begin{cases} \Delta V_1^c = 0, \text{ in } \mathcal{O}, \\ \partial_n V_1^c|_{\partial\mathcal{O}} = (q_m/q_c)\partial_t^2 V_0^c, \end{cases}$$

with gauge conditions:

$$\int_{\partial\mathcal{O}} V_0^c \, d\sigma = 0, \quad \int_{\partial\mathcal{O}} V_1^c \, d\sigma = 0.$$

$V_0^m$  and  $V_1^c$  are defined in  $[0, 1] \times \mathbb{T}$  by:

$$\begin{aligned} \forall(\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad V_0^m(\eta, \theta) &= V_0^c \circ \Phi_0(\theta), \\ \forall(\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad V_1^m(\eta, \theta) &= \eta f + V_1^c \circ \Phi_0(\theta). \end{aligned}$$

Let  $W$  be the function defined on  $\Omega_h$  by:

$$W = \begin{cases} V - (V_0^c + hV_1^c), \text{ in } \mathcal{O}, \\ V - (V_0^m \circ \Phi^{-1} + hV_1^m \circ \Phi^{-1}), \text{ in } \mathcal{O}_h. \end{cases}$$

Then, there exists a constant  $C_{\mathcal{O}} > 0$  depending only on the domain  $\mathcal{O}$  such that

$$\|W\|_{H^1(\Omega_h)} \leq C_{\mathcal{O}} \frac{1 + |q_m/q_c|}{\min(\Re(q_c/q_m), 1)} h^{3/2} \|f\|_{H^3(\partial\mathcal{O})}.$$

*Proof.* The proof of Theorem 4.1 is based on estimates of the tangential and the second tangential derivatives of  $V_0^m|_{\eta=0}$  and  $V_1^m|_{\eta=0}$ .

Denote by  $W^c$  and  $W^m$  the following functions:

$$(19) \quad W^c = V^c - (V_0^c + hV_1^c), \text{ in } \mathcal{O},$$

$$(20) \quad W^m = V^m - (V_0^m + hV_1^m), \text{ in } [0, 1] \times \mathbb{T}.$$

In order to simplify the notations, we introduce  $\mathcal{L}_{\eta, \theta}$ , the Laplacian in the local coordinates  $(\eta, \theta)$  given by (6):

$$\mathcal{L}_{\eta, \theta} = \frac{1}{h(1 + h\eta\kappa)} \left( \frac{1}{h} \partial_\eta ((1 + h\eta\kappa) \partial_\eta) + \partial_\theta \left( \frac{h}{1 + h\eta\kappa} \partial_\theta \right) \right).$$



Let us write the problem satisfied by  $(W^c, W^m)$ . We use the expressions of  $V_0^c$ ,  $V_1^c$ ,  $V_0^m$  and  $V_1^m$  found in Section 3 to obtain, by a simple calculation:

$$(21a) \quad \Delta W^c = 0, \text{ in } \mathcal{O},$$

$$\forall(\eta, \theta) \in [0, 1] \times \mathbb{T},$$

$$(21b) \quad \begin{aligned} \mathcal{L}_{\eta, \theta} W^m = & -\frac{1}{h(1+h\eta\kappa)} \left\{ h\kappa f + \frac{h}{1+h\eta\kappa} \partial_\theta^2 V_0^m \right. \\ & \left. + h^2 \left( -\frac{\eta\kappa'}{(1+h\eta\kappa)^2} \partial_\theta V_0^m + \partial_\theta \left( \frac{\partial_\theta V_1^m}{1+h\eta\kappa} \right) \right) \right\}. \end{aligned}$$

with the transmission conditions coming from (7c)–(7d)

$$(21c) \quad q_c \partial_n W^c \circ \Phi_0 = \frac{q_m}{h} \left( \partial_\eta W^m|_{\eta=0} - h^2 \partial_\theta^2 V_0^m \right),$$

$$(21d) \quad W^c \circ \Phi_0 = W^m|_{\eta=0},$$

and the boundary condition

$$(21e) \quad (1+h\kappa) \partial_\eta W^m|_{\eta=1} = -h^2 \kappa f.$$

Let us denote by  $V_2$  the primitive with respect to  $\eta$  of  $\partial_\eta V_2^m$  defined in (15), which vanishes in  $\eta = 0$ :

$$(22) \quad \forall(\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad V_2(\eta, \theta) = -(\eta^2/2)\kappa f + \eta(1-\eta/2)\partial_\theta^2 V_0^m.$$

