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► **To cite this version:**

Christophe Chesneau. Regression in random design and wavelet block thresholding. 2006. hal-00083141v2

**HAL Id: hal-00083141**

**<https://hal.science/hal-00083141v2>**

Preprint submitted on 15 Oct 2006 (v2), last revised 25 Nov 2006 (v3)

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# Regression with Random Design and Wavelet Block Thresholding

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*01 October 2006*

## **Abstract**

In the framework of regression model with (known) random design, we prove that estimators of wavelet coefficients of the unknown regression function satisfy a strong large deviation inequality. This result can be used to show several statistical properties concerning a wavelet block thresholding estimator.

**Key Words:** Regression with random design, Large deviation inequality, Wavelet, Block thresholding.

**AMS subject classification:** 62G07, 62G20.

## **1 Motivation**

Wavelets have been shown to be a very successful tool in the framework of nonparametric function estimation. They provide adaptive estimators which enjoy good theoretical and practical properties.

In the present paper, we focus our attention on the regression model with (known) random design. We show that estimators of wavelet coefficients of the unknown regression function satisfy a strong large deviation inequality.

Thanks to this inequality, we can apply several general results established in the literature concerning the performances of a  $L^p$  version of an adaptive wavelet block thresholding estimator. It has been initially developed by [Cai \(1997, 2002\)](#) in the framework of the regression model with deterministic equispaced data. Among the numerous consequences of our inequality, we can show that the considered BlockShrink construction provides adaptive

confidence balls under the local  $\mathbb{L}^p$  risk and that it achieves (near) minimax rates of convergence over Besov balls under the global  $\mathbb{L}^p$  risk.

The paper is organized as follows. Section 2 describes wavelet bases. Section 3 presents the model and the main result of the paper. Some applications are described in Section 4. Section 5 contains a detailed proof of the main result.

## 2 Wavelets

We consider an orthonormal wavelet basis generated by dilation and translation of a compactly supported "father" wavelet  $\phi$  and a "mother" wavelet  $\psi$ . For the purposes of this paper, we use the periodized wavelets bases on the unit interval. Let us set

$$\phi_{j,k} = 2^{j/2}\phi(2^jx - k), \quad \psi_{j,k} = 2^{j/2}\psi(2^jx - k).$$

And let us denote the periodized wavelets by  $\phi_{j,k}^p(x) = \sum_{l \in \mathbb{Z}} \phi_{j,k}(x - l)$ ,  $\psi_{j,k}^p(x) = \sum_{l \in \mathbb{Z}} \psi_{j,k}(x - l)$ ,  $x \in [0, 1]$ . If  $\tau$  large enough then the collection  $\zeta = \{\phi_{j,k}^p(x), k = 0, \dots, 2^j - 1; \psi_{j,k}^p(x), j = \tau, \dots, \infty, k = 0, \dots, 2^j - 1\}$  constitutes an orthonormal basis of  $\mathbb{L}^2([0, 1])$ . The superscript "p" will be suppressed from the notations for convenience.

For any  $l \in \{\tau, \dots\}$ , a square-integrable function on  $[0, 1]$  can be expanded into a wavelet series

$$f(x) = \sum_{k=0}^{2^l-1} \alpha_{l,k} \phi_{l,k}(x) + \sum_{j=l}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k} \psi_{j,k}(x),$$

where  $\alpha_{j,k} = \int_0^1 f(x) \phi_{j,k}(x) dx$ ,  $\beta_{j,k} = \int_0^1 f(x) \psi_{j,k}(x) dx$ .

For further details about wavelets, see [Meyer \(1990\)](#) and [Cohen et al. \(1993\)](#).

Since the wavelet basis  $\zeta$  is compactly supported, it satisfies the following property of concentration : there exists a constant  $C > 0$  such that, for any  $m > 0$  and any  $x \in [0, 1]$ , we have

$$\sum_{k=0}^{2^j-1} |\psi_{j,k}(x)|^m \leq C 2^{jm/2}. \quad (2.1)$$

### 3 The model and the main result

In this study, we consider the regression model with random design described as follows : suppose that we observe  $n$  pairs of random variables i.i.d  $((X_1, Y_1), \dots, (X_n, Y_n))$  governed by the equation:

$$Y_i = f(X_i) + z_i, \quad i \in \{1, \dots, n\}, \quad (3.1)$$

where the  $z_i$ 's are Gaussian i.i.d with mean zero, variance one and are independent of the design  $(X_1, \dots, X_n)$ . We assume that the design is known with  $X_1 \in [0, 1]$ . We denote by  $g$  the density of  $X_1$  and we assume that  $g$  is bounded from above and below. The function  $f$  is unknown and bounded from above ( $\|f\|_\infty$  is supposed to be known). The objective is to estimate  $f$  from observations  $((X_1, Y_1), \dots, (X_n, Y_n))$ .

This statistical problem has been investigated by many authors via various approaches. Numerous minimax results can be found in the book of [Tsybakov \(2004\)](#).

If we consider the framework of wavelet analysis, the first step to estimate  $f$  consists in estimating the associated wavelet coefficients  $\beta_{j,k}$ . Here, we consider the following unbiased estimator :

$$\hat{\beta}_{j,k} = n^{-1} \sum_{i=1}^n Y_i g(X_i)^{-1} \psi_{j,k}(X_i). \quad (3.2)$$

Theorem [3.1](#) below shows that  $(\hat{\beta}_{j,k})_k$  satisfies a strong large deviation inequality.

(It is important to mention that all the constants of our study are independent of  $f$  and  $n$ .)

