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# Model-completion of scaled lattices\*

Luck Darnière<sup>†</sup>

## Abstract

It is known from Grzegorzczuk's paper [Grz51] that the lattice of real semi-algebraic closed subsets of  $\mathbb{R}^n$  is undecidable for every integer  $n \geq 2$ . More generally, if  $X$  is any definable set over a real or algebraically closed field  $K$ , then the lattice  $L(X)$  of all definable subsets of  $X$  closed in  $X$  is undecidable whenever  $\dim X \geq 2$ . Nevertheless, we investigate in this paper the model theory of the class  $\text{SC}_{\text{def}}(K, d)$  of all such lattices  $L(X)$  with  $\dim X \leq d$  and  $K$  as above or a henselian valued field of characteristic zero.

We show that the universal theory of  $\text{SC}_{\text{def}}(K, d)$ , in a natural expansion by definition of the lattice language, is the same for every such field  $K$ . We give a finite axiomatization of it and prove that it is locally finite and admits a model-completion, which turns to be decidable as well as all its completions. We expect  $L(\mathbb{Q}_p^d)$  to be a model of (a little variant of) this model-completion. This leads us to a new conjecture in  $p$ -adic semi-algebraic geometry which, combined with the results of this paper, would give decidability (via a natural recursive axiomatization) and elimination of quantifiers for the complete theory of  $L(\mathbb{R}_p^d)$ , uniformly in  $p$ .

## 1 Introduction

In this paper we study the model-theory of a class of lattices coming from the following examples.

**Example 1.1** Let  $K$  be a henselian valued field of characteristic zero, a real closed field or an algebraically closed field. There exists a good notion of dimension for definable sets  $A$  over  $K$  (see [vdD89] for the henselian case, and any book of real or complex algebraic geometry for the other cases). For any positive integer  $i$  let:

$$C^i(A) = \overline{\{a \in A \mid \dim(A, a) = i\}}$$

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where  $\dim(A, a)$  is the maximal dimension of definable neighborhood of  $a$  in  $A$ , and the overline stands for the topological closure in  $A$ . This is a definable subset of  $A$  which we call the  $i$ -pure<sup>1</sup> component of  $A$ . Given a definable set  $X$  over  $K$  of dimension  $d$ , let  $L_{\text{def}}(X)$  be the lattice of all definable subsets of  $X$  closed in  $X$ , enriched<sup>2</sup> with the unary functions  $(C^i)_{i \leq d}$  and the binary function:

$$A - B = \overline{A \setminus B}$$

Eventually let  $\text{SC}_{\text{def}}(K, d)$  denote the class of lattices  $L_{\text{def}}(X)$  for all definable sets over  $K$  of dimension at most  $d$ .

**Example 1.2** Let  $K$  be any infinite field,  $X$  a constructible subset over  $K$  (that is a boolean combination of Zariski closed subsets of  $K^n$  for some positive integer  $n$ ) and  $L_{\text{Zar}}(X)$  be the lattice of all constructible subsets of  $X$  which are Zariski closed in  $X$  enriched with the following structure. For any  $A, B$  in  $L_{\text{Zar}}(X)$  let  $A - B$  be as in the above example and  $C^i(A)$  be the union of the irreducible components of  $A$  of Krull dimension  $i$  (in the usual sense for topological spaces).

Eventually let  $\text{SC}_{\text{Zar}}(K, d)$  denote the class of lattices  $L_{\text{Zar}}(X)$  for all definable sets over  $K$  of dimension at most  $d$ . Of course  $\text{SC}_{\text{Zar}}(K, d) = \text{SC}_{\text{def}}(K, d)$  when  $K$  is algebraically closed.

It is known from an argument of [Grz51] that the complete theory of  $L_{\text{def}}(K^n)$  is undecidable for every real closed field  $K$  and every integer  $n \geq 2$ , and the argument can easily be adapted to algebraically closed fields  $K$ . This paper gives some reason to believe that the complete theory of  $L_{\text{def}}(K^n)$  is decidable for every  $p$ -adically closed field  $K$  and every  $n$ . It is organized as follows.

We first give in Section 2 a finite list of universal axioms of a theory  $T_d$  in a language  $\mathcal{L}_{\text{SC}_d}$  extending the language of lattices, the model of which we call  $d$ -subscaled lattices. The examples given above are all  $d$ -subscaled lattices. After some preliminar technical results in Section 3 we prove in Section 4 that every finitely generated  $d$ -subscaled lattice is finite. Combining this result with a linear representation for finite  $d$ -subscaled lattices and with the model-theoretic compactness theorem, we then prove in Section 5 that  $T_d$  is precisely the universal theory of  $\text{SC}_{\text{Zar}}(K, d)$ . In particular this theory is finitely axiomatizable and, remarkably enough, does not depend on  $K$ . Eventually a detailed study of finitely generated extensions of finite  $d$ -subscaled lattices, achieved in Sections 6 and 7 allows us to exhibit in Section 8 a model-completion  $\bar{T}_d$  for  $T_d$ , having a finite axiomatisation. Moreover we show that  $\bar{T}_d$  has finitely many completions, each of which is finitely axiomatizable and  $\aleph_0$ -categorical. It follows that  $\bar{T}_d$  is decidable.

It is difficult to find a model of  $\bar{T}_d$  coming from geometry because such models are atomless. We present in the last section a similar model completion

<sup>1</sup>If  $C^i(A)$  is non-empty it has pure dimension  $i$ , that is the local dimension  $\dim(A, a) = i$  at every point of  $A$ .

<sup>2</sup>The additional functions are definable in the lattice structure of  $L_{\text{def}}(X)$ .

and decidability result for a theory  $\bar{T}_d^*$  authorizing atoms. These results lead us to the following conjecture (or question):

**Conjecture 1.3** *Let  $K$  be a  $p$ -adically closed field and  $A$  be an infinite definable subset of  $K^n$  which is open in its closure. Let  $(B_k)_{k \leq q}$  be a finite collection of closed definable subsets of  $\bar{A} \setminus A$ . Then there exists a collection  $(A_k)_{k \leq q}$  of non-empty definable subsets of  $A$  clopen in  $A$  such that:*

$$\forall k \leq q, \quad \overline{A_k} = A_k \cup B_k$$

If this conjecture is true then it follows immediatly that  $L_{\text{def}}(K^n)$  is a model of  $T_n^*$  hence has a decidable complete theory (not depending on  $p$ ).

**Remark 1.4** Since 0-subscaled lattices are exactly non-trivial boolean algebras (with the  $\mathcal{L}_{\text{SC}_0}$ -structure and the boolean structure being quantifier-free bi-definable) our model-completion result for subscaled lattices is a generalisation to arbitrary finite dimension  $d$  of the well known theorem on the model-completion of boolean algebras.

**Remark 1.5** The duals of  $d$ -subscaled lattices form an elementary class of Heyting algebras so this paper may also be considered as a contribution to the model-theory of Heyting algebras. However the usual geometric objects whose study motivated this paper are closed sets (points, curves, surfaces, and so on). From this point of view the lattice  $L_{\text{Zar}}(K^d)$  is a more natural object to consider than its dual, the Heyting algebra of *open* algebraic sets. This is the reason why we had to present our results in this settings and not in terms of Heyting algebra.

## 2 Notation and definitions

The set of all positive integers is denoted by  $\mathbb{N}$ , and  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ . If  $\mathcal{N}$  is an unbounded subset of  $\mathbb{N}$  (resp. the empty subset) we set  $\max \mathcal{N} = \infty$  (resp.  $\max \mathcal{N} = -1$ ). The symbols  $\subseteq$  and  $\subset$  denote respectively the inclusion and the strict inclusion.

### 2.1 Lattices

Let  $\mathcal{L}_{\text{lat}} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge\}$  be the language of lattices. An upper semi-lattice is an  $\mathcal{L}_{\text{up}}$ -substructure of a lattice, with  $\mathcal{L}_{\text{up}} = \{\mathbf{0}, \vee\}$ . As usually  $b \leq a$  is an abbreviation for  $a \vee b = a$  and similarly for  $b < a$ ,  $b \geq a$  and  $b > a$ . Iterated  $\vee$  and  $\wedge$  operations are denoted by  $\bigvee_{i \in I} a_i$  and  $\bigwedge_{i \in I} a_i$  respectively. If the index set  $I$  is empty then of course  $\bigvee_{i \in I} a_i = \mathbf{0}$  and  $\bigwedge_{i \in I} a_i = \mathbf{1}$ . The logical connectives ‘or’, ‘and’ and their iterated forms will be denoted by  $\bigvee$ ,  $\bigwedge$  and  $\bigvee$  respectively. We consider the following relation, definable in any lattice:

$$\begin{aligned} b \ll a &\iff b < a \text{ and } \forall c (c < a \Rightarrow b \vee c < a) \\ &\iff b \leq a \neq \mathbf{0} \text{ and } \forall c (c < a \Rightarrow b \vee c < a) \end{aligned}$$

The **spectrum** of a distributive lattice  $L$  is the set  $\text{Spec}(L)$  of all prime filters of  $L$ , endowed with the so-called Zarisky topology, defined by taking as a basis of closed sets the family:

$$P(a) = \{\mathfrak{p} \in \text{Spec}(L) \mid a \in \mathfrak{p}\}$$

for  $a$  ranging over  $L$ . Stone-Priestley's duality asserts that  $a \mapsto P(a)$  is an isomorphism between  $L$  and the lattice of closed subsets of  $\text{Spec}(L)$  whose complement in  $\text{Spec}(L)$  is compact. We call a lattice **noetherian** if it is isomorphic to the lattice of closed sets of a noetherian topological space. By Stone-Priestley's duality a lattice  $L$  is noetherian if and only if its spectrum is a noetherian topological space. In such a lattice every filter is principal and every element  $a$  writes uniquely as the join of its  **$\vee$ -irreducible components**, which are the (finitely many) maximal elements in the set of non-zero  $\vee$ -irreducible elements of  $L$  smaller than  $a$ . We denote by  $\mathcal{I}(L)$  the set of all non-zero  $\vee$ -irreducible elements of  $L$ .

We define the **lattice dimension of an element  $a$  in a lattice  $L$**  as the least upper bound (in  $\mathbb{N} \cup \{-1, \infty\}$ ) of the set of positive integers  $n$  such that:

$$\exists \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n \in P(a)$$

This is nothing but the ordinary topological dimension (defined by chains of irreducible closed subsets) of the spectral space  $P(a)$ . We denote this dimension by  $\dim_L a$ . By construction  $\dim_L a = -1$  if and only if  $a = \mathbf{0}_L$ . The index  $L$  is necessary since  $\dim_L a$  is not preserved by  $\mathcal{L}_{\text{lat}}$ -embeddings, nevertheless we omit it whenever the ambient lattice is clear from the context and we often call the lattice dimension simply the dimension. We let the **lattice dimension of  $L$**  be the lattice dimension of  $\mathbf{1}_L$  in  $L$ .

**Fact 2.2** *If  $L = L(X)$  is any of the lattices of the introduction, then for any  $A \in L(X)$ ,  $\dim_{L(X)} A$  is exactly the usual dimension of  $A$  as a definable (or constructible) set over  $K$ .*

Fact 2.4 below is a key argument in the proof of this result, which is non-obvious because the definition of  $\dim_{L(X)} A$  lies on prime filters of closed definable subsets of  $A$ , an object which do not have a natural geometric meaning.

### 2.3 TC-lattices

Let  $\mathcal{L}_{\text{TC}} = \mathcal{L}_{\text{lat}} \cup \{-\}$  with  $'-\'$  a binary function symbol. A **topologically complemented lattice**, or **TC-lattice** for short, is an  $\mathcal{L}_{\text{TC}}$ -structure which is a lattice and in which the **relative topological complement**  $a - b$  is defined as the least element  $c$  such that  $a \leq b \vee c$  (or equivalently  $P(a - b)$  is the topological closure of the relative complement  $P(a) \setminus P(b)$ , so the name). This is clearly the dual of a Heyting algebra with  $a - b$  in the TC-lattice becoming  $b \rightarrow a$  in its dual. So we know from the theory of Heyting algebra (see for example [Joh82]) that every TC-lattice is distributive, and that the class of all

TC-lattices is a variety (in the sense of universal algebra). Observe that in TC-lattices the  $\ll$  relation is quantifier-free definable since:

$$a - b = a \iff b \wedge a \ll a \text{ or } a = \mathbf{0}$$

So it will be preserved by  $\mathcal{L}_{\text{TC}}$ -embeddings. On the other hand the lattice dimension will not be preserved in general by  $\mathcal{L}_{\text{TC}}$ -embeddings of TC-lattices.

We will use the following rules, the proof of which are elementary exercises (using either dual properties, if known, of Heyting algebras or, more directly, Stone-Priestley's duality).

$$\mathbf{TC}_1 : a = (a \wedge b) \vee (a - b).$$

In particular if  $a$  is  $\vee$ -irreducible then  $b < a \implies b \ll a$ .

$$\mathbf{TC}_2 : (a_1 \vee a_2) - b = (a_1 - b) \vee (a_2 - b).$$

$$\mathbf{TC}_3 : (a - b) - b = a - b.$$

So either  $a - b = \mathbf{0}$  or  $(a - b) \wedge b \ll a - b \leq a$ .

$$\mathbf{TC}_4 : \text{More generally } a - (b_1 \vee b_2) = (a - b_1) - b_2.$$

So if  $a - b_1 = a$  then  $a - (b_1 \vee b_2) = a - b_2$ .

