Modeling financial assets without semimartingale

Rosanna Coviello, Francesco Russo

Modeling financial assets without semimartingale

Rosanna COVIELLO,[a,b] **Francesco RUSSO**[a]

a Université Paris 13, Institut Galilée, Mathématiques, 99, avenue J.B. Clément, F-93430 Villetaneuse, France

b Modal'X, Université Paris 10, Avenue de la République 92001 Nanterre Cedex, France

June 26, 2006

Abstract: This paper does not suppose a priori that the evolution of the price of a financial asset is a semimartingale. Since possible strategies of investors are self-financing, previous prices are forced to be finite quadratic variation processes. The non-arbitrage property is not excluded if the class $\mathcal{A}$ of admissible strategies is restricted. The classical notion of martingale is replaced with the notion of $\mathcal{A}$-martingale. A calculus related to $\mathcal{A}$-martingales with some examples is developed. Some applications to the maximization of the utility of an insider are expanded.

MSC: 60G48, 60H05, 60H07, 60H10.
JEL: G10, G11, G12, G13.
Keywords and phrases: $\mathcal{A}$-martingale, weak k-order Brownian motion, utility maximization, insider investor.

1. Introduction

According to the fundamental theorem of asset pricing of Delbaen and Schachermayer in [7], in absence of *free lunches with vanishing risk* (NFLVR), when investing possibilities run only trough simple predictable strategies with respect to some filtration $\mathbb{G}$, the price process of the risky asset S is forced to be a semimartingale. However (NFLVR) condition could not be reasonable in several situations. In that case S may not be a semimartingale. We illustrate here some of those circumstances.

Generally, admissible strategies are let vary in a quite large class of predictable processes with respect to some filtration $\mathbb{G}$, representing the information flow available to the investor. As a matter of fact, the class of admissible strategies could be reduced because of different market regulations or for practical reasons. For instance, the investor could not be allowed to hold more than a certain number of stock shares. On the other hand it could be realistic to impose a minimal delay between two possible transactions as suggested by Cheridito ([5]): when the logarithmic price $\log(S)$ is a geometric fractional Brownian motion (fbm),

it is impossible to realize arbitrage possibilities satisfying that minimal requirement. We remind that without that restriction, the market admits arbitrages, see for instance [24]. When the logarithmic price of S is a geometric fbm or some particular strong Markov process, arbitrages can be excluded taking into account proportional transactions costs: Guasoni ([15]) has shown that, in that case, the class of admissible strategies has to be restricted to bounded variation processes and this rules out arbitrages.

Besides the restriction of the class of admissible strategies, the adoption of non-semimartingale models finds its justification when the no-arbitrage condition itself is not likely.

Empirical observations reveal, indeed, that S could fail to be a semimartingale because of market imperfections due to micro-structure noise, as intra-day effects. A model which considers those imperfections would add to W, the Brownian motion describing log-prices, a zero quadratic variation process, as a fractional Brownian motion of Hurst index greater than $\frac{1}{2}$, see for instance [32]. Theoretically arbitrages in very small time interval could be possible, which would be compatible with the lack of semimartingale property.

At the same way if (FLVR) are not possible for an *honest investor*, an *inside trader* could realize a free lunch with respect to the enlarged filtration $\mathbb{G}$ including the one generated by prices and the extra-information. Again in that case S may not be a semimartingale. The literature concerning inside trading and asymmetry of information has been extensively enriched by several papers in the last ten years; among them we quote Pikowski and Karatzas ([21]), Grorud and Pontier ([14]), Amendinger, Imkeller and Schweizer ([1]). They adopt enlargement of filtration techniques to describe the evolution of stock prices in the insider filtration.

Recently, some authors approached the problem in a new way using in particular forward integrals, in the framework of stochastic calculus via regularizations. For a comprehensive survey of that calculus see [29]. Indeed, forward integrals could exist also for non-semimartingale integrators. Leon, Navarro and Nualart in [18], for instance, solve the problem of maximization of expected logarithmic utility of an agent who holds an initial information depending on the future of prices. They operate under technical conditions which, a priori, do not imply the classical assumption (H') for enlargement considered in [16]. Using forward integrals, they determine the utility maximum. However, a posteriori, they found out that their conditions let $\log(S)$ be a semimartingale.

Biagini and Øksendal ([3]) considered somehow the converse implication. Supposing that the maximum utility is attained, they proved that S is a semimartingale. Ankkirchner and Imkeller ([2]) continue to develop the enlargement of filtrations techniques and show, among the others, a similar result as [3] using the fundamental theorem of asset pricing of Delbaen-Schachermayer. In particular they establish a link between that fundamental theorem and finite utility.

In our paper we treat a market where there are one risky asset, whose price is a strictly positive process S, and a *less risky* asset with price S^0, possibly

riskless but a priori only with bounded variation. A class $\mathcal{A}$ of admissible trading strategies is specified. If $\mathcal{A}$ is not large enough to generate all predictable simple strategies, then S has no need to be a semimartingale, even requiring the absence of free lunches among those strategies. We try to build the basis of a corresponding financial theory which allows to deal with several problems as hedging and non-arbitrage pricing, viability and completeness as well as with utility maximization.

For the sake of simplicity in this introduction we suppose that the less risky asset S^0 is constant and equal to 1.

As anticipated, a natural tool to describe the self-financing condition is the forward integral of an integrand process Y with respect to an integrator X, denoted by $\int_0^t Y d^- X$; see section 2 for definitions. Let $\mathbb{G} = (\mathcal{G}_t)_{0\leq t\leq 1}$ be a filtration on an underlying probability space $(\Omega, \mathcal{F}, P)$, with $\mathcal{F} = \mathcal{G}_1$; $\mathbb{G}$ represents the flow of information available to the investor. A **self-financing portfolio** is a pair (X_0, h) where X_0 is the initial value of the portfolio and h is a $\mathbb{G}$-adapted and S-forward integrable process specifying the number of shares of S held in the portfolio. The market value process X of such a portfolio, is given by $X_0 + \int_0^{\cdot} h_s d^- S_s$, while $h_t^0 = X_t - S_t h_t$ constitutes the number of shares of the less risky asset held.

This formulation of self-financing condition is coherent with the discrete-time case. Indeed, let we consider a *buy-and-hold strategy*, i.e. a pair (X_0, h) with $h = \eta I_{(t_0,t_1]}, 0 \leq t_0 \leq t_1 \leq 1$, and η being a $\mathcal{G}_{t_0}$-measurable random variable. Using the definition of forward integral it is not difficult to see that: $X_{t_0} = X_0$, $X_{t_1} = X_0 + \eta(S_{t_1} - S_{t_0})$. This implies $h^0_{t_0^+} = X_0 - \eta S_{t_0}$, $h^0_{t_1^+} = X_0 + \eta(S_{t_1} - S_{t_0})$ and

$$X_{t_0} = h_{t_0^+} S_{t_0} + h^0_{t_0^+}, \quad X_{t_1} = h_{t_1^+} S_{t_1} + h^0_{t_1^+} : \tag{1}$$

at the *re-balancing* dates t_0 and t_1, the value of the old portfolio must be reinvested to build the new portfolio without exogenous withdrawal of money.

In this paper $\mathcal{A}$ will be a real linear subspace of all self-financing portfolios and it will constitute, by definition, the class of all **admissible** portfolios. $\mathcal{A}$ will depend on the kind of problems one has to face: hedging, utility maximization, modeling inside trading. If we require that S belongs to $\mathcal{A}$, then the process S is forced to be a finite quadratic variation process. In fact, $\int_0^{\cdot} S d^- S$ exists if and only if the quadratic variation $[S]$ exists, see [29]; in particular one would have

$$\int_0^{\cdot} S_s d^- S_s = S^2 - S_0^2 - \frac{1}{2}[S].$$

$\mathcal{L}$ will be the sub-linear space of $L^0(\Omega)$ representing a set of **contingent claims of interest** for one investor. An **$\mathcal{A}$-attainable contingent claim** will be a random variable C for which there is a self-financing portfolio (X_0, h) with $h \in \mathcal{A}$ and

$$C = X_0 + \int_0^1 h_s d^- S_s.$$

X_0 will be called **replication price** for C. The market will be said $(\mathcal{A}, \mathcal{L})$-**complete** if every element of $\mathcal{L}$ is $\mathcal{A}$-attainable.

In these introductory lines we will focus only on one particular elementary situation.

For simplicity we illustrate the case where $[\log(S)]_t = \sigma^2 t$. We choose as $\mathcal{L}$ the set of all *European* contingent claims $C = \psi(S_1)$ where ψ is continuous with polynomial growth. We consider the case $\mathcal{A} = \mathcal{A}_S$, where

$$\begin{aligned} \mathcal{A}_S &= \{(u(t, S_t)), 0 \leq t < 1 \,|\, u : [0,1] \times \mathbb{R} \to \mathbb{R}, \text{ Borel-measurable} \\ &\qquad \text{with polynomial growth and lower bounded}\}\,. \end{aligned}$$

Such a market is $(\mathcal{A}, \mathcal{L})$-complete: in fact, a random variable $C = \psi(S_1)$ is an $\mathcal{A}$-attainable contingent claim. To build a replicating strategy the investor has to choose v as solution of the following problem

$$\left\{ \begin{array}{lcl} \partial_t v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial^{(2)}_{xx} v(t,x) & = & 0 \\ v(1,x) & = & \psi(x) \end{array} \right.$$

and $X_0 = v(0, S_0)$. This follows easily after application of Itô formula contained in proposition 2.11, see proposition 5.29.

We highlight that this method can be adjusted to hedge also *Asian* contingent claims.

A crucial concept is the one of $\mathcal{A}$-martingale processes. Those processes naturally intervene in utility maximization, arbitrage and uniqueness of hedging prices.

A process M is said to be an $\mathcal{A}$-**martingale** if for any process $Y \in \mathcal{A}$,

$$E\left[\int_0^{\cdot} Y d^- M\right] = 0.$$

If for some filtration $\mathbb{F}$ with respect to which M is adapted, $\mathcal{A}$ contains the class of all bounded $\mathbb{F}$-predictable processes, then M is an $\mathbb{F}$-martingale.

An example of $\mathcal{A}$-martingale is the so called **weak Brownian motion of order** $k = 1$ and quadratic variation equal to t. That notion was introduced in [13]: a weak Brownian motion of order 1 is a process X such that the law of X_t is $N(0, t)$ for any $t \geq 0$.

A portfolio (X_0, h) is said to be an $\mathcal{A}$-arbitrage if $h \in \mathcal{A}$, $X_1 \geq X_0$ almost surely and $P\{X_1 - X_0 > 0\} > 0$. We denote by $\mathcal{M}$ the set of probability measures being equivalent to the initial probability P under which S is an $\mathcal{A}$-martingale. If $\mathcal{M}$ is non empty then the market is $\mathcal{A}$-arbitrage free. In fact if $Q \in \mathcal{M}$, given a pair (X_0, h) which is an $\mathcal{A}$-arbitrage, then $E^Q[X_1 - X_0] = E^Q[\int_0^1 h d^- S] = 0$. In that case the replication price X_0 of an $\mathcal{A}$-attainable contingent claim C is unique, provided that the process $h\eta$, for any bounded random variable η in $\mathcal{G}_0$ and h in $\mathcal{A}$, still belongs to $\mathcal{A}$. Moreover $X_0 = E^Q[C|\mathcal{G}_0]$. In reality, under the weaker assumption that the market is $\mathcal{A}$-arbitrage free, the replication price is still unique, see proposition 5.27. Furthermore if $\mathcal{M}$ is non empty and $\mathcal{A} = \mathcal{A}_S$, as assumed in this section, the law of S_t has to be equivalent to Lebesgue measure for every $0 \leq t \leq 1$, see proposition 5.21.

If the market is $(\mathcal{A}, \mathcal{L})$-complete then all the probabilities measures in $\mathcal{M}$ coincide on $\sigma(\mathcal{L})$, see proposition 5.28. If $\sigma(\mathcal{L}) = \mathcal{F}$ then $\mathcal{M}$ is a singleton: this result recovers the classical case.

Given an utility function satisfying usual assumptions, it is possible to show that the maximum π is attained on a class of portfolios fulfilling conditions related to assumption 5.37, if and only if there exists a probability measure under which $\log(S) - \int_0^{\cdot} \left(\sigma^2 \pi_t - \frac{1}{2}\sigma^2\right) dt$ is an $\mathcal{A}$-martingale, see proposition 5.44. Therefore if $\mathcal{A}$ is big enough to fulfill conditions related to assumption $\mathcal{D}$ in Definition 4.6, then S is a classical semimartingale.

Those considerations show that most of the classical results of basic financial theory admit a natural extension to non-semimartingale models.

