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MIXING ANGLES AND NON-DEGENERATE COUPLED SYSTEMS OF PARTICLES

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Abstract: Defining, in the framework of quantum field theory, their mass eigenstates through their matrix propagator, we show why the mixing matrices of non-degenerate coupled systems should not be parametrized as unitary. This is how, for leptonic binary systems, two-angles solutions with discrete values $\pi/4 \bmod \pi/2$ and $\pi/2 \bmod \pi$ arise when weak leptonic currents of mass eigenstates approximately satisfy the two properties of universality and vanishing of their non-diagonal neutral components. Charged weak currents are also discussed, which leads to a few remarks concerning oscillations. We argue that quarks, which cannot be defined on shell because of the confinement property, are instead more naturally endowed with unitary Cabibbo-like mixing matrices, involving a single unconstrained mixing angle. The similarity between neutrinos and neutral kaons is outlined, together with the role of the symmetry by exchange of families.

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1 Introduction

The observed large mixing angles in the neutrino sector have already long been a matter of surprise and questioning [1][2]. Symmetries [3] have been invoked, which should be approximate since the mixing is only close to maximal, various “textures” of mass matrices have been proposed [4], [5] which are not fully satisfying either, and furthermore unstable by unitary transformations on flavour eigenstates [6].

In the while, the CP violating parameters ϵ_L and ϵ_S of the physical neutral kaons K_L and K_S have been shown [7][8], using the propagator formalism of quantum field theory, not to be rigorously identical. This amounts to a tiny lack of unitarity in the mixing matrices linking “in” (or “out”) mass eigenstates to the orthonormal basis of flavour eigenstates. Phrased in another way, mixing matrices of physical kaons cannot be parametrized with a single mixing angle. It was also shown that systems of this type should not be described by a single constant mass matrix and that their mass eigenstates cannot be correctly determined by a bi-unitary transformation.

In this note we establish a link between these two peculiarities and show they are among general properties of non-degenerate coupled systems of particles.

2 General framework

2.1 Flavour and mass eigenstates

Since their couplings to the Higgs boson are not flavor-diagonal, massive fermions in the standard model form coupled systems (like neutral kaons). The usual approach to such systems makes use of a mass matrix (see appendix A). It was however shown in [7][8] that it is an inadequate procedure: indeed, a (constant) mass matrix can only be introduced as a linear approximation to the inverse propagator in the vicinity of each of its poles, such that, in the case under study, as many mass matrices as there are poles should be considered. This is why we stick below to basic principles of quantum field theory, which state in particular that the physical masses of particles, bosons or fermions, can only be the poles of their full propagator. The corresponding eigenstates – the propagating states – are the mass / spin eigenstates of the Lorentz-Poincaré group.

Two bases play a fundamental role, in particular in the electroweak physics of leptons: the $2n_f$ flavour eigenstates ¹ ($e_f^-, \mu_f^-, \nu_{e,f}, \nu_{\mu,f} \dots$) which, by convention, couple to weak vector bosons, and the $2n_f$ propagating eigenstates ($e_m^-, \mu_m^-, \nu_{e,m}, \nu_{\mu,m} \dots$), which are also the mass eigenstates. At the classical level only left-handed flavour fermions weakly couple, but the situation changes when quantum corrections are included.

The *physical masses* $z_i = m_i^2$ satisfy by definition the gauge invariant pole equation (the variable z is used for q^2)

$$\det \Delta^{-1}(z) = 0, \text{ for } z = z_i, \quad (1)$$

where $\Delta(z)$ is the full (renormalized) $2n_f \times 2n_f$ matrix propagator in momentum space. The solutions of (1) are independent of the renormalization procedure. The propagating (mass) eigenstates φ_m^i are the corresponding eigenvectors with vanishing eigenvalues

$$\Delta^{-1}(z = z_i)\varphi_m^i = 0. \quad (2)$$

It is also convenient to introduce $\Delta^{-1}(z) = L^{(2)}(z)$ as the renormalized quadratic Lagrangian operator. (1) then reads

$$\det L^{(2)}(z) = 0, \quad (3)$$

and the mass eigenstates satisfy the equation (equivalent to (2))

$$L^{(2)}(z = z_i)\varphi_m^i = 0. \quad (4)$$

¹ n_f denotes here the number of families.

The situation is accordingly that of a z dependent $2n_f \times 2n_f$ matrix $L^{(2)}(z)$, the $2n_f$ eigenvalues $\lambda_j(z), j = 1 \dots 2n_f$ of which are supposed non-degenerate and satisfy, by definition of the poles z_i of $\Delta(z)$, $\lambda_i(z_i) = 0$. At any z it has $2n_f$ eigenvectors $\psi^j(z), j = 1 \dots 2n_f$. When $z \rightarrow z_i, \psi^i(z) \rightarrow \varphi_m^i$ and $\psi^j(z), j \neq i \rightarrow \omega_{i \neq j}^j$. So, among the $2n_f$ eigenvectors of $L^{(2)}(z_i)$ lies the mass eigenstate φ_m^i corresponding to the vanishing eigenvalue and $2n_f - 1$ other eigenstates $\omega_{i \neq j}^j$, that we call spurious [7][8], and which correspond to non-vanishing eigenvalues $\lambda_j(z_i), j \neq i$. They just represent off-mass-shell states. The case of two flavours ($n_f = 1$) is depicted on Fig. 1 below.

Mixing matrices link flavour (Ψ_f) to mass (Ψ_m) eigenstates: simplifying to $n_f = 2$ (4 flavors)

$$\Psi_f = J\Psi_m, \quad \Psi_f = \begin{pmatrix} \nu_{e,f} \\ \nu_{\mu,f} \\ e_f^- \\ \mu_f^- \end{pmatrix}, \quad \Psi_m = \begin{pmatrix} \nu_{e,m} \\ \nu_{\mu,m} \\ e_m^- \\ \mu_m^- \end{pmatrix}, \quad J = \left(\begin{array}{c|c} K_\nu & \\ \hline & K_\ell \end{array} \right), \quad (5)$$

where we have split every $2n_f \times 2n_f$ matrix into four $n_f \times n_f$ sub-blocks. The entries of Ψ_m are the φ_m^i 's of (2) and (4).