**Fact 2.4** For any TC-lattice  $L$  and any  $a \in L$ ,  $\dim_L a$  is exactly the least upper bound of the set of positive integers  $n$  such that there exists  $a_0, \dots, a_n \in L$  such that:

$$\mathbf{0} \neq a_0 \ll a_1 \ll \dots \ll a_n \leq a$$

The proof is a good exercise that we leave to the reader. The fact that  $\dim_L a$  is at least equal to the above least upper bound is true in any lattice. Equality holds in TC-lattices because they are min-compact (that is for any  $a \in L$  the set of elements of  $P(a)$  which are minimal with respect to the inclusion, is compact).

## 2.5 (Sub)scaled lattices.

For a given positive integer  $d$  let  $\mathcal{L}_{\text{SC}_d} = \mathcal{L}_{\text{TC}} \cup \{C^i\}_{0 \leq i \leq d}$  where the  $C^i$ 's are unary function symbols. With the examples of the introduction in mind, define the **sc-dimension of an element  $a$  of an  $\mathcal{L}_{\text{SC}_d}$ -structure  $L$**  as:

$$\text{sc-dim } a = \min\{l \leq d / a = \bigvee_{0 \leq i \leq l} C^i(a)\} \in \mathbb{N} \cup \{-1, \infty\}$$

The **sc-dimension of  $L$** , denoted  $\text{sc-dim}(L)$ , is by definition the sc-dimension of  $\mathbf{1}_L$ . In general the lattice dimension of an element is not preserved by  $\mathcal{L}_{\text{lat}}$ -embeddings neither by  $\mathcal{L}_{\text{TC}}$ -lattices. On the other hand the sc-dimension of an element is obviously preserved by  $\mathcal{L}_{\text{SC}_d}$ -embeddings.

A  **$d$ -subscaled lattice** is then an  $\mathcal{L}_{\text{SC}_d}$ -structure which is a topologically complemented lattice and which satisfies the following list of axioms:

$$\mathbf{SC}_1 : \bigvee_{0 \leq i \leq d} C^i(a) = a$$

**SC<sub>2</sub>** :  $\forall I \subseteq \{0, \dots, d\}, \forall k$ :

$$C^k\left(\bigvee_{i \in I} C^i(a)\right) = \begin{cases} \mathbf{0} & \text{if } k \notin I \\ C^k(a) & \text{if } k \in I \end{cases}$$

**SC<sub>3</sub>** :  $\forall k \geq \max(\text{sc-dim}(a), \text{sc-dim}(b)), C^k(a \vee b) = C^k(a) \vee C^k(b)$

**SC<sub>4</sub>** :  $\forall i \neq j, \text{sc-dim}(C^i(a) \wedge C^j(b)) < \min(i, j)$

**SC<sub>5</sub>** :  $\forall k \geq \text{sc-dim}(b), C^k(a) - b = C^k(a) - C^k(b)$

In particular, by **SC<sub>3</sub>**:  $\text{sc-dim } b < a \implies C^k(a) - b = C^k(a)$ .

**SC<sub>6</sub>** : If  $b \ll a$  then  $\text{sc-dim } b < \text{sc-dim } a$ .

Axioms **SC<sub>1</sub>** to **SC<sub>5</sub>** are easily seen to be equivalent to a finite set of equations in  $\mathcal{L}_{\text{SC}_d}$ . On the other hand **SC<sub>6</sub>** is expressible by a universal formula in  $\mathcal{L}_{\text{SC}_d}$  but not by an equation (indeed the class of  $d$ -subscaled lattices is not preserved by  $\mathcal{L}_{\text{SC}_d}$ -projections, hence is not a variety). We call  **$d$ -scaled lattices** the  $d$ -subscaled lattices satisfying the following additional property:

**SC<sub>0</sub>** :  $\text{sc-dim } a = \dim a$

This is an elementary class, which is not preserved by  $\mathcal{L}_{\text{SC}_d}$ -substructures. By Fact 2.2 all the examples of  $\mathcal{L}_{\text{SC}_d}$ -structures given in the introduction are  $d$ -scaled lattices in which the lattice dimension and the sc-dimension coincide with the usual geometric dimension. As the terminology suggests, we will see that  $d$ -subscaled lattices are precisely the  $\mathcal{L}_{\text{SC}_d}$ -substructures of  $d$ -scaled lattices.

The finite language  $\mathcal{L}_{\text{SC}_d}$  allows to write *finite* axiomatisations. When this is not essential we consider the language  $\mathcal{L}_{\text{SC}} = \mathcal{L}_{\text{TC}} \cup \{C^i\}_{i \in \mathbb{N}}$  and define **subscaled lattices** (resp. **scaled lattices**) as those  $\mathcal{L}_{\text{SC}}$ -structures whose  $\mathcal{L}_{\text{SC}_d}$ -reduct is a  $d$ -subscaled lattice (resp. a  $d$ -scaled lattice) for every  $d$  large enough. This is not an elementary class, but for any fixed integer  $d$  the class of (sub)scaled lattices of sc-dimension at most  $d$  (resp. exactly  $d$ ) is elementary. Any  $d$ -(sub)scaled lattice expands uniquely to a (sub)scaled lattice of dimension at most  $d$  by realizing  $C^i$  as the constant map  $x \mapsto \mathbf{0}$  for every  $i > d$ . Conversely every (sub)scaled lattice  $L$  is of that kind for every  $d \geq \text{sc-dim}(L)$ .

## 2.6 Basic properties

The next additional properties follow easily from the axioms of subscaled lattices.

**SC<sub>7</sub>** :  $\text{sc-dim } a = \max\{k / C^k(a) \neq \mathbf{0}\}$

In particular  $\forall k, \text{sc-dim } C^k(a) = k \iff C^k(a) \neq \mathbf{0}$

**SC<sub>8</sub>** :  $\text{sc-dim } a \vee b = \max(\text{sc-dim } a, \text{sc-dim } b)$

In particular  $b \leq a \implies \text{sc-dim } b \leq \text{sc-dim } a$ .

$$\mathbf{SC}_9 : \forall k, \quad C^k(a) = \mathbb{W} \left\{ b / b \leq \bigvee_{0 \leq i \leq k} C^i(a) \text{ and } C^k(b) = b \right\}$$

$$\mathbf{SC}_{10} : \text{sc-dim } a \leq \dim a$$

$$\mathbf{SC}_{11} : \forall I \subseteq \{0, \dots, d\}, \quad a - \bigvee_{i \in I} C^i(a) = \bigvee_{i \notin I} C^i(a)$$

In particular  $\text{sc-dim}(a - C^d(a)) < d$ .

$$\mathbf{SC}_{12} : \forall k, \quad C^k(a) - b = C^k(C^k(a) - b)$$

$$\mathbf{SC}_{13} : \forall k, \quad C^k(a) = a \iff \forall b (a - b \neq \mathbf{0} \Rightarrow \text{sc-dim } a - b = k)$$

*Proof:* (Sketch)  $\mathbf{SC}_7$  follows from  $\mathbf{SC}_1$  and  $\mathbf{SC}_2$ ;  $\mathbf{SC}_8$  from  $\mathbf{SC}_2$ ,  $\mathbf{SC}_3$  and  $\mathbf{SC}_7$ ;  $\mathbf{SC}_9$  from  $\mathbf{SC}_2$  and  $\mathbf{SC}_3$ ; eventually  $\mathbf{SC}_{10}$  is equivalent to  $\mathbf{SC}_6$  modulo the other axioms. Only the three last properties require a little effort.

$\mathbf{SC}_{11}$ : For every  $l \in I$ ,  $C^l(a) \leq \bigvee_{i \in I} C^i(a)$  hence  $C^l(a) - \bigvee_{i \in I} C^i(a) = \mathbf{0}$ . On the other hand for every  $l \notin I$  and every  $i \in I$ ,  $C^l(a) - C^i(a) = C^l(a)$  by  $\mathbf{SC}_4$  and  $\mathbf{SC}_5$ . So  $C^l(a) - \bigvee_{i \in I} C^i(a) = C^l(a)$  by  $\mathbf{TC}_4$ . Eventually by  $\mathbf{SC}_1$  and  $\mathbf{TC}_2$ :

$$a - \bigvee_{i \in I} C^i(a) = \bigvee_{l \leq d} \left( C^l(a) - \bigvee_{i > k} C^i(a) \right) = \bigvee_{l \notin I} C^l(a)$$

In particular if  $I = \{d\}$  then  $a - C^d(a) = \bigvee_{k < d} C^k(a)$  hence  $\text{sc-dim}(a - C^d(a)) < d$ .

$\mathbf{SC}_{12}$ : By  $\mathbf{TC}_1$ ,  $C^k(a) = (C^k(a) - b) \vee (C^k(a) \wedge b)$ , and by  $\mathbf{SC}_8$  both  $C^k(a) - b$  and  $C^k(a) \wedge b$  have sc-dimension at most  $k$ , so by  $\mathbf{SC}_3$ :

$$C^k(C^k(a)) = C^k(C^k(a) - b) \vee C^k(C^k(a) \wedge b)$$

By  $\mathbf{SC}_1$ ,  $C^k(C^k(a) \wedge b) \leq C^k(a) \wedge b \leq b$ , and by  $\mathbf{SC}_2$ ,  $C^k(a) = C^k(C^k(a))$  so:

$$C^k(a) \leq C^k(C^k(a) - b) \vee b$$

It follows that  $C^k(a) - b \leq C^k(C^k(a) - b)$ , and equality holds by  $\mathbf{SC}_1$ .

$\mathbf{SC}_{13}$ : If  $a = C^k(a)$  then  $a - b = C^k(a - b)$  for every  $b$  by  $\mathbf{SC}_{12}$ . If moreover  $a - b \neq \mathbf{0}$  then  $\text{sc-dim}(a - b) = k$  by  $\mathbf{SC}_7$ . Conversely assume that  $a \neq C^k(a)$  (hence  $a \neq \mathbf{0}$ ). If  $\text{sc-dim } a \neq k$  then  $a - \mathbf{0} = a \neq \mathbf{0}$  hence  $\text{sc-dim } a - \mathbf{0} \neq k$ . If  $\text{sc-dim } a = k$ , let  $b = C^k(a)$ . Then by assumption  $a - b \neq \mathbf{0}$ , and by  $\mathbf{SC}_{11}$ ,  $a - b = \bigvee_{i < k} C^i(a)$  hence  $\text{sc-dim } a - b < k$ .

■

## 2.7 Miscellanies

Given an integer  $k$  we say that an element  $a$  of a distributive lattice  $L$  is  **$k$ -pure in  $L$**  if and only if<sup>3</sup>:

$$\forall b \in L (a - b \neq \mathbf{0} \Rightarrow \dim_L a - b = k)$$

<sup>3</sup>Remember Example 1.1 and Footnote 1.



Then either  $a = \mathbf{0}$  or  $\dim_L a = k$ . In the latter case we say that  $a$  **has pure dimension  $k$  in  $L$** .

Similarly in any  $d$ -subscaled lattice we say that an element  $a$  is  **$k$ -sc-pure** if and only if:

$$\forall b \in L (a - b \neq \mathbf{0} \Rightarrow \text{sc-dim } a - b = k)$$

By SC<sub>13</sub>,  $a$  is  $k$ -sc-pure if and only if  $a = C^k(a)$ . Then by SC<sub>7</sub>, either  $a = \mathbf{0}$  or  $\text{sc-dim } a = k$ . In the latter case we say that  $a$  **has pure sc-dimension  $k$** . For any  $a$ , the element  $C^k(a)$  is called the  **$k$ -sc-pure component of  $a$** , and simply its  **$k$ -pure component** if  $L$  is a scaled lattice.

**Proposition 2.8** *The  $\mathcal{L}_{\text{SC}_d}$ -structure of a  $d$ -scaled lattice  $L$  is uniformly definable in the  $\mathcal{L}_{\text{lat}}$ -structure of  $L$ . In particular it is uniquely determined by this  $\mathcal{L}_{\text{lat}}$ -structure.*

*Proof:* Clearly the TC-structure is an extension by definition of the lattice structure of  $L$ . For every positive integer  $k$  the class of  $k$ -pure elements is uniformly definable, using the definability of  $\ll$ . Then so is the function  $C^k$  for every  $k$ , by decreasing induction on  $k$ . Indeed by SC<sub>9</sub> and SC<sub>11</sub>,  $C^k(a)$  is the largest  $k$ -pure element  $c$  such that  $c \leq a - \bigvee_{i>k} C^i(a)$ .

■

Our study of (sub)scaled lattices is motivated by the examples given in the introduction. Although they are less natural, the following examples which will be needed further in this paper.