The trick of the proof consists in introducing the function  $A^m$  defined on  $[0, 1] \times \mathbb{T}$  by:

$$(23) \quad A^m = W^m - h^2 V_2.$$

It is obvious that  $A^m$  satisfies the following equalities:

$$(24a) \quad \partial_\eta A^m|_{\eta=0} = \partial_\eta W^m|_{\eta=0} - h^2 \partial_\theta^2 V_0^m,$$

$$(24b) \quad A^m|_{\eta=0} = W^m|_{\eta=0},$$

$$(24c) \quad \partial_\eta A^m|_{\eta=1} = 0.$$

According to equalities (21b)–(23) a simple calculation shows that for all  $(\eta, \theta)$  in  $[0, 1] \times \mathbb{T}$ :

$$(24d) \quad \begin{aligned} \mathcal{L}_{\eta, \theta} A^m = & -\frac{1}{h(1+h\eta\kappa)} \left\{ \frac{-h^2 \eta \kappa \partial_\theta^2 V_0^m}{1+h\eta\kappa} - h^2 \kappa (2\kappa \eta f - (1-2\eta)\partial_\theta^2 V_0^m) \right. \\ & \left. + h^2 \left( -\frac{\eta\kappa' \partial_\theta V_0^m}{(1+h\eta\kappa)^2} + \partial_\theta \left( \frac{\partial_\theta V_1^m + h\partial_\theta V_2}{1+h\eta\kappa} \right) \right) \right\}. \end{aligned}$$

Let us denote by  $g$  the right-hand side of equality (24d) multiplied by  $(1+h\eta\kappa)/h$ :

$$\begin{aligned} \forall(\eta, \theta) \in [0, 1] \times \mathbb{T}, \\ g(\eta, \theta) = & \left\{ \frac{\eta\kappa \partial_\theta^2 V_0^m}{1+h\eta\kappa} - \kappa (2\kappa \eta f - (1-2\eta)\partial_\theta^2 V_0^m) \right. \\ & \left. + \frac{\eta\kappa' \partial_\theta V_0^m}{(1+h\eta\kappa)^2} - \partial_\theta \left( \frac{\partial_\theta V_1^m + h\partial_\theta V_2}{1+h\eta\kappa} \right) \right\}. \end{aligned}$$

According to (22) it is obvious that there exists a constant  $C > 0$ , depending only on the geometry of  $\partial\mathcal{O}$  such that:

$$(25) \quad \forall \theta \in \mathbb{T}, \sup_{\eta \in [0,1]} |g(\eta, \theta)| \leq C(|f| + |f'| + |f''| + |\partial_\theta V_0^m| + |\partial_\theta^2 V_0^m| + |\partial_\theta^3 V_0^m| + |\partial_\theta^4 V_0^m| + |\partial_\theta V_1^m|_{|\eta=0}| + |\partial_\theta^2 V_1^m|_{|\eta=0}|).$$

Let us denote by  $D$  the unit disc:

$$D = [0, 1] \times \mathbb{T}.$$

We remember that the  $L^2$  norm of a 0-form  $u$  in the  $[0, 1] \times \mathbb{T}$  with the metric (3), denoted by  $\|u\|_{\Lambda^0 L_m^2(D)}$ , is equal to:

$$\begin{aligned} \|u\|_{\Lambda^0 L_m^2(D)} &= \left( \int_0^1 \int_0^{2\pi} h(1 + h\eta\kappa) |u(\eta, \theta)|^2 d\eta d\theta \right)^{1/2}, \\ &= \|u\|_{L^2(\mathcal{O}_h)}, \end{aligned}$$

and the  $L^2$  norm of its exterior derivative  $du$ , denoted by  $\|du\|_{\Lambda^1 L_m^2}$  is equal to

$$\begin{aligned} \|du\|_{\Lambda^1 L_m^2(D)} &= \left( \int_0^1 \int_0^{2\pi} \frac{1 + h\eta\kappa}{h} |\partial_\eta u(\eta, \theta)|^2 + \frac{h}{1 + h\eta\kappa} |\partial_\theta u(\eta, \theta)|^2 d\eta d\theta \right)^{1/2}, \\ &= \|\text{grad } u\|_{L^2(\mathcal{O}_h)}. \end{aligned}$$