The (renormalized) quadratic Lagrangian density is $\mathcal{L}^{(2)}(x) = \overline{\Psi}_f L^{(2)}(x) \Psi_f$.

Fermions are usually considered as bi-spinors (of Dirac or Majorana types) built from two Weyl spinors of different chiralities which are also orthogonal to each other. Two sets of different mixing matrices therefore generally occur, respectively for left and right spinors (see also appendix A). K_ν and K_ℓ in (5) must then be also attributed a subscript L or R depending on which chirality is considered. In order not to overload the notations, these subscripts will be understood in the following.

2.2 Mixing matrices for mass-split on-shell fermions are not unitary

As already mentioned in [7][8], the connection between flavour eigenstates and non-degenerate mass eigenstates is not a unitary transformation. Indeed:

in flavour space, $L^{(2)}(z)$ being, at each z , a hermitian $2n_f \times 2n_f$ operator (matrix), its $2n_f$ eigenstates form an orthonormal basis $\Psi(z)$ (because it is in particular normal, left and right eigenstates coincide). At $z = z_i = m_i^2, \langle \varphi_m^i | \varphi_m^i \rangle = 1$ and $\langle \varphi_m^i | \omega_{i \neq j}^j \rangle = 0$. Thus, at the $2n_f$ values $z = z_i, 2n_f$ different orthonormal bases (of $2n_f$ eigenstates) occur. Since two non-degenerate mass eigenstates φ_m^i and φ_m^k belong to two different orthonormal bases, they are in general not orthogonal:

$$\langle \varphi_m^i | \varphi_m^k \rangle \neq 0, \quad i \neq k. \quad (6)$$

This being true in the neutral and charged sectors, both K_ν and K_ℓ , which connect the flavour basis to a non-orthonormal one, have no reasons to be unitary

$$K_\ell^\dagger K_\ell \neq 1, \quad K_\nu^\dagger K_\nu \neq 1, \quad q.e.d. \quad (7)$$

The non-unitarity of mixing matrices does not however jeopardize the unitarity of the theory (see appendix B). It simply states that, at a given q^2 , all physical states cannot be simultaneously on-shell when they are non-degenerate.

It may happen, for example to describe unstable particles (like neutral kaons), that one is led to introduce an (effective) Hamiltonian, or Lagrangian, which is non-hermitian, and even non-normal. Then, at each z , the set of eigenstates $\psi^j(z)$ do not form any more an orthonormal basis. Spurious states still accompany the mass eigenstate at $z = z_i$. Different mass eigenstates, corresponding to different z_i 's, have no reason either in this case to form an orthonormal basis, as explicitly checked in [7].

The simplest case of two flavours ($n_f = 1$) is depicted on Fig. 1 which represents either the neutral kaon system, or, in the cases of two lepton families, the neutrino sector or the charged lepton sector. The

z -independent flavour basis (ψ_1, ψ_2) (for example $(K^0, \overline{K^0})$ for neutral kaons) has been represented by the two horizontal lower lines.

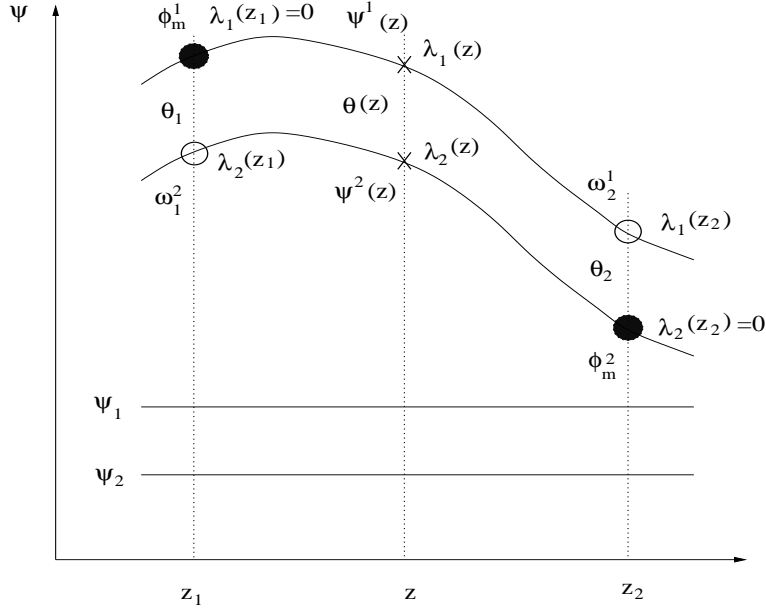


Fig. 1: Eigenstates of a binary coupled system

The two eigenstates of the hermitian (normal) renormalized propagator $\psi^1(z)$ and $\psi^2(z)$ form a z -dependent orthonormal basis. When z varies, this builds up an infinite set of orthonormal bases which is depicted by the two (parallel) curved lines. At a given z , the orthonormal basis $(\psi^1(z), \psi^2(z))$ is connected to the orthonormal flavour basis by a unitary mixing matrix with angle $\theta(z)$. At $z = z_1$, φ_m^1 and ω_1^2 form an orthonormal basis, and so do, at $z = z_2$, φ_m^2 and ω_2^1 . They are respectively related to the basis of flavour eigenstates by two different unitary matrices, with respective angles θ_1 and θ_2 . φ_m^1 and φ_m^2 do not form in general an orthonormal basis, and it intuitively appears on the picture that the mixing matrix connecting them to the flavour basis cannot be parametrized with a single angle (both angles θ_1 and θ_2 obviously play a role).

2.3 The case of quarks

(7) applies to states for which the full propagator has poles, corresponding to physical (“on-shell”) propagating states which can be identified with particles. In contrast, quarks are never produced on shell: the poles of their full propagator are ill-defined and so are accordingly their “physical” masses and mass splittings. The only unambiguous orthonormal basis which then occurs in $L^{(2)}$ (supposed to be hermitian) is the z dependent basis $\psi^j(z)$. At each z are associated two unitary, z -dependent mixing matrices $K_u(z)$ and $K_d(z)$. Their unitary product $K(z) = K_u^\dagger(z)K_d(z)$ we propose to consider as the equivalent of the renormalized unitary Cabibbo-Kobayashi-Maskawa (CKM) matrix of the standard approach, in which the complex mass matrices M_u and M_d generated by the couplings of quarks to the Higgs boson are diagonalized by bi-unitary transformations (see also appendix A).