**Example 2.9** In an arbitrary noetherian lattice  $L$  an  $\mathcal{L}_{\text{SC}}$ -structure can be defined as follows. For every  $a, b \in L$ , if  $\mathcal{C}(a)$  denotes the set of all  $\vee$ -irreducible components of  $a$ , let:

$$\begin{aligned} a - b &= \bigvee \{c \in \mathcal{C}(a) / c \not\leq b\} \\ (\forall k) \quad C^k(a) &= \bigvee \{c \in \mathcal{C}(a) / \dim_L c = k\} \end{aligned}$$

This  $\mathcal{L}_{\text{TC}}$ -structure (resp.  $\mathcal{L}_{\text{SC}}$ -structure) is the only one (by Proposition 2.8) which turns  $L$  into a TC-lattice (resp. a scaled lattice). On the other hand, for any strictly increasing map  $D: \mathcal{I}(L) \rightarrow \mathbb{N}$  and any  $a, b \in L$  define  $a - b$  as above and:

$$(\forall k) \quad C_D^k(a) = \bigvee \{c \in \mathcal{C}(a) / D(c) = k\}$$

This  $\mathcal{L}_{\text{SC}}$ -structure turns  $L$  into a subscaled lattice which is not a scaled lattice (except if  $D$  coincides with the map  $\dim_L$ ). We will use without further mention the following obvious fact:

**Fact 2.10** *Every noetherian (hence in particular every finite) subscaled lattice is of the above kind.*

Eventually the following notation will be convenient in induction arguments. If  $\mathcal{L}$  is any of our languages  $\mathcal{L}_{\text{lat}}$ ,  $\mathcal{L}_{\text{TC}}$ ,  $\mathcal{L}_{\text{SC}_d}$  or  $\mathcal{L}_{\text{SC}}$  we let  $\mathcal{L}^* = \mathcal{L} \setminus \{\mathbf{1}\}$ . Given an  $\mathcal{L}$ -structure  $L$  whose reduct to  $\mathcal{L}_{\text{lat}}$  is a lattice, for any  $a \in L$  we denote by:

$$L(a) = \{b \in L / b \leq a\}$$

$L(a)$  is a typical example of  $\mathcal{L}^*$ -substructure of  $L$ .

### 3 Embeddings of subscaled lattices

We need a reasonably easy criterion for an  $\mathcal{L}_{\text{lat}}$ -embedding of subscaled lattices to be an  $\mathcal{L}_{\text{SC}}$ -embedding. The special case of a noetherian embedded lattice, presented in the next proposition, is sufficient for this paper. However combining the model-theoretic compactness theorem with the local finiteness Theorem 4.1, one can easily derive from Proposition 3.1 that an  $\mathcal{L}_{\text{lat}}$ -embedding  $\varphi : L \rightarrow L'$  between arbitrary subscaled lattices is an  $\mathcal{L}_{\text{SC}}$ -embedding if and only if it preserves the sc-dimension and sc-purity, that is for every  $a \in L$  and every  $k \in \mathbb{N}$ :

$$C^k(a) = a \implies C^k(\varphi(a)) = \varphi(a)$$

**Proposition 3.1** *Let  $L_0$  be a noetherian subscaled lattice and  $\varphi : L_0 \rightarrow L$  an  $\mathcal{L}_{\text{lat}}$ -embedding such that for every  $a \in \mathcal{I}(L_0)$ ,  $\varphi(a)$  is sc-pure and has the same sc-dimension as  $a$ . Then  $\varphi$  is an  $\mathcal{L}_{\text{SC}}$ -embedding.*

**Remark 3.2** Clearly the same statement remains true with  $\mathcal{L}_{\text{lat}}$  and  $\mathcal{L}_{\text{SC}}$  replaced respectively by  $\mathcal{L}_{\text{lat}}^*$  and  $\mathcal{L}_{\text{SC}}^*$  (or  $\mathcal{L}_{\text{SC}_d}^*$ ). We will freely use these variants.

*Proof:* Let  $d = \text{sc-dim } L$ , and for any  $a \in L_0$ :

$$(a_0, \dots, a_d) = (\varphi(C^0(a)), \dots, \varphi(C^d(a)))$$

For every positive integer  $k$ ,  $C^k(a)$  is  $k$ -sc-pure by  $\text{SC}_{13}$ , hence each  $\vee$ -irreducible component  $c$  of  $C^k(a)$  in  $L_0$  has pure sc-dimension  $k$ . By assumption each such  $\varphi(c)$  then has pure sc-dimension  $k$ . The join of finitely many elements of pure sc-dimension  $k$  is easily seen to be  $k$ -sc-pure by definition and by  $\text{TC}_2$ , so we have proved:

$$\forall k, a_k \text{ is } k\text{-sc-pure.} \quad (1)$$

Moreover for any  $k \neq l$ ,  $\text{sc-dim}(C^k(a) \wedge C^l(a)) < \min(k, l)$  by  $\text{SC}_4$ . It follows that each  $\vee$ -irreducible component  $c$  of  $C^k(a) \wedge C^l(a)$  has sc-dimension strictly less than  $\min(k, l)$ , hence so does  $\varphi(c)$  by assumption. By  $\text{SC}_8$  we conclude that:

$$\forall k \neq l, \text{sc-dim}(a_k \wedge a_l) < \min(k, l) \quad (2)$$

For every  $k > d$ ,  $\varphi(C^k(a)) = \varphi(\mathbf{0}) = \mathbf{0}$ . Each  $\vee$ -irreducible component of  $a$  has sc-dimension at most  $d$  hence so does  $\varphi(a)$  by assumption, so by  $\text{SC}_8$ ,  $\text{sc-dim}(\varphi(a)) \leq d$ . It follows that  $C^k(\varphi(a)) = \mathbf{0} = C^k(\varphi(a))$ .

For every  $k \leq d$  let:

$$b_k = \bigvee_{0 \leq l \leq k} a_l \quad \text{and} \quad c_k = \bigvee_{0 \leq l \leq k} C^l(\varphi(a))$$

By  $\text{SC}_{11}$ ,  $c_k - C^k(\varphi(a)) = c_{k-1}$ . Moreover the proof of  $\text{SC}_{11}$  proves as well that  $b_k - a_k = \bigvee_{l \leq k} (a_l - a_k) = b_{k-1}$ , thanks to (1) and (2) above.

Now assume that  $b_k = c_k$  for some  $k \leq d$ . Then  $C^k(\varphi(a)) = C^k(c_k)$  by SC<sub>2</sub>, so  $C^k(\varphi(a)) = C^k(b_k)$  by assumption. It follows by SC<sub>5</sub> that:

$$C^k(\varphi(a)) - a_k = C^k(b_k) - C^k(a_k) = C^k(b_k - a_k) = C^k(b_{k-1})$$

SC<sub>8</sub> implies that  $\text{sc-dim}(b_{k-1}) \leq k - 1$  so  $C^k(b_{k-1}) = \mathbf{0}$  by SC<sub>2</sub>. It follows that  $C^k(\varphi(a)) - a_k = \mathbf{0}$  that is  $C^k(\varphi(a)) \leq a_k$ . On the other hand  $a_k \leq \bigvee_{l \leq k} C^k(\varphi(a))$  and  $C^k(a_k) = a_k$  by (1) so  $a_k \leq C^k(\varphi(a))$  by SC<sub>9</sub>. We have proved:

$$b_k = c_k \implies a_k = C^k(\varphi(a))$$

Then  $b_{k-1} = b_k - a_k = c_k - C^k(\varphi(a)) = c_{k-1}$ . Since  $b_d = c_d = a$  it follows by decreasing induction that  $b_k = c_k$  for every  $k$ , that is  $\varphi(C^k(a)) = C^k(\varphi(a))$ . Incidentally, since  $\varphi$  is injective, this implies by SC<sub>7</sub> that for every  $a \in L_0$ :

$$\text{sc-dim } a = \text{sc-dim } \varphi(a) \tag{3}$$

Now let  $a, b \in L_0$ , and  $a', b'$  be their images by  $\varphi$ . We have to show that  $\varphi(a - b) = a' - b'$ . By TC<sub>2</sub>, replacing if necessary  $a$  by its  $\vee$ -irreducible components, we may assume w.l.o.g. that  $a$  itself is  $\vee$ -irreducible in  $L_0$ . This implies that  $a = C^k(a)$  for some  $k$ . It then remains two possibilities for  $a - b$ :

- If  $b \geq a$  then  $a - b = \mathbf{0}$  and  $b' \geq a'$  hence  $\varphi(a - b) = \mathbf{0} = a' - b'$ .
- Otherwise  $c = b \wedge a < a$  hence  $c \ll a$  by TC<sub>1</sub>, so  $\text{sc-dim } c < \text{sc-dim } a$  by SC<sub>6</sub>. Let  $c' = \varphi(c) = b' \wedge a'$ , by assumption  $a' = C^k(a)$  and by (3)  $\text{sc-dim}(c') < \text{sc-dim}(a') = k$ , hence by SC<sub>5</sub>  $a' - c' = a'$ . We conclude that  $a - b = a$  and  $a' - b' = a'$  hence  $\varphi(a - b) = \varphi(a) = a' = a' - b'$ .

We have proved that  $\varphi$  preserves  $-$  and the  $C^k$ 's operations, so  $\varphi$  is an  $\mathcal{L}_{\text{SC}}$ -embedding.

■

**Corollary 3.3** *Let  $L_0$  be a noetherian sublattice of a subscaled lattice  $L$ . If for any  $b < a \in \mathcal{I}(L)$ ,  $a$  is sc-pure in  $L$  and  $\text{sc-dim } b < \text{sc-dim } a$  in  $L$ , then  $L_0$  is an  $\mathcal{L}_{\text{SC}}$ -substructure of  $L_0$ .*

*Proof:* The assumptions imply that the map  $D : a \mapsto \text{sc-dim } a$  is a strictly increasing map from  $\mathcal{I}(L_0)$  to  $\mathbb{N}$ . Endow  $L_0$  with the structure of subscaled lattice determined by  $D$  as in Example 2.9. Proposition 3.1 then applies to the inclusion map  $\varphi$  from  $L_0$  to  $L$ .

■

## 4 Local finiteness

We prove in this section that every finitely generated subscaled lattice is finite. This result is far non-obvious, due to the lack of any known normal form for terms in  $\mathcal{L}_{\text{SC}}$ . It contrasts with the situation in TC-lattices, which can be both

infinite and generated by a single element. Our main ingredient, which explains this difference, is the uniform bound given *a priori* for the sc-dimension of any element in a given subscaled lattice.

**Theorem 4.1** *Any subscaled lattice  $L$  of sc-dimension  $d$  generated by  $n$  elements is finite. More precisely, the cardinality of  $\mathcal{I}(L)$  is then bounded by the fonction  $\mu(n, d)$  defined by:*

$$\begin{aligned}\mu(n, -1) &= 0 & (\forall n) \\ \mu(n, d) &= 2^n + \mu(2^{n+1}, d-1) & (\forall n, \forall d \geq 0)\end{aligned}$$

*Proof:* If  $d = -1$  the only subscaled lattice of dimension  $-1$  is the one-element lattice  $\{\mathbf{0}\}$ , so the result is trivial.

Assume the  $d \geq 0$  and that the result is proved for every  $d' < d$  and every positive integer  $n$ . Let  $L$  be a subscaled lattice of sc-dimension  $d$  generated by elements  $x_1, \dots, x_n$ . Let  $\Omega_n$  be the family of all subsets of  $\{1, \dots, n\}$  (so  $\Omega_0 = \{\emptyset\}$ ). For every  $I \in \Omega_n$  let  $I^c = \Omega_n \setminus I$  and:

$$y_I = \left( \bigwedge_{i \in I} x_i \right) - \left( \bigvee_{i \in I^c} x_i \right), \quad z_I = C^d(y_I)$$

The family of all  $\mathcal{Y}_I = \bigcap_{i \in I} P(x_i) \cap \bigcap_{i \in I^c} P(x_i)^c$  is a partition of  $\text{Spec}(L)$ . Indeed the  $\mathcal{Y}_i$ 's are the atoms of the boolean algebra generated in the power set  $\mathcal{P}(\text{Spec}(L))$  by the  $P(x_i)$ 's. Moreover each  $P(y_I)$  is the topological closure  $\overline{\mathcal{Y}}_I$  of  $\mathcal{Y}_I$  in  $\text{Spec}(L)$  hence for every  $x \in L$ :

$$P(x) = \bigcup_{I \in \Omega_n} P(x) \cap \mathcal{Y}_I \subseteq \bigcup_{I \in \Omega_n} P(x) \cap \overline{\mathcal{Y}}_I = P\left(\bigvee_{I \in \Omega_n} x \wedge y_I\right)$$

So  $x \leq \bigvee_{I \in \Omega_n} (x \wedge y_I)$  by Stone-Priestley's duality. The reverse inequality being obvious we have proved:

$$\forall x \in L, \quad x \leq \bigvee_{I \in \Omega_n} (x \wedge y_I) \tag{4}$$

In particular  $\text{SC}_3$  also gives:

$$C^d(\mathbf{1}) = C^d\left(\bigvee_{I \in \Omega_n} y_I\right) = \bigvee_{I \in \Omega_n} z_I \tag{5}$$

For every  $I \neq J \in \Omega_n$ , if for example  $I \not\subseteq J$  choose any  $i \in I \setminus J$  and observe that  $y_I \leq x_i$  and  $y_J \leq \mathbf{1} - x_i$  so  $y_I \wedge y_J \ll \mathbf{1} - x_i$  by  $\text{TC}_3$ . By  $\text{SC}_6$  and the  $d$ -sc-purity of the  $z_I$ 's it follows that:

$$\text{sc-dim } z_I \wedge z_J < d \quad \text{hence} \quad z_I - z_J = z_I \tag{6}$$

It follows from  $\text{SC}_8$ ,  $\text{SC}_{11}$  and (6) above, that the element:

$$a = (\mathbf{1} - C^d(\mathbf{1})) \vee \left( \bigvee_{I \in \Omega_n} (y_I - z_I) \right) \vee \left( \bigvee_{I \neq J \in \Omega_n} (z_I \wedge z_J) \right)$$

has sc-dimension strictly smaller than  $d$ . So the induction hypothesis applies to the  $\mathcal{L}_{\text{SC}}$ -substructure  $L_0^-$  of  $L(a)$  generated by the  $(y_I - z_I)$ 's and the  $(z_I \wedge a)$ 's:  $L_0^-$  is finite, with at most  $\mu(2|\Omega_n|, d - 1)$  non-zero  $\vee$ -irreducible elements. Eventually let  $L_1$  be the upper semi-lattice generated in  $L$  by  $L_0^- \cup \{z_I\}_{I \in \Omega_n}$ . By construction  $L_1$  is finite and  $\mathcal{I}(L_1) \subseteq \mathcal{I}(L_0^-) \cup \{z_I\}_{I \in \Omega_n}$ , so  $|\mathcal{I}(L_1)| \leq 2^n + \mu(2^{n+1}, d - 1) = \mu(n, d)$ . It is then sufficient to show that  $L_1 = L$ .