The paper is organized as follows. After some preliminaries about stochastic calculus via regularizations for forward integrals, we provide in section 3 examples of integrators and integrands for which forward integrals exist and realize some important properties in view of financial applications: those examples appear in three essential situations coming from Malliavin calculus, substitution formulae and Itô-fields. Regarding finance applications, the class of strategies defined using Malliavin calculus are useful when $\log(S)$ is a geometric Brownian motion with respect to a filtration $\mathbb{F}$ contained in $\mathbb{G}$; the use of substitution formulae naturally appear when trading with an initial extra information, already available at time 0; Itô fields apply whenever S is a generic finite quadratic variation process.

Section 4 is devoted to the study of $\mathcal{A}$-martingales: after having defined and established basic properties, we explore the relation between $\mathcal{A}$-martingales and weak Brownian motion; later we discuss the link between the existence of a maximum for a an optimization problem and the $\mathcal{A}$-martingale property.

In Section 5 we finally deal with applications to mathematical finance. We define self-financing portfolio strategies and we provide examples. Moreover we face technical problems related to the use of forward integral in order to describe the evolution of the wealth process. Those problems arise because of the lack of chain rule properties. Later, we discuss absence of $\mathcal{A}$-arbitrages, $(\mathcal{A}, \mathcal{L})$-completeness and hedging. We conclude the section analyzing the problem of maximizing expected utility from terminal wealth. We obtain results about the existence of an *optimal portfolio* generalizing those of [18] and [3].

2. Preliminaries

For the convenience of the reader we give some basic concepts and fundamental results about stochastic calculus with respect to finite quadratic variation processes which will be extensively used later. For more details we refer the reader to [29].

In the whole paper $(\Omega, \mathcal{F}, P)$ will be a fixed probability space. For a stochastic process $X = (X_t, 0 \leq t \leq 1)$ defined on $(\Omega, \mathcal{F}, P)$ we will adopt the convention $X_t = X_{(t \vee 0) \wedge 1}$, for t in $\mathbb{R}$. Let $0 \leq T \leq 1$. We will say that a sequence of

processes $(X_t^n, 0 \leq t \leq T)_{n\in\mathbb{N}}$ **converges uniformly in probability (ucp) on** $[0,T]$ toward a process $(X_t, 0 \leq t \leq T)$, if $\sup_{t\in[0,T]} |X_t^n - X_t|$ converges to zero in probability.

Definition 2.1. *1. Let $X = (X_t, 0 \leq t \leq T)$ and $Y = (Y_t, 0 \leq t \leq T)$ be processes with paths respectively in $C^0([0,T])$ and $L^1([0,T])$. Set, for every $0 \leq t \leq T$,*

$$I(\varepsilon, Y, X, t) = \frac{1}{\varepsilon}\int_0^t Y_s\,(X_{s+\varepsilon} - X_s)\,ds,$$

and

$$C(\varepsilon, X, Y, t) = \frac{1}{\varepsilon}\int_0^t (Y_{s+\varepsilon} - Y_s)\,(X_{s+\varepsilon} - X_s)\,ds.$$

If $I(\varepsilon, Y, X, t)$ converges in probability for every t in $[0,T]$, and the limiting process admits a continuous version $I(Y, X, t)$ on $[0,T]$, Y is said to be ***X-forward integrable*** *on $[0,T]$. The process $(I(Y, X, t), 0 \leq t \leq T)$ is denoted by $\int_0^{\cdot} Y d^- X$. If $I(\varepsilon, Y, X, \cdot)$ converges ucp on $[0,T]$ we will say that the forward integral $\int_0^{\cdot} Y d^- X$ is the* ***limit ucp of its regularizations****.*

2. If $(C(\varepsilon, X, Y, t), 0 \leq t \leq T)$ converges ucp on $[0,T]$ when ε tends to zero, the limit will be called the ***covariation process*** *between X and Y and it will be denoted by $[X, Y]$. If $X = Y$, $[X, X]$ is called the* ***finite quadratic variation*** *of X: it will also be denoted by $[X]$, and X will be said to be a* ***finite quadratic variation process*** *on $[0,T]$.*

Definition 2.2. *We will say that a process $X = (X_t, 0 \leq t \leq T)$, is* ***localized by the sequence*** $\left(\Omega_k, X^k\right)_{k\in\mathbb{N}^*}$, *if $P\left(\cup_{k=0}^{+\infty}\Omega_k\right) = 1$, $\Omega_h \subseteq \Omega_k$, if $h \leq k$, and $I_{\Omega_k} X^k = I_{\Omega_k} X$, almost surely for every k in $\mathbb{N}$.*

Remark 2.3. *Let $(X_t, 0 \leq t \leq T)$ and $(Y, 0 \leq t \leq T)$ be two stochastic processes. The following statements are true.*

1. *Let Y and X be localized by the sequences $\left(\Omega_k, X^k\right)_{k\in\mathbb{N}}$ and $\left(\Omega_k, Y^k\right)_{k\in\mathbb{N}}$, respectively, such that Y^k is X^k-forward integrable on $[0,T]$ for every k in $\mathbb{N}$. Then Y is X-forward integrable on $[0,T]$ and*

$$\int_0^{\cdot} Y d^- X = \int_0^{\cdot} Y^k d^- X^k, \qquad \text{on } \Omega_k, \quad a.s..$$

2. *If Y is X-forward integrable on $[0,T]$, then $Y I_{[0,t]}$ is X-forward integrable for every $0 \leq t \leq T$, and*

$$\int_0^{\cdot} Y_s I_{[0,t]} d^- X_s = \int_0^{\cdot\wedge t} Y_s d^- X_s.$$

3. *If the covariation process $[X, Y]$ exists on $[0,T]$, then the covariation process $[X I_{[0,t]}, Y I_{[0,t]}]$ exists for every $0 \leq t \leq T$, and*

$$\left[X_{[0,t]}, Y I_{[0,t]}\right] = [X, Y]_{t\wedge T}.$$

Definition 2.4. *Let $X = (X_t, 0 \leq t \leq T)$ and $Y = (Y_t, 0 \leq t < T)$ be processes with paths respectively in $C^0([0,T])$ and $L^1_{loc}([0,T))$, i.e. $\int_0^t |Y_s|\,ds < +\infty$ for any $t < T$.*

1. *If $YI_{[0,t]}$ is X-forward integrable for every $0 \leq t < T$, Y is said* ***locally X-forward integrable on*** *$[0,T)$. In this case there exists a continuous process, which coincides, on every compact interval $[0,t]$ of $[0,1)$, with the forward integral of $YI_{[0,t]}$ with respect to X. That process will still be denoted with $I(\cdot,Y,X) = \int_0^{\cdot} Y d^- X$.*
2. *If Y is locally X-forward integrable and $\lim_{t\to T} I(t,Y,X)$ exists almost surely, Y is said X-****improperly forward integrable*** *on $[0,T]$.*
3. *If the covariation process $[X, YI_{[0,t]}]$ exists, for every $0 \leq t < T$, we say that the* ***covariation process*** *$[X,Y]$* ***exists locally*** *on $[0,T)$ and it is still denoted by $[X,Y]$. In this case there exists a continuous process, which coincides, on every compact interval $[0,t]$ of $[0,1)$, with the covariation process $\left[X, YI_{[0,t]}\right]$. That process will still be denoted with $[X,Y]$. If $X = Y$, $[X,X]$ we will say that the* ***quadratic variation of*** *X* ***exists locally*** *on $[0,T]$.*
4. *If the covariation process $[X,Y]$ exists locally on $[0,T)$ and $\lim_{t\to T}[X,Y]_t$ exists, the limit will be called the* ***improper covariation process*** *between X and Y and it will still be denoted by $[X,Y]$. If $X = Y$, $[X,X]$ we will say that the* ***quadratic variation of*** *X* ***exists improperly*** *on $[0,T]$.*

Remark 2.5. *Let $X = (X_t, 0 \leq t \leq T)$ and $Y = (Y_t, 0 \leq t \leq T)$ be two stochastic processes being in $C^0([0,1])$ and $L^1([0,1])$, respectively. If Y is X-forward integrable on $[0,T]$ then its restriction to $[0,1)$ is X-improperly forward integrable and the improper integral coincides with the forward integral of Y with respect to X.*

Definition 2.6. *A vector $\left(\left(X_t^1, ..., X_t^m\right), 0 \leq t \leq T\right)$ of continuous processes is said to have all its* ***mutual brackets*** *on $[0,T]$ if $\left[X^i, X^j\right]$ exists on $[0,T]$ for every $i,j = 1, ..., m$.*

In the sequel if $T = 1$ we will omit to specify that objects defined above exist on the interval $[0,1]$ (or $[0,1)$, respectively).

Proposition 2.7. *Let $M = (M_t, 0 \leq t \leq T)$ be a continuous local martingale with respect to some filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ of $\mathcal{F}$. Then the following properties hold.*

1. *The process M is a finite quadratic variation process on $[0,T]$ and its quadratic variation coincides with the classical bracket appearing in the Doob decomposition of M^2.*
2. *Let $Y = (Y_t, 0 \leq t \leq T)$ be an $\mathbb{F}$-adapted process with left continuous and bounded paths. Then Y is M-forward integrable on $[0,T]$ and $\int_0^{\cdot} Y d^- M$ coincides with the classical Itô integral $\int_0^{\cdot} Y dM$.*

Proposition 2.8. *Let $V = (V_t, 0 \leq t \leq T)$ be a bounded variation process and $Y = (Y_t, 0 \leq t \leq T)$, be a process with paths being bounded and with at most countable discontinuities. Then the following properties hold.*

1. *The process Y is V-forward integrable on $[0,T]$ and $\int_0^{\cdot} Y d^- V$ coincides with the Lebesgue-Stieltjes integral denoted with $\int_0^{\cdot} Y dV$.*

2. *The covariation process $[Y,V]$ exists on $[0,T]$ and it is equal to zero. In particular a bounded variation process has zero quadratic variation.*

Corollary 2.9. *Let $X=(X_t,0\leq t\leq T)$ be a continuous process and $Y=(Y_t,0\leq t\leq T)$ a bounded variation process. Then*

$$XY-X_0Y_0=\int_0^{\cdot}X_sdY_s+\int_0^{\cdot}Y_sd^-X_s.$$

Proposition 2.10. *Let $X=(X_t,0\leq t\leq T)$ be a continuous finite quadratic variation process, and f a function in $C^1(\mathbb{R})$. Then $Y=f(X)$ has a finite quadratic variation on $[0,T]$ and $[Y]=\int_0^{\cdot}f'(X)^2d[X]$.*

Proposition 2.11. *Let $X=(X_t,0\leq t\leq T)$ be a continuous finite quadratic variation process and $V=((V_t^1,\ldots,V_t^m),0\leq t\leq T)$ be a vector of continuous bounded variation processes. Then for every u in $C^{1,2}(\mathbb{R}^m\times\mathbb{R})$, the process $(\partial_x u(V_t,X_t),0\leq t\leq T)$ is X-forward integrable on $[0,T]$ and*

$$\begin{aligned}u(V,X) &= u(V_0,X_0)+\sum_{i=1}^m\int_0^{\cdot}\partial_{v_i}u(V_t,X_t)dV_t^i+\int_0^{\cdot}\partial_x u(V_t,X_t)d^-X_t\\ &+ \frac{1}{2}\int_0^{\cdot}\partial^{(2)}_{xx}u(V_t,X_t)d[X]_t.\end{aligned}$$

Lemma 2.12. *Let $X=(X_t^1,\ldots,X_t^m,0\leq t\leq T)$ be a vector of continuous processes having all its mutual brackets. Let $\psi:\mathbb{R}^m\to\mathbb{R}$ be of class $C^2(\mathbb{R}^m)$ and $Y=\psi(X)$. Then Z is Y-forward integrable on $[0,T]$, if and only if $Z\partial_{x^i}\psi(X)$ is X^i-forward integrable on $[0,T]$, for every $i=1,...,m$ and*

$$\int_0^{\cdot}Zd^-Y=\sum_{i=1}^m\int_0^{\cdot}Z\partial_{x^i}\psi(X)d^-X^i+\frac{1}{2}\sum_{i,j=0}^m\int_0^{\cdot}Z\partial^{(2)}_{x^ix^j}\psi(X)d\left[X^i,X^j\right].$$

Proof. The proof derives from proposition 4.3 of [28]. The result is a slight modification of that one. It should only be noted that there forward integral of a process Y with respect to a process X was defined as limit ucp of its regularizations. □

3. Existence of forward integrals and related properties: some examples

In this section we illustrate examples of processes for which forward integrals exist and we list some related properties which will be extensively used in further applications to finance.