**Theorem 3.1.** *Let us consider the regression model with random design [\(3.1\)](#). Let  $p \in [2, \infty[$  and  $L \asymp (\log n)^{p/2}$ . Let  $j_1$  and  $j_2$  be integers satisfying*

$$2^{j_1} \asymp L, \quad 2^{j_2} \asymp (n/\log n)^{1/2}.$$

*For any  $j \in \{j_1, \dots, j_2\}$ , let us set  $\mathcal{A}_j = \{1, \dots, 2^j L^{-1}\}$  and*

$$\mathcal{B}_{j,K} = \{k \in \{0, \dots, 2^j - 1\} : (K-1)L \leq k \leq KL - 1\}, \quad K \in \mathcal{A}_j.$$

Then, there exist two constants  $\mu_1 > 0$  and  $C > 0$  such that, for any  $j \in \{j_1, \dots, j_2\}$ ,  $K \in \mathcal{A}_j$  and  $n$  large enough, the estimators  $(\hat{\beta}_{j,k})_k$  defined by (3.2) satisfies

$$\mathbb{P}_f^n \left( (L^{-1} \sum_{k \in \mathcal{B}_{j,K}} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{1/p} \geq \mu_1 2^{-1} n^{-1/2} \right) \leq C n^{-p}. \quad (3.3)$$

The proof of Theorem 3.1 uses several well-known large deviation inequalities as the Talagrand inequality, the Cirelson inequality, the Bernstein inequality and the Hoeffding inequality.

**Remark 3.1.** Let us adopt the statistical framework of Theorem 3.1. For any  $k \in \{0, \dots, 2^j - 1\}$ , there exist three constants  $K^* > 0$ ,  $\mu_2 > 0$  and  $C > 0$  such that

$$\begin{aligned} & \mathbb{P}_f^n (|\hat{\beta}_{j,k} - \beta_{j,k}| \geq \mu_2 \sqrt{(\log n/n)}) \\ \leq & \mathbb{P}_f^n \left( (L^{-1} \sum_{k \in \mathcal{B}_{j,K^*}} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{1/p} \geq \mu_1 2^{-1} n^{-1/2} \right) \leq C n^{-p}. \end{aligned}$$

Therefore, the inequality (3.3) is stronger than an usual large deviation inequality generally used to investigate the performance of the hard thresholding procedure introduced by Donoho and Johnstone (1995). See, for instance, (Kerkycharian and Picard, 2000, Theorems 5.1 and 5.2).

## 4 Some applications of Theorem 3.1

In the framework of the regression model with (known) random design (3.1), the inequality of large deviation proved in Theorem (3.1) can be applied to show several statistical properties concerning the following adaptive estimator of  $f$  :

$$\hat{f}(x) = \sum_{k=0}^{2^{j_1}-1} \hat{\alpha}_{j_1,k} \phi_{j_1,k}(x) + \sum_{j=j_1}^{j_2} \sum_{K \in \mathcal{A}_j} \sum_{k \in \mathcal{B}_{j,K}} \hat{\beta}_{j,k} \mathbf{1}_{\{\hat{b}_{j,K} \geq \mu_1 n^{-1/2}\}} \psi_{j,k}(x), \quad (4.1)$$

where  $j_1$ ,  $j_2$ ,  $\hat{\beta}_{j,k}$ ,  $\mathcal{A}_j$ ,  $\mathcal{B}_{j,K}$  and  $\mu_1$  are defined in Theorem 3.1, the estimator  $\hat{\alpha}_{j,k}$  is defined by  $\hat{\alpha}_{j,k} = n^{-1} \sum_{i=1}^n Y_i g(X_i)^{-1} \phi_{j,k}(X_i)$  and  $\hat{b}_{j,K}$  is the

normalized  $l_p$ -norm of  $(\hat{\beta}_{j,k})_{k \in \mathcal{B}_{j,K}}$  i.e:

$$\hat{b}_{j,K} = (L^{-1} \sum_{k \in \mathcal{B}_{j,K}} |\hat{\beta}_{j,k}|^p)^{1/p}.$$

This construction is a  $\mathbb{L}^p$  version of the BlockShrink estimator adapted to our statistical problem. Such estimator has been introduced by [Cai \(1997, 2002\)](#) in the framework of the regression model with deterministic equispaced data. Two applications of [Theorem 3.1](#) are briefly described below.

- *Adaptive confidence balls* : if we apply [Picard and Tribouley \(2000, Proposition 1\)](#) then the estimator [\(4.1\)](#) provides adaptive confidence intervals around  $f(x_0)$  with  $x_0 \in [0, 1]$  under the local  $\mathbb{L}^p$  risk.
- *Optimality result* : if we use the Bernstein inequality, one can show that there exists a constant  $C > 0$  such that the estimator  $\hat{\beta}_{j,k}$  defined by [\(3.2\)](#) satisfies the following moments inequality :

$$\mathbb{E}_f^n(|\hat{\beta}_{j,k} - \beta_{j,k}|^{2p}) \leq Cn^{-p},$$

(see for instance [\(Kerkyacharian and Picard, 2005, Proof of Proposition 3\)](#)). Combining this result with [Theorem 3.1](#), [Chesneau \(2006, Theorem 4.2\)](#) determines the rates of convergence achieved by the estimator [\(4.1\)](#) over Besov balls under the global  $\mathbb{L}^p$  risk. More precisely, by considering the Besov balls  $B_{\pi,r}^s(M)$  defined by :

$$B_{\pi,r}^s(M) = \{f \in \mathbb{L}^\pi([0, 1]); [\sum_{j=\tau-1}^{\infty} [2^{j(s+1/2-1/\pi)} (\sum_{k=0}^{2^j-1} |\beta_{j,k}|^\pi)^{1/\pi}]^r]^{1/r} \leq M\},$$