In  $\mathcal{O}$  parameterized by Euclidean coordinates, the  $L^2$  norm of a 0-form  $v$ , denoted by  $\|v\|_{\Lambda^0 L^2}$ , is equal to:

$$\|v\|_{\Lambda^0 L^2(\mathcal{O})} = \|v\|_{L^2(\mathcal{O})},$$

and the  $L^2$  norm of its exterior derivative  $dv$ , denoted by  $\|dv\|_{\Lambda^1 L^2}$  is equal to

$$\|dv\|_{\Lambda^1 L^2(\mathcal{O})} = \|\text{grad } v\|_{L^2(\mathcal{O})}.$$

According to (24) and (21),  $(W^c, A^m)$  is the solution of the following problem:

$$(26a) \quad \Delta W^c = 0, \text{ in } \mathcal{O},$$

$$\forall (\eta, \theta) \in [0, 1] \times \mathbb{T},$$

$$(26b) \quad \mathcal{L}_{\eta, \theta} A^m = \frac{hg(\eta, \theta)}{(1 + h\eta\kappa)},$$

with the transmission conditions coming from (21c)–(21d) and from equalities (24)

$$(26c) \quad q_c \partial_n W^c \circ \Phi_0 = \frac{q_m}{h} \partial_\eta A^m \Big|_{\eta=0},$$

$$(26d) \quad W^c \circ \Phi_0 = A^m \Big|_{\eta=0},$$

and the boundary condition

$$(26e) \quad \partial_\eta A^m \Big|_{\eta=1} = 0.$$

and with gauge condition:

$$(26f) \quad \int_0^{2\pi} A^m(0, \theta) d\theta = 0.$$

We multiply equality (26a) by  $q_c \overline{W^c}$ , we integrate by parts, and we multiply (26b) by  $q_m h(1 + h\eta\kappa) \overline{A^m}$  and we integrate by parts. Using the transmission conditions (26c)–(26d) and the boundary condition (26e) we obtain:

$$\begin{aligned} -q_c \int_{\mathcal{O}} |\text{grad } W^c|^2 \, \text{dvol}_{\mathcal{O}} - q_m \int_0^{2\pi} \int_0^1 \left( \frac{1 + h\eta\kappa}{h} |\partial_{\eta} A^m|^2 \right. \\ \left. + \frac{h}{1 + h\eta\kappa} |\partial_{\theta} A^m|^2 \right) \, \text{d}\eta \, \text{d}\theta \\ = q_m h^2 \int_0^{2\pi} \int_0^1 g(\eta, \theta) \overline{A^m} \, \text{d}\eta \, \text{d}\theta. \end{aligned}$$

Since  $q_m \neq 0$ , and using hypothesis (18) we infer:

$$\begin{aligned} \Re(q_c/q_m) \|dW^c\|_{\Lambda^1 L^2(\mathcal{O})}^2 \\ + \|dA^m\|_{\Lambda^1 L_m^2(D)}^2 \leq h^{3/2} \left( \int_0^{2\pi} \int_0^1 \frac{|g(\eta, \theta)|^2}{1 + h\eta\kappa} \, \text{d}\eta \, \text{d}\theta \right)^{1/2} \|A^m\|_{\Lambda^0 L_m^2(D)}. \end{aligned}$$

Since  $\partial\mathcal{O}$  is of class  $\mathcal{C}^2$ , using equalities (13)–(17b) and problems (12)–(16) there exists a constant  $C > 0$  depending only on  $\mathcal{O}$  such that:

$$(27) \quad \forall i \in \{0, 1, 2, 3, 4\}, \quad \|\partial_{\theta}^i V_0^m\|_{L^2(\mathbb{T})} \leq C \left| \frac{q_m}{q_c} \right| \|f\|_{H^3(\partial\mathcal{O})},$$

$$(28) \quad \forall j \in \{0, 1, 2\}, \quad \|\partial_{\theta}^j V_1^m|_{\eta=0}\|_{L^2(\mathbb{T})} \leq C \left| \frac{q_m}{q_c} \right| \|f\|_{H^3(\partial\mathcal{O})}.$$

Using (25) we infer

$$(29) \quad \min(\Re(q_c/q_m), 1) \left( \|dW^c\|_{\Lambda^1 L^2(\mathcal{O})}^2 + \|dA^m\|_{\Lambda^1 L_m^2(D)}^2 \right) \leq Ch^{3/2} (1$$

$$(30) \quad + |q_m/q_c|) \|f\|_{H^3(\partial\mathcal{O})} \|A^m\|_{\Lambda^0 L_m^2(D)}.$$

It remains to use Poincaré-Wirtinger inequality. Actually, according to gauge condition (26f) and according to (26d), it is obvious that:

$$\int_{\partial\mathcal{O}} W^c \, \text{dvol}_{\partial\mathcal{O}} = 0.$$

Thus, using Poincaré-Wirtinger inequality, there exists a constant  $C$  depending on the domain  $\mathcal{O}$  such that:

$$(31) \quad \|W^c\|_{\Lambda^0 L^2(\mathcal{O})} \leq C_{\mathcal{O}} \|dW^c\|_{\Lambda^1 L^2(\mathcal{O})}.$$