3.1. Forward integrals of Itô fields

In this subsection ξ will be a $\mathbb{G}$-adapted process with finite quadratic variation, where $\mathbb{G}$ is some filtration of $\mathcal{F}$. The following definitions and results are extracted from [10].

Definition 3.1. *Let k be in $\mathbb{N}^*$. A random field $(H(t,x), 0 \leq t \leq 1, x \in \mathbb{R})$ is called a C^k $\mathbb{G}$-**Itô-semimartingale field** driven by the vector $N = \left(N^1, ..., N^n\right)$, if N is a vector of semimartingales with respect to $\mathbb{G}$, and*

$$H(t,x) = f(x) + \sum_{i=1}^{n} \int_0^t a^i(s,x) dN_s^i, \quad 0 \leq t \leq 1, \tag{2}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ belongs to $C^k(\mathbb{R})$ almost surely and it is $\mathbb{G}^0$-measurable for every x, H and $a^i : [0,1] \times \mathbb{R} \times \Omega \to \mathbb{R}$, $i = 1, ..., n$ are $\mathbb{G}$-adapted for every x, almost surely continuous with their partial derivatives with respect to x in (t,x) up to order k, and for every index $h \leq k$ it holds

$$\partial_x^{(h)} H(t,x) = \partial_x^{(h)} f(x) + \sum_{i=1}^{n} \int_0^t \partial_x^{(h)} a^i(s,x) dN_s^i, \quad 0 \leq t \leq 1.$$

Definition 3.2. *We denote with $\mathcal{C}_\xi^k(\mathbb{G})$ the set of processes of the form*

$$\left(H(t, \xi_t), 0 \leq t \leq 1\right),$$

being $(H(t,x), 0 \leq t \leq 1, x \in \mathbb{R})$ a C^k $\mathbb{G}$-Itô-semimartingale field driven by the vector $N = \left(N^1, ..., N^n\right)$, such that $\left(N^1, ..., N^n, \xi\right)$ has all its mutual brackets.

Remark 3.3. *1. The set $\mathcal{C}_\xi^1(\mathbb{G})$ is an algebra.*

2. Let ψ be in $C^\infty(\mathbb{R})$ and h in $\mathcal{C}_\xi^2(\mathbb{G})$. Itô formula implies that $\psi(h)$ belongs to $\mathcal{C}_\xi^2(\mathbb{G})$.

Proposition 3.4. *Let h and k be in $\mathcal{C}_\xi^1(\mathbb{G})$. Then the following statements are true.*

1. *The process h is ξ-forward integrable, the forward integral $\int_0^\cdot h_t d^- \xi_t$ is the limit ucp of its regularizations and it belongs to $\mathcal{C}_\xi^2(\mathbb{G})$.*
2. *The covariation process $\left(\left[\int_0^\cdot h_t d^- \xi_t, \int_0^\cdot k_t d^- \xi_t\right]\right)$ exists and it is equal to $\int_0^\cdot h_t k_t d\left[\xi\right]_t$.*
3. *The process $\int_0^\cdot h_t d^- \xi_t$ is forward integrable with respect to the process $\int_0^\cdot k_t d^- \xi_t$ and*

$$\int_0^\cdot h_t d^- \int_0^t k_s d^- \xi_s = \int_0^\cdot h_t k_t d^- \xi_t.$$

Using remark 2.3 it is not difficult to prove that proposition 3.4 extends to processes which are *simple combinations* of processes in $\mathcal{C}_\xi^1(\mathbb{G})$. We illustrate this result below.

Definition 3.5. *Let $\mathcal{S}(\mathcal{C}_\xi^k(\mathbb{G}))$ be the set of all processes h of type $h = h^0 I_{\{0\}} + \sum_{i=1}^m h^i I_{(t_{i-1}, t_i]}$ where $0 = t_0 \leq t_1, \cdots, t_m = 1$, and h^i belongs to $\mathcal{C}_\xi^k(\mathbb{G})$, for $i = 1, ..., m$.*

Remark 3.6. *Thanks to remark 3.3, if h belongs to $\mathcal{S}(\mathcal{C}_\xi^k(\mathbb{G}))$ and ψ is of class $C^\infty(\mathbb{R})$, then $\psi(h)$ is still in $\mathcal{S}(\mathcal{C}_\xi^k(\mathbb{G}))$.*

Proposition 3.7. *Let h and k be in $\mathcal{S}(\mathcal{C}^1_\xi(\mathbb{G}))$. Then the we can state the following.*

1. *The process h is ξ-forward integrable and it belongs to $\mathcal{S}(\mathcal{C}^2_\xi(\mathbb{G}))$.*
2. *The covariation process $\left(\left[\int_0^\cdot h_t d^-\xi_t, \int_0^\cdot k_t d^-\xi_t\right]\right)$ exists and it is equal to $\int_0^\cdot h_t k_t d\left[\xi\right]_t$.*
3. *The process $\left(\int_0^\cdot h_t d^-\xi_t, 0 \le t \le 1\right)$ is forward integrable with respect to the process $\left(\int_0^\cdot k_t d^-\xi_t, 0 \le t \le 1\right)$ and*

$$\int_0^\cdot h_t d^- \int_0^t k_s d^-\xi_s = \int_0^\cdot h_t k_t d^-\xi_t.$$

Proof. By linearity of forward integral and bilinearity of covariation it is sufficient to prove the statement for processes of type $hI_{[0,t]}$ and $kI_{[0,t]}$, with h and k in $\mathcal{C}^1_\xi(\mathbb{G})$ and $0 \le t \le 1$. The proof is a consequence of remark 2.3 and proposition 3.4. □

3.2. Forward integrals via Malliavin calculus

We work in the Malliavin calculus framework. To this extent we recall some basic notations and definitions from [20] and [19].

We suppose that $(\Omega, \mathbb{F}, \mathcal{F}, P)$ is the canonical probability space, meaning that $\Omega = C\left([0,1], \mathbb{R}\right)$, P is the Wiener measure, W is the Wiener process, $\mathbb{F}$ is the filtration generated by W and the P-null sets and $\mathcal{F}$ is the completion of the Borel σ-algebra with respect to P.

Let $\mathcal{S}$ be the space of all random variables on $(\Omega, \mathcal{F}, P)$, of the form

$$F = f(W(t_1), ..., W(t_n)), \quad 0 \le t_0, \cdots, t_n \le 1,$$

with f in $C^\infty(\mathbb{R}^n)$ being bounded with its derivatives of all orders. The iterated derivative of order k operator is denoted by D^k. Then $D^k : \mathbb{D}^{k,p} \to L^p\left(\Omega \times [0,1]^k\right)$, where $\mathbb{D}^{k,p}$, $p \ge 2$, $k \in \mathbb{N}^*$, is the closure of $\mathcal{S}$ with respect to the norm

$$|| F ||^p_{\mathbb{D}^{k,p}} = ||F||^p_{L^p(\Omega)} + \sum_{j=1}^k \left|\left|\left|\left|D^j F\right|\right|_{L^2([0,1]^j)}\right|\right|^p_{L^p(\Omega)}.$$

For any $p \ge 2$, $L^{1,p}$ denotes the space of all functions u in $L^p\left(\Omega \times [0,1]\right)$ such that u_t belongs to $\mathbb{D}^{1,p}$ for every $0 \le t \le 1$ and there exists a measurable version of $(D_s u_t, 0 \le s, t \le 1)$ with $\int_0^1 \mathbb{E}\left[||Du_t||^p_{L^2([0,1])}\right] dt < \infty$. For every u in $L^{1,p}$ we denote $||u||^p_{L^{1,p}} = \int_0^1 ||u_t||^p_{\mathbb{D}^{1,p}} dt$. Similarly, for $p \ge 2$, $L^{2,p}$ denotes the space of all functions u in $L^p\left(\Omega \times [0,1]\right)$, such that u_t belongs to $\mathbb{D}^{2,p}$ for every $0 \le t \le 1$ and there exist measurable versions of $(D_s u_t, 0 \le s, t \le 1)$ and $(D_r D_s u_t, 0 \le s, t, r \le 1)$ with

$$\int_0^1 \mathbb{E}\left[||Du_t||^p_{L^2([0,1])}\right] + \mathbb{E}\left[\left|\left|D^2 u_t\right|\right|^p_{L^2([0,1]^2)}\right] dt < \infty.$$

For every u in $L^{2,p}$ we denote $||u||^p_{L^{2,p}} = \int_0^1 ||u_t||^p_{\mathbb{D}^{2,p}}\, dt$.

The Skorohod integral δ is the adjoint of the derivative operator D; its domain is denoted by $Dom\delta$. An element u belonging to $Dom\delta$ is said Skorohod integrable. We recall that $\mathbb{D}^{1,2}$ is dense in $L^2(\Omega)$, $L^{1,2} \subset Dom\delta$, and that if u belongs to $L^{1,2}$ then, for each $0 \leq t \leq 1$, $uI_{[0,t]}$ is still in $L^{1,2}$. In particular it is Skorohod integrable. We will use the notation $\delta\left(uI_{[0,t]}\right) = \int_0^t u_s \delta W_s$, for each u in $L^{1,2}$. The process $\left(\int_0^t u_s \delta W_s, 0 \leq t \leq 1\right)$ is mean square continuous and then it admits a continuous version, which will be still denoted by $\int_0^{\cdot} u_t \delta W_t$. We finally recall that for every u in $L^{1,p}$ there exists a positive constant c_p such that

$$\begin{aligned} ||\delta(u)||^p_{L^p(\Omega)} &\leq c_p \left[\left(\int_0^1 \left| \mathbb{E}\left[u_t\right] \right|^2 dt \right)^{\frac{p}{2}} + \left|\left| ||Du||_{L^2([0,1]^2)} \right|\right|^p_{L^p(\Omega)} \right] \qquad (3) \\ &\leq c_p \, ||u||^p_{L^{1,p}} \, . \end{aligned}$$

It is useful to remind the following result contained in [19], exercise 1.2.13.

Lemma 3.8. *Let F and G be two random variables in $\mathbb{D}^{1,2}$. Suppose that G and $||DG||_{L^2([0,1])}$ are bounded. Then FG is still in $\mathbb{D}^{1,2}$ and $D(FG) = FDG + GDF$.*

Remark 3.9. 1. *Let u be a process in $L^{1,p}$, for some $p \geq 2$, and v in $L^{1,2}$ such that the random variable*

$$\sup_{t\in[0,1]} \left(|v_t| + \int_0^1 (D_s v_t)^2 ds \right)$$

is bounded. By lemma 3.8 the process uv belongs to $L^{1,p}$ and $Duv = uDv + vDu$.

2. *Let u be a process in $L^{2,p}$, for some $p \geq 2$ and v in $L^{2,2}$ such that the random variable*

$$\sup_{t\in[0,1]} \left(|v_t| + \int_0^1 (D_s v_t)^2 ds + \int_0^1 \int_0^1 (D_r D_s v_t)^2 dr ds \right)$$

is bounded. Then the process uv belongs to $L^{2,p}$.

In order to state a chain rule formula we will need the *Fubini*-type lemma below.

Lemma 3.10. *Let u be in $L^2\left(\Omega \times [0,1]^2\right)$. Assume that for every $0 \leq t \leq 1$, the process $u(\cdot, t)$ belongs to $L^{1,2}$, that there exist measurable versions of the two processes $(\delta(u(\cdot,t), 0 \leq t \leq 1)$ and $(D_r u(s,t), 0 \leq r, s, t \leq 1)$ and that*

$$\mathbb{E}\left[\int_0^1 ||Du(\cdot,t)||^2_{L^2([0,1]^2)}\, dt \right] < +\infty. \qquad (4)$$

Then the process $\left(\int_0^1 u(s,t)dt, 0 \leq s \leq 1\right)$ belongs to $L^{1,2}$ and

$$\delta\left(\int_0^1 u(\cdot,t)dt \right) = \int_0^1 \delta(u(\cdot,t))dt.$$

Proof. Consider the process $(g_s, 0 \leq s \leq 1)$ so defined: $g_s = \int_0^1 u(s,t)dt$. Let $0 \leq s \leq 1$ be fixed. Since $(u(s,t), 0 \leq t \leq 1)$ is in $L^{1,2}$, g_s is in $\mathbb{D}^{1,2}$ and $Dg_s = \int_0^1 Du(s,t)dt$. By Fubini theorem $\left(\int_0^1 D_r u(s,t)dt, 0 \leq r, s \leq 1\right)$ admits a measurable version. Thanks to inequality (4), $\int_0^1 \mathbb{E}\left[||Dg_s||_{L^2([0,1])}\right] ds < +\infty$. This implies that g is in $L^{1,2}$. The conclusion of the proof is achieved using exercise 3.2.8, page 174 of [19]. □

Definition 3.11. *For every $p \geq 2$, $L_-^{1,p}$ will be the space of all processes u belonging to $L^{1,p}$ such that $\lim_{\varepsilon \to 0} D_t u_{t-\varepsilon}$ exists in $L^p(\Omega \times [0,1])$. The limiting process will be denoted by $\left(D_t^- u_t, 0 \leq t \leq 1\right)$.*

Remark 3.12. *1. If u belongs to $L_-^{1,p}$ then*

$$\mathbb{E}\left[\int_0^1 \left(\frac{1}{\varepsilon}\int_s^{s+\varepsilon} \left|D_r u_s - D_r^- u_r\right|^p dr\right) ds\right] \tag{5}$$

converges to zero when ε tends to zero. Indeed term (5) equals $\frac{1}{\varepsilon}\int_0^\varepsilon f(z)dz$, with $f(z) = \mathbb{E}\left[\int_0^1 |D_r u_{r-z} - D_r^- u_r|^p dr\right]$, and $\lim_{\nu \to 0} f(z) = 0$.