3 Leptonic weak currents

3.1 Fermion coupling to weak gauge bosons

In the flavour basis Ψ_f , the weak Lagrangian reads

$$\mathcal{L}_{weak} = \bar{\Psi}_f \gamma^\mu \frac{1 - \gamma^5}{2} [W_\mu^+ T^+ + W_\mu^- T^- + W_\mu^3 T^3] \Psi_f,$$

$$T^+ = \left(\begin{array}{c|c} & 1 \\ \hline - & \end{array} \right), T^- = \left(\begin{array}{c|c} & \\ \hline 1 & \end{array} \right), T^3 = \left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array} \right). \quad (8)$$

T^\pm, T^3 form a representation of the $SU(2)$ group of weak interactions: $[T^+, T^-] = T^3$, etc. In the orthonormal basis $\Psi(z)$ one finds another $SU(2)$ representation $[\hat{T}^+(z), \hat{T}^-(z)] = \hat{T}^3(z)$:

$$\mathcal{L}_{weak} = \bar{\Psi}(z) \gamma^\mu \frac{1 - \gamma^5}{2} [W_\mu^+ \hat{T}^+(z) + W_\mu^- \hat{T}^-(z) + W_\mu^3 \hat{T}^3(z)] \Psi(z),$$

$$\hat{T}^+ = \left(\begin{array}{c|c} & K_\nu^\dagger(z) K_\ell(z) \\ \hline - & \end{array} \right), \hat{T}^- = \left(\begin{array}{c|c} & \\ \hline K_\ell^\dagger(z) K_\nu(z) & \end{array} \right), \hat{T}^3 = \left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array} \right). \quad (9)$$

In the non-orthonormal basis Ψ_m of mass eigenstates, (8) becomes

$$\mathcal{L}_{weak} = \bar{\Psi}_m \gamma^\mu \frac{1 - \gamma^5}{2} [W_\mu^+ \mathfrak{T}^+ + W_\mu^- \mathfrak{T}^- + W_\mu^3 \mathfrak{T}^3] \Psi_m, \quad \mathfrak{T}^i = J^\dagger T^i J,$$

$$\mathfrak{T}^+ = \left(\begin{array}{c|c} & K_\nu^\dagger K_\ell \\ \hline - & \end{array} \right), \mathfrak{T}^- = \left(\begin{array}{c|c} & \\ \hline K_\ell^\dagger K_\nu & \end{array} \right), \mathfrak{T}^3 = \left(\begin{array}{c|c} K_\nu^\dagger K_\nu & \\ \hline & -K_\ell^\dagger K_\ell \end{array} \right) \quad (10)$$

and, because of (7), the $SU(2)$ commutation relations are not systematically satisfied by the \mathfrak{T} 's (and Ψ_m does not simply decompose into two $SU(2)$ doublets).

K_ν is related to $K_\nu(z_1)$ and $K_\nu(z_2)$ by

$$K_\nu = \frac{1}{K_{\nu[22]}(z_1) K_{\nu[11]}(z_2) - K_{\nu[12]}(z_1) K_{\nu[21]}(z_2)} \begin{pmatrix} \mathcal{D}_\nu(z_1) K_{\nu[11]}(z_2) & \mathcal{D}_\nu(z_2) K_{\nu[12]}(z_1) \\ \mathcal{D}_\nu(z_1) K_{\nu[21]}(z_2) & \mathcal{D}_\nu(z_2) K_{\nu[22]}(z_1) \end{pmatrix},$$

$$\mathcal{D}_\nu(z) = \det K_\nu(z), \quad z_1 = m_{\nu_{em}}^2, \quad z_2 = m_{\nu_{\mu m}}^2. \quad (11)$$

The following exact relations

$$\langle \varphi_m^1 | \psi_1 \rangle = K_{[11]}(z_1), \langle \varphi_m^2 | \psi_1 \rangle = K_{[12]}(z_2), \langle \varphi_m^1 | \psi_2 \rangle = K_{[21]}(z_1), \langle \varphi_m^2 | \psi_2 \rangle = K_{[22]}(z_2), \quad (12)$$

also hold, which become compatible with the approximate formula

$$K_\nu \simeq \begin{pmatrix} K_{\nu[11]}(z_1) & K_{\nu[12]}(z_2) \\ K_{\nu[21]}(z_1) & K_{\nu[22]}(z_2) \end{pmatrix} \quad (13)$$

when one neglects the scalar products $\langle \varphi_m^1 | \varphi_m^2 \rangle$ supposed to be small. When $z_2 \rightarrow z_1$, (12) and (13) give back a unitary mixing matrix $K_\nu(z_1) \equiv \lim_{z_2 \rightarrow z_1} K_\nu$. This shows the role of the non-degeneracy in the non-unitarity of K_ν .

3.2 Weak currents

It is remarkable, but often unnoticed that, in the quark sector of the standard model, with unitary K_u and K_d ², the built-in characteristic of the weak Lagrangian that the couplings of flavour fermions to gauge bosons are symmetric by the exchange of families, translates into a similar property for mass states, without any constraint on the (then unique) mixing angle. The same holds concerning the absence of flavour changing neutral currents.

² K_u and K_d are the mixing matrices respectively for u -type and d -type quarks.

The situation becomes different for mass-split physical states. Considering the case of two families, when a single mixing angle is not enough to describe the system, the symmetry by exchange of families for mass states (that we call universality for mass states) and the absence of “mass changing neutral currents” (MCNC’s) are no longer automatically achieved. They instead require well defined relations between mixing angles. We demonstrate below that, within each 2×2 subspace of given electric charge, they constrain the two mixing angles to be: either $\pi/2 \pmod{\pi}$, or $\pi/4 \pmod{\pi/2}$. In the first case the family indices of flavour and mass states are either identical or crossed. The last case corresponds to the “maximal mixing” (approximately) observed for neutrinos and for neutral kaons. Mass eigenstates are then symmetric or antisymmetric by family exchange (for example so are the PC eigenstates $K^0 \pm \overline{K}^0$). Maximal mixing accordingly realizes universality in both spaces of flavour and mass eigenstates.