(with the usual modification if  $r = \infty$ ) we can set the following theorem

**Theorem 4.1** (Application of [Theorem 4.2](#) proved by [Chesneau \(2006\)](#)). *Let us consider the regression model with random design [\(3.1\)](#). Let  $p \in [2, \infty[$ . Let us consider the estimator  $\hat{f}$  defined by [\(4.1\)](#). Then there exists a constant  $C > 0$  such that, for any  $\pi \in [1, \infty]$ ,  $r \in [1, \infty]$ ,  $s \in [1/\pi + 1/2, N]$  and  $n$  large enough, we have*

$$\sup_{f \in B_{\pi,r}^s(M)} \mathbb{E}_f^n \left( \int_0^1 |\hat{f}(x) - f(x)|^p dx \right) \leq C\varphi_n,$$

where

$$\varphi_n = \begin{cases} n^{-\alpha_1 p} (\log n)^{\alpha_1 p 1_{\{p > \pi\}}}, & \text{when } \epsilon > 0, \\ (\log n/n)^{\alpha_2 p} (\log n)^{(p-\pi/r)+1_{\{\epsilon=0\}}}, & \text{when } \epsilon \leq 0, \end{cases}$$

with  $\alpha_1 = s/(2s+1)$ ,  $\alpha_2 = (s - 1/\pi + 1/p)/(2(s - 1/\pi) + 1)$  and  $\epsilon = \pi s + 2^{-1}(\pi - p)$ .

These rates of convergence are minimax and better than those achieved by the well-known hard thresholding estimator. Theorem 4.1 can be viewed as a generalization of a result proved by [Chicken \(2003, Theorem 2\)](#) for the uniform design, the  $\mathbb{L}^2$  risk and the Hölder balls  $B_{\infty, \infty}^s(M)$ .

## 5 Proofs

First of all, let us recall the Talagrand inequality and the Cirelson inequality.

**Lemma 5.1** ([Talagrand \(1994\)](#)). *Let  $(V_1, \dots, V_n)$  be i.i.d random variables and  $(\epsilon_1, \dots, \epsilon_n)$  be independent Rademacher variables, also independent of  $(V_1, \dots, V_n)$ . Let  $\mathcal{F}$  be a class of functions uniformly bounded by  $T$ . Let  $r_n : \mathcal{F} \rightarrow \mathbb{R}$  be the operator defined by :*

$$r_n(h) = n^{-1} \sum_{i=1}^n h(V_i) - \mathbb{E}(h(V_1)).$$

*Suppose that  $\sup_{h \in \mathcal{F}} \text{Var}(h(V_1)) \leq v$  and  $\mathbb{E}(\sup_{h \in \mathcal{F}} \sum_{i=1}^n \epsilon_i h(V_i)) \leq nH$ . Then, there exist two constants  $C_1 > 0$  and  $C_2 > 0$  such that, for any  $t > 0$ , we have :*

$$\mathbb{P}(\sup_{h \in \mathcal{F}} r_n(h) \geq t + C_2 H) \leq \exp(-nC_1 (t^2 v^{-1} \wedge tT^{-1})).$$

**Lemma 5.2** ([Cirelson et al. \(1976\)](#)). *Let  $\mathcal{D}$  be a subset of  $\mathbb{R}$ . Let  $(\eta_t)_{t \in \mathcal{D}}$  be a centered Gaussian process. Suppose that  $\mathbb{E}(\sup_{t \in \mathcal{D}} \eta_t) \leq N$  and  $\sup_{t \in \mathcal{D}} \text{Var}(\eta_t) \leq Q$ . Then, for any  $x > 0$ , we have*

$$\mathbb{P}(\sup_{t \in \mathcal{D}} \eta_t \geq x + N) \leq \exp(-x^2/(2Q)). \quad (5.1)$$

We are now in position to prove Theorem 3.1. Here and latter,  $C$  represents a constant which may be different from one term to the other. We suppose that  $n$  is large enough.

*Proof of Theorem 3.1.* We have the following decomposition

$$\hat{\beta}_{j,k} - \beta_{j,k} = A_{j,k} + B_{j,k}$$

where

$$A_{j,k} = n^{-1} \sum_{i=1}^n f(X_i)g(X_i)^{-1}\psi_{j,k}(X_i) - \mathbb{E}_f^n(f(X_1)g(X_1)^{-1}\psi_{j,k}(X_1)),$$

$$B_{j,k} = n^{-1} \sum_{i=1}^n g(X_i)^{-1}\psi_{j,k}(X_i)z_i.$$

By the  $l_p$  Minkowski inequality, for any  $\mu > 0$ , we have

$$\mathbb{P}_f^n\left(\left(L^{-1} \sum_{k \in \mathcal{B}_{j,K}} |\hat{\beta}_{j,k} - \beta_{j,k}|^p\right)^{1/p} \geq 2^{-1}\mu n^{-1/2}\right) \leq \mathcal{U} + \mathcal{V},$$

where

$$\mathcal{U} = \mathbb{P}_f^n\left(\left(L^{-1} \sum_{k \in \mathcal{B}_{j,K}} |A_{j,k}|^p\right)^{1/p} \geq 4^{-1}\mu n^{-1/2}\right),$$

$$\mathcal{V} = \mathbb{P}_f^n\left(\left(L^{-1} \sum_{k \in \mathcal{B}_{j,K}} |B_{j,k}|^p\right)^{1/p} \geq 4^{-1}\mu n^{-1/2}\right).$$

Let us investigate separately the upper bounds of  $\mathcal{U}$  and  $\mathcal{V}$ .