2. Let u and v be two left continuous processes respectively in $L_-^{1,p}$ and $L_-^{1,2}$ with $p \geq 2$. Suppose, furthermore, that $\sup_{t\in[0,1]} |u_t|$ belongs to $L^p(\Omega)$ and that the random variable $\sup_{t\in[0,1]} \left(|v_t| + \sup_{s\in[0,1]} |D_s v_t|\right)$ is bounded. Then uv belongs to $L_-^{1,q}$, for every $2 \leq q < p$. Moreover $D^-uv = uD^-v + vD^-u$. In particular v belongs to $L_-^{1,q}$, for every $q \geq 2$.

The hypothesis on the left continuity of u and v on point 2. of previous remark allows us to show that

$$\lim_{\varepsilon \to 0}\left[\int_0^1 |z_{t-\varepsilon} - z_t|^p dt\right] = 0, \quad z = u, v, \quad a.s. \tag{6}$$

That condition could be relaxed. It would be enough to suppose that,

$$\lambda((0 \leq t \leq 1, s.t. |z_t - z_{t^-}| \neq 0)) = 0,$$

almost surely, for $z = u, v$, being λ the Lebesgue measure on $\mathcal{B}([0,1])$. Nevertheless, convergence in (6) does not hold for every bounded process. To see this it is sufficient to consider, for instance, $z = I_{\mathbb{Q}\cap[0,1]}$.

Lemma 3.13. *Let u and v be respectively in $L_-^{1,p}$, $p \geq 2$, and $L_-^{1,2}$. Suppose that the random variable $\sup_{t\in[0,1]} \left(|v_t| + \sup_{s\in[0,1]} |D_s v_t|\right)$ is bounded. Then the sequence of processes*

$$\left(\frac{1}{\varepsilon}\int_0^1 u_t v_t (W_{t+\varepsilon} - W_t)dt - \frac{1}{\varepsilon}\int_0^1 u_t \left(\int_t^{t+\varepsilon} v_s \delta W_s\right) dt\right)_{\varepsilon > 0}$$

converges in $L^q(\Omega)$ to $\int_0^1 u_t D_t^- v_t dt$, for every $2 \leq q < p$.

Proof. Set

$$A_\varepsilon = \frac{1}{\varepsilon}\int_0^1 u_t v_t \left(W_{t+\varepsilon} - W_t\right) dt, \quad B_\varepsilon = \frac{1}{\varepsilon}\int_0^1 u_t \left(\int_t^{t+\varepsilon} v_s \delta W_s\right) dt.$$

Proposition 1.3.4 in section 1.3 of [19] permits to rewrite A_ε in the following way:

$$A_\varepsilon = \frac{1}{\varepsilon}\int_0^1 u_t v_t \left(\int_t^{t+\varepsilon} I_{[0,1]}(s)\delta W_s\right) dt.$$

Moreover, by point 1. of remark 3.9, $Duv = vDu + uDv$. For every $0 \leq t \leq 1$, the random variables $\int_t^{t+\varepsilon} D_s(u_t v_t)ds$ and $\int_t^{t+\varepsilon} D_s u_t v_s ds$ are square integrable. Therefore property (4) in section 1.3 of [19] can be exploited to write

$$A_\varepsilon = \frac{1}{\varepsilon}\int_0^1 \left(\int_t^{t+\varepsilon} u_t v_t \delta W_s\right) dt + \frac{1}{\varepsilon}\int_0^1 \left(\int_t^{t+\varepsilon} D_s(u_t v_t) ds\right) dt,$$

and

$$B_\varepsilon = \frac{1}{\varepsilon}\int_0^1 \left(\int_t^{t+\varepsilon} u_t v_s \delta W_s\right) dt + \frac{1}{\varepsilon}\int_0^1 \left(\int_t^{t+\varepsilon} D_s u_t v_s ds\right) dt.$$

This implies

$$\begin{aligned} A_\varepsilon - B_\varepsilon &= \int_0^1 \left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon} u_t(v_t - v_s)\delta W_s\right) dt \\ &+ \int_0^1 \left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon} (v_t - v_s) D_s u_t ds\right) dt + \int_0^1 \left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon} u_t D_s v_t ds\right) dt \\ &= I_\varepsilon^1 + I_\varepsilon^2 + I_\varepsilon^3. \end{aligned}$$

We observe that the function $(\omega, s, t) \longmapsto I_{(t,t+\varepsilon]}(s) u_t(\omega)(v_t - v_s)(\omega)$, for every (ω, s, t) in $\Omega \times [0,1]^2$, satisfies the hypotheses of lemma 3.10. Therefore I_ε^1 can be rewritten as follows

$$I_\varepsilon^1 = \int_0^1 \left(\frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} u_t(v_t - v_s) dt\right) \delta W_s.$$

Using inequality (3) it is possible to prove that there exists a positive constant c such that

$$\mathbb{E}\left[\left|I_\varepsilon^1\right|^p\right] \leq c\mathbb{E}\left[\int_0^1 \left(|u_t|^p + \left(\int_0^1 (D_r u_t)^2 dr\right)^{\frac{p}{2}}\right) h_t^\varepsilon dt\right],$$

with

$$h_t^\varepsilon = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \left(|v_t - v_s|^p + \left(\int_0^1 |D_r v_t - D_r v_s|^2 dr\right)^{\frac{p}{2}}\right) ds.$$

Since $\sup_{t\in[0,1]} \left(|v_t| + \sup_{s\in[0,1]} |D_s v_t|\right)$ is a bounded random variable, for almost all (ω, t), h_t^ε converges to zero when ε goes to zero. Consequently, Lebesgue

dominated convergence theorem applies to conclude that $\mathbb{E}\left[\left|I_\varepsilon^1\right|^p\right]$ converges to zero.

Considering the term I_ε^2, Hölder inequality and the boundedness of v lead to

$$\begin{aligned}\mathbb{E}\left[\left|I_\varepsilon^2\right|^p\right] &\leq c\mathbb{E}\left[\int_0^1\left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\left|D_s u_t - D_s^- u_s\right|^p ds\right)dt\right] \\ &+ c\mathbb{E}\left[\int_0^1\left(\frac{1}{\varepsilon}\int_{s-\varepsilon}^{s}|v_t - v_s|^p\, dt\right)\left|D_s^- u_s\right|^p ds\right],\end{aligned}$$

for some positive constant c. The first term of previous sum converges to zero by point 1. of remark 3.12; the second by Lebesgue dominated convergence theorem.

Finally I_ε^3 may be rewritten as follows:

$$\begin{aligned}I_\varepsilon^3 &= \int_0^1\left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon}(D_s v_t - D_s^- v_s)ds\right)u_t dt \\ &+ \int_0^1\frac{1}{\varepsilon}\int_t^{t+\varepsilon}D_s^- v_s(u_t - u_s)dsdt + \int_0^1 u_s D_s^- v_s ds.\end{aligned}$$

Hölder inequality and again remark 3.12 implies the convergence to zero in $L^q(\Omega)$ of the first term of the sum for every $2 \leq q < p$. The convergence to zero of the second term of the sum in $L^p(\Omega)$ is due to the boundedness of $|D^- v|$ and the following maximal inequality contained in [31], theorem 1.:

$$\int_0^1 \sup_{\varepsilon>0}\left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon}|z_s|^p\, ds\right)dt \leq \int_0^1 |z_t|^p\, dt, \quad z \in L^p(\Omega\times[0,1]).$$

This leads to the conclusion. □

We omit the proof of the following lemma which is, indeed, a slight modification of the proof of previous one.

Lemma 3.14. *Let v be in $L_-^{1,p}$, $p \geq 2$. Then the sequence of processes*

$$\left(\frac{1}{\varepsilon}\int_0^1 v_t(W_{t+\varepsilon} - W_t)dt - \frac{1}{\varepsilon}\int_0^1\left(\int_t^{t+\varepsilon}v_s\delta W_s\right)dt\right)_{\varepsilon>0}$$

converges in $L^p(\Omega)$ to $\int_0^1 D_t^- v_t dt$.

Lemma 3.15. *Let u be a process in $L^{1,p}$ with $p \geq 2$. Then the process*

$$\left(\int_0^t u_s ds, 0 \leq t \leq 1\right)$$

belongs to $L_-^{1,p}$, and $D^-\left(\int_0^\cdot u_t dt\right) = \int_0^\cdot Du_t dt$.

Proof. We set $g_t = \int_0^t u_s ds$. Clearly g is in $L^p(\Omega\times[0,1])$. As already observed for the proof of lemma 3.10, since the process u belongs to $L^{1,2}$ for every $0 \leq t \leq 1$,

g_t is in $\mathbb{D}^{1,2}$ and $Dg_t = \int_0^t Du_s ds$. Moreover Hölder inequality implies

$$\mathbb{E}\left[\int_0^1 ||Dg_t||^p_{L^2([0,1])}\, dt\right] \leq \mathbb{E}\left[\int_0^1 ||Du_s||^p_{L^2([0,1])}\, ds\right] < +\infty.$$

Then g belongs to $L^{1,p}$. To conclude it is sufficient to observe that

$$\mathbb{E}\left[\int_0^1 \int_{t-\varepsilon}^t |D_t u_s|^p\, dsdt\right] = \mathbb{E}\left[\int_0^1 \int_s^{s+\varepsilon} |D_t u_s|^p\, dtds\right],$$

and that the right hand side of previous equality converges to zero when ε goes to zero by point 1. of remark 3.12. □

Lemma 3.16. *Let u be a process in $L^{2,p}$, with $p \geq 2$. Suppose furthermore that*

$$\int_0^1 \left(\mathbb{E}\left[||Du_t||^p_{L^p([0,1])}\right] + \mathbb{E}\left[\left|\left|D^2 u_t\right|\right|^p_{L^p([0,1]^2)}\right]\right) dt < +\infty. \tag{7}$$

Then the process $\left(\int_0^t u_s \delta W_s, 0 \leq t \leq 1\right)$ is in $L^{1,p}_-$, and

$$D^-\left(\int_0^{\cdot} u_t \delta W_t\right) = \int_0^{\cdot} Du_t \delta W_t.$$

Proof. We set $g = \int_0^{\cdot} u_t \delta W_t$. By proposition 5.5 of [20], for every t in $[0,1]$, g_t belongs to $\mathbb{D}^{1,2}$ and $D_r g_t = \delta\left(D_r u I_{[0,t]}\right) + u_r I_{[0,t]}(r)$, for every r, almost surely. Using inequality (3) it is possible to find a positive constant c such that

$$||g||^p_{L^p(\Omega\times[0,1])} \leq c\, ||u||^p_{L^{1,p}} < +\infty.$$

To prove that g belongs to $L^{1,p}$ we still have to show that $\mathbb{E}\left[\int_0^1 ||Dg_t||^p_{L^2([0,1])}\, dt\right]$ is finite. Clearly, $\mathbb{E}\left[\int_0^1 \left|\left|uI_{[0,t]}\right|\right|^p_{L^2([0,1])}\, dt\right] \leq ||u||_{L^p(\Omega\times[0,1])}$, which is finite. It remains to prove that $\mathbb{E}\left[\int_0^1 \left(\int_0^1 \left|\delta\left(D_r u I_{[0,t]}\right)\right|^2 dr\right)^{\frac{p}{2}} dt\right] < +\infty$. Applying again inequality (3) we obtain, for some $c > 0$,

$$\begin{aligned}
\mathbb{E}\left[\int_0^1 \left(\int_0^1 \left|\delta\left(D_r u I_{[0,t]}\right)\right|^2 dr\right)^{\frac{p}{2}} dt\right] &\leq c\int_0^1 \mathbb{E}\left[\int_0^1 \left|\delta\left(D_r u I_{[0,t]}\right)\right|^p dt\right] dr \\
&\leq c\int_0^1 \int_0^1 \left|\left|D_r u I_{[0,t]}\right|\right|^p_{L^{1,p}} drdt \\
&\leq c\int_0^1 ||D_r u||^p_{L^{1,p}}\, dr.
\end{aligned}$$

Last term in the expression above is bounded by the integral appearing in inequality (7). This permits to get the result. □

Proposition 3.17. *Let v be a process in $L_-^{1,p}$, with $p > 4$. Then v is both forward and Skorohod integrable with respect to W and*

$$\int_0^{\cdot} v_t d^- W_t = \int_0^{\cdot} v_t \delta W_t + \int_0^{\cdot} D_t^- v_t dt.$$

Furthermore, if v is also left continuous with right limit, then $\int_0^{\cdot} v_t d^- W_t$ has finite quadratic variation equal to $\int_0^{\cdot} v_t^2 dt$.