3.2.1 Neutral weak currents of mass eigenstates

We deal with weak currents of mass eigenstates and investigate the property that MCNC’s are very small and that their diagonal counterparts are quasi-universal. Allowing a lack of unitarity (7), we parametrize, with transparent notations (preserving a unit norm for all states and discarding irrelevant global phases)

$$K_\nu = \begin{pmatrix} e^{i\alpha}c_1 & e^{i\delta}s_1 \\ -e^{i\beta}s_2 & e^{i\gamma}c_2 \end{pmatrix}, \quad K_\ell = \begin{pmatrix} e^{i\theta}c_3 & e^{i\zeta}s_3 \\ -e^{i\chi}s_4 & e^{i\omega}c_4 \end{pmatrix}. \quad (14)$$

Note that $[K_\nu, K_\nu^\dagger] \neq 0$: K_ν and K_ℓ are not normal. (14) entails

$$K_\nu^\dagger K_\nu = \begin{pmatrix} c_1^2 + s_2^2 & c_1s_1e^{i(\delta-\alpha)} - c_2s_2e^{i(\gamma-\beta)} \\ c_1s_1e^{i(\alpha-\delta)} - c_2s_2e^{i(\beta-\gamma)} & s_1^2 + c_2^2 \end{pmatrix},$$

$$K_\ell^\dagger K_\ell = \begin{pmatrix} c_3^2 + s_4^2 & c_3s_3e^{i(\zeta-\theta)} - c_4s_4e^{i(\omega-\chi)} \\ c_3s_3e^{i(\theta-\zeta)} - c_4s_4e^{i(\chi-\omega)} & s_3^2 + c_4^2 \end{pmatrix}.$$

The two requests of (quasi) universality and absence of MCNC’s, equivalent to $K_\nu^\dagger K_\nu \approx 1 \approx K_\ell^\dagger K_\ell$, translate into, respectively, the identity of diagonal elements and the vanishing of non-diagonal elements. The condition $K_\nu^\dagger K_\nu = 1$, in the simple case where the phases $\alpha, \beta, \gamma, \delta$ are vanishing, can be visualized on Fig. 2, which is drawn in the orthonormal flavour basis:

- the two unit vectors (c_1, s_1) and $(-s_2, c_2)$ are the two dotted vectors;
- the mass eigenstates proportional to $(c_2, -s_1)$ and (s_2, c_1) are the two green vectors;
- all vectors are uniquely determined by θ_1 and θ_2 ; the condition under scrutiny is that of finding these two angles such that the red and blue vectors $(c_1, -s_2)$ and (s_1, c_2) (or, equivalently, the mass eigenstates) become *orthonormal*.

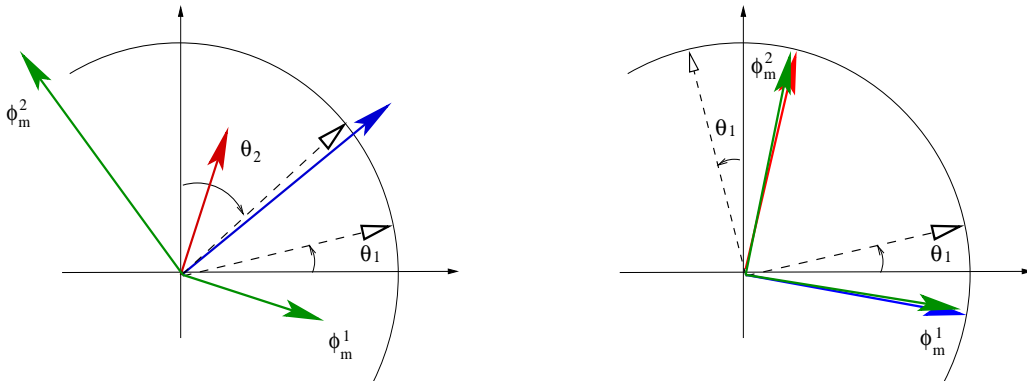


Fig. 2: graphical representation of the two conditions of universality and absence of MCNC’s; on the right is a “Cabibbo-like” solution $\theta_2 = \theta_1$.

The discussions for neutrinos and for charged leptons being similar, we proceed with the former.

- Quasi-universality is satisfied for $c_1^2 + s_2^2 = c_2^2 + s_1^2 \Leftrightarrow c_1^2 = c_2^2$ which requires $\theta_2 \approx \pm\theta_1 + k\pi$ (a).

- The quasi-absence of MCNC's requires

- * either: $c_1 s_1 = c_2 s_2$ (f_1) and $e^{i(\alpha-\delta)} = e^{i(\beta-\gamma)}$ (f_2),
- * or: $c_1 s_1 = -c_2 s_2$ (g_1) and $e^{i(\alpha-\delta)} = -e^{i(\beta-\gamma)}$ (g_2).

(f_1) requires either $\theta_2 \approx \theta_1 + n\pi$ (b), or $\theta_2 \approx -\theta_1 + \pi/2 + n\pi$ (c), while (g_1) requires either $\theta_2 = -\theta_1 + n\pi$ (d) or $\theta_2 = \theta_1 + \pi/2 + n\pi$ (e). The different cases to consider are accordingly (a) \cup (b) \cup (f_2), (a) \cup (c) \cup (f_2), (a) \cup (d) \cup (g_2) and (a) \cup (e) \cup (g_2).