By (5),  $\mathbf{1} = \text{C}^d(\mathbf{1}) \vee a = \bigvee_{I \in \Omega_n} z_I \vee a \in L_1$ . For every  $I \in \Omega_n$  and every  $b \in L_0^-$ ,  $z_I \wedge b = (z_I \wedge a) \wedge b \in L_0^-$ . Eventually for every  $I \neq J \in \Omega_n$ ,  $z_I \wedge z_J = (z_I \wedge a) \wedge (z_J \wedge a) \in L_0^-$ . So by the distributivity law,  $L_1$  is a sublattice of  $L$ .

Since  $\mathcal{I}(L_1) \subseteq \mathcal{I}(L_0^-) \cup \{z_I\}_{I \in \Omega_n}$  it is immediate that for any  $b' < b$  in  $\mathcal{I}(L_1)$ ,  $\text{sc-dim } b' < \text{sc-dim } b$ . So  $L_1$  is an  $\mathcal{L}_{\text{SC}}$ -substructure of  $L$  by Corollary 3.3. Moreover each  $y_I = (y_I - z_I) \vee z_I \in L_1$  and for every  $i \leq n$ , (4) gives:

$$x_i = \bigvee_{I \in \Omega_n} x_i \wedge y_I \leq \bigvee_{\substack{I \in \Omega_n \\ i \in I}} y_I \leq x_i$$

So equality holds, hence each  $x_i \in L_1$  and eventually  $L = L_1$ .

■

**Corollary 4.2** *For every  $n, d$  there are finitely many non-isomorphic subscaled lattices of sc-dimension  $d$  generated by  $n$  elements.*

*Proof:* Any such subscaled lattice  $L$  is finite, with  $|\mathcal{I}(L)| \leq \mu(n, d)$  by Theorem 4.1. Clearly there are finitely many non-isomorphic lattices such that  $|\mathcal{I}(L)| \leq \mu(n, d)$  and each of them admits finitely many non-isomorphic  $\mathcal{L}_{\text{SC}_d}$ -structures. The conclusion follows.

■

## 5 Linear representation

In this section we prove that the theory of  $d$ -subscaled lattices is the universal theory of various natural classes of  $\mathcal{L}_{\text{SC}_d}$ -structures, including  $\text{SC}_{\text{Zar}}(K, d)$ . The argument is based on an elementary representation theorem for  $d$ -subscaled lattices, combined with the local finiteness result of Section 4.

Given an arbitrary field  $K$ , a non-empty linear variety  $X \subseteq K^m$  is determined by the data of an arbitrary point  $P \in X$  and the vector subspace  $\overline{X}$  of  $K^m$ , via the relation  $X = P + \overline{X}$  (the orbit of  $P$  under the action of  $\overline{X}$  by translation). We call  $X$  a **special linear variety** (resp. a **special linear set**) if  $X$  is a linear variety such that  $\overline{X}$  is generated by a subset of the canonical basis of  $K^m$  (resp. if  $X$  is a finite union of special linear varieties). For example the empty set is a special linear set, as the union of an empty family of special linear varieties. The family  $L_{\text{lin}}(X)$  of all special linear subsets of  $X$  is a noetherian lattice (because it is the family of closed sets of a noetherian topology on  $X$ ). So it has a natural structure of scaled lattice defined as in Example 2.9.

**Remark 5.1** For every  $A \in \mathbf{L}_{\text{lin}}(X)$ ,  $\text{sc-dim } A = \dim_{\mathbf{L}_{\text{lin}}(X)} A =$  the dimension of  $A$  as defined in linear algebra. If  $K$  is infinite then this dimension coincides with the Krull dimension of  $A$ . Moreover if  $A$  is  $\vee$ -irreducible in  $\mathbf{L}_{\text{lin}}(X)$  then it is pure dimensionnal, hence it is sc-pure both in  $\mathbf{L}_{\text{lin}}(X)$  and  $\mathbf{L}_{\text{zar}}(X)$ . By Proposition 3.1 it follows that if  $K$  is infinite then  $\mathbf{L}_{\text{lin}}(X)$  is an  $\mathcal{L}_{\text{SC}}$ -substructure of  $\mathbf{L}_{\text{zar}}(X)$ . Similarly if  $K$  is a henselian valued field of characteristic zero, a real closed field or an algebraically closed field then  $\mathbf{L}_{\text{lin}}(X)$  is an  $\mathcal{L}_{\text{SC}}$ -substructure of  $\mathbf{L}_{\text{def}}(X)$ .

In the following proposition  $K^m$  is identified to the subset  $K^m \times \{0\}^r$  of  $K^{m+r}$ .

**Proposition 5.2** *Given any two special linear sets  $C \subseteq B \subseteq K^m$  and any  $N \geq \dim C$  there exists a positive integer  $r$  and a special linear set  $A \subseteq K^{m+r}$  of pure dimension  $n$  such that  $A \cap B = C$ .*

*Proof:* For any integer  $n$  let  $(e_1, \dots, e_n)$  be the canonical basis of  $K^n$ . If  $I$  is a subset of  $\{1, \dots, n\}$  we denote  $\vec{E}(I)$  the vector subspace of  $K^n$  generated by  $(e_i)_{i \in I}$ .

Decompose  $C$  as a union of special linear varieties:  $C_1, \dots, C_p$  with each  $C_i = P_i + \vec{E}(J_i)$  and  $|J_i| = \dim C_i \leq n$ . Let  $r$  be larger than 0,  $N - m$  and every  $N - |J_i|$ . For each  $i$  take an arbitrary subset  $J'_i$  of cardinality  $N - |J_i|$  inside  $\{m + 1, \dots, m + r\}$  and let  $I_i = J_i \cup J'_i$ . Choose an arbitrary point  $P_0 \in K^{m+r} \setminus K^m$  and let  $I_0 = \{1, \dots, N\}$ . Eventually let  $A_i = P_i + \vec{E}(I_i)$  for every  $i \leq p$ , and  $A = A_0 \cup \dots \cup A_p$ . By construction each  $A_i$  has dimension  $|I_i| = N$ , hence  $A$  is  $N$ -pure. At least  $A_0$  is non empty (it is the only one in case  $p = 0$  that is if  $C$  is empty) hence  $A$  has pure dimension  $N$ . Clearly  $A \cap K^m = C$  hence *a fortiori*  $A \cap B = C$ .

■

**Proposition 5.3 (Linear representation)** *Let  $K$  be an infinite field and let  $L$  be any finite subscaled lattice. Then there exists a linear set  $X$  over  $K$  and an  $\mathcal{L}_{\text{SC}}$ -embedding  $\varphi: L \rightarrow \mathbf{L}_{\text{lin}}(X)$ .*

**Remark 5.4** Since  $\varphi$ , in the above proposition, is an  $\mathcal{L}_{\text{SC}}$ -embedding it preserves the sc-dimension hence  $\dim X = \text{sc-dim } L$ . So if  $\text{sc-dim } L \leq d$  we can indentify  $L$  and  $X$  with their respective  $\mathcal{L}_{\text{SC}_d}$ -reduct, and  $\varphi$  is then also an  $\mathcal{L}_{\text{SC}_d}$ -embedding.

*Proof:* Remember that  $\mathcal{L}_{\text{SC}}^* = \mathcal{L}_{\text{SC}} \setminus \{\mathbf{1}\}$ . We prove by induction on the number  $n$  of non-zero  $\vee$ -irreducible elements of a subscaled lattice  $L$  that there exists an  $\mathcal{L}_{\text{SC}}^*$ -embedding  $\varphi$  of  $L$  into  $\mathbf{L}_{\text{lin}}(K^m)$  for some  $m$  depending on  $L$ .

For  $n = 0$ ,  $L$  is the one-element lattice  $\{\mathbf{0}\}$  hence it is an  $\mathcal{L}_{\text{SC}}^*$ -substructure of  $\mathbf{L}_{\text{lin}}(K)$ .

Let  $n \geq 1$ , assume the result proved for  $n - 1$  and take a subscaled lattice  $L$  with non-zero  $\vee$ -irreducible elements  $a_1, \dots, a_n$ . Reordering if necessary we

may assume that  $a_n$  is maximal among the  $a_i$ 's. Let  $a = a_n$ ,  $b = \bigvee_{1 \leq i < n} a_i$ ,  $c = a \wedge b$  and  $\varphi$  an  $\mathcal{L}_{\text{SC}}^*$ -embedding of  $L(b)$  into some  $\text{L}_{\text{lin}}(K^m)$  given by induction hypothesis. Since  $a$  is  $\vee$ -irreducible in  $L$  it is sc-pure. Moreover  $c \ll a$  by  $\text{TC}_1$ , hence  $a$  has pure sc-dimension  $N$  for some  $N > \text{sc-dim}(c)$  by  $\text{SC}_6$ . Let  $B, C$  be the respective images of  $b, c$  by  $\varphi$ . Proposition 5.2 gives a positive integer  $r$  and a special linear set  $A \subseteq K^{m+r}$  of pure dimension  $N$  such that  $A \cap B = C$ . Since we identified  $K^m$  with  $K^m \times \{0\}^r$  we consider  $\text{L}_{\text{lin}}(K^m)$  as an  $\mathcal{L}_{\text{SC}}^*$ -substructure of  $\text{L}_{\text{lin}}(K^{m+r})$ . So  $\varphi$  actually embeds  $L(b)$  into  $\text{L}_{\text{lin}}(K^{m+r})$ .

Every element  $x$  of  $L$  writes uniquely  $x_a \vee x_b$  with  $x_a \in \{\mathbf{0}, a\}$  and  $x_b \in L(b)$  (group appropriately the  $\vee$ -irreducible components of  $x$ , using the maximality of  $a_n$ ) hence:

$$\bar{\varphi}(x) = \begin{cases} \varphi(x_b) & \text{if } x_a = \mathbf{0} \\ A \cup \varphi(x_b) & \text{if } x_a = a \end{cases}$$

is a well-defined  $\mathcal{L}_{\text{lat}}^*$ -embedding of  $L$  into  $\text{L}_{\text{lin}}(K^{m+r})$ . Moreover  $\bar{\varphi}$  is an  $\mathcal{L}_{\text{SC}}^*$ -embedding by Proposition 3.1. This finishes the induction.

We have constructed an  $\mathcal{L}_{\text{SC}}^*$ -embedding  $\varphi$  of  $L$  into  $\text{L}_{\text{lin}}(K^m)$  for some  $m$ . Then  $X = \varphi(\mathbf{1}_L)$  is a special linear set, so  $\varphi$  induces an  $\mathcal{L}_{\text{SC}}$ -embedding of  $L$  into  $\text{L}_{\text{lin}}(X)$ .

■

For any infinite field  $K$  and positive integer  $d$  let  $\text{SC}_{\text{lin}}(K, d)$  be the class of  $d$ -scaled lattices  $\text{L}_{\text{lin}}(X)$  for every special linear variety  $X$  over  $K$  of dimension at most  $d$ .

**Theorem 5.5** *For any infinite field  $K$  and positive integer  $d$ , the universal theories of  $\text{SC}_{\text{lin}}(K, d)$  and  $\text{SC}_{\text{Zar}}(K, d)$  are exactly the theory of  $d$ -subscaled lattices.*

*If moreover  $K$  is a henselian valued field of characteristic zero, a real closed field or an algebraically closed field then the same holds for  $\text{SC}_{\text{def}}(K, d)$ .*

*Proof:* By Remark 5.1 it suffices to prove the theorem for  $\text{SC}_{\text{lin}}(K, d)$ . Let  $T(K, d)$  be its universal theory.