Proof. First of all we observe that if a process v belongs to $L_-^{1,p}$ then $I_{[0,t]}v$ inherits the property for every t in $[0,1]$. Using lemma 3.14 and lemma 3.10 we find that $I(\varepsilon, v, W, t) - \int_0^t v_s \delta W_s$ converges in $L^p(\Omega)$ toward $\int_0^t D_s^- v_s ds$, for every $0 \leq t \leq 1$. If v belongs to $L_-^{1,p}$, with $p > 4$, by theorem 5.2 in [20], the Skorohod integral process $\int_0^{\cdot} v_t \delta W_t$ admits a continuous version. At the same time, thanks to theorem 1.1 of [26] we know that $\int_0^{\cdot} v_t \delta W_t$ has finite quadratic variation equal to $\int_0^{\cdot} v_t^2 dt$. The proof is complete. □

Proposition 3.18. *Let u and v be left continuous processes, respectively in $L_-^{1,p}$ and $L_-^{1,2}$, with $p > 4$. Suppose that $\sup_{t\in[0,1]} |u_t|$ belongs to $L^p(\Omega)$, and that the random variable $\sup_{t\in[0,1]} \left(|v_t| + \sup_{s\in[0,1]} |D_s v_t| \right)$ is bounded. Then uv and v are forward integrable with respect to W. Furthermore u is forward integrable with respect to $\int_0^{\cdot} v_t d^- W_t$ and*

$$\begin{aligned} \int_0^{\cdot} u_t d^- \left(\int_0^t v_s d^- W_s \right) &= \int_0^{\cdot} u_t v_t d^- W_t \\ &= \int_0^{\cdot} u_t v_t \delta W_t + \int_0^{\cdot} (v_t D_t^- u_t + u_t D_t^- v_t) dt. \end{aligned}$$

Proof. By point 2. of remark 3.12 the process uv belongs to $L_-^{1,q}$, for every $4 < q < p$, and $D^-uv = vD^-u + uD^-v$. Proposition 3.17 immediately implies that

$$\int_0^{\cdot} u_t v_t d^- W_t = \int_0^{\cdot} u_t v_t \delta W_t + \int_0^{\cdot} (v_t D_t^- u_t + u_t D_t^- v_t) dt.$$

Lemma 3.13 permits to write, for every $0 \leq t \leq 1$,

$$\begin{aligned} I\left(\varepsilon, u, \int_0^{\cdot} v_t d^- W_t, t\right) &= \frac{1}{\varepsilon} \int_0^t u_s \int_s^{s+\varepsilon} v_r d^- W_r ds \\ &= \frac{1}{\varepsilon} \int_0^t u_s \left(\int_s^{s+\varepsilon} v_r \delta W_r \right) ds \\ &+ \frac{1}{\varepsilon} \int_0^t u_s \left(\int_s^{s+\varepsilon} D_r^- v_r dr \right) ds. \end{aligned}$$

Since $\sup_{t\in[0,1]} \left| D_t^- v_t \right|$ belongs to $L^p(\Omega)$ the second term of previous sum converges toward $\int_0^t u_s D_s^- v_s ds$ in $L^q(\Omega)$, for every $2 \leq q < p$. As a consequence of this, by lemma 3.13, $I\left(\varepsilon, u, \int_0^{\cdot} v_t d^- W_t, t\right)$ converges toward $\int_0^t u_s v_s d^- W_s$ in $L^2(\Omega)$. The proof is then complete. □

Definition 3.19. *We say that a process u belongs to $L^{1,p}_{-,loc}$ if it is localized by a sequence $\left(\Omega_k, u^k\right)_{k\in\mathbb{N}}$, with u^k belonging to $L^{1,p}_-$ for every k in $\mathbb{N}$.*

Lemma 3.20. *Let $u = \left(u^1, \ldots, u^n\right)$, $n > 1$, be a vector of left continuous processes with bounded paths and in $L^{1,p}_-$, for some $p \geq 2$. Then, for every ψ in $C^1\left(\mathbb{R}^n\right)$ the process $\psi(u)$ belongs to $L^{1,p}_{-,loc}$. Moreover, the localizing sequence $\left(\Omega_k, \psi(u)^k\right)_{k\in\mathbb{N}}$ can be chosen such that $\psi(u)^k$ is left continuous, and $\sup_{t\in[0,1]}\left|\psi(u)^k_t\right|$ belongs to $L^p(\Omega)$ for every k in $\mathbb{N}$.*

Proof. For k in $\mathbb{N}^*$, set $\Omega_k = \left\{\sup_{0\leq t\leq 1}||u_t||_{\mathbb{R}^n} \leq k\right\}$ and $\psi(u)^k = \psi(u)f_k(u)$, being $f_k(u) = f(\frac{u}{k})$, and f a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$, with compact support and $f(x) = 1$, for every $||x|| \leq 1$. Clearly $\psi(u)$ is localized by $\left(\Omega_k, \psi(u)^k\right)_{k\in\mathbb{N}}$. By [20], proposition 4.8, $\psi(u)^k$ belongs to $L^{1,2}$, for every k in $\mathbb{N}^*$. Since $\psi \circ f_k$ has bounded first partial derivatives, proposition 1.2.2 of [19] implies that

$$D\psi(u)^k_s = \sum_{i=1}^{n} \partial_i(\psi \circ f_k)(u_s)Du^i_s.$$

In particular $\psi(u)^k$ belongs to $L^{1,p}$. Using the continuity of all first partial derivatives of $\psi \circ f_k$ and the left continuity of u^i for every $i = 1, ..., n$, it is possible to prove that $\psi(u)^k$ belongs indeed to $L^{1,p}_-$, and $D^-\psi(u)^k = \sum_{i=1}^n \partial_i(\psi \circ f_k(u))D^-u^i$. The proof is then complete. □

We conclude this section giving a generalization of proposition 3.18.

Proposition 3.21. *Let $u = \left(u^1, \ldots, u^n\right)$, $n > 1$, be a vector of left continuous processes with bounded paths and in $L^{1,p}_-$, with $p > 4$. Let v be a process in $L^{1,2}_-$ with left continuous paths such that the random variable $|v_t| + \sup_{s\in[0,1]}|D_s v_t|$ is bounded. Then for every ψ in $C^1(\mathbb{R}^n)$ $\psi(u)v$ and v are forward integrable with respect to W. Furthermore $\psi(u)$ is forward integrable with respect to $\int_0^{\cdot} v_t d^-W_t$ and*

$$\int_0^{\cdot} \psi(u_t)d^-\left(\int_0^t v_s d^-W_s\right) \quad = \quad \int_0^{\cdot} \psi(u_t)v_t d^-W_t.$$

Proof. Let $\left(\Omega^k, \psi(u)^k\right)_{k\in\mathbb{N}}$ be a localizing sequence for $\psi(u)$ such that $\psi(u)^k$ is left continuous and $\sup_{t\in[0,1]}\left|\psi(u)^k_t\right|$ belongs to $L^p(\Omega)$ for every k in $\mathbb{N}$. Such a sequence exists thanks to lemma 3.20. Clearly $\left(\Omega^k, \psi(u)^k v\right)_{k\in\mathbb{N}}$ localizes $\psi(u)v$. For every k in $\mathbb{N}$, thanks to proposition 3.18, $\psi(u)^k$ and $\psi(u)^k v$ are forward integrable with respect to W and

$$\int_0^{\cdot} \psi(u)^k_t d^-\int_0^t v_s d^-W_s = \int_0^{\cdot} \psi(u)^k_t v_t d^-W_t.$$

The conclusion follows by remark 2.3. □

3.3. Forward integrals of anticipating processes: substitution formulae

Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$ be a filtration on $(\Omega, \mathcal{F}, P)$, with $\mathcal{F}_1 = \mathcal{F}$, and L an $\mathcal{F}$ measurable random variable with values in $\mathbb{R}^d$. We set $\mathcal{G}_t = (\mathcal{F}_t \vee \sigma(L))$, and we suppose that $\mathbb{G}$ is right continuous:

$$\mathcal{G}_t = \bigcap_{\varepsilon>0} (\mathcal{F}_{t+\varepsilon} \vee \sigma(L)).$$

In this section $\mathcal{P}^{\mathbb{F}}$ ($\mathcal{P}^{\mathbb{G}}$, resp.) will denote the σ-algebra of $\mathbb{F}$(of $\mathbb{G}$, resp.)-predictable processes. E will be the Banach space of all continuous functions on $[0,1]$ equipped with the uniform norm $||f||_E = \sup_{t\in[0,1]} |f(t)|$.

3.3.1. Preliminary results

We state in the sequel some results about forward integrals involving processes that are random field evaluated at L. To be more precise we will establish conditions to insure the existence of such integrals, their quadratic variation and a related associativity property.

Definition 3.22. *An increasing sequence of random times $(T_k)_{k\in\mathbb{N}}$ is said **suitable** if $P\left(\cup_{k=0}^{+\infty}\{T_k = 1\}\right) = 1$.*

Definition 3.23. *For every random time $0 \le S \le 1$, $p > 0$, and $\gamma > 0$, we define $\mathcal{C}_S^{p,\gamma}$ as the set of all families of continuous processes $((F(t,x), 0 \le t \le 1); x \in \mathbb{R}^d)$ such that for each compact set C of $\mathbb{R}^d$ there exists a constant $c > 0$ such that*

$$\mathbb{E}\left[\sup_{t\in[0,S]} |F(t,x) - F(t,y)|^p\right] \le c\,|x-y|^\gamma, \quad \forall x, y \in C.$$

If $S = 1$, $\mathcal{C}^{p,\gamma}$ will stand for $\mathcal{C}_S^{p,\gamma}$.

We begin recalling a result stated in [26], lemma 1.2, page 93.

Lemma 3.24. *Let $\left\{(F_n(t,x), 0 \le t \le 1), (F(t,x), 0 \le t \le 1); n \ge 1, x \in \mathbb{R}^d\right\}$ be a family of continuous processes such that F_n and F are $\mathcal{F}\otimes\mathcal{B}([0,1])\otimes\mathcal{B}(\mathbb{R}^d)$-measurable. Suppose that for each x in $\mathbb{R}^d$, $F_n(\cdot,x)$ converges to $F(\cdot,x)$ ucp and that there exist $p > 1$, $\gamma > d$, with*

$$\mathbb{E}\left[\sup_{t\in[0,1]} |F_n(t,x) - F_n(t,y)|^p\right] \le c\,|x-y|^\gamma, \quad \forall x, y \in C, \quad \forall n \in \mathbb{N}.$$

Then $x \mapsto F(\cdot,x)$ admits a continuous version $\bar{F}(\cdot,x)$, from $\mathbb{R}^d$ to E and $F_n(\cdot,L)$ converges toward $F(\cdot,L)$ ucp.

Definition 3.25. *$\mathtt{bL}(\mathcal{P}^{\mathbb{F}}\otimes\mathcal{B}(\mathbb{R}^d))$ will denote the set of all functions*

$$((h(t,x),0\le t\le 1);x\in\mathbb{R}^d)$$

which are $\mathcal{P}^{\mathbb{F}}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable, such that for every x in $\mathbb{R}^d$, $h(\cdot,x)$ has left continuous and bounded paths.

Definition 3.26. *Let $p>1,\gamma>0$. We define $\mathcal{A}^{p,\gamma}$ as the set of all functions h in $\mathtt{bL}(\mathcal{P}^{\mathbb{F}}\otimes\mathcal{B}(\mathbb{R}^d))$ satisfying the following assumption. There exists a suitable sequence of stopping times $(T_k)_{k\in\mathbb{N}}$ such that h belongs to $\bigcap_{k\in\mathbb{N}}\mathcal{C}^{p,\gamma}_{T_k}$.*

We state this lemma which will be useful later.