• *Upper bound for  $\mathcal{U}$ .* Our goal is to apply the Talagrand inequality described in Lemma 5.1. Let us consider the set  $\mathcal{C}_q$  defined by  $\mathcal{C}_q = \{a = (a_{j,k}) \in \mathbb{Z}^*; \sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^q \leq 1\}$  and the functions class  $\mathcal{F}$  defined by  $\mathcal{F} = \{h; h(x) = f(x)g(x)^{-1} \sum_{k \in \mathcal{B}_{j,K}} a_{j,k}\psi_{j,k}(x), a \in \mathcal{C}_q\}$ . By an argument of duality, we have

$$\left(\sum_{k \in \mathcal{B}_{j,K}} |A_{j,k}|^p\right)^{1/p} = \sup_{a \in \mathcal{C}_q} \sum_{k \in \mathcal{B}_{j,K}} a_{j,k} A_{j,k} = \sup_{h \in \mathcal{F}} r_n(h),$$

where  $r_n$  denotes the operator defined in Lemma 5.1. Let us evaluate the parameters  $T$ ,  $H$  and  $v$  of the Talagrand inequality.

First of all, notice that, for  $p \geq 2$  (and a fortiori  $q = 1 + (p - 1)^{-1} \leq 2$ ), an elementary inequality of  $l_p$  norm gives  $\sup_{a \in \mathcal{C}_q} (\sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^2)^{1/2} \leq \sup_{a \in \mathcal{C}_q} (\sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^q)^{1/q} \leq 1$ .

– *Value of  $T$ .* Let  $h$  be a function in  $\mathcal{F}$ . By the Hölder inequality, the assumptions of boundedness of  $f$  and  $g$  and the property of concentration (2.1), we find

$$\begin{aligned} |h(x)| &\leq |f(x)||g(x)|^{-1} \left( \sum_{k \in \mathcal{B}_{j,K}} |\psi_{j,k}(x)|^2 \right)^{1/2} \sup_{a \in \mathcal{C}_q} \left( \sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^2 \right)^{1/2} \\ &\leq \|f\|_\infty \|1/g\|_\infty \left( \sum_{k \in \mathcal{B}_{j,K}} |\psi_{j,k}(x)|^2 \right)^{1/2} \leq C2^{j/2}, \quad x \in [0, 1]. \end{aligned}$$

Hence  $T = C2^{j/2}$ .

– *Value of  $H$ .* The  $l_p$ -Hölder inequality and the Hölder inequality imply

$$\begin{aligned} &\mathbb{E}_f^n \left( \sup_{a \in \mathcal{C}_q} \sum_{i=1}^n \left( \sum_{k \in \mathcal{B}_{j,K}} a_{j,k} \epsilon_i f(X_i) g(X_i)^{-1} \psi_{j,k}(X_i) \right) \right) \\ &\leq \sup_{a \in \mathcal{C}_q} \left( \sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^q \right)^{1/q} \left( \sum_{k \in \mathcal{B}_{j,K}} \mathbb{E}_f^n \left( \left| \sum_{i=1}^n \epsilon_i f(X_i) g(X_i)^{-1} \psi_{j,k}(X_i) \right|^p \right) \right)^{1/p} \\ &\leq \left( \sum_{k \in \mathcal{B}_{j,K}} \mathbb{E}_f^n \left( \left| \sum_{i=1}^n \epsilon_i f(X_i) g(X_i)^{-1} \psi_{j,k}(X_i) \right|^p \right) \right)^{1/p}. \end{aligned} \quad (5.2)$$

Since  $(\epsilon_1, \dots, \epsilon_n)$  are independent Rademacher variables, also independent of  $(X_1, \dots, X_n)$ , the Khintchine inequality yields

$$\begin{aligned} &\mathbb{E}_f^n \left( \left| \sum_{i=1}^n \epsilon_i f(X_i) g(X_i)^{-1} \psi_{j,k}(X_i) \right|^p \right) \\ &= \mathbb{E}_f^n \left( \mathbb{E}_f^n \left( \left| \sum_{i=1}^n \epsilon_i f(X_i) g(X_i)^{-1} \psi_{j,k}(X_i) \right|^p \mid \mathbb{X} \right) \right) \\ &\leq C \mathbb{E}_f^n \left( \left| \sum_{i=1}^n |f(X_i)|^2 |g(X_i)|^{-2} |\psi_{j,k}(X_i)|^2 \right|^{p/2} \right) \\ &= CI. \end{aligned} \quad (5.3)$$

Let us consider the i.i.d random variables  $(N_1, \dots, N_n)$  with

$$N_i = |f(X_i)|^2 |g(X_i)|^{-2} |\psi_{j,k}(X_i)|^2, \quad i \in \{1, \dots, n\}.$$

An elementary inequality of convexity implies  $I \leq 2^{p/2-1}(I_1 + I_2)$  where

$$I_1 = \mathbb{E}_f^n(|\sum_{i=1}^n (N_i - \mathbb{E}_f^n(N_1))|^{p/2}), \quad I_2 = n^{p/2} \mathbb{E}_f^n(N_1)^{p/2}.$$

Let us analyze the upper bounds for  $I_1$  and  $I_2$ , in turn.

– *Upper bound for  $I_1$ .* The Bernstein inequality applied to  $(N_1, \dots, N_n)$  and the Cauchy-Schwartz inequality imply

$$\begin{aligned} I_1 &\leq C(n\mathbb{E}_f^n(|N_1 - \mathbb{E}_f^n(N_1)|^{p/2}) + (n\mathbb{E}_f^n(|N_1 - \mathbb{E}_f^n(N_1)|^2))^{p/4}) \\ &\leq C(n\mathbb{E}_f^n(|N_1|^{p/2}) + (n\mathbb{E}_f^n(|N_1|^2))^{p/4}). \end{aligned}$$

For any  $m \geq 1$ ,  $j \in \{j_1, \dots, j_2\}$  and  $k \in \{0, \dots, 2^j - 1\}$ , the assumptions of boundedness of  $f$  and  $g$  give

$$\begin{aligned} \mathbb{E}_f^n(|N_1|^m) &= \int_0^1 |f(x)|^{2m} |g(x)|^{-2m+1} |\psi_{j,k}(x)|^{2m} dx \\ &\leq C2^{j(m-1)} \|\psi\|_\infty^{2m-2} \int_0^1 |\psi_{j,k}(x)|^2 dx \leq C2^{j(m-1)} \leq Cn^{m-1}. \end{aligned}$$

We deduce that  $I_1 \leq Cn^{p/2}$ .