The linear representation Proposition 5.3 shows that every finite  $d$ -subscaled lattice embeds into some  $L \in \text{SC}_{\text{lin}}(K, d)$  hence is a model of  $T(K, d)$ . Since every finitely generated  $d$ -subscaled lattice is finite by Theorem 4.1, the model-theoretic compactness argument then implies that any  $d$ -subscaled lattice is a model of  $T(K, d)$ .

Conversely every  $L \in \text{SC}_{\text{lin}}(K, d)$  is a  $d$ -scaled lattice hence obviously its universal theory contains the theory of  $d$ -subscaled lattices.

■

## 6 Primitive extensions

This section is devoted to the study of minimal proper extensions of finite subscaled lattices. Let  $L_0$  be a finite subscaled lattice,  $L$  an  $\mathcal{L}_{\text{SC}}$ -extension of  $L_0$  and  $x \in L$ . We introduce the following notation.

- For every  $a \in L_0$ ,  $a^- = \mathbb{W}\{b \in L_0 / b < a\}$ .
- $L_0\langle x \rangle$  denotes the  $\mathcal{L}_{\text{SC}}$ -substructure of  $L$  generated by  $L_0 \cup \{x\}$ .
- $g(x, L_0) = \mathbb{M}\{a \in L_0 / x \leq a\}$ .

Clearly  $a \in \mathcal{I}(L_0)$  if and only if  $a^-$  is the unique predecessor of  $a$  in  $L_0$  (otherwise  $a^- = a$ ). We say that a tuple  $(x_1, x_2)$  of elements of  $L$  is **primitive over  $L_0$**  if there exists  $g \in \mathcal{I}(L_0)$  such that:

1.  $x_1, x_2$  are sc-pure of the same sc-dimension.
2. Each  $g^- \wedge x_i \in L_0$ .
3. One of the following happens:
  - $x_1 = x_2$  and  $g^- \wedge x_1 \ll x_1 \ll g$ .
  - $x_1 \neq x_2$ ,  $x_1 \wedge x_2 \in L_0$  and  $g - x_1 = x_2$ ,  $g - x_2 = x_1$ .

The above conditions imply that each  $x_i \notin L_0$  and:

$$g = g(x_1, L_0) = g(x_2, L_0)$$

We say that  $L$  is **primitively generated over  $L_0$** , or simply that it is a **primitive extension** of  $L_0$ , if there exists  $(x_1, x_2)$  primitive over  $L_0$  such that  $L = L_0\langle x_1, x_2 \rangle$  (then clearly  $L = L_0\langle x \rangle_1 = L_0\langle x \rangle_2$ ). By the following proposition such a tuple is necessarily unique.

**Proposition 6.1** *Let  $L_0$  be a finite subscaled lattice,  $L$  an extension generated over  $L_0$  by a primitive tuple  $(x_1, x_2)$ , and let  $g = g(x_1, L_0)$ .*

*Then  $L$  is exactly the upper semi-lattice generated over  $L_0$  by  $x_1, x_2$ . It is a finite subscaled lattice and one of the following happens:*

1.  $x_1 = x_2$ ,  $\text{sc-dim } x_1 < \text{sc-dim } g$  and  $\mathcal{I}(L) = \mathcal{I}(L_0) \cup \{x_1\}$ .
2.  $x_1 \neq x_2$ ,  $\text{sc-dim } x_1 = \text{sc-dim } g$  and  $\mathcal{I}(L) = (\mathcal{I}(L_0) \setminus \{g\}) \cup \{x_1, x_2\}$ .

*Proof:* Let  $L_1$  be the upper semi-lattice generated over  $L_0$  by  $x_1, x_2$ . In order to show that  $L_1 = L$  it is sufficient to check that  $L_1$  is an  $\mathcal{L}_{\text{SC}}$ -substructure of  $L$ .

Let  $p = \text{sc-dim } g$  and  $q = \text{sc-dim } x_1 = \text{sc-dim } x_2$ . By  $\text{TC}_1$ ,  $g^- \ll g$  hence  $g^- \wedge x_i \ll g$  since  $g^- \wedge x_i \in L_0$  by assumption. So  $\text{sc-dim } g^- < p$ . We will need the following facts, for every  $a \in L_0$ :

$$g \not\leq a \quad \implies \quad a \wedge x_i \in L_0 \quad \text{and} \quad a \wedge x_i \ll x_i \quad (7)$$

Indeed  $g \wedge a = g^- \wedge a$  hence  $a \wedge x_i = (a \wedge g) \wedge x_i = a \wedge (g^- \wedge x_i)$ . By assumption  $x_i \wedge g^- \in L_0$  hence  $a \wedge x_i \in L_0$  is proved. Moreover  $x_i \wedge (g^- \wedge a) \leq x_i \wedge g^-$  hence it suffices to check that  $g^- \wedge x_i \ll x_i$ . If  $x_1 = x_2$  this is an assumption. Otherwise  $x_1 = g - x_2$  and  $x_2 = g - x_1$  are  $p$ -sc-pure by  $\text{SC}_{13}$ . Then  $x_i - g^- = x_i$  by  $\text{SC}_5$ , since  $\text{sc-dim } g^- < p$ , so  $g^- \wedge x_i \ll x_i$ .



$L_1$  is a sublattice of  $L$  then follows easily from (7), the distributivity law, and the fact that by assumption  $x_1 \wedge x_2 \in L_0 \cup \{x_1\}$ .

Since  $L_0$  is finite and  $L_1$  is generated by  $L_0 \cup \{x_1, x_2\}$  as an upper semi-lattice, it follows immediatly that  $L_1$  is finite and:

$$\mathcal{I}(L_1) \subseteq \mathcal{I}(L_0) \cup \{x_1, x_2\} \quad (8)$$

So for any  $b' < b \in cI(L_1)$  it is easily seen that  $\text{sc-dim}(b') < \text{sc-dim } b$  in  $L$ . Corollary 3.3 then shows that  $L_1$  is an  $\mathcal{L}_{\text{SC}}$ -substructure of  $L$ , hence  $L_1 = L$ .

We turn now to the description of  $\mathcal{I}(L)$ . If  $x_1 \neq x_2$  then of course  $g = x_1 \vee x_2 \notin \mathcal{I}(L)$ . Conversely if  $x_1 = x_2$  then (8) implies that  $g$  is  $\vee$ -irreducible in  $L$ , so:

$$g \in \mathcal{I}(L) \iff x_1 \neq x_2 \quad (9)$$

Assume that  $\mathcal{I}(L_0) \not\subseteq \mathcal{I}(L)$ . Let  $b \in \mathcal{I}(L_0) \setminus \mathcal{I}(L)$  and let  $y_1, \dots, y_r$  ( $r \geq 2$ ) be its  $\vee$ -irreducible components in  $L_1$ . By (8), each  $y_i$  either belongs to  $L_0$  or to  $\{x_1, x_2\}$ , and at least one of them does not belong to  $L_0$ . We may assume without loss of generality that  $y_1 = x_1$ . Then  $x_1 \leq b$  hence  $g \leq b$ . If  $g < b$  then  $g \ll b$  since  $b \in \mathcal{I}(L_0)$  so  $b - g = b$ , but then we have a contradiction:

$$y_1 \leq b - g \leq b - x_1 = \bigvee_{i=2}^r y_i$$

So  $b = g$ . We have proved that:

$$\mathcal{I}(L_0) \setminus \{g\} \subseteq \mathcal{I}(L) \quad (10)$$

The conclusion follows by combining (8), (9), (10) and an obvious argument of cardinality.

■

It is not difficult to deduce from Proposition 6.1 that any primitively generated extension  $L$  of a finite subscaled lattice  $L_0$  is minimal, in the sense that there is no intermediate proper extension  $L_0 \subset L_1 \subset L$ . Only the converse, which we prove now, is actually needed in the remaining of this paper.

**Proposition 6.2** *Any finitely generated proper extension  $L$  of a finite  $d$ -subscaled lattice  $L_0$  is the union of a finite chain of primitively generated extensions of  $L_0$ .*

*Proof:* Since  $L$  is finite by the local finiteness Theorem 4.1, it suffices to show that  $L$  contains a primitive extension of  $L_0$ . Take any element  $x$  minimal in  $\mathcal{I}(L) \setminus L_0$ . Observe that if  $y$  is any element of  $L$  strictly smaller than  $x$  then all the  $\vee$ -irreducible components of  $y$  in  $L$  actually belong to  $L_0$ , so  $y \in L_0$ .

Let  $g = g(x, L_0)$ . For every  $a \in L_0$ , if  $a < g$  then  $a \not\leq x$  hence  $a \wedge x < x$ , so  $a \wedge x \in L_0$ . It follows that  $g^- \wedge x \in L_0$ , hence  $g^- < g$ . In particular  $g \in \mathcal{I}(L_0)$ .

Since  $x \in \mathcal{I}(L)$  it is sc-pure, and moreover  $g^- \wedge x < x$  implies that  $g^- \wedge x \ll x$  by  $\text{TC}_1$ . So if moreover  $x \ll g$  then we have proved that  $(x, x)$  is primitive over  $L_0$ .

On the other hand if  $x \not\ll g$  let  $x_1 = x$  and  $x_2 = g - x$ . Since  $g \in \mathcal{I}(L_0)$  it is sc-pure, and so are  $x_1$  and  $x_2$ , with the same sc-dimension  $p = \text{sc-dim } g$  (indeed  $g - x \neq \mathbf{0}$  since  $x < g$ ). Moreover  $x_1 \wedge x_2 \ll x_2$  by  $\text{TC}_3$  so  $\text{sc-dim } x_1 \wedge x_2 < p$  hence  $g - x_1 = x_2$  and  $g - x_2 = x_1$  by  $\text{TC}_1$ ,  $\text{TC}_2$  and the  $p$ -sc-purity of  $x_1$  and  $x_2$ . Eventually  $x_1 \wedge x_2 < x$  imply that  $x_1 \wedge x_2 \in L_0$ . We conclude that  $(x_1, x_2)$  is primitive over  $L_0$ .  
 ■

## 7 Signatures

A triple  $(g, q, H)$  will be called a **signature** in a finite subscaled lattice  $L_0$  if and only if  $g \in \mathcal{I}(L_0)$ ,  $q \leq \text{sc-dim } g$  is a positive integer, and  $H$  is a subset of two (not necessarily distinct) elements  $h_1, h_2 \in L_0$  such that:

- $h_1 \vee h_2 < g$  (hence  $h_1 \vee h_2 \ll g$  by  $\text{TC}_1$ )
- $\text{sc-dim}(h_1 \vee h_2) < q$
- If  $q < \text{sc-dim } g$  then  $h_1 = h_2$ .

By Proposition 6.1 and the definition, any extension  $L$  primitively generated over  $L_0$  by  $x_1, x_2$  determines a unique signature:

$$\sigma(L) = (g(x_1, L_0), \text{sc-dim } x_1, \{x_1 \wedge g^-, x_2 \wedge g^-\})$$

which we call the **signature of  $L$  in  $L_0$** . It determines the extension  $L|L_0$  as follows.

**Proposition 7.1** *Two primitively generated extensions of a finite subscaled lattice  $L_0$  are isomorphic over  $L_0$  if and only if they have the same signature in  $L_0$ .*

*Proof:* Let  $L$  (resp  $L'$ ) be an extension of  $L_0$  generated over  $L_0$  by a primitive tuple  $(x_1, x_2)$  (resp.  $x'_1, x'_2$ ). If they are isomorphic over  $L_0$  then obviously  $\sigma(L) = \sigma(L')$ . Conversely assume that:

$$\sigma(L) = \sigma(L') = (g, q, \{h_1, h_2\})$$

Reordering if necessary we may assume that each  $h_i = x_i \wedge g^- = x'_i \wedge g^-$ . Let  $\varphi(x_i) = x'_i$  for each  $i \in \{1, 2\}$ , and  $\varphi(a) = a$  for every  $a \in \mathcal{I}(L_0)$ . By Proposition 6.1 this is a well defined bijection from  $\mathcal{I}(L)$  to  $\mathcal{I}(L')$  which preserves the order, hence it extends uniquely to an isomorphism of upper semi-lattice  $\varphi: L' \rightarrow L$ . The  $\mathcal{L}_{\text{lat}}$ -structure of a lattice being an extension by definition of its upper semi-lattice structure, this is an  $\mathcal{L}_{\text{lat}}$ -isomorphism. Moreover by construction  $\varphi$  preserves the sc-dimensions of the  $\vee$ -irreducible elements of  $L$  and  $L'$ , hence by Proposition 3.1 it is an  $\mathcal{L}_{\text{SC}}$ -isomorphism, whose restriction to  $L_0$  is the identity.  
 ■

**Proposition 7.2** *Let  $\sigma = (g, q, \{h_1, h_2\})$  be a signature in a finite subscaled lattice  $L_0$ . There exists a primitive extension  $L$  of  $L_0$  whose signature in  $L_0$  is precisely  $\sigma$ .*

*Proof:* We only treat the case when  $q = \text{sc-dim } g$ . The case when  $q < \text{sc-dim } g$  (hence  $h_1 = h_2$ ) is similar, and left to the reader.