The solutions of (a) \cup (b) \cup (f_2) and (a) \cup (d) \cup (g_2) are

$$\begin{aligned} (a) \cup (b) \cup (f_2) &: \theta_2 = \theta_1 + k\pi \text{ or } \left\{ \theta_1 = (k-n)\frac{\pi}{2}, \theta_2 = (k+n)\frac{\pi}{2} = -\theta_1 + k\pi \right\}, \\ (a) \cup (d) \cup (g_2) &: \theta_2 = -\theta_1 + k\pi \text{ or } \left\{ \theta_1 = (n-k)\frac{\pi}{2}, \theta_2 = (n+k)\frac{\pi}{2} = \theta_1 + k\pi \right\}. \end{aligned} \quad (15)$$

The solutions of (a) \cup (c) \cup (f_2) and (a) \cup (e) \cup (g_2) are

$$\begin{aligned} (a) \cup (c) \cup (f_2) &: \left\{ \theta_1 = \frac{\pi}{4} + (n-k)\frac{\pi}{2}, \theta_2 = \frac{\pi}{4} + (n+k)\frac{\pi}{2} = \theta_1 + k\pi \right\}, \\ (a) \cup (e) \cup (g_2) &: \left\{ \theta_1 = -\frac{\pi}{4} + (k-n)\frac{\pi}{2}, \theta_2 = \frac{\pi}{4} + (k+n)\frac{\pi}{2} = -\theta_1 + k\pi \right\}. \end{aligned} \quad (16)$$

There exist accordingly two sets of solutions:

- * Cabibbo-like solutions $\theta_2 = \pm\theta_1 + k\pi$ for which the equations for universality and for the absence of MCNC's coincide: the (θ_1, θ_2) surfaces defined by $c_1^2 = c_2^2$ and $c_1 s_1 = c_2 s_2$ intersect along a line, which yields a one parameter solution (with a single, unconstrained, mixing angle).

- * Cases for which the two equations are independent: the two surfaces (one of which can be checked to always present a saddle point in the vicinity of the intersections) intersect at discrete points.

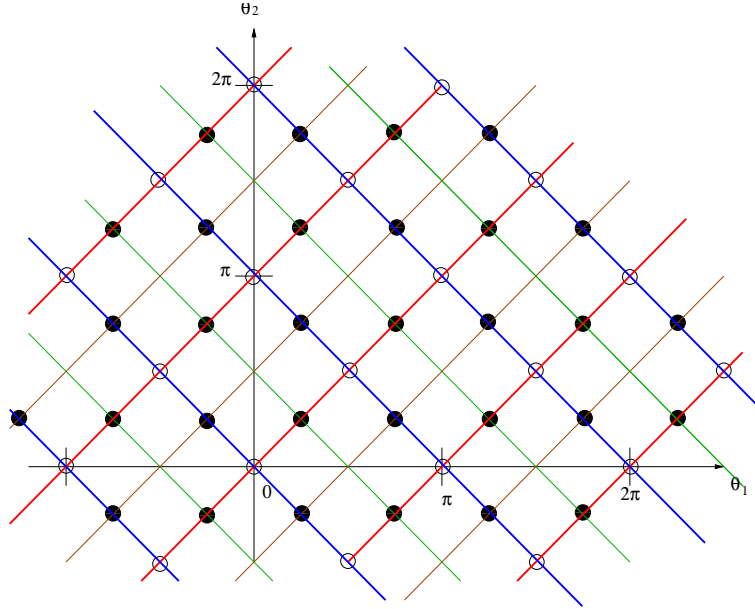


Fig. 3: graphical solution of the two conditions of universality and absence of MCNC

The set of all solutions is depicted on Fig. 3 (in which the conditions (f_2) or (g_2) are supposed to be realized). It is made of the entire red and blue lines + all black and white dots. The blue and red lines correspond respectively to $\theta_2 = \theta_1 + k\pi$ and $\theta_2 = -\theta_1 + k\pi$ (conditions (a), (b) or (d)). They represent the Cabibbo-like situations. They cross (white dots) at the discrete values $\pi/2 + k\pi$ of θ_1 and θ_2 . The

brown and green lines correspond respectively to $\theta_2 = \pi/2 + \theta_1 + k\pi$ (c) and $\theta_2 = \pi/2 - \theta_1 + k\pi$ (e). Their intersections (black dots) with the blue or red lines (on which in particular condition (a) holds) provide the maximal mixing solutions $\pm\pi/4 + n\pi/2$.

Note that $\theta_{1,2} \approx 0, \pi$ are allowed for physical non-degenerate particles. The case $\theta = 0$ means that mass and flavour eigenstates are exactly aligned (which is usually assumed for charged leptons).

Though discrete solutions are also located on red or blue lines, they should not be mixed up with Cabibbo-like solutions: the former are one-parameter solutions, while the latter depend on two parameters.

Both types, when exact, can be shrunk, by rephasing the fermions, to a single mixing angle which is unconstrained for Cabibbo-like cases and has fixed values for others. We give below a Cabibbo-like example³. For $\theta_2 = \theta_1 + \pi$ and $e^{i(\delta-\alpha)} = e^{i(\gamma-\beta)} = e^{i\xi}$ (i.e. for conditions (b) \cup (f₂)), or $\theta_2 = -\theta_1$ and $e^{i(\delta-\alpha)} = -e^{i(\gamma-\beta)} = e^{i\xi}$ (i.e. for conditions (d) \cup (g₂)), one has

$$\begin{pmatrix} e^{-i\alpha}\nu_{em} \\ e^{-i\beta}\nu_{\mu m} \end{pmatrix} = \begin{pmatrix} \cos(-\theta_1) & \sin(-\theta_1) \\ -\sin(-\theta_1) & \cos(-\theta_1) \end{pmatrix} \begin{pmatrix} \nu_{ef} \\ -e^{i\xi}\nu_{\mu f} \end{pmatrix}. \quad (17)$$

So doing, for all *exact* solutions, K_ν becomes unitary.

However, exact solutions are purely academic since, for example, the absence of MCNC's is expected to be only approximate⁴. As for exact universality (a), it is by itself not enough to have a unique mixing angle since, in particular, (f₂) or (g₂) may not be satisfied. Moreover, (a) may be only approximately realized. So, in the vicinity of the solutions above, a single mixing angle is not enough to describe the system.