Lemma 3.27. *Let h and g be respectively in $\mathcal{A}^{p,\gamma_p}$ and $\mathcal{A}^{q,\gamma_q}$ for some $p,q>1$, $\gamma_p,\gamma_q>0$. Then the following statements hold.*

1. *If ψ belongs to $C^1(\mathbb{R})$ and it has bounded derivative, then $\psi(h)$ belongs to $\mathcal{A}^{p,\gamma}$.*
2. *The process hg belongs to $\mathcal{A}^{\alpha,\frac{\gamma\alpha}{p}\wedge\frac{\gamma_q\alpha}{q}}$ with $\alpha=\frac{pq}{p+q}$.*
3. *If N is a continuous $\mathbb{F}$-semimartingale, then the function*

$$\left(\int_0^t h(s,x)dN_s,0\le t\le 1,x\in\mathbb{R}\right)$$

belongs to $\mathcal{A}^{p,\gamma}$.

Proof. Let $(T_k)_{k\in\mathbb{N}}$ be a suitable sequence of stopping times such that h belongs to $\bigcap_{k\in\mathbb{N}}\mathcal{C}^{p,\gamma}_{T_k}$.

The conclusion of the first point is straightforward.

Concerning the second point we set, for every k in $\mathbb{N}$,

$$S_k=\inf\{0\le t\le 1,|\,|h(t,0)|+|g(t,0)|\ge k\}\wedge T_k.$$

If C is a compact set of $\mathbb{R}^d$, using Hölder inequality we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,S_k]}|hg(t,x)-hg(t,y)|^\alpha\right] &\le c\left(\mathbb{E}\left[\sup_{t\in[0,S_k]}|h(t,x)-h(t,y)|^p\right]\right)^{\frac{\alpha}{p}}\\ &+ c\left(\mathbb{E}\left[\sup_{t\in[0,S_k]}|g(t,x)-g(t,y)|^q\right]\right)^{\frac{\alpha}{q}}\\ &\le c\,|x-y|^{\frac{\gamma\alpha}{p}\wedge\frac{\gamma_q\alpha}{q}},\end{aligned}$$

where $c=\sup_{x,y\in C}\left(\mathbb{E}\left[\sup_{t\in[0,S_k]}|h(t,x)|^p\right]\right)^{\frac{\alpha}{p}}+\left(\mathbb{E}\left[\sup_{t\in[0,S_k]}|g(t,x)|^q\right]\right)^{\frac{\alpha}{q}}$ is bounded thanks to the choice of the sequence $(S_k)_{k\in\mathbb{N}}$ and the compactness of C.

To prove point 3. it is sufficient to define $S_k=\inf\{0\le t\le 1,|\,|V|_t+[M]_t\ge k\}\wedge T_k$, for every k in $\mathbb{N}$, where $N=M+V$, M is an $\mathbb{F}$-local martingale, V is a bounded variation process, and $|V|$ denotes the total variation of V. □

3.3.2. Existence

Let $M, V : \left(\Omega \times [0,1] \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}([0,1]) \otimes \mathcal{B}(\mathbb{R}^d)\right) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be measurable functions such that for each x in $\mathbb{R}^d$, $M(0,x) = V(0,x) = 0$, $(M(t,x), 0 \leq t \leq 1)$ is an $\mathbb{F}$-continuous local martingale, and $(V(t,x), 0 \leq t \leq 1)$, a continuous bounded variation process.

Remark 3.28. *If h is in $\mathfrak{bL}(\mathcal{P}^{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}))$, the process $(h(t,L), 0 \leq t \leq 1)$ is left continuous with bounded paths and the process $(V(t,L), 0 \leq t \leq 1)$ is continuous with bounded variation. Then, by proposition 2.8, $\int_0^{\cdot} h(t,L)d^-V(t,L)$ exists and coincides with the Lebesgue-Stieltjes integral $\int_0^{\cdot} h(t,L)dV(t,L)$. Moreover*

$$\int_0^{\cdot} h(t,L)dV(s,L) = \left(\int_0^{\cdot} h(t,x)dV(t,x)\right)_{x=L}.$$

Lemma 3.29. *Let h be in $\mathfrak{bL}(\mathcal{P}^{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}))$. Suppose that both $\sup_{t\in[0,1]} |h(t,0)|$ and $\sup_{t\in[0,1]} |M(t,0)|$ are bounded, and that there exist $p > 1$, $q > \frac{p}{p-1}$, $\gamma_p > \frac{d(q+p)}{q}$, $\gamma_q > \frac{d(q+p)}{p}$ such that M belongs to $\mathcal{C}^{p,\gamma_p}$ and h to $\mathcal{C}^{q,\gamma_q}$. Then the function $x \mapsto \int_0^{\cdot} h(s,x)dM(s,x)$ admits a continuous version, $\int_0^{\cdot} h(s,L)d^-M(s,L)$ exists as limit ucp of its regularizations and*

$$\int_0^{\cdot} h(t,L)d^-M(t,L) = \left(\int_0^{\cdot} h(t,x)dM(t,x)\right)_{x=L}.$$

Proof. For every $x \in \mathbb{R}^d$ and $0 \leq t \leq 1$ we set

$$F_\varepsilon(t,x) = \frac{1}{\varepsilon}\int_0^t h(s,x)(M(s+\varepsilon,x) - M(s,x))ds,$$

and

$$F(t,x) = \int_0^t h(s,x)dM(s,x).$$

To prove our statement we verify that lemma 3.24 applies to the families defined above.

Let x and y in $\mathbb{R}^d$ be fixed. Point 2. of proposition 2.7 implies that $F_\varepsilon(\cdot,x)$ converges *ucp* to $F(\cdot,x)$. Set $\alpha = \frac{pq}{p+q}$. Using theorem 45 in chapter IV of [22], we can write, for every $\varepsilon > 0$,

$$\begin{aligned} F_\varepsilon(\cdot,x) - F_\varepsilon(\cdot,y) &= \int_0^{\cdot}\left(\frac{1}{\varepsilon}\int_{r-\varepsilon}^r (h(s,x) - h(s,y))ds\right)dM(r,x) \\ &+ \int_0^{\cdot}\left(\frac{1}{\varepsilon}\int_{r-\varepsilon}^r h(s,y)ds\right)d\left(M(r,y) - M(r,x)\right). \end{aligned}$$

Thanks to theorem 2 in chapter V of [22], we find a positive constant a, depending only on p and q such that, for every $\varepsilon > 0$,

$$\mathbb{E}\left[\sup_{t\in[0,1]} |F(\varepsilon,t,x) - F(\varepsilon,t,y)|^\alpha\right] \leq a(\delta_1 + \delta_2)$$

with

$$\delta_1 = \mathbb{E}\left[\sup_{t\in[0,1]} |h(t,x)-h(t,y)|^q\right]^{\frac{\alpha}{q}} \mathbb{E}\left[\sup_{t\in[0,1]} |M(t,x)|^p\right]^{\frac{\alpha}{p}}$$

and

$$\delta_2 = \mathbb{E}\left[\sup_{t\in[0,1]} |M(t,x)-M(t,y)|^p\right]^{\frac{\alpha}{p}} \mathbb{E}\left[\sup_{t\in[0,1]} |h(t,y)|^q\right]^{\frac{\alpha}{q}}.$$

Thanks to the hypotheses on M and h it is possible to find a constant b depending on C such that

$$\delta_1 \le b\,|x-y|^{\frac{\gamma_q \alpha}{q}}, \quad \delta_2 \le b\,|x-y|^{\frac{\gamma_p \alpha}{p}}, \quad \forall \varepsilon > 0.$$

Consequently, there will exist $c > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,1]} |F(\varepsilon,t,x)-F(\varepsilon,t,y)|^\alpha\right] \le c\,|x-y|^\gamma, \qquad \forall x,y \in C, \quad \forall \varepsilon > 0,$$

with $\gamma = \frac{\gamma_p \alpha}{p} \wedge \frac{\gamma_q \alpha}{q} > d$ and proof is complete. □

The following proposition represents a generalization of previous lemma.

Proposition 3.30. *Suppose that M belongs to $\mathcal{A}^{p,\gamma_p}$ and that h belongs to $\mathcal{A}^{q,\gamma_q}$ for some $p > 1$, $q > \frac{p}{p-1}$, $\gamma_p > \frac{d(q+p)}{q}$ and $\gamma_q > \frac{d(q+p)}{p}$. Then $x \mapsto \int_0^\cdot h(s,x)dM(s,x)$ admits a continuous version, $\int_0^\cdot h(s,L)d^-M(s,L)$ exists as limit ucp of its regularizations and*

$$\int_0^\cdot h(s,L)d^-M(s,L) = \left(\int_0^\cdot h(s,x)dM(s,x)\right)_{x=L}.$$

Proof. We observe that we do not loose generality assuming that there exists a suitable sequence of $\mathbb{F}$-stopping times $(T_k)_{k\in\mathbb{R}}$ such that M and h belong respectively to $\bigcap_{k\in\mathbb{N}} \mathtt{C}^{p,\gamma_p}_{T_k}$ and $\bigcap_{k\in\mathbb{N}} \mathtt{C}^{q,\gamma_q}_{T_k}$. Let $(S_k)_{k\in\mathbb{N}}$ be a suitable sequence of $\mathbb{F}$-stopping times such that, for every k in $\mathbb{M}$, S^k is the first instant, between 0 and 1, the process $|M(\cdot,0)| + |h(\cdot,0)|$ is greater than k. Set, for every k in $\mathbb{N}$,

$$R_k = S_k \wedge T_k, \quad M^k = M^{R_k}, \quad h^k = h^{R_k},$$

and

$$\Omega_k = \left\{\sup_{t\in[0,1]} |M(t,0)| \le k\right\} \cap \left\{\sup_{t\in[0,1]} |h(t,0)| \le k\right\} \cap \{R_k = 1\}.$$

Let k be fixed. It is clear that $\sup_{t\in[0,1]} \left|h^k(t,0)\right|$ and $\sup_{t\in[0,1]} \left|M^k(\cdot,0)\right|$ are bounded and that M^k and h^k belong, respectively, to $\mathtt{C}^{p,\gamma_p}$ and $\mathtt{C}^{q,\gamma_q}$. We can thus apply lemma 3.29 to state that the function $x \mapsto \int_0^\cdot h^k(t,x)dM^k(t,x)$ admits a continuous version and

$$\int_0^\cdot h^k(t,L)dM^k(t,L) = \left(\int_0^\cdot h^k(t,x)dM^k(t,x)\right)_{x=L}.$$

By the *local character* of the classical stochastic integral, see [22], theorem 26, $F^k(\cdot,x) = F^h(\cdot,x) = \int_0^{\cdot} h(t,x)dM(t,x)$, for every x in $\mathbb{R}^d$, almost surely on Ω_h, for every $h \leq k$. Therefore it is possible to define $\left(\bar{F}(t,x), 0 \leq t \leq 1, x \in \mathbb{R}^d\right)$ such that $x \mapsto \bar{F}(\cdot,x)$ is continuous, and for every k in $\mathbb{N}$

$$\bar{F}(\cdot,x) = \bar{F}^k(\cdot,x) = \int_0^{\cdot} h(t,x)dM(t,x), \quad \forall x \in \mathbb{R}^d, \quad \text{on } \Omega_k.$$

Furthermore, remark 2.3 implies that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{\cdot} h(s,L)(M(s+\varepsilon,L) - M(s,x))ds = \bar{F}(\cdot,L),$$

ucp, since the convergence holds on every on Ω_k, for every k in $\mathbb{N}$. □

Previous proposition combined with lemma 3.27 implies directly the following corollary.

Corollary 3.31. *Let N be a continuous $\mathbb{F}$-local martingale. Let h be in $\mathcal{A}^{q,\gamma_q}$, for $q > 1$, $\gamma_q > d$. Then $x \mapsto \int_0^{\cdot} h(t,x)dN_t$ admits a continuous version, $\int_0^{\cdot} h(t,L)d^-N_t$ exists as limit ucp of its regularizations and*

$$\int_0^{\cdot} h(t,L)d^-N_t = \left(\int_0^{\cdot} h(t,x)dN_t\right)_{x=L}.$$

3.3.3. Quadratic variation

We examine the existence of the quadratic variation of forward integrals of the anticipating processes considered in the present subsection. We start giving a generalization of a substitution formula proved in [26], proposition 1.3. We furnish, in fact, a *localized* version of that result in view of further applications to finance. We omit the details of its proof, which are indeed similar to those used in the proof of proposition 3.30.

Proposition 3.32. *Suppose that M belongs to $\mathcal{A}^{p,\gamma}$ with $p > 2$ and $\gamma > 2d$. Then $x \mapsto [M(\cdot,x), M(\cdot,x)]$ admits a continuous version, the process $M(\cdot,L)$ has finite quadratic variation and*

$$[M(\cdot,L), M(\cdot,L)] = [M(\cdot,x), M(\cdot,x)]_{x=L}.$$

A consequence of previous proposition is the following.