Let  $x_1, x_2$  be any two distinct elements in a set disjoint from  $\mathcal{I}(L_0)$  and let:

$$\mathcal{I} = (\mathcal{I}(L_0) \setminus \{g\}) \cup \{x_1, x_2\}$$

The order on  $\mathcal{I}(L_0) \setminus \{g\}$  inherited from  $L_0$  can be extended to  $\mathcal{I}$  by stating that  $x_1 \not\leq x_2$ ,  $x_2 \not\leq x_1$ , and for every  $b \in \mathcal{I}(L_0) \setminus \{g\}$  and every  $j \in \{1, 2\}$ :

- $b < x_j \iff b \leq h_j$ .
- $x_j < b \iff g \leq b$

For every  $z \in \mathcal{I}$  let  $z\uparrow = \{y \in \mathcal{I} / z \leq y\}$ . Let  $L$  be the upper-semilattice generated in the power set  $\mathcal{P}(\mathcal{I})$  of  $\mathcal{I}$  by all the  $z\uparrow$ 's with  $z$  ranging over  $\mathcal{I}$ . This is a sublattice of  $\mathcal{P}(\mathcal{I})$  such that  $\mathcal{I}(L) = \{z\uparrow / z \in \mathcal{I}\}$ , that is:

$$\mathcal{I}(L) = (\{b\uparrow / b \in \mathcal{I}(L_0)\} \setminus \{g\uparrow\}) \cup \{x_1\uparrow, x_2\uparrow\}$$

Let  $\varphi(g) = x_1\uparrow \cup x_2\uparrow$ , and  $\varphi(b) = b\uparrow$  for every  $b \in \mathcal{I}(L_0) \setminus \{g\}$ . This application uniquely extends to an embedding of upper semi-lattice that we still denote  $\varphi$  from  $L_0$  to  $L$ , which is easily seen to be an  $\mathcal{L}_{\text{lat}}$ -embedding. For every  $z \in \mathcal{I}$  let:

$$D(z\uparrow) = \begin{cases} \text{sc-dim}(z) & \text{if } z \in \mathcal{I}(L_0) \setminus \{g\}, \\ \text{sc-dim}(g) & \text{otherwise.} \end{cases}$$

This is a strictly increasing map from  $\mathcal{I}(L)$  to  $\mathbb{N}$  hence it determines an  $\mathcal{L}_{\text{SC}_d}$ -structure on  $L$  as in Example 2.9. Proposition 3.1 asserts that  $\varphi$  is an  $\mathcal{L}_{\text{SC}_d}$ -embedding. By construction  $L$  is primitively generated over  $L_0$  by  $x_1\uparrow, x_2\uparrow$ , with signature  $\sigma$  in  $L_0$ .

The case when  $q < \text{sc-dim } g$  (hence  $h_1 = h_2$ ) is similar, and left to the reader.

■

## 8 Model-completion

We call **super  $d$ -scaled lattice** (resp. **super scaled lattice**) any  $d$ -subscaled lattice (resp. subscaled lattice)  $L$  which satisfy the following additional properties.

**Scaling:**  $L$  is a scaled lattice.

**Catenarity:** For every positive integers  $r \leq q \leq p$  and every elements  $c \leq a$ , if  $a$  has pure dimension  $p$  and  $c$  has pure dimension  $r$  then there exists a  $q$ -pure element  $b$  such that  $c \leq b \leq a$ .

**Splitting:** For every elements  $b_1, b_2, a$ , if  $b_1 \vee b_2 \ll a$  then there exists non-zero elements  $a_1 \geq b_1$  and  $a_2 \geq b_2$  such that:

$$\begin{cases} a_1 = a - a_2 \\ a_2 = a - a_1 \\ a_1 \wedge a_2 = b_1 \wedge b_2 \end{cases}$$

All these properties are clearly axiomatizable in  $\mathcal{L}_{\text{SC}_d}$ , using only finitely many  $\forall\exists$ -formulas. The name of the second one comes from the fact that in a subscaled lattice whose spectrum is a noetherian topological space, this property is equivalent to the usual notion of catenarity, namely that any two maximal chains in  $\text{Spec}(L)$  having the same first and last element have the same length.

**Proposition 8.1** *Every subscaled lattice  $\mathcal{L}_{\text{SC}}$ -embeds in a superscaled lattice.*

*Proof:* Let  $L_0$  be a finitely generated subscaled lattice and  $a, b_1, b_2 \in L_0$  such that  $b_1 \vee b_2 \ll a$ . By Theorem 4.1,  $L_0$  is finite. Let  $g_1, \dots, g_r$  be the  $\vee$ -irreducible components of  $a$  in  $L_0$ , and for every  $i \leq r$  let  $g_i^-$  be the unique predecessor of  $g_i$  in  $L_0$ ,  $h_{i,1} = b_1 \wedge g_i^-$  and  $h_{i,2} = b_2 \wedge g_i^-$ .

$\sigma_1 = (g_1, \text{sc-dim } g_1, \{h_{1,1}, h_{1,2}\})$  is a signature in  $L_0$ . Proposition 7.2 gives an extension  $L_1$  primitively generated over  $L_0$  by  $x_{1,1}, x_{1,2}$ , with signature  $\sigma_1$  in  $L_0$ . By construction  $x_{1,1} \wedge x_{1,2}$  belongs to  $L_0$  and is strictly smaller than  $g_1$  hence  $x_{1,1} \wedge x_{1,2} \leq g_1^-$ . For every  $i \geq 2$ ,  $g_i \in \mathcal{I}(L_0) \setminus \{g_1\} \subseteq \mathcal{I}(L_1)$ , and moreover  $g_i^-$  is still the unique predecessor of  $g_i$  in  $L_1$ . Indeed let  $g_i^\dagger$  be the union of every  $c \in \mathcal{I}(L_1)$  strictly smaller than  $g_i$ . Neither  $x_{1,1}$  nor  $x_{1,2}$  are smaller than  $g_i$  because  $g_1 \not\leq g_i$ , so every such  $c$  must belong to  $\mathcal{I}(L_0)$  by Proposition 6.1. It follows that  $g_i^\dagger = g_i^-$ .

So we can repeat in  $L_1$  the same construction applied to  $g_2, h_{2,1}, h_{2,2}$ , and after  $r$  steps we obtain a chain of extensions  $(L_i)_{i \leq r}$  and non-zero elements  $x_{i,j} \in L = L_r$  such that for every  $i$  ( $1 \leq i \leq r$ ):

$$g_i - x_{i,1} = x_{i,2} \quad \text{and} \quad g_i - x_{i,2} = x_{i,1} \quad (11)$$

$$x_{i,1} \wedge g_i^- = h_{i,1} \quad \text{and} \quad x_{i,2} \wedge g_i^- = h_{i,2} \quad (12)$$

$$x_{i,1} \wedge x_{i,2} \leq g_i^- \quad (13)$$

Let  $x_1 = \bigvee_{i \leq r} x_{i,1}$  and  $x_2 = \bigvee_{i \leq r} x_{i,2}$ . For every  $i \leq r$  and every  $k \neq i$ ,  $g_i \wedge x_{k,1} \leq g_i \wedge g_k \ll g_i$ , so by  $\text{TC}_4$  and (11):

$$g_i - x_1 = g_i - \bigvee_{1 \leq k \leq r} x_{k,1} = g_i - x_{i,1} = x_{i,2}$$

So by  $\text{TC}_2$ ,  $a - x_1 = \bigvee_{i \leq r} (g_i - x_1) = x_2$ . Symmetrically  $a - x_2 = x_1$ .

(12) and (13) imply that  $x_{i,1} \wedge x_{i,2} = x_{i,1} \wedge g_i^- \wedge x_{i,2} = h_{i,1} \wedge h_{i,2}$ . Similarly for every  $k \neq i$ :

$$x_{i,1} \wedge x_{k,2} = x_{i,1} \wedge g_i \wedge g_k \wedge x_{k,2} = x_{i,1} \wedge g_i^- \wedge g_k^- \wedge x_{k,2} = h_{i,1} \wedge h_{k,2}$$

So eventually:

$$x_1 \wedge x_2 = \bigvee_{1 \leq i \leq r} \bigwedge_{1 \leq k \leq r} x_{i,1} \wedge x_{k,2} = \bigvee_{1 \leq i \leq r} \bigwedge_{1 \leq k \leq r} h_{i,1} \wedge h_{k,2} = b_1 \wedge b_2$$

We have proved that for every finitely generated scaled lattice  $L_0$  and every  $a, b_1, b_2 \in L_0$  such that  $b_1 \vee b_2 \ll a$ , there exists an  $\mathcal{L}_{\text{SC}}$ -embedding  $\varphi$  from  $L_0$  to a subscaled lattice  $L$  in which there exists non-zero elements  $x_1, x_2$  such that, after identifying  $L_0$  to its image:

$$\begin{cases} x_1 = a - a_2 \\ x_2 = a - a_1 \\ x_1 \wedge x_2 = \varphi(b_1) \wedge \varphi(b_2) \end{cases}$$

On the other hand, Theorem 4.1 and Proposition 5.3, show that every finitely generated subscaled lattice also  $\mathcal{L}_{\text{SC}}$ -embeds into some  $L_{\text{lin}}(X)$ , which is a catenary scaled lattice. The model-theoretic compactness argument then implies that every subscaled lattice  $\mathcal{L}_{\text{SC}}$ -embeds in a superscaled lattice.

■

**Proposition 8.2** *Let  $L_0$  be a finite  $\mathcal{L}_{\text{SC}}$ -substructure of a super scaled lattice  $\hat{L}$ . Then for every signature  $\sigma$  in  $L_0$  there exists a primitive tuple  $(x_1, x_2) \in \hat{L}$  such that  $\sigma$  is the signature of  $L_0\langle x_1, x_2 \rangle$  in  $L_0$ .*

*Proof:* Let  $\sigma = (g, q, \{h_1, h_2\})$  be a signature in  $L_0$ , let  $p = \text{sc-dim } g$  and  $r = \text{sc-dim}(h_1 \vee h_2)$ . Observe that since  $g$  is  $\vee$ -irreducible in  $L_0$  it is sc-pure, and admits a unique predecessor  $g^-$  in  $L_0$ . Let  $y_1, y_2 \in \hat{L}$  given by the splitting property applied to  $g, h_1, h_2$ . By construction  $y_1 \vee y_2 = g$ , and since  $g$  has pure sc-dimension  $p$  so does each  $y_i$ . Moreover:

$$y_1 \wedge h_2 \leq y_1 \wedge y_2 = h_1 \wedge h_2$$

hence  $y_1 \wedge (h_1 \vee h_2) = h_1 \vee (y_1 \wedge h_2) = h_1$ . Since  $h_1 \vee h_2 < g$  it follows that  $y_1 \wedge g^- = h_1 \in L_0$ , and symmetrically  $y_2 \wedge g^- = h_2 \in L_0$ . So  $(y_1, y_2)$  is primitive over  $L_0$ , and the signature of  $L_1 = L_0\langle y_1, y_2 \rangle$  in  $L_0$  is  $(g, p, \{h_1, h_2\})$ .

If  $p = q$  then we are done. Assume now that  $q < p$ , hence  $h_1 = h_2$ . For every  $i \leq r$ , the catenarity gives an element  $x_i \in \hat{L}$  of pure sc-dimension  $p$  such that  $C^i(h_1) \leq x_i \leq y_1$ . Even in case  $r = -1$  the catenarity gives  $x_{-1} \in \hat{L}$  of pure sc-dimension  $p$  such that  $x_{-1} \leq y_1$ . Let  $x = \bigvee_{-1 \leq i \leq r} x_i$ , by construction  $x$  has pure sc-dimension  $p$  and  $h_1 \leq x \leq y_1$ . Then  $x \leq g$ , and in view of their sc-dimension  $x \ll g$ . Moreover:

$$h_1 \leq x \wedge g^- \leq y_1 \wedge g^- = h_1$$

hence  $h_1 \wedge x = h_1 \in L_0$  and in view of the sc-dimensions,  $h_1 \ll x$ . So  $(x, x)$  is a primitive tuple over  $L_0$ , of signature  $\sigma$  in  $L_0$ .

■

**Theorem 8.3** *The theory of super  $d$ -scaled lattices is the model-completion of the theory of  $d$ -subscaled lattices*

**Remark 8.4** Obviously the theorem remains true by replacing everywhere  $d$ -(sub)scaled lattices by (sub)scaled lattices of sc-dimension at most  $d$  (resp. exactly  $d$ ).

*Proof:* Since the axioms of super  $d$ -scaled lattices are  $\forall\exists$ , it follows from Proposition 8.1 that every existentially closed  $d$ -subscaled lattice is super  $d$ -scaled. By classical model-theoretic arguments, it is then sufficient to show that: given a  $d$ -superscaled lattice  $\hat{L}$ , a finitely generated  $d$ -subscaled lattice  $L$ , and a common  $\mathcal{L}_{\text{SC}_d}$ -substructure  $L_0$  of  $L$  and  $\hat{L}$ , there exists an  $\mathcal{L}_{\text{SC}_d}$ -embedding of  $L$  into  $\hat{L}$  whose restriction to  $L_0$  is the identity.

By the local finiteness Theorem 4.1,  $L_0$  is finite. By Proposition 6.2 an immediate induction allows us to assume w.l.o.g. that  $L$  is primitively generated over  $L_0$ . Let  $\sigma$  be the signature of  $L$  over  $L_0$ . By Proposition 8.2 there exists a primitively generated extension  $L_1$  of  $L_0$  in  $\hat{L}$  whose signature is  $\sigma$ . Eventually  $L$  is isomorphic to  $L_1$  over  $L_0$  by Proposition 7.1.