We are going to prove the existence of a constant  $C_{\mathcal{O}}$  depending on the domain  $\mathcal{O}$  such that:

$$(32) \quad \|A^m\|_{\Lambda^0 L_m^2(D)} \leq C_{\mathcal{O}} \|dA^m\|_{\Lambda^1 L_m^2(D)}.$$

Suppose that (32) holds. Thus, according to (31)–(32), and according to hypothesis (18), we deduce from (30) the existence of a constant  $C > 0$  depending on  $\mathcal{O}$  such that:

$$\begin{aligned} \|W^c\|_{H^1(\mathcal{O})} + (\|A^m\|_{\Lambda^0 L_m^2(D)} \\ + \|dA^m\|_{\Lambda^1 L_m^2(D)}) \leq C \frac{1 + |q_m/q_c|}{\min(\Re(q_c/q_m), 1)} h^{3/2} \|f\|_{H^3(\partial\mathcal{O})}. \end{aligned}$$

According to (23) and to (27), it is obvious that there exists a constant  $C$  depending only on  $\mathcal{O}$  such that:

$$\begin{aligned} & (\|A^m - W^m\|_{\Lambda^0 L_m^2(D)} \\ & + \|dA^m - dW^m\|_{\Lambda^1 L_m^2(D)}) \leq Ch^{3/2} (1 + |q_m/q_c|) \|f\|_{H^3(\partial\mathcal{O})}, \end{aligned}$$

thus according to (18), and since we supposed that  $q_m \neq 0$ , we have proved the existence of  $C_{\mathcal{O}} > 0$  such that:

$$\|W\|_{H^1(\Omega_h)} \leq C_{\mathcal{O}} \frac{1 + |q_m/q_c|}{\min(\Re(q_c/q_m), 1)} h^{3/2} \|f\|_{H^3(\partial\mathcal{O})},$$

which ends the proof of Theorem 4.1.

It remains to prove (32). According to the definition of  $h_0$  in (2) there exists two constants  $C_{\mathcal{O}}^1$  and  $C_{\mathcal{O}}^2$  depending on the domain  $\mathcal{O}$  such that the following inequalities hold:

$$(33a) \quad \|A^m\|_{\Lambda^0 L_m^2(D)}^2 \leq C_{\mathcal{O}}^1 h \int_0^1 \int_0^{2\pi} |A^m(\eta, \theta)|^2 d\theta d\eta,$$

$$(33b) \quad \|dA^m\|_{\Lambda^1 L_m^2(D)}^2 \geq C_{\mathcal{O}}^2 \left( \int_0^1 \int_0^{2\pi} \frac{|\partial_{\eta} A^m(\eta, \theta)|^2}{h} + h |\partial_{\theta} A^m|^2 d\theta d\eta \right).$$

Let us denote by  $(\widehat{A^m})_k$  for  $k \in \mathbb{Z}$  the  $k^{\text{th}}$ -Fourier coefficient (with respect to  $\theta$ ) of  $A^m$ :

$$(\widehat{A^m})_k = \int_0^{\pi} A^m(\theta) e^{-2i\pi k/L} d\theta.$$

Since  $(\widehat{\partial_{\theta} A^m})_k = 2i\pi k (\widehat{A^m})_k$ , it is easy to see that:

$$\forall k \neq 0, \quad \int_0^1 |(\widehat{A^m})_k(\eta)|^2 d\eta \leq 4\pi^2 \int_0^1 |(\widehat{\partial_{\theta} A^m})_k(\eta)|^2 d\eta.$$

According to gauge condition (26f), we have:

$$(\widehat{A^m})_0(0) = 0,$$

thus, using the equality

$$(\widehat{A^m})_0(\eta) = \int_0^{\eta} (\widehat{\partial_{\eta} A^m})_0(s) ds,$$

we infer

$$\int_0^1 |(\widehat{A^m})_0(\eta)|^2 d\eta \leq \int_0^1 |(\widehat{\partial_{\eta} A^m})_0(\eta)|^2 d\eta.$$

Therefore,

$$\sum_{k \in \mathbb{Z}} \int_0^1 |(\widehat{A^m})_k(\eta, \theta)|^2 d\eta \leq \sum_{k \in \mathbb{Z}} \left\{ \int_0^1 |(\widehat{\partial_{\theta} A^m})_k(\eta)|^2 d\eta + \int_0^1 |(\widehat{\partial_{\eta} A^m})_k(\eta)|^2 d\eta \right\}.$$

We end the proof of (32) by using Parseval inequality and inequalities (33).  $\square$