Another characteristic of the discrete solutions is their low sensitivity to small translations in the (θ_2, θ_1) plane. If one varies, for example, θ_2 , by ϵ close to a specific point, the l.h.s.'s of the universality condition, $(c_2^2 + s_1^2) - (c_1^2 + s_2^2) = 0$, and of the condition for the absence of MCNC's, $c_2s_2 \pm c_1s_1 = 0$, vary respectively by $-4\epsilon c_2s_2$ and $\epsilon(c_2^2 - s_2^2)$. Hence:

- at the discrete values $m\pi/2 + n\pi$, the universality condition is satisfied at $\mathcal{O}(\epsilon^2)$ while the MCNC condition is only satisfied at $\mathcal{O}(\epsilon)$. Referring to Fig. 2, this means that, if one varies by ϵ the angle (which is then a right angle) between the two white dotted vectors, the (right) angle between the blue and red vectors (and the one between the green mass eigenstates) also vary by ϵ , while their (unit) lengths are only altered at $\mathcal{O}(\epsilon^2)$;

- at the ‘‘maximal mixing’’ values, the reverse holds: the absence of MCNC's is specially enforced; by the same variation as above, it is now the angle between the red and blue vectors (and the one between the mass eigenstates) which only varies by ϵ^2 , while their lengths are altered at $\mathcal{O}(\epsilon)$;

- outside the set of discrete solutions, in particular for Cabibbo-like solutions, both variations are instead $\mathcal{O}(\epsilon)$.

The absence of MCNC's is thus specially enforced at maximal mixing, while universality is at angles $m\pi/2 + n\pi$.

As seen on Fig. 1, Cabibbo-like systems, characterized by a single unconstrained mixing angle, can only be:

- degenerate particles $z_1 = z_2$ (in which case $\omega_1^2 = \varphi_m^2$ and $\omega_2^1 = \varphi_m^1$ such that $(\varphi_m^1, \varphi_m^2)$ form an orthonormal basis);

- ‘‘off shell’’ systems $(\psi^1(z), \psi^2(z))$ evaluated at a common scale $z = q^2$, like quarks, for which the mixing angle is $\theta(z)$;

- very special systems like the ones satisfying eqs. (81)(82) of [7].

Physical non-degenerate mesonic systems like $K^0 - \bar{K}^0$ correspond to the other category (non Cabibbo-like): when CP (or exact family symmetry) holds, mixing angles are identical and maximum, but when CP is broken, two angles occur, as shown in [7], which are only close to maximum. Such systems

³Since they lie on the trajectories of Cabibbo-like solutions, this is also a general property of all exact discrete solutions.

⁴Note that (b) \cup (f₂) or (c) \cup (g₂), which entail the exact absence of MCNC, is enough to have a single mixing angle, since they also entail exact universality (a).

are expected to lie inside the small (2-dimensional) areas in the vicinity of the discrete solutions (the extended dots of Fig. 3), and not inside 1-dimensional deformations of exact Cabibbo-like systems, that stay on the red or blue lines.

3.2.2 Charged weak currents of mass eigenstates. Short comments on oscillations.

Charged weak currents are coupled through $K_\nu^\dagger K_\ell$, the so-called PMNS matrix [9]. Since charged leptons are non-degenerate coupled fermions too, we expect, like previously obtained for neutrinos, the occurrence of a discrete set of mixing angles $\pi/4 \bmod \pi/2$ and $\pi/2 \bmod \pi$. K_ℓ , like K_ν , lies accordingly close to one of the “academic” unitary matrices evoked above, such that $K_\nu^\dagger K_\ell$ should also be close to a unitary matrix with a mixing angle in the same set of discrete values⁵. Several cases arise, the relevance of which with respect to oscillations we would like to briefly discuss:

- * if one among K_ν and K_ℓ is close to “maximal” and the other close to a multiple of $\pi/2$, the PMNS matrix is close to “maximal”;
- * if both K_ℓ and K_ν are close to “maximal” with respective mixing angles $(2k+1)\pi/4$ and $(2n+1)\pi/4$, the PMNS matrix is close to a matrix with mixing angle $(k-n)\pi/2$; this includes the diagonal unit matrix (up to an irrelevant sign) and the antidiagonal unit matrix;
- * if both mixing angles of K_ℓ and K_ν are close to a multiple of $\pi/2$, the same result holds.

Let us first stress that, while neutrino oscillations are determined by K_ν alone, the detection of neutrinos on earth always goes through their coupling to charged leptons, which involves the PMNS matrix. We will consider two configurations for the latter which both seem able to reproduce the observed solar electron neutrino deficit on earth.

“Measuring” a PMNS matrix close to maximal for two generations favors the first possibility. One among the two sets (ν_e, ν_μ) and (e^-, μ^-) of leptons has then a maximal mixing, while the mixing angle of the second is a multiple of $\pi/2$ (in which case only simple mass-flavour alignment or nearly perfect “crossing” can occur). The following picture may then be conceived. Let us suppose that the flux of neutrinos stays unperturbed during its travel from the center of the sun to the surface of the earth, where it is detected. This can for example happen if the Mikheyev-Smirnov-Wolfenstein (MSW) effect [11] does not operate inside the sun and if, then, vacuum oscillations do not modify the neutrino spectrum. Its detection through the charged currents (and, so, through the maximal PMNS matrix) introduces a coefficient $\approx \pm 1/\sqrt{2}$, which yields a factor 1/2 in the square of the corresponding amplitude. A 1/2 “deficit” occurs though, in reality, no oscillation took place.

Mass-flavour alignment for one fermion species, which is one of the two alternatives leading to this first possibility, rules out the corresponding oscillations. It is natural to assume this property for charged leptons (as usually done), since such oscillations cannot anyhow be observed as soon as one measures their energy with a precision much higher than their mass-splitting [10]. The emerging picture may appear coherent, though the asymmetry arising between the two species of leptons raises questions concerning the role of the electric charge. Another possible weakness of this point of view lies in the importance acquired by the measuring process through which, furthermore, the determination of the PMNS matrix cannot be truly asserted.