– *Upper bound for  $I_2$ .* Since  $\mathbb{E}_f^n(N_1) \leq C$ , we have  $I_2 \leq Cn^{p/2}$ .

Combining the obtained upper bounds for  $I_1$  and  $I_2$ , we find

$$I \leq C(I_1 + I_2) \leq Cn^{p/2}. \quad (5.4)$$

Putting (5.2), (5.3) and (5.4) together, we see that

$$\mathbb{E}_f^n(\sup_{a \in \mathcal{C}_q} \sum_{i=1}^n (\sum_{k \in \mathcal{B}_{j,K}} a_{j,k} \epsilon_i f(X_i) g(X_i)^{-1} \psi_{j,k}(X_i))) \leq (\sum_{k \in \mathcal{B}_{j,K}} I)^{1/p} \leq Cn^{1/2} L^{1/p}.$$

Hence  $H = Cn^{-1/2} L^{1/p}$ .

– *Value of  $v$ .* By the assumptions of boundedness of  $f$  and  $g$  and the

orthonormality of  $\zeta$ , we obtain

$$\begin{aligned}
\sup_{h \in \mathcal{F}} \text{Var}(h(X_1)) &\leq \sup_{a \in \mathcal{C}_q} \mathbb{E}_f^n(|f(X_1)|^2 |g(X_1)|^{-2} \sum_{k \in \mathcal{B}_{j,K}} a_{j,k} \psi_{j,k}(X_1)|^2) \\
&\leq C \sup_{a \in \mathcal{C}_q} \mathbb{E}_f^n \left( \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} g(X_1)^{-1} \psi_{j,k}(X_1) \psi_{j,k'}(X_1) \right) \\
&= C \sup_{a \in \mathcal{C}_q} \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} \int_0^1 \psi_{j,k}(x) \psi_{j,k'}(x) dx \\
&= C \sup_{a \in \mathcal{C}_q} \sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^2 \leq C.
\end{aligned}$$

Hence  $v = C$ .

Now, let us notice that, for any  $j \in \{j_1, \dots, j_2\}$ , we have  $n2^j \leq n2^{j_2} \leq Cn^{3/2}(\log n)^{-1/2}$  and  $L^{1/p} \asymp (\log n)^{1/2}$ . Therefore, if  $t = 8^{-1}\mu L^{1/p}n^{-1/2}$  then

$$(t^2 v^{-1} \wedge tT^{-1}) \geq C \left( \mu^2 (\log n/n) \wedge \mu \sqrt{(\log n/n2^j)} \right) \geq C \mu^2 (\log n/n).$$

So, for  $\mu$  large enough and  $t = 8^{-1}\mu L^{1/p}n^{-1/2}$ , the Talagrand inequality yields

$$\begin{aligned}
\mathcal{U} &= \mathbb{P}_f^n \left( (L^{-1} \sum_{k \in \mathcal{B}_{j,K}} |A_{j,k}|^p)^{1/p} \geq 4^{-1} \mu n^{-1/2} \right) \\
&\leq \mathbb{P}_f^n \left( (L^{-1} \sum_{k \in \mathcal{B}_{j,K}} |A_{j,k}|^p)^{1/p} \geq 8^{-1} \mu n^{-1/2} + Cn^{-1/2} \right) \\
&\leq \mathbb{P}_f^n \left( \sup_{h \in \mathcal{F}} r_n(h) \geq t + C_2 H \right) \\
&\leq \exp(-nC_1 (t^2 v^{-1} \wedge tT^{-1})) \leq \exp(-nC \mu^2 (\log n/n)) \leq n^{-p}.
\end{aligned}$$

We obtain the desired inequality for  $\mathcal{U}$ .

• *Upper bound for  $\mathcal{V}$ .* Our goal is to apply the Cirelson inequality described in Lemma 5.2. Let us consider the set  $\mathcal{C}_q$  defined by  $\{a = (a_{j,k}) \in \mathbb{Z}^*; \sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^q \leq 1\}$  and the process  $\mathcal{Z}(a)$  defined by  $\mathcal{Z}(a) = \sum_{k \in \mathcal{B}_{j,K}} a_{j,k} B_{j,k}$ . Let us notice that, conditionally to  $\mathbb{X} = (X_1, \dots, X_n)$ ,  $\mathcal{Z}(a)$  is a gaussian centered process. Moreover, by an argument of duality, we have

$$\sup_{a \in \mathcal{C}_q} \mathcal{Z}(a) = \sup_{a \in \mathcal{C}_q} \sum_{k \in \mathcal{B}_{j,K}} a_{j,k} B_{j,k} = \left( \sum_{k \in \mathcal{B}_{j,K}} |B_{j,k}|^p \right)^{1/p}.$$

Now, let us investigate the bounds of  $\mathbb{E}_f^n(\sup_{a \in \mathcal{C}_q} \mathcal{Z}(a) | \mathbb{X})$  and  $\sup_{a \in \mathcal{C}_q} \text{Var}_f^n(\mathcal{Z}(a) | \mathbb{X})$ .