■

The completions of the theory of super  $d$ -scaled lattices are easy to classify. Let us say that a  $d$ -subscaled lattice is **prime** if it does not contain any proper  $d$ -subscaled lattice, or equivalently if it is generated by the empty set. Every prime  $d$ -subscaled lattice is finite. By Corollary 4.2 there exists finitely many prime  $d$ -subscaled lattices up to isomorphism.

**Corollary 8.5** *The theory of super  $d$ -scaled lattices containing (a copy of) a given prime  $d$ -subscaled lattice is  $\aleph_0$ -categorical, hence complete. It is also finitely axiomatisable, hence decidable. Since every completion of the theory of super  $d$ -scaled lattices is of that kind, the theory of super  $d$ -scaled lattices is decidable.*

*Proof:* Let  $L, L'$  be any two countable super  $d$ -scaled lattices containing isomorphic prime  $d$ -subscaled lattices  $L_0$  and  $L'_0$ . Any partial isomorphism between  $L$  and  $L'$  (extending the given isomorphism between  $L_0$  and  $L'_0$ ) can be extended by va-et-vient, using exactly the same argument as in the proof of Theorem 8.3. So the first statement is proved. The other ones are immediate consequences.

■

## 9 Atomic scaled lattices

A natural example of a super scaled lattice is not easy to find. Indeed if  $X$  is any topological space in which points are closed, the points of  $X$  are the atoms of  $L(X)$ , and the splitting axiom imply that a super scaled lattice has no atom. In this section we explore a possible solution to this problem, which lead us to a new conjecture in  $p$ -adic semi-algebraic geometry.

Let  $\mathcal{L}_{\text{ASC}} = \mathcal{L}_{\text{SC}} \cup \{\text{At}_k\}_{k \in \mathbb{N}^*}$  with each  $\text{At}_k$  a new unary predicate symbol. We call **sub-ASC-lattices** the  $\mathcal{L}_{\text{ASC}}$ -structures whose  $\mathcal{L}_{\text{SC}}$ -reduct is a subscaled lattice and which satisfy the following lists of universal axioms:

$$\mathbf{ASC}_1 : (\forall k \neq l), \quad \forall a, \text{At}_k(a) \rightarrow \neg \text{At}_l(a)$$

$$\mathbf{ASC}_2 : (\forall k), \quad \forall a, a_0, \dots, a_{2^k}, \quad \text{At}_k(a) \rightarrow \\ \text{sc-dim } a = 0 \wedge \left[ \left( \bigvee_{0 \leq i \leq 2^k} a_i = a \right) \rightarrow \left( \bigvee_{0 \leq i < j \leq 2^k} a_i = a_j \right) \right]$$

$$\mathbf{ASC}_3 : (\forall k, n), \quad \forall a, a_1, \dots, a_n, \\ \left[ \left( a = \bigvee_{1 \leq i \leq n} a_i \right) \wedge \left( \bigwedge_{1 \leq j < i \leq n} a_i \wedge a_j = \mathbf{0} \right) \right] \rightarrow \\ \left[ \text{At}_k(a) \longleftrightarrow \bigvee_{l_1 + \dots + l_n = k} \bigwedge_{1 \leq i \leq n} \text{At}_{l_i}(a_i) \right]$$

For any  $\mathcal{L}_{\text{ASC}}$ -structure  $L$  we denote by  $\text{At}_k(L)$  the set of elements  $a$  in  $L$  such that  $L \models \text{At}_k(a)$ , and we let  $\text{At}_0(L) = L \setminus \bigcup_{k > 0} \text{At}_k(L)$ . If  $L$  is a sub-ASC-lattice then  $\mathbf{ASC}_1$  asserts that  $(\text{At}_k(L))_{k \in \mathbb{N}}$  is a partition of  $L$ . For any  $a \in L$  we then define  $\text{asc}(a)$  as the unique  $k \in \mathbb{N}$  such that  $a \in \text{At}_k(L)$ . The other axioms of sub-ASC-lattices have the following meaning:

- If  $\text{asc}(a) = k > 0$  then  $\mathbf{ASC}_2$  asserts that  $L(a)$  is a boolean algebra generated by  $n$  atoms  $a_1, \dots, a_n$  with  $n \leq k$ , and  $\mathbf{ASC}_3$  then implies that each  $\text{asc}(a_i) > 0$  and for every  $b \in L(a)$ :

$$\text{asc}(b) = \sum_{c \in \mathcal{C}(b)} \text{asc}(c)$$

where  $\mathcal{C}(b)$  is the set of atoms in  $L(a)$  (as well as in  $L$ ) smaller than  $b$ .

- Conversely  $\mathbf{ASC}_3$  implies that if  $L(a)$  is finite and  $\text{asc}(c) > 0$  for each atom  $c$  in  $L(a)$ :

$$\text{asc}(a) = \sum_{c \in \mathcal{I}(L(a))} \text{asc}(c)$$

In particular if  $a \neq \mathbf{0}$  then  $\text{asc}(a) > 0$ .

**Remark 9.1** It follows immediatly that an  $\mathcal{L}_{\text{SC}}$ -embedding of sub-ASC-lattices  $\varphi: L \rightarrow L'$  is an  $\mathcal{L}_{\text{ASC}}$ -embedding if and only if for every  $k > 0$  and every atom  $a \in L$ :

$$L \models \text{At}_k(a) \iff L' \models \text{At}_k(\varphi(a))$$

Every natural example of sub-ASC-lattice  $L$  satisfy the following additionnal property, which imply  $\mathbf{ASC}_1$  to  $\mathbf{ASC}_3$ :

$\mathbf{ASC}_0$  :  $L$  is a scaled lattices, and for every  $k > 0$ ,  $\text{At}_k(L)$  is the set of elements of  $L$  which are the join of exactly  $k$  atoms in  $L$ .

We call **ASC-lattices**<sup>4</sup> the sub-ASC-lattices which satisfy  $\text{ASC}_0$ . Every sub-scaled lattice  $L$  admits a unique structure of ASC-lattice which is an extension by definition of its lattice structure. We denote by  $L^{\text{At}}$  this expansion of  $L$ .

**Proposition 9.2 (Linear representation)** *Let  $K$  be an infinite field and let  $L_0$  be a finite sub-ASC-lattice. For any positive integer  $N$  there exists a positive integer  $m$ , not depending on  $N$ , and an  $\mathcal{L}_{\text{SC}}^*$ -embedding  $\varphi_N: L_0 \rightarrow L_{\text{lin}}^{\text{At}}(K^m)$  such that for every element  $a$  of  $L_0$  of sc-dimension zero:*

$$\begin{aligned} \text{asc}(a) > 0 &\implies \text{asc}(\varphi_N(a)) = \text{asc}(a) \\ \text{asc}(a) = 0 &\implies \text{asc}(\varphi_N(a)) \geq N \end{aligned}$$

*Proof:* We prove by induction on tuples  $(r, s)$  ordered lexicographically that the proposition holds for every finite sub-ASC-lattice  $L_0$  having  $r$  non-zero  $\vee$ -irreducible elements,  $s$  of which have the same sc-dimension as  $L_0$ .

If  $r = 0$  then  $s = 0$  and the unique embedding of  $L_0 = \{\mathbf{0}\}$  into  $L_{\text{lin}}^{\text{At}}(K^0)$  has the required property. So let us assume that  $r \geq 1$  and that the result is proved for every  $(r', s') < (r, s)$ . Let  $a_1, \dots, a_r$  be the elements of  $\mathcal{I}(L_0)$  ordered by increasing sc-dimension, so  $d = \text{sc-dim } L = \text{sc-dim } a_r \geq 0$ .

**Case  $d = 0$ .** Then  $L$  is a boolean algebra, and  $a_1, \dots, a_r$  are its atoms. For every  $i \leq r$  we choose a finite subset  $A_i$  of  $K \setminus \bigcup_{j < i} A_j$  such that:

- If  $\text{asc}(a_i) > 0$ ,  $A_i$  has  $\text{asc}(a_i)$  elements, so  $\text{asc}(A_i) = \text{asc}(a_i)$ .
- If  $\text{asc}(a_i) = 0$ ,  $A_i$  has  $N$  elements, so  $\text{asc}(A_i) = N$ .

Clearly  $m = 1$  does not depend on  $N$ , and the map  $\varphi$  which maps each  $a_i$  to  $A_i$  extends uniquely to an  $\mathcal{L}_{\text{SC}}^*$ -embedding of  $L_0$  into  $L_{\text{lin}}^{\text{At}}(K)$  which has the required properties.

**Case  $d > 0$ .** The upper semi-lattice  $L_0^-$  generated by  $a_1, \dots, a_{r-1}$  is an  $\mathcal{L}_{\text{ASC}}^*$ -substructure of  $L_0$  to which the induction hypothesis applies. This gives a positive integer  $m$  not depending on  $N$  and an  $\mathcal{L}_{\text{SC}}^*$ -embedding  $\varphi: L_0^- \rightarrow L_{\text{lin}}^{\text{At}}(K^m)$  having the required properties. One can extend  $\varphi$  to an embedding  $\bar{\varphi}$  of  $L_0$  into some  $L_{\text{lin}}^{\text{At}}(K^{m+p})$  exactly like in the proof of Proposition 5.3. The integer  $m+p$  does not depend on  $N$  and  $\bar{\varphi}$  inherits from  $\varphi$  the required properties because all the elements of  $L$  with sc-dimension zero belong to  $L_0^-$ .

■

Let  $\text{ASC}_{\text{Zar}}(K, d)$  (resp.  $\text{ASC}_{\text{lin}}(K, d)$ , resp.  $\text{ASC}_{\text{def}}(K, d)$ ) the class of all ASC-lattices  $L^{\text{At}}$  with  $L$  ranging over  $\text{SC}_{\text{Zar}}(K, d)$ , (resp.  $\text{SC}_{\text{lin}}(K, d)$ , resp.  $\text{SC}_{\text{def}}(K, d)$ ).

**Corollary 9.3** *For any infinite field  $K$  and positive integer  $d$ , the universal theories of  $\text{ASC}_{\text{Zar}}(K, d)$  and  $\text{ASC}_{\text{lin}}(K, d)$  are exactly the theory of sub-ASC-lattices.*

*The same holds for  $\text{ASC}_{\text{def}}(K, d)$  if moreover  $K$  is a henselian valued field of characteristic zero, a real closed field or an algebraically closed field.*

<sup>4</sup>Of course we will show that the sub-ASC-lattices are precisely the  $\mathcal{L}_{\text{ASC}}$ -substructures of ASC-lattices.



*Proof:* Since  $\text{ASC}_{\text{lin}}(K, d)$  is contained in the other classes, all of which are contained in the class of ASC-lattices, it suffices to prove that every sub-ASC-lattice  $\mathcal{L}_{\text{ASC}}$ -embeds into an ultraproduct of elements of  $\text{ASC}_{\text{lin}}(K, d)$ .

For any positive integer  $N$  let  $\varphi_N : L_0 \rightarrow L_{\text{lin}}^{\text{At}}(K^m)$  be the  $\mathcal{L}_{\text{SC}}^*$ -embedding given by Proposition 9.2. We still denote by  $\varphi_N$  the induced  $\mathcal{L}_{\text{SC}}$ -embedding of  $L_0$  into  $L_{\text{lin}}^{\text{At}}(X_N)$  where  $X_N = \varphi_N(\mathbf{1}_{L_0})$ . Let  $\mathcal{U}$  be a non principal ultrafilter in the boolean algebra of subsets of  $\mathbb{N}$ , and  $L = \prod_{N \in \mathbb{N}} L_{\text{lin}}^{\text{At}}(X_N) / \mathcal{U}$ . Then  $\varphi = \prod_{N \in \mathbb{N}} \varphi_N / \mathcal{U}$  is an  $\mathcal{L}_{\text{SC}}$ -embedding of  $L_0$  into the ultraproduct  $L$ . Let  $a$  be an atom of  $L_0$  and  $k = \text{asc}(a)$ .

If  $k > 0$  then for every  $N \geq k$ ,  $L_{\text{lin}}^{\text{At}}(X_N) \models \text{At}_k(\varphi_N(a))$  by construction. So  $L \models \text{At}_k(\varphi(a))$ , that is  $\text{asc}(\varphi(a)) = k$ .

If  $k = 0$ , let  $l$  be any strictly positive integer. For every  $N \geq l$ ,  $L_{\text{lin}}^{\text{At}}(X_N) \models \text{At}_N(\varphi_N(a))$  by construction, hence  $\text{alin}(X_N) \not\models \text{At}_l(\varphi_N(a))$ . So  $L \not\models \text{At}_l(\varphi(a))$ , and this being true for every  $l > 0$  it follows that  $\text{asc}(\varphi(a)) = 0$ .

By Remark 9.1,  $\varphi$  is then an  $\mathcal{L}_{\text{ASC}}$ -embedding.

■

Let us call **super ASC-lattices** those ASC-lattices which satisfy the following axioms:

**Atomicity:** Every element  $x$  is the least upper bound in  $L$  of the set of atoms of  $L$  smaller than  $x$ .

**Catenarity:** For every positive integers  $r \leq q \leq p$  and every  $c \leq a$ , if  $a$  has pure dimension  $p$  and  $c$  has pure dimension  $r$  then there exists a  $q$ -pure element  $b$  such that  $c \leq b \leq a$ .