Now, we would like to point out that the following scenario, with a PMNS matrix $\approx \text{diag}(1, 1)$, is possible as well. This belongs to the second possibility in the original list, and accordingly provides a symmetric treatment of neutral and charged leptons, which both have maximal mixing. In this case, we are led instead to consider that neutrinos do oscillate in their travel from the core of the sun to the earth. So, with respect to what is expected from solar models, a modified flux of ν_{em} reaches the earth, which can for example be altered by a factor $\approx \pm 1/\sqrt{2}$. These neutrinos then diagonally couple, in the detector, to charged leptons with the coefficient 1 occurring now in the PMNS matrix, such that a global factor 1/2 again occurs in the (amplitude)². It can rightly be interpreted as “neutrino oscillations”.

⁵After convenient rephasing of the fermions (see (17)), both K_ℓ and K_ν become close to unitary matrices with respective mixing angles θ_ℓ and θ_ν ; the PMNS matrix is then close to a unitary matrix with angle $(\theta_\ell - \theta_\nu)$.

This treatment avoids the slightly opportunistic eviction of electron-muon oscillations that occurred before, and which is not mandatory: it indeed undoubtedly leads to potential such oscillations, which can appear problematic, but can be argued away, as already explained, according to [10]. Charged currents now differ from those in the quark sector by a stronger suppression of their off-diagonal components as compared with the ones obtained from the CKM matrix.

The two scenarii just described, which differ, seem nevertheless to lead to the same conclusion, *i.e.* the observed depletion of electronic solar neutrinos on earth. Discriminating between them (and others ?) needs a more careful investigation which lies beyond the scope of this work.

As for the third possibility, it can easily be shown never to lead to any neutrino deficit. Thus, the maximal character of K_ν , which is common to the first two cases, appears as the essential ingredient for the occurrence of this phenomenon.

4 Neutral mesons

Neutral kaons are composite states, and any Lagrangian that limits to their description can only be effective. A similar propagator formalism can nevertheless be applied [7][8].

The second order electroweak transitions, which couple K^0 to \overline{K}^0 , are family changing transitions in which d and s quarks get swapped. When CP is conserved, the K_1^0 and K_2^0 mass eigenstates, respectively symmetric and antisymmetric with respect to $d \leftrightarrow s$ family exchange, correspond to exact maximal mixing. They form an orthonormal basis and no spurious state occurs. CP violation alters this situation: the CP violating parameters ϵ_L and ϵ_S for K_L and K_S mass eigenstates slightly differ due to their mass splitting and the Hamiltonian is no longer normal. Inside each *in* or *out* space mass eigenstates no longer form orthonormal basis while *in* and *out* mass eigenstates, which differ, form a bi-orthogonal basis.

The striking similarity between the latter and neutrinos suggests that the symmetry by exchange of families (universality) plays an important role in the nature of physical states.

Composite states (mesons) are however more complex than fundamental particles. Indeed, while the underlying electroweak theory for quarks does satisfy the criteria of universality and absence of flavour / mass changing neutral currents, the corresponding two types of conditions are not directly available in an effective theory for neutral kaons alone. Whether or not they could be implemented in a larger frame of an effective theory for all scalar and pseudoscalar mesons, in which a general mass matrix in flavour space should be diagonalized (see for example [12]), is a forthcoming matter of investigation.

5 Conclusion

In this short note, we have proposed an enlargement of the mixing scheme between mass and flavour eigenstates, which incorporates the peculiarity of both neutrinos and neutral kaons that their mixing angles are close to maximal. In continuation of [7][8] we have shown that, in quantum field theory, the mixing matrices of on-shell coupled mass-split fermions should not be parametrized as unitary. The physics of two massive neutrinos is then not that of a single mixing angle, but of two. A new family of discrete mixing angles then springs out, among which lies the quasi-maximal mixing observed for neutrinos and neutral kaons. When two different mixing angles are concerned, the naive $\theta_2 \rightarrow \theta_1$ (“Cabibbo”) limit does not exist, which explains how discrete solutions can easily be overlooked.

The role of family exchange symmetry has been emphasized. The generalization of this simple exercise to more than two flavours will be the subject of a subsequent work. Other aspects of coupled fermionic systems will also appear in [13].

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Appendix

A The mass matrix in electroweak physics

The Yukawa couplings between fermions and the $SU(2)$ Higgs doublet yield mass terms which write (we take the example of u -type quarks) $\overline{\psi_{ufL}} M_u \psi_{ufR}$, to which are always added their complex conjugate $\overline{\psi_{ufR}} M_u^\dagger \psi_{ufL}$ (in order to make the corresponding operator in the Lagrangian hermitian). ψ_{ufL} denotes the vector $(u_{fL}, c_{fL}, t_{fL})^T$ of left-handed flavour eigenstates, ψ_{ufR} its right counterpart.

The usual way to make the CKM matrix appear in the weak bare Lagrangian is to diagonalize the complex and potentially non-hermitian M_u by a bi-unitary transformation $U_u^\dagger M_u V_u = D_u$. One accordingly *defines* quark masses as the square roots of the (real positive) eigenvalues of the hermitian $M_u M_u^\dagger$ and $M_u^\dagger M_u$. They accordingly differ from the eigenvalues of M_u or M_u^\dagger and they are real positive (thus presumably not suitable to describe unstable fermions). The CKM matrix, which acts on left-handed quarks, is $K = U_u^\dagger U_d$ (the d subscript is attached to d -like quarks). The mixing matrices V_u and V_d for right-handed quarks do not appear in the bare electroweak Lagrangian (right handed quarks do not couple to weak gauge bosons and their vectorial couplings to other gauge fields are, like their kinetic terms, invariant by any unitary transformation). Both initial mass term and its complex conjugate are diagonalized by the same bi-unitary transformation, and they rewrite (D_u being real) $(U_u^\dagger \psi_{ufL}) D_u (V_u^\dagger \psi_{ufR}) + \overline{(V_u^\dagger \psi_{ufR})} D_u (U_u^\dagger \psi_{ufL})$. Two different mass bases then occur, $\psi_{umL} = U_u^\dagger \psi_{ufL}$ for left fermions and $\chi_{umR} = V_u^\dagger \psi_{ufR}$ for right fermions, in terms of which the mass term finally takes the following form: $\psi_{umL} D_u \chi_{umR} + \overline{\chi_{umR}} D_u \psi_{umL}$, which is that of a set of Dirac fermions built from the two sets of Weyl fermions ψ_{umL} and χ_{umR} . Kinetic terms can also be rewritten as the ones of these Dirac fermions.