– *Upper bound for  $\mathbb{E}_f^n(\sup_{a \in \mathcal{C}_q} \mathcal{Z}(a) | \mathbb{X})$ .* Let us consider the set  $\mathcal{B}_\mu$  defined by

$$\mathcal{B}_\mu = \left\{ \left| n^{-1} \sum_{i=1}^n g(X_i)^{-1} |\psi_{j,k}(X_i)|^2 - 1 \right| \geq \mu \right\}.$$

Let us work on the set  $\mathcal{B}_\mu^c$ . By the Jensen inequality, the fact that  $\mathcal{Z}(a) | \mathbb{X} \sim \mathcal{N}(0, n^{-2} \sum_{i=1}^n |g(X_i)|^{-2} |\psi_{j,k}(X_i)|^2)$  and the fact that  $g$  is bounded from below, we find

$$\begin{aligned} \mathbb{E}_f^n(\sup_{a \in \mathcal{C}_q} \mathcal{Z}(a) | \mathbb{X}) &\leq \left( \sum_{k \in \mathcal{B}_{j,K}} \mathbb{E}_f^n(|B_{j,k}|^p | \mathbb{X}) \right)^{1/p} \\ &= C \left( \sum_{k \in \mathcal{B}_{j,K}} \left( n^{-2} \sum_{i=1}^n |g(X_i)|^{-2} |\psi_{j,k}(X_i)|^2 \right)^{p/2} \right)^{1/p} \\ &\leq C \|1/g\|_\infty \left( \sum_{k \in \mathcal{B}_{j,K}} \left( n^{-2} \sum_{i=1}^n g(X_i)^{-1} |\psi_{j,k}(X_i)|^2 \right)^{p/2} \right)^{1/p} \\ &= C n^{-1/2} \left( \sum_{k \in \mathcal{B}_{j,K}} \left( n^{-1} \sum_{i=1}^n g(X_i)^{-1} |\psi_{j,k}(X_i)|^2 - 1 + 1 \right)^{p/2} \right)^{1/p} \\ &\leq C n^{-1/2} \left( \sum_{k \in \mathcal{B}_{j,K}} (\mu + 1)^{p/2} \right)^{1/p} \leq C (\mu + 1)^{1/2} L^{1/p} n^{-1/2}. \end{aligned}$$

Hence  $N = N(\mathbb{X}) = C(\mu + 1)^{1/2} L^{1/p} n^{-1/2}$ .

– *Upper bound for  $\sup_{a \in \mathcal{C}_q} \text{Var}_f^n(\mathcal{Z}(a) | \mathbb{X})$ .* Let us define the set  $\mathcal{A}_\mu$  by

$$\mathcal{A}_\mu = \left\{ \sup_{a \in \mathcal{C}_q} \left( \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} \left( n^{-1} \sum_{i=1}^n g(X_i)^{-1} \psi_{j,k}(X_i) \psi_{j,k'}(X_i) \right) - \sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^2 \right) \geq \mu \right\}.$$

Let us work on  $\mathcal{A}_\mu^c$ . By using the assumptions of boundedness of  $g$ , we have

$$\begin{aligned} G &= \sup_{a \in \mathcal{C}_q} \left( \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} \left( n^{-1} \sum_{i=1}^n |g(X_i)|^{-2} \psi_{j,k}(X_i) \psi_{j,k'}(X_i) \right) \right) \\ &\leq C \left[ \sup_{a \in \mathcal{C}_q} \left( \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} \left( n^{-1} \sum_{i=1}^n g(X_i)^{-1} \psi_{j,k}(X_i) \psi_{j,k'}(X_i) \right) \dots \right) \right] \end{aligned}$$

$$- \sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^2) + \sup_{a \in \mathcal{C}_q} \sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^2] \leq C(\mu + 1).$$

Since  $\mathbb{E}_f^n(z_i z_{i'}) = 1$  if  $i = i'$  and 0 otherwise, we have

$$\begin{aligned} & \sup_{a \in \mathcal{C}_q} \text{Var}_f^n(\mathcal{Z}(a) | \mathbb{X}) = \sup_{a \in \mathcal{C}_q} \mathbb{E}_f^n(|\mathcal{Z}(a)|^2 | \mathbb{X}) \\ = & \sup_{a \in \mathcal{C}_q} \mathbb{E}_f^n\left(\sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} B_{j,k} B_{j,k'} \mid \mathbb{X}\right) \\ = & \sup_{a \in \mathcal{C}_q} \left(n^{-2} \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} \sum_{i=1}^n \sum_{i'=1}^n |g(X_i)|^{-2} \psi_{j,k}(X_i) \psi_{j,k'}(X_{i'}) \mathbb{E}_f^n(z_i z_{i'})\right) \\ = & n^{-1} \sup_{a \in \mathcal{C}_q} \left(\sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} \left(n^{-1} \sum_{i=1}^n |g(X_i)|^{-2} \psi_{j,k}(X_i) \psi_{j,k'}(X_i)\right)\right) \\ = & n^{-1} G \leq C n^{-1} (\mu + 1). \end{aligned}$$

Hence  $Q = Q(\mathbb{X}) = C n^{-1} (\mu + 1)$ .