**Remark 4.2.** *If we suppose that  $q_m = 0$ , which is equivalent to consider a perfectly conducting cytoplasm, from Theorem 4.1 it is obvious to obtain:*

$$\begin{aligned} V_1^m &= \kappa f, \text{ in } [0, 1] \times \mathbb{T}, \\ \|V - hV_1^m \circ \Phi^{-1}\|_{H^1(\mathcal{O}_h)} &\leq C_{\mathcal{O}} h^{3/2} \|f\|_{H^3(\partial\mathcal{O})}. \end{aligned}$$

**Remark 4.3.** *Since  $q_c$  and  $q_m$  are complex permittivities, there are both of the same form (see Balanis and Constantine [3]) :*

$$(34) \quad q_c = a_c - ib_c, \text{ and } q_m = a_m - ib_m,$$

*with  $a_c, a_m, b_c$  and  $b_m$  positive. Thus the hypothesis (18) is always satisfied for dielectric materials.*

## APPENDIX

Let  $\star$  denote the Hodge star operator, which maps 0-forms to 2-forms, 1-forms to 1-forms and 2-forms to 0-forms (see Flanders [10]). We give explicit formulae for the operators  $d, \delta, \text{ext}$  and  $\text{int}$ . These formulae can be easily obtained from their definitions and from the operators  $\star, d$  and  $\delta = \star^{-1} d \star$ . We refer the reader to Dubrovin, Fomenko and Novikov [8].

We consider the metric given by the following matrix  $G$

$$(35) \quad G = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}.$$

We denote by  $|G|$  the determinant of  $G$ . The inverse of  $G$  is denoted by  $G^{-1} = (g^{ij})_{ij}$ , and we suppose that the signature of  $G$  is equal to 1. Thereby, the operator  $\star^2$  is equal to  $\text{Id}$  on the space of 0-forms and 2-forms and it is equal to  $-\text{Id}$  on 1-forms.

### 4.1. Star operator in $\mathbb{R}^2$ .

4.1.1. *On 0-forms and on 2-forms.* Let  $T$  be a 0-form and let  $S$  be the 2-form  $\nu dy^1 dy^2$ . Then  $\star T$  is the 2-form  $\mu dy^1 dy^2$  and  $\star S$  is the 0-form  $f$ . Then, we have:

$$\begin{aligned} \mu &= \sqrt{|G|} T, \\ f &= \frac{1}{\sqrt{|G|}} \nu. \end{aligned}$$

4.1.2. *On 1-forms.* Let  $T$  be the 1-form  $T_1 dy^1 + T_2 dy^2$ . Then  $\star T$  is the 1-form  $\mu_1 dy^1 + \mu_2 dy^2$ , and we have:

$$\begin{aligned} \mu_1 &= -\sqrt{|G|} (g^{12} T_1 + g^{22} T_2), \\ \mu_2 &= \sqrt{|G|} (g^{11} T_1 + g^{12} T_2). \end{aligned}$$

4.2. **The action of  $d$  acting on 0-forms in  $\mathbb{R}^2$ .** Let  $\mu$  be a 0 form, then we have :

$$d\mu = \frac{\partial \mu}{\partial y^1} dy^1 + \frac{\partial \mu}{\partial y^2} dy^2.$$

**4.3. The action of  $\delta$  acting on 1-forms on  $\mathbb{R}^2$ .** Let  $\mu$  be the 1-form  $\mu_1 dy^1 + \mu_2 dy^2$ , and define  $\delta\mu = \alpha$ . Then, we have :

$$\alpha = -\frac{1}{\sqrt{|G|}} \left\{ \frac{\partial}{\partial y_1} \left( \sqrt{|G|} (g^{11}\mu_1 + g^{12}\mu_2) \right) + \frac{\partial}{\partial y_2} \left( \sqrt{|G|} (g^{12}\mu_1 + g^{22}\mu_2) \right) \right\}.$$

**4.4. The exterior product of a 1-form with a 0-form.** Let  $N$  be the 1-form  $N_1 dy^1 + N_2 dy^2$  and  $f$  be a 0-form. We have :

$$\text{ext}(N)f = fN_1 dy^1 + fN_2 dy^2.$$

**4.5. The interior product of a 1-form with a 1-form.** Let  $N$  and  $\mu$  be the 1-forms  $N_1 dy^1 + N_2 dy^2$ , and  $\mu_1 dy^1 + \mu_2 dy^2$ . Then, we have :

$$\text{int}(N)\mu = N_1 (\mu_1 g^{11} + \mu_2 g^{12}) + N_2 (\mu_1 g^{12} + \mu_2 g^{22}).$$

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