Another (equivalent) argument uses the *polar decomposition theorem* stating that any complex matrix can always be written as the product of an hermitian matrix times a unitary matrix $M_u = H_u \mathcal{W}_u$. One then absorbs the unitary matrix \mathcal{W}_u into the right-handed fermions $\tilde{\psi}_{ufR} = \mathcal{W}_u \psi_{ufR}$, arguing again that they do not couple to the weak gauge bosons, and one is left with a hermitian mass matrix [14] H_u which can be diagonalized with a single unitary matrix U_u . The mass term then becomes $(U_u^\dagger \psi_{ufL}) D_u (U_u^\dagger \tilde{\psi}_{ufR})$. As already mentioned, the kinetic terms for right-handed fermions, which are insensitive to any unitary transformation, are in particular unaffected by their redefinition. Like before, the mass bases for right- and left-handed fermions differ.

Thus, when the mass matrix is non-hermitian, two types of mixing matrices have to be introduced for left and right fermions. The flavour content of the two spinors making up mass eigenstates is then different, unlike what occurs (by convention) for bispinor flavour eigenstates. If one rather sticks to identical transformations for left and right fermions, the mass term $\overline{\psi_{ufL}} M_u \psi_{ufR} + \overline{\psi_{ufR}} M_u^\dagger \psi_{ufL}$ generated by the coupling to the Higgs doublet includes a γ^5 term proportional to $M_u - M_u^\dagger$ which must be taken into account. A certain type of left-right symmetry is thus seen to be broken.

The necessary transformation of right fermions plays a role at the quantum level (in the renormalized Lagrangian) because they do couple to weak gauge bosons when radiative corrections are included (at order g^4) [15].

B Unitarity for an effective CP -violating theory of neutral kaons [7]

The following simple exercise can be done for neutral kaons, which proves that the $K_L \rightarrow K_L$ transition amplitude stays unaltered by off-diagonal transitions and the non-orthogonality of mass eigenstates. This happens, as proved in [7], when their two CP violation parameters are different $\epsilon_L \neq \epsilon_S$.

Consider first the simplest possible theory of an (uncoupled) unique generic kaon K ; the only possible transition is that of K into itself, with, of course, probability 1. The propagator Δ of K is that of a free field.

Consider now the effective theory for the coupled $K^0 - \overline{K}^0$ system with CP violation like in [7]. On one side, since K_L and K_S are mass-split and no particle can carry away the missing energy in a $K_L \rightarrow K_S$ on-shell decay, the only open channel for the evolution of K_L is itself. On the other side, the Lagrangian includes a $K_L - K_S$ operator $\left[|K_S \rangle_{in} \langle K_L| - |K_L \rangle_{in} \langle K_S| \right]$ with a coefficient $V(z)$ given in eq. (114) of [7], and K_L and K_S are not orthogonal $\langle K_S | K_L \rangle_{in} = -\langle K_L | K_S \rangle_{in} = \frac{1}{2} \left(\frac{\xi_S}{\xi_L} - \frac{\xi_L}{\xi_S} \right) = v \neq 0$, $\xi_{L,S} = \sqrt{\frac{1-\epsilon_{L,S}^{in}}{1+\epsilon_{L,S}^{in}}}$.⁶

A_{LL} and A_{LS} (see Fig. 4) are the V -resummed diagonal $K_L - K_L$ and mixed $K_L - K_S$ propagators; $A_{LL} = \Delta \left(\frac{1}{1+V^2\Delta^2} \right)$ and $A_{LS} = \Delta \left(\frac{-V\Delta}{1+V^2\Delta^2} \right)$, where $\Delta \approx \Delta_L \approx \Delta_S$ is the ‘‘average’’ neutral kaon propagator.

The non-orthogonality of K_L and K_S entails that any ingoing or outgoing K_L has a non-zero K_S component, and *vice versa*. This brings additional corrections to A_{LL} and A_{LS} , such that the $K_L - K_L$ and $K_L - K_S$ full propagators are finally given by B_{LL} and B_{LS} (we limited the expansion at $\mathcal{O}(v^2)$ for B_{LL}).

Fig. 4: $K_L \rightarrow K_L$ transitions

In the expression of $V(z)$ (eq. (114) in [7]):

- $b(z)$ and $c(z)$, which describe $K^0 - \overline{K}^0$ transitions, are of second order (g^4) in the weak interactions;
- $e^{i\alpha}c(z) \stackrel{CP}{=} e^{-i\alpha}b(z)$ (the corrections due to CP violation are proportional to $(\epsilon_S^{in} + \epsilon_L^{in})(e^{i\alpha}b(z) + e^{-i\alpha}c(z))$);
- $D(z) = 1 + \mathcal{O}(\epsilon^2)$, $a(z) \approx \frac{1}{\Delta(z)}$;
- the coefficient which factorizes $a(z)$ is the same $2v$ as above.

Since $a(z)$ in the diagonal part of the inverse propagator $a(z) \approx \frac{1}{\Delta(z)}$ and since one can approximate $V(z) \approx \frac{1}{\Delta(z)}v$, one gets $A_{LL} \approx \Delta \left(\frac{1}{1+v^2} \right)$ and $A_{LS} \approx -\Delta \left(\frac{v}{1+v^2} \right)$; likewise, one finds $A_{SS} \approx A_{LL}$ and $A_{SL} \approx -A_{LS}$, which finally gives $B_{LL} \approx \Delta$: it is the ‘‘free’’ kaon propagator, like in a theory where none of the intricacies due to mass splitting and CP violation (in particular the non-unitarity of the mixing matrix) occurs. The transition probability for $K_L \rightarrow K_L$ thus stays also unchanged and equal to 1: no violation of unitarity occurs. If neutral kaons are given complex $(mass)^2$ to account for their instability, the probability is no longer conserved, but this violation of unitarity is unrelated with the peculiarities of the mixing matrix.

⁶No confusion should arise with asymptotic states of the S-matrix: *in* and *out* states denote here the right and left eigenvectors of the kaon propagator, which is non-normal when CP is broken.

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