The obtained values of  $N$  and  $Q$  will allow us to conclude. For any  $x > 0$ , we have

$$\begin{aligned} & \mathbb{P}_f^n(\sup_{a \in \mathcal{C}_q} \mathcal{Z}(a) \geq x + C(1 + \mu)^{1/2} L^{1/p} n^{-1/2}) \\ = & \mathbb{E}_f^n(\mathbb{P}_f^n(\sup_{a \in \mathcal{C}_q} \mathcal{Z}(a) \geq x + C(1 + \mu)^{1/2} L^{1/p} n^{-1/2} | \mathbb{X})(1_{\mathcal{B}_\mu} + 1_{\mathcal{B}_\mu^c})) \\ \leq & \mathbb{P}_f^n(\mathcal{B}_\mu) + \mathbb{E}_f^n(\mathbb{P}_f^n(\sup_{a \in \mathcal{C}_q} \mathcal{Z}(a) \geq x + N(\mathbb{X}) | \mathbb{X})). \end{aligned} \quad (5.5)$$

The Cirelson inequality described in Lemma 5.2 implies

$$\mathbb{E}_f^n(\mathbb{P}_f^n(\sup_{a \in \mathcal{C}_q} \mathcal{Z}(a) \geq x + N(\mathbb{X}) | \mathbb{X})) \leq \mathbb{E}_f^n(\exp(-x^2/(2Q(\mathbb{X}))).) \quad (5.6)$$

Moreover, by definition of  $\mathcal{A}_\mu$ , we have

$$\begin{aligned} \mathbb{E}_f^n(\exp(-x^2/(2Q(\mathbb{X})))) &= \mathbb{E}_f^n(\exp(-x^2/(2Q(\mathbb{X}))(1_{\mathcal{A}_\mu} + 1_{\mathcal{A}_\mu^c})) \\ &\leq \mathbb{P}_f^n(\mathcal{A}_\mu) + \exp(-n x^2/(2(\mu + 1))). \end{aligned} \quad (5.7)$$

Putting the inequalities (5.5), (5.6) and (5.7) together, for  $x = 8^{-1} \mu L^{1/p} n^{-1/2}$

and  $\mu$  large enough, we obtain

$$\begin{aligned}
\mathcal{V} &= \mathbb{P}_f^n(\sup_{a \in \mathcal{C}_q} \mathcal{Z}(a) \geq 4^{-1} \mu L^{1/p} n^{-1/2}) \\
&\leq \mathbb{P}_f^n(\sup_{a \in \mathcal{C}_q} \mathcal{Z}(a) \geq 8^{-1} \mu L^{1/p} n^{-1/2} + C(1 + \mu)^{1/2} L^{1/p} n^{-1/2}) \\
&\leq C(\mathbb{P}_f^n(\mathcal{A}_\mu) + \mathbb{P}_f^n(\mathcal{B}_\mu) + \exp(-C\mu^2 L^{2/p}/(\mu + 1))). \tag{5.8}
\end{aligned}$$

Lemma 5.3 below investigates the asymptotic properties of  $\mathbb{P}_f^n(\mathcal{A}_\mu)$  and  $\mathbb{P}_f^n(\mathcal{B}_\mu)$ .

**Lemma 5.3.** *For  $\mu$  and  $n$  large enough, there exists a constant  $C > 0$  such that*

$$\max(\mathbb{P}_f^n(\mathcal{A}_\mu), \mathbb{P}_f^n(\mathcal{B}_\mu)) \leq Cn^{-p}.$$

By the inequality (5.8), the fact that  $L \asymp (\log n)^{p/2}$  and Lemma 5.3, for  $\mu$  large enough, we have

$$\mathcal{V} \leq Cn^{-p}.$$

Combining the obtained upper bounds for  $\mathcal{U}$  and  $\mathcal{V}$ , we end Theorem 3.1.  $\square$

*Proof of Lemma 5.3.* Let us investigate the upper bounds for  $\mathbb{P}_f^n(\mathcal{B}_\mu)$  and  $\mathbb{P}_f^n(\mathcal{A}_\mu)$ .

- *Upper bound for  $\mathbb{P}_f^n(\mathcal{B}_\mu)$ .* First of all, notice that the random variables

$$(|\psi_{j,k}(X_1)|^2 g(X_1)^{-1}, \dots, |\psi_{j,k}(X_n)|^2 g(X_n)^{-1}),$$

are i.i.d. and, since  $g$  is bounded from below, we have

$$|\psi_{j,k}(X_i)|^2 g(X_i)^{-1} \leq \|1/g\|_\infty \|\psi\|_\infty^2 2^j, \quad \mathbb{E}_f^n(|\psi_{j,k}(X_1)|^2 g(X_1)^{-1}) = 1.$$

So, for any  $j \in \{j_1, \dots, j_2\}$ , the Hoeffding inequality implies the existence of a constant  $C > 0$  such that

$$\mathbb{P}_f^n(\mathcal{B}_\mu) \leq 2 \exp(-Cn\mu^2 2^{-2j}) \leq 2 \exp(-Cn\mu^2 2^{-2j_2}) \leq 2n^{-C\mu^2}.$$

We obtain the desired upper bound by taking  $\mu$  large enough.

• *Upper bound for  $\mathbb{P}_f^n(\mathcal{A}_\mu)$ .* The goal is to apply the Talagrand inequality described in Lemma 5.1. Let us consider the set  $\mathcal{C}_q$  defined by  $\mathcal{C}_q = \{a = (a_{j,k}) \in \mathbb{Z}^*; \sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^q \leq 1\}$  and the functions class  $\mathcal{F}'$  defined by

$$\mathcal{F}' = \{h; h(x) = g(x)^{-1} \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} \psi_{j,k}(x) \psi_{j,k'}(x), a \in \mathcal{C}_q\}.$$

We have

$$\begin{aligned} & \sup_{a \in \mathcal{C}_q} \left( \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} \left( n^{-1} \sum_{i=1}^n g(X_i)^{-1} \psi_{j,k}(X_i) \psi_{j,k'}(X_i) \right) - \sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^2 \right) \\ &= \sup_{h \in \mathcal{F}'} r_n(h), \end{aligned}$$

where  $r_n$  denotes the operator defined in Lemma 5.1. Thus, it suffices to determine the parameter  $T$ ,  $H$  and  $v$  of the Talagrand inequality.