Proposition 3.33. *Suppose that $M(t,x) = \int_0^{\cdot} h(t,x)dN(t,x)$, for every x in $\mathbb{R}^d$, where h and N verify the following assumption. The functions h and N are, respectively, in $\mathcal{A}^{q,\gamma_q}$ and $\mathcal{A}^{p,\gamma_p}$ with $p > 2$, $q > \frac{2p}{p-2}$, $\gamma_p > \frac{2d(q+p)}{q}$ and $\gamma_q > \frac{2d(q+p)}{p}$; for every x in $\mathbb{R}^d$, $N(0,x) = 0$, and $(N(t,x), 0 \leq t \leq 1)$ is a continuous $\mathbb{F}$-local martingale. Then $M(\cdot,L)$ has finite quadratic variation and $[M(\cdot,L)] = [M(\cdot,x)]_{x=L}$.*

Proof. We have to show that hypotheses of proposition 3.32 are satisfied. We denote with $(T_k)_{k\in\mathbb{N}}$ a suitable sequence of $\mathbb{F}$-stopping times such that N and h belong respectively to $\bigcap_{k\in\mathbb{N}} C^{p,\gamma_p}_{T_k}$ and $\bigcap_{k\in\mathbb{N}} C^{q,\gamma_q}_{T_k}$. Let $(S_k)_{k\in\mathbb{N}}$ be a suitable sequence of $\mathbb{F}$-stopping times such that, for every k in $\mathbb{N}$, S^k is the first instant, between 0 and 1, the process $|N(\cdot,0)| + |h(\cdot,0)|$ is greater than k. Set, for every k in $\mathbb{N}$, $R_k = S_k \wedge T_k$, $N^k = N^{R_k}$, $M^k = M^{R_k}$ $h^k = h^{R_k}$, and $\Omega_k = \left\{\sup_{t\in[0,1]} |N(t,0)| \leq k\right\} \cap \left\{\sup_{t\in[0,1]} |h(t,0)| \leq k\right\} \cap \{R_k = 1\}$. Let C be a compact set of $\mathbb{R}^d$, x, y in C, and k in $\mathbb{N}$. Using arguments already employed in the proof of lemma 3.29, it is not difficult to show that if $\alpha = \frac{pq}{p+q} > 2$, there exists a constant $d_k > 0$, depending on C and k, such that,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,R_k]} |M(t,x) - M(t,y)|^\alpha\right] &= \mathbb{E}\left[\sup_{t\in[0,1]} \left|M^k(t,x) - M^k(t,y)\right|^\alpha\right] \\ &\leq d_k\,|x-y|^\gamma\,,\end{aligned}$$

with $\gamma = \frac{\gamma_p\alpha}{p} \wedge \frac{\gamma_q\alpha}{q} > 2d$. This concludes the proof. □

From proposition 3.32 and lemma 3.27 we can easily derive the following corollary.

Corollary 3.34. *Suppose that $M(\cdot,x) = \int_0^\cdot h(t,x)dN$ where N is a continuous $\mathbb{F}$-local martingale and h belongs to $\mathcal{A}^{q,\gamma_q}$, $q > 2$, $\gamma_q > 2d$. Then the function $x \mapsto [M(\cdot,x), M(\cdot,x)]$ admits a continuous version, the process $M(\cdot,L)$ has finite quadratic variation and $[M(\cdot,L), M(\cdot,L)] = [M(\cdot,x), M(\cdot,x)]_{x=L}$.*

3.3.4. Chain rule formula

We conclude this section by proving the associative property of forward integrals for the processes studied in this part of the paper.

Proposition 3.35. *Let h and k be respectively in $\mathcal{A}^{p,\gamma_p}$ and $\mathcal{A}^{q,\gamma_q}$, with $p > 1$, $q > \frac{p}{p-1}$, $\gamma_p > \frac{d(q+p)}{q}$, $\gamma_q > \frac{d(q+p)}{p}$. Let N be a continuous $\mathbb{F}$-local martingale. Then $x \mapsto \int_0^\cdot k(t,x)d^-\int_0^t h(s,x)dN_s = \int_0^\cdot h(t,x)k(t,x)dN_s$ admits a continuous version, $\int_0^\cdot k(t,L)d^-\int_0^t h(s,L)d^-N_s$ exists as limit ucp of its regularizations and*

$$\int_0^\cdot k(t,L)d^-\int_0^t h(s,L)dN_s = \int_0^\cdot k(t,L)h(t,L)d^-N_t.$$

Proof. By point 3. of lemma 3.27 we know that $\int_0^\cdot h(t,x)dN_t$ belongs to $\mathcal{A}^{p,\gamma_p}$. Then, by proposition 3.30, $x \mapsto \int_0^\cdot k(t,x)d\int_0^t h(s,x)dN_s = \int_0^\cdot k(t,x)h(t,x)dN_t$ admits a continuous version and

$$\begin{aligned}\int_0^\cdot k(t,L)d^-\int_0^t h(s,L)dN_s &= \left(\int_0^\cdot k(t,x)d^-\int_0^t h(s,x)dN_s\right)_{x=L} \\ &= \left(\int_0^\cdot k(t,x)h(t,x)dN_t\right)_{x=L}.\end{aligned}$$

Point 2. of lemma 3.27 again shows that hk satisfies the hypotheses of corollary 3.31. As a consequence of of this

$$\int_0^{\cdot} h(t,L)k(t,L)d^-N_t = \left(\int_0^{\cdot} k(t,x)h(t,x)dN_t\right)_{x=L},$$

and we achieve the end of the proof. □

Definition 3.36. *1. For $p > 1$ and $\gamma > d$, we define $\mathcal{A}^{p,\gamma}(L)$ as the set of all processes $(h(t,L), 0 \le t \le 1)$ with h belonging to $\mathcal{A}^{p,\gamma}$.*
2. $\mathcal{S}(\mathcal{A}^{p,\gamma}(L))$ will be the set of all processes $h = h^0 I_{\{0\}} + \sum_{i=1}^m h^i I_{(t_{i-1},t_i]}$ where $0 = t_0 \le t_1, \cdots, t_m = 1$, and h^i belongs to $\mathcal{A}^{p,\gamma}(L)$ for $i = 1, ..., m$.

Using similar arguments employed in the proof of proposition 3.7, it is possible to demonstrate the following one.

Proposition 3.37. *Let N be a continuous $\mathbb{F}$-local martingale and h be a process in $\mathcal{S}(\mathcal{A}^{p,\gamma}(L))$ for some $p > 1$, and $\gamma > d$. Then the following statements are true.*

1. *h is N-forward integrable and the forward integral $\int_0^{\cdot} h_t d^-N_t$ still belongs to $\mathcal{S}(\mathcal{A}^{p,\gamma}(L))$.*
2. *If $p > 2$ and $\gamma > 2d$, $\int_0^{\cdot} h_t d^-N_t$ has finite quadratic variation equal to $\int_0^{\cdot} h_t^2 d[N]_t$.*
3. *If k belongs to $\mathcal{S}(\mathcal{A}^{q,\gamma_q}(L))$ with $q > \frac{p}{p-1}$, $\gamma_p > \frac{d(p+q)}{q}$ and $\gamma_q > \frac{d(p+q)}{p}$, then k is forward integrable with respect to $\int_0^{\cdot} h_t d^-N_t$ and*

$$\int_0^{\cdot} k_t d^- \int_0^t h_s d^-N_s = \int_0^{\cdot} k_t h_t d^-N_t.$$

4. $\mathcal{A}$-martingales

Throughout this section $\mathcal{A}$ will be a real linear set of measurable processes indexed by $[0,1)$ with paths which are bounded on each compact interval of $[0,1)$.

We will denote with $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$ a filtration indexed by $[0,1]$ and with $\mathcal{P}(\mathbb{F})$ the σ-algebra generated by all left continuous and $\mathbb{F}$-adapted processes. In the remainder of the paper we will adopt the notations $\mathbb{F}$ and $\mathcal{P}(\mathbb{F})$ even when the filtration $\mathbb{F}$ is indexed by $[0,1)$. At the same way, if X is a process indexed by $[0,1]$, we shall continue to denote with X its restriction to $[0,1)$.

4.1. Definitions and properties

Definition 4.1. *A process $X = (X_t, 0 \le t \le 1)$ is said $\mathcal{A}$-**martingale** if every θ in $\mathcal{A}$ is X-improperly forward integrable and $\mathbb{E}\left[\int_0^t \theta_s d^-X_s\right] = 0$ for every $0 \le t \le 1$.*

Definition 4.2. *A process $X = (X_t, 0 \leq t \leq 1)$ is said* ***$\mathcal{A}$-semimartingale*** *if it can be written as the sum of an $\mathcal{A}$-martingale M and a bounded variation process V, with $V_0 = 0$.*

Remark 4.3.
1. *If X is a continuous $\mathcal{A}$-martingale with X belonging to $\mathcal{A}$, its quadratic variation exists improperly. In fact, if $\int_0^{\cdot} X_t d^- X_t$ exists improperly, it is possible to show that $[X, X]$ exists improperly and $[X, X] = X^2 - X_0^2 - 2\int_0^{\cdot} X_s d^- X_s$. We refer to proposition 4.1 of [28] for details.*
2. *Let X a continuous square integrable martingale with respect to some filtration $\mathbb{F}$. Suppose that every process in $\mathcal{A}$ is the restriction to $[0, 1)$ of a process $(\theta_t, 0 \leq t \leq 1)$ which is $\mathbb{F}$-adapted, it has left continuous with right limit paths and $\mathbb{E}\left[\int_0^1 \theta_t^2 d\,[X]_t\right] < +\infty$. Then X is an $\mathcal{A}$-martingale.*
3. *In [13] the authors introduced the notion of* ***weak-martingale****. A semimartingale X is a weak-martingale if $\mathbb{E}\left[\int_0^t f(s, X_s) dX_s\right] = 0$, $0 \leq t \leq 1$, for every $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, bounded Borel-measurable. Clearly we can affirm the following. Suppose that $\mathcal{A}$ contains all processes of the form $f(\cdot, X)$, with f as above. Let X be a semimartingale X which is an $\mathcal{A}$-martingale. Then X is a weak-martingale.*

Proposition 4.4. *Let X be a continuous $\mathcal{A}$-martingale. The following statements hold true.*

1. *If X belongs to $\mathcal{A}$, $X_0 = 0$ and $[X, X] = 0$. Then $X \equiv 0$.*
2. *Suppose that $\mathcal{A}$ contains all bounded $\mathcal{P}(\mathbb{F})$-measurable processes. Then X is an $\mathbb{F}$-martingale.*

Proof. From point 1. of remark 4.3, $\mathbb{E}\left[X_t^2\right] = 0$, for all $0 \leq t \leq 1$.

Regarding point 2. it is sufficient to observe that processes of type $I_A I_{(s,t]}$, with $0 \leq s \leq t \leq 1$, and A in $\mathcal{F}_s$ belong to $\mathcal{A}$. Moreover $\int_0^1 I_A I_{(s,t]}(r) d^- X_r = I_A(X_t - X_s)$. This imply $\mathbb{E}[X_t - X_s \mid \mathcal{F}_s] = 0$, $0 \leq s \leq t \leq 1$. □

Corollary 4.5. *The decomposition of an $\mathcal{A}$-semimartingale X in definition 4.2 is unique among the class of processes of type $M + V$, being M a continuous $\mathcal{A}$-martingale in $\mathcal{A}$ and V a bounded variation process.*

Proof. If $M+V$ and $N+W$ are two decompositions of that type, then $M-N$ is a continuous $\mathcal{A}$-martingale in $\mathcal{A}$ starting at zero with zero quadratic variation. Point 1. of proposition 4.4 permits to conclude. □

The following proposition gives sufficient conditions for an $\mathcal{A}$-martingale to be a martingale with respect to some filtration $\mathbb{F}$, when $\mathcal{A}$ is made up of $\mathcal{P}(\mathbb{F})$-measurable processes. It constitutes a generalization of point 2. in proposition 4.4.