**ASC-Splitting:** For every  $b_1, b_2, a$ , if  $b_1 \vee b_2 \ll a$  and  $C^0(a) = \mathbf{0}$  there exists non-zero elements  $a_1 \geq b_1$  and  $a_2 \geq b_2$  such that:

$$\begin{cases} a_1 = a - a_2 \\ a_2 = a - a_1 \\ a_1 \wedge a_2 = b_1 \wedge b_2 \end{cases}$$

The class of super-ASC-lattices of sc-dimension at most  $d$  (resp. exactly  $d$ ) is clearly axiomatisable by  $\forall\exists$ -formulas in  $\mathcal{L}_{\text{ASC}}$ . We are going to show that its theory is the model-completion of the theory of sub-ASC-lattices of dimension at most  $d$  (resp. exactly  $d$ ).

**Remark 9.4** An immediate consequence of the atomicity axiom is that for every elements  $x, y$  in a super ASC-lattice  $L$  such that  $y \ll x$ , there are infinitely many atoms  $a \in L$  such that  $a \leq x$  and  $a \wedge y = \mathbf{0}$ . Indeed  $y < x$  hence by the atomicity axiom there is an atom  $a_1 \in L$  such that  $a_1 \leq x$  and  $a_1 \not\leq y$ . Then  $a_1 \leq x$  and  $a_1 \wedge y < a_1$  hence  $a_1 \wedge y = \mathbf{0}$  (because  $a_1$  is an atom). Moreover  $\text{sc-dim } y \vee a_1 < \text{sc-dim } x$  because  $y \ll x$ , hence  $y \vee a_1 < x$  so the same argument applies to  $x$  and  $y \vee a_1$ . It gives another atom  $a_2 \leq x$  such that  $a_2 \wedge y = \mathbf{0}$ , and so on.

Primitive tuples and primitive extensions are defined for sub-ASC-lattices exactly like for subscaled lattices. If  $L$  is an extension of a sub-ASC-lattice  $L_0$  and  $x \in L$  we denote now  $L_0\langle x \rangle$  the  $\mathcal{L}_{\text{ASC}}$ -substructure of  $L$  generated by  $L_0 \vee \{x\}$  (that is the subscaled lattice generated by  $L_0 \cup \{x\}$  endowed with the  $\mathcal{L}_{\text{ASC}}$ -structure induced by  $L$ ).

We define **ASC-signatures** in a finite sub-ASC-lattice  $L_0$  as triples  $(g, p, H)$  with  $H$  a set of non-necessarily distinct ordered pairs  $(h_1, k_1), (h_2, k_2)$  such that  $(g, q, \{h_1, h_2\})$  is a signature in the  $\mathcal{L}_{\text{SC}}$ -reduct of  $L_0$ ,  $k_1, k_2$  are positive integers and:

1. If  $q < \text{sc-dim } g$  then  $(h_1, k_1) = (h_2, k_2)$ .
2. If  $q \neq 0$  then  $k_1 = k_2 = 0$ .
3. If  $k_1 \neq 0, k_2 \neq 0$  and  $\text{sc-dim } g = 0$  then  $\text{asc}(g) = k_1 + k_2$
4. If  $k_1 = 0$  or  $k_2 = 0$  then  $\text{asc}(g) = 0$ .

**Example 9.5** Let  $L_0$  be a finite sub-ASC-lattice, and  $L$  an extension of  $L_0$  generated by a primitive tuple  $(x_1, x_2)$ . Let  $(g, p, \{h_1, h_2\})$  be the signature of  $L$  in  $L_0$  (in the sense of subscaled lattices) and  $k_i = \text{asc}(x_i)$ . Then  $(g, p, \{(h_1, k_1), (h_2, k_2)\})$  is easily seen to be an ASC-signature in  $L_0$ , uniquely determined by  $L$ . We call it the **ASC-signature of  $L$  in  $L_0$** .

The same argument as in Proposition 7.1 shows (using Remark 9.1 in addition to Proposition 3.1) that two primitively generated extensions of a finite sub-ASC-lattice  $L_0$  are isomorphic over  $L_0$  if and only if they have the same signature in  $L_0$ .

**Proposition 9.6** *Let  $L_0$  be a finite  $\mathcal{L}_{\text{ASC}}$ -substructure of an  $\aleph_0$ -saturated super ASC-lattice  $\hat{L}$ . Then for every ASC-signature  $\sigma_{\text{At}}$  in  $L_0$  there exists a primitive tuple  $(x_1, x_2) \in \hat{L}$  such that  $\sigma_{\text{At}}$  is the signature of  $L_0\langle x_1, x_2 \rangle$  in  $L_0$ .*

*Proof:* Let  $\sigma_{\text{At}} = (g, q, \{(h_1, k_1), (h_2, k_2)\})$  be an ASC-signature in  $L_0$ , let  $\sigma = (g, q, \{h_1, h_2\})$  and  $p = \text{sc-dim } g$ . In each of the following cases, the fact that the constructed tuple  $(x_1, x_2)$  is primitive over  $L_0$  and generates an extension whose signature in  $L_0$  is precisely  $\sigma_{\text{At}}$  is straightforward.

**Case  $p \geq 1$ .** Then  $g$  is not an atom, so the ASC-splitting property applies to  $g, h_1, h_2$ . Moreover by Remark 9.4, there are infinitely many atoms  $a$  in  $L$  smaller than  $g$  and not in  $L_0$ . By  $\aleph_0$ -saturation it follows that there exists  $x_0 \in L \setminus L_0$  such that  $\text{sc-dim } x_0 = 0$  and  $L \not\equiv \text{At}_k(a)$  for every  $k > 0$ , that is  $\text{asc}(x_0) = 0$ .

If  $q \geq 1$  then the construction in the proof of Proposition 8.2 applies here. It gives a primitive tuple  $(x_1, x_2)$  in  $L$  such that the  $\mathcal{L}_{\text{SC}}$ -substructure  $L_1$  of  $L$  generated by  $(x_1, x_2)$  over  $L_0$  has signature  $\sigma$  in  $L_0$ . Moreover each  $k_i = 0$  since  $p \neq 0$ , and on the other hand each  $\text{asc } x_i = 0$  since  $\text{sc-dim } x_i = p \geq 1$ .

Otherwise  $q = 0 < p$  hence  $(h_1, k_1) = (h_2, k_2)$ . If  $k_1 \neq 0$  let  $x_1 = x_2 = a_1 \vee \dots \vee a_{k_1}$  with the  $a_i$ 's being any distinct atoms in  $L$  smaller than  $g$  and not belonging to  $L_0$ . If  $k_1 = 0$  let  $x_1 = x_2 = x_0$ .

**Case  $p = 0$ .** Then  $q = 0$  and  $h_1 = h_2 = \mathbf{0}$ .

If  $k_1$  and  $k_2$  are non-zero then  $\text{asc}(g) = k_1 + k_2$  hence  $L(g)$  contains  $k_1 + k_2$  atoms. Let  $x_1$  be the join of  $k_1$  of them, and  $x_2$  be the join of the others.

Eventually if  $k_1 = 0$  or  $k_2 = 0$  then  $\text{asc}(g) = 0$  so  $L(g)$  contains infinitely many atoms, hence by saturation there exists an element  $x$  in  $L$  smaller than  $g$  such that both  $x$  and  $g - x$  are non-zero and  $\text{asc}(x) = \text{asc}(g - x) = 0$ . If  $k_1 = k_2 = 0$  let  $(x_1, x_2) = (x, g - x)$ . If  $k_1 \neq 0$  let  $x_1 = a_1 \vee \dots \vee a_{k_1}$  with the  $a_i$ 's atoms in  $L$  smaller than  $g$  and  $x_2 = g - x_1$ . If  $k_2 \neq 0$  exchange  $k_1$  and  $k_2$ .

■

**Proposition 9.7** *Every sub-ASC-lattice embeds in a super-ASC-lattice.*

*Proof:* Obviously every finitely generated substructure of a sub-ASC-lattices is finite by the local finiteness Theorem 4.1 because  $\mathcal{L}_{\text{SC}}$  and  $\mathcal{L}_{\text{ASC}}$  have the same function symbols.

Let  $L_0$  be a finitely generated sub-ASC-lattice, and  $a, b_1, b_2 \in L_0$  such that  $b_1 \vee b_2 \ll a$  and  $\text{C}^0(a) = \mathbf{0}$ . So  $L_0$  is finite, let  $g_1, \dots, g_r$  be the  $\vee$ -irreducible components of  $a$  in  $L_0$ . As in the proof of Proposition 8.1 we construct a tower of  $\mathcal{L}_{\text{SC}}$ -extensions  $(L_i)_{i \leq r}$  so that each  $L_r$  contains elements  $x_1, x_2$  satisfying

$$\begin{cases} x_1 = a - a_2 \\ x_2 = a - a_1 \\ x_1 \wedge x_2 = \varphi(b_1) \wedge \varphi(b_2) \end{cases}$$

and each  $L_i$  is generated over  $L_{i-1}$  by a primitive tuple of elements  $x_{i,1} \neq x_{i,2}$  such that  $g_i = g(x_{i,1}, L_{i-1}) = g(x_{i,2}, L_{i-1})$ . Proposition 6.1 then implies that:

$$\mathcal{I}(L_r) = (\mathcal{I}(L_0) \setminus \{g_1, \dots, g_r\}) \cup \{x_{i,1}, x_{i,2}\}_{1 \leq i \leq r}$$

By assumption all the  $g_i$ 's have non-zero sc-dimension, hence the  $x_{i,1}$ 's and  $x_{i,2}$ 's do the same. In particular  $L_0$  and  $L_r$  have the same elements of sc-dimension zero. Defining  $\text{At}_k(L_r) = \text{At}_k(L_0)$  for every  $k$  then endows  $L_r$  with an  $\mathcal{L}_{\text{ASC}}$ -structure which makes it a sub-ASC-lattice and an  $\mathcal{L}_{\text{ASC}}$ -extension of  $L_0$ .

On the other hand every  $\text{L}_{\text{lin}}^{\text{At}}(X)$  is an ASC-lattice satisfying the atomicity and catenarity properties, hence Corollary 9.3 implies that every finitely generated sub-ASC-lattice embeds in an ASC-lattice having these two properties.

The conclusion follows by the model-theoretic compactness argument.

■

**Theorem 9.8** *The theory of super ASC-lattices of sc-dimension at most  $d$  (resp. exactly  $d$ ) is the model-completion of the theory of ASC-lattices of dimension at most  $d$  (resp. exactly  $d$ ). It admits  $\aleph_0$  completions, each of which is decidable, and it is decidable.*

*Proof:* The proof of the first statement is very similar to Theorem 8.3 and Corollary 8.5, with the only difference that in the embedding argument we have to assume the super ASC-lattice  $\hat{L}$  to be  $\aleph_0$ -saturated, in order to apply Proposition 9.6 in place of Proposition 8.2.

The last statement then follows from the remark that there are  $\aleph_0$  prime sub-ASC-lattices of dimension at most  $d$  (resp. exactly  $d$ ). Indeed there are finitely many subscaled lattices of dimension at most  $d$  (resp. exactly  $d$ ) and on each of them  $\aleph_0$  different structures of sub-ASC-lattices.

■

Let  $L$  be any super ASC-lattice and  $L'$  the quotient of  $L$  by the equivalence relation  $x \sim y$  if and only if  $x - y$  and  $y - x$  are the join of finitely many atoms. Then it is easily seen that  $L'$  is a super scaled lattice. So the problem of finding a natural example of super scaled lattice boils down to finding a natural example of super ASC-lattice. This and related ideas lead us to the following conjecture:

**Conjecture 9.9** *Let  $K$  be a  $p$ -adically closed field and  $A$  be an infinite definable subset of  $K^n$  which is open in its closure. Let  $(B_k)_{k \leq q}$  be a finite collection of closed definable subsets of  $\overline{A} \setminus A$ . Then there exists a collection  $(A_k)_{k \leq q}$  of non-empty definable subsets of  $A$  clopen in  $A$  such that:*

$$\forall k \leq q, \quad \overline{A_k} = A_k \cup B_k$$

It is an elementary exercise to deduce from this conjecture that  $L_{\text{def}}^{\text{At}}(X)$  models the ASC-splitting property for every definable set  $X$  over  $K$ . Moreover the  $\mathcal{L}_{\text{SC}}$ -substructure of  $L_{\text{def}}^{\text{At}}(K^n)$  generated by the empty set is simply the two element lattice with the obvious  $\mathcal{L}_{\text{SC}}$ -structure (because  $K^d$  is  $d$ -pure). This gives an explicit recursive axiomatisation of  $L_{\text{def}}^{\text{At}}(K^n)$ .

**Corollary 9.10 (Modulo Conjecture 9.9)** *Let  $K$  be a  $p$ -adically closed field, then  $L_{\text{def}}^{\text{At}}(K^n)$  is a super-ASC-lattice. In particular its complete theory is decidable and eliminates the quantifier in  $\mathcal{L}_{\text{ASC}}$ .*

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