– *Value of  $T$ .* Let  $h$  be a function of  $\mathcal{F}'$ . Using the Hölder inequality, the fact that  $g$  is bounded from below and the concentration property (2.1), we find

$$|h(x)| \leq \|1/g\|_\infty \sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^2 \sum_{k' \in \mathcal{B}_{j,K}} |\psi_{j,k}(x)|^2 \leq C2^j, \quad x \in [0, 1].$$

Hence  $T = C2^j$ .

– *Value of  $H$ .* The  $l_2$ -Hölder inequality and the Hölder inequality imply

$$\begin{aligned} & \mathbb{E}_f^n \left( \sup_{a \in \mathcal{C}_q} \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} \left( \sum_{i=1}^n \epsilon_i g(X_i)^{-1} \psi_{j,k}(X_i) \psi_{j,k'}(X_i) \right) \right) \\ & \leq \sup_{a \in \mathcal{C}_q} \left( \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} |a_{j,k}|^2 |a_{j,k'}|^2 \right)^{1/2} \dots \\ & \quad \left( \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} \mathbb{E}_f^n \left( \left| \sum_{i=1}^n \epsilon_i (g(X_i)^{-1} \psi_{j,k}(X_i) \psi_{j,k'}(X_i)) \right|^2 \right) \right)^{1/2} \\ & \leq C \left( \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} \mathbb{E}_f^n \left( \left| \sum_{i=1}^n \epsilon_i (g(X_i)^{-1} \psi_{j,k}(X_i) \psi_{j,k'}(X_i)) \right|^2 \right) \right)^{1/2}. \quad (5.9) \end{aligned}$$

Since  $(\epsilon_1, \dots, \epsilon_n)$  are independent Rademacher variables, also independent of  $(X_1, \dots, X_n)$ , the Khintchine inequality and the fact that  $g$  is bounded from below imply

$$\begin{aligned}
& \mathbb{E}_f^n \left( \left| \sum_{i=1}^n \epsilon_i (g(X_i)^{-1} \psi_{j,k}(X_i) \psi_{j,k'}(X_i)) \right|^2 \right) \\
&= \mathbb{E}_f^n \left( \mathbb{E}_f^n \left( \left| \sum_{i=1}^n \epsilon_i (g(X_i)^{-1} \psi_{j,k}(X_i) \psi_{j,k'}(X_i)) \right|^2 \middle| \mathbb{X} \right) \right) \\
&\leq C \mathbb{E}_f^n \left( \sum_{i=1}^n |g(X_i)|^{-2} |\psi_{j,k}(X_i)|^2 |\psi_{j,k'}(X_i)|^2 \right) \\
&\leq C \|1/g\|_\infty^2 n \mathbb{E}_f^n (|\psi_{j,k}(X_1)|^2 |\psi_{j,k'}(X_1)|^2). \tag{5.10}
\end{aligned}$$

Using the property of concentration (2.1) and the inequalities (5.9) and (5.10), we find

$$\begin{aligned}
& \mathbb{E}_f^n \left( \sup_{a \in \mathcal{C}_q} \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} \left( \sum_{i=1}^n \epsilon_i g(X_i)^{-1} \psi_{j,k}(X_i) \psi_{j,k'}(X_i) \right) \right) \\
&\leq C (n \mathbb{E}_f^n \left( \sum_{k \in \mathcal{B}_{j,K}} |\psi_{j,k}(X_i)|^2 \right))^2)^{1/2} \leq C n^{1/2} 2^j.
\end{aligned}$$

Hence  $H = C 2^j n^{-1/2}$ .

– *Value of  $v$ .* By using the fact that  $g$  is bounded from below, the Hölder inequality and the property of concentration (2.1), we have

$$\begin{aligned}
\sup_{h \in \mathcal{F}} \text{Var}(h(X_1)) &\leq \sup_{a \in \mathcal{C}_q} \mathbb{E}_f^n \left( |g(X_1)|^{-2} \left| \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} \psi_{j,k}(X_1) \psi_{j,k'}(X_1) \right|^2 \right) \\
&\leq C \|1/g\|_\infty^2 \sup_{a \in \mathcal{C}_q} \left( \sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^2 \right)^2 \mathbb{E}_f^n \left( \left( \sum_{k \in \mathcal{B}_{j,K}} |\psi_{j,k}(X_1)|^2 \right)^2 \right) \\
&\leq C 2^{2j}.
\end{aligned}$$

Hence  $v = C 2^{2j}$ .

Now, let us notice that if  $t = 2^{-1} \mu$  then

$$(t^2 v^{-1} \wedge t T^{-1}) \geq C (\mu^2 2^{-2j} \wedge \mu 2^{-j}) = C \mu^2 2^{-2j}.$$

For any  $j \in \{j_1, \dots, j_2\}$ ,  $\mu$  large enough and  $t = 2^{-1}\mu$ , the Talagrand inequality gives

$$\begin{aligned}
& \mathbb{P}_f^n(\mathcal{A}_\mu) \\
& \leq \mathbb{P}_f^n(\sup_{a \in \mathcal{C}_q} (\sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} (n^{-1} \sum_{i=1}^n g(X_i)^{-1} \psi_{j,k}(X_i) \psi_{j,k'}(X_i)) - \dots \\
& \quad \sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^2) \geq 2^{-1}\mu + C2^j n^{-1/2}) \leq \mathbb{P}(\sup_{h \in \mathcal{F}} r_n(h) \geq t + C_2 H) \\
& \leq \exp(-nC_1 (t^2 v^{-1} \wedge tT^{-1})) \leq C \exp(-nC\mu^2 2^{-2j}) \\
& \leq C \exp(-nC\mu^2 2^{-2j_2}) \leq Cn^{-p}.
\end{aligned}$$

This ends the proof of Lemma 5.3.  $\square$

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