Definition 4.6. *We will say that $\mathcal{A}$ satisfies* ***assumption*** *$\mathcal{D}$ with respect to a filtration $\mathbb{F}$ if*

1. *Every θ in $\mathcal{A}$ is $\mathbb{F}$-adapted;*
2. *For every $0 \leq s < 1$ there exists a basis $\mathcal{B}_s$ for $\mathcal{F}_s$, with the following property. For every A in $\mathcal{B}_s$ there exists a sequence of $\mathcal{F}_s$-measurable random variables $(\Theta_n)_{n\in\mathbb{N}}$, such that for each n the process $\Theta_n I_{[0,1)}$ belongs to $\mathcal{A}$, $\sup_{n\in\mathbb{N}} |\Theta_n| \leq 1$, almost surely and*

$$\lim_{n\to+\infty} \Theta_n = I_A, \quad a.s.$$

Proposition 4.7. *Let $X = (X_t, 0 \leq t \leq 1)$ be a continuous $\mathcal{A}$-martingale adapted to some filtration $\mathbb{F}$, with X_t belonging to $L^1(\Omega)$ for every $0 \leq t \leq 1$. Suppose that $\mathcal{A}$ satisfies assumption $\mathcal{D}$ with respect to $\mathbb{F}$. Then X is an $\mathbb{F}$-martingale.*

Proof. We have to show that for all $0 \leq s \leq t \leq 1$, $\mathbb{E}[I_A (X_t - X_s)] = 0$, for all A in $\mathcal{F}_s$. We fix $0 \leq s < t \leq 1$ and A in $\mathcal{B}_s$. Let $(\Theta_n)_{n\in\mathbb{N}}$ be a sequence of random variables converging almost surely to I_A as in the hypothesis. Since X is an $\mathcal{A}$-martingale, $\mathbb{E}[\Theta_n (X_t - X_s)] = 0$, for all n in $\mathbb{N}$. We note that $X_t - X_s$ belongs to $L^1(\Omega)$, then, by Lebesgue dominated convergence theorem,

$$|\mathbb{E}[I_A (X_t - X_s)]| \leq \lim_{n\to+\infty} \mathbb{E}[|I_A - \Theta_n| |X_t - X_s|] = 0.$$

Previous result extends to the whole σ-algebra $\mathcal{F}_s$ and this permits to achieve the end of the proof. □

Some interesting properties can be derived taking inspiration from [13].

For a process X, we will denote

$$\mathcal{A}_X = \{(\psi(t, X_t)), 0 \leq t < 1 \mid \psi : [0,1] \times \mathbb{R} \to \mathbb{R}, \text{ Borel-measurable with polynomial growth and lower bounded}\}. \quad (8)$$

Remark 4.8. *At this stage we could avoid to impose a lower bound on functions in $\mathcal{A}_X$. Nevertheless, we prefer to consider this qualitative restriction in view of further applications to finance. Indeed, $\mathcal{A}_X$ will play the role of a possible class of admissible portfolios and we are interested in excluding among them the so called doubling strategies. Generally speaking, a doubling strategy is an arbitrage which can be realized if unbounded accumulation of losses are allowed. For more details about this arguments the reader is referred to Harrison and Pliska (1979).*

Proposition 4.9. *Let X be a continuous $\mathcal{A}$-martingale with $\mathcal{A} = \mathcal{A}_X$.*

Then, for every ψ in $C^2(\mathbb{R})$ with bounded first and second derivatives, the process

$$\psi(X) - \frac{1}{2}\int_0^{\cdot} \psi''(X_s) d[X, X]_s$$

is an $\mathcal{A}$-martingale.

Proof. The process X belongs to $\mathcal{A}$. In particular, X admits improper quadratic variation. We set $Y = \psi(X) - \frac{1}{2}\int_0^{\cdot} \psi''(X_s) d[X,X]_s$. Let θ in $\mathcal{A}_X$. By lemma 2.12, for every $0 \leq t < 1$

$$\int_0^t \theta_s d^- Y_s = \int_0^t \theta_s \psi'(X_s) d^- X_s.$$

Since $\theta\psi'(X)$ still belongs to $\mathcal{A}$, θ is Y-improperly integrable and

$$\int_0^{\cdot} \theta_t d^- Y_t = \int_0^{\cdot} \theta_t \psi'(X_t) d^- X_t. \tag{9}$$

We conclude taking the expectation in equality (9). □

Proposition 4.10. *Suppose that $\mathcal{A}$ is an algebra. Let X and Y be two continuous $\mathcal{A}$-martingales with X and Y in $\mathcal{A}$.*

Then the process $XY - [X,Y]$ is an $\mathcal{A}$-martingale .

Proof. Since $\mathcal{A}$ is a real linear space, $(X+Y)$ belongs to $\mathcal{A}$. In particular by point 1. of remark 4.3, $[X+Y, X+Y]$, $[X,X]$ and $[Y,Y]$ exist improperly. This implies that $[X,Y]$ exists improperly too and that it is a bounded variation process. Therefore the vector (X,Y) admits all its mutual brackets on each compact set of $[0,1)$. Let θ be in $\mathcal{A}$. Since $\mathcal{A}$ is an algebra, θX and θY belong to $\mathcal{A}$ and so both $\int_0^{\cdot} \theta_s X_s d^- Y_s$ and $\int_0^{\cdot} \theta_s Y_s d^- X_s$ locally exist. By lemma 2.12 $\int_0^{\cdot} \theta_t d^-\left(X_t Y_t - [X,Y]\right)$ exists improperly too and

$$\int_0^{\cdot} \theta_t d^-\left(X_t Y_t - [X,Y]_t\right) = \int_0^{\cdot} Y_t \theta_t d^- X_t + \int_0^{\cdot} X_t \theta_t d^- Y_t.$$

Taking the expectation in the last expression we then get the result. □

We recall a notion and a related result of [6].

A process R is *strongly predictable* with respect to a filtration $\mathbb{F}$, if

$$\exists\ \delta > 0, \text{ such that } R_{\varepsilon+\cdot} \text{ is } \mathbb{F}\text{-adapted}, \quad \text{for every } \varepsilon \leq \delta.$$

Proposition 4.11. *Let R be an $\mathbb{F}$-strongly predictable continuous process. Then for every continuous $\mathbb{F}$-local martingale Y, $[R,Y] = 0$.*

Proposition 4.11 combined with proposition 4.10 implies proposition 4.12 and corollary 4.13.

Proposition 4.12. *Let $\mathcal{A}$, X and Y be as in proposition 4.10. Assume, moreover, that X is an $\mathbb{F}$-local martingale, and that Y is strongly predictable with respect to $\mathbb{F}$. Then XY is an $\mathcal{A}$-martingale.*

Corollary 4.13. *Let $\mathcal{A}$, X and Y be as in proposition 4.10. Assume that X is a local martingale with respect to some filtration $\mathbb{G}$ and that Y is either $\mathbb{F}$-independent, or $\mathcal{G}_0$-measurable. Then XY is an $\mathcal{A}$-martingale.*

Proof. If Y is $\mathbb{G}$-independent, it is sufficient to apply previous proposition with $\mathbb{F} = \left(\bigcap_{\varepsilon>0} \mathcal{G}_{t+\varepsilon} \vee \sigma(Y)\right)_{t\in[0,1]}$. □

4.2. $\mathcal{A}$-martingales and Weak Brownian motion

We proceed defining and discussing processes which are *weak-Brownian motions* in order to exhibit explicit examples of $\mathcal{A}$-martingales.

Definition 4.14. *([13]) A stochastic process $(X_t, 0 \leq t \leq 1)$ is a **weak Brownian motion of order** k if for every k-tuple $(t_1, t_2, ..., t_k)$*

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_k}) \stackrel{law}{=} (W_{t_1}, W_{t_2}, \ldots, W_{t_k})$$

where $(W_t, 0 \leq t \leq 1)$ is a Brownian motion.

We set, for a process $(X_t, 0 \leq t \leq 1)$,

$$\begin{aligned} \mathcal{A}_X^1 &= \{(\psi(t,x), 0 \leq t \leq 1, \text{ with polynomial growth s.t } \psi = \partial_x \Psi \\ &\quad \Psi \in C^{1,2}([0,1] \times \mathbb{R}) \text{ with } |\partial_t \Psi| + |\partial_{xx} \Psi| \text{ bounded } \}. \end{aligned}$$

Assumption 4.15. *We suppose that $\sigma : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Borel-measurable and bounded function such that the following equation has a unique solution $(\nu_t)_{t \in [0,1]}$ in the sense of distribution*

$$\begin{cases} \partial_t \nu_t(dx) = \frac{1}{2} \left(\sigma(t,x)^2 \nu_t(dx)\right)'' \\ \nu_0(dx) = \delta_0. \end{cases} \tag{10}$$

Remark 4.16. *Assumption 4.15 is verified for $\sigma(t,x) \equiv \sigma$, being σ a positive real constant and, in that case, $\nu_t = N(0, \sigma^2 t)$, for every $0 \leq t \leq 1$.*

Proposition 4.17. *Let $(X_t, 0 \leq t \leq 1)$ be a continuous finite quadratic variation process with $X_0 = 0$, and $d[X]_t = (\sigma(t, X_t))^2 dt$, where σ fulfills assumption 4.15. Then the following statements are true.*

1. *Suppose that $\mathcal{A} = \mathcal{A}_X^1$. Then X is an $\mathcal{A}$-martingale if and only if, for every $0 \leq t \leq 1$, $X_t \stackrel{law}{=} Z_t$, for every (Z, B) solution of equation $dZ = \sigma(\cdot, Z)dB, Z_0 = 0$, in the sense of definition 1.2 in chapter IX of [23]. In particular, if $\sigma \equiv 1$, X is a weak Brownian motion of order 1, if and only if it is an $\mathcal{A}_X^1$-martingale.*
2. *Suppose that $d[X]_t = f_t dt$, with f $\mathcal{B}([0,1])$-measurable and bounded. If X is a weak Brownian motion of order $k = 1$, then X is an $\mathcal{A}$-semimartingale. Moreover the process*

$$X + \int_0^{\cdot} \frac{(1 - f_s) X_s}{2s} ds.$$

is an $\mathcal{A}$-martingale.

Proof. 1. Using Itô inverse formula recalled in proposition 2.11 we can write, for every $0 \leq t \leq 1$ and $\psi = \partial_x \Psi$ in $\mathcal{A}_X^1$

$$\begin{aligned} \int_0^t \psi(s, X_s) d^- X_s &= \Psi(t, X_t) - \Psi(0, X_0) \\ &- \int_0^t \left(\partial_s \Psi + \frac{1}{2} \partial_{xx}^{(2)} \Psi \sigma^2\right)(s, X_s) ds. \end{aligned} \tag{11}$$

For every $0 \leq t \leq 1$, we denote with $\mu_t(dx)$ the law of X_t. If X is an $\mathcal{A}^1_X$-martingale, from (11) we derive

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \Psi(t,x)d\mu_t(x) - \int_{\mathbb{R}} \Psi(0,x)\mu_0(dx) - \int_0^t \int_{\mathbb{R}} \partial_s \Psi(s,x)\mu_s(dx)ds \\ &- \frac{1}{2}\int_0^t \int_{\mathbb{R}} \partial^{(2)}_{xx}\Psi(s,x)\sigma(s,x)^2\mu_s(dx)ds. \end{aligned} \tag{12}$$

In particular, the law of X solves equation (10).
On the other hand, let (Z,B) be a solution of equation $Z = \int_0^{\cdot} \sigma(s,Z_s)dB_s$. The law of Z fulfills equation (12) too. Indeed, Z is a finite quadratic variation process with $d[Z]_t = (\sigma(t,Z_t))^2 dt$ which is an $\mathcal{A}^1_X$-martingale by point 2. of remark 4.3. By assumption (4.15) X_t must have the same law of Z_t. This establishes the converse implication of point 1.
Suppose, on the contrary, that X_t has the same law of Z_t, for every $0 \leq t \leq 1$. Using the fact that Z is an $\mathcal{A}^1_X$-martingale which solves equation (11) we get

$$\mathbb{E}\left[\Psi(t,Z_t) - \Psi(0,Z_0) - \int_0^t \left(\partial_s\Psi + \frac{1}{2}\partial^{(2)}_{xx}\Psi\sigma^2\right)(s,X_s)ds\right] = 0.$$

for every Ψ in $C^{1,2}([0,1]\times\mathbb{R})$ with $\partial_x\Psi = \psi$ in $\mathcal{A}^1_X$. Since X_t has the same law of Z_t, for every $0 \leq t \leq 1$, equality (11) implies that

$$\mathbb{E}\left[\int_0^{\cdot} \psi(t,X_t)d^-X_t\right] = \mathbb{E}\left[\int_0^{\cdot} \psi(t,Z_t)d^-Z_t\right] = 0,$$

The proof of the first point is now achieved.

2. Suppose that $\sigma(t,x)^2 = f_t$, for every (t,x) in $[0,1]\times\mathbb{R}$. Let Ψ be in $C^{1,2}([0,1]\times\mathbb{R})$ such that $\psi = \partial_x\Psi$ belongs to $\mathcal{A}^1_X$. Proposition 2.11 yields

$$\int_0^t \psi(s,X_s)d^-X_s = Y^{\Psi}_t + \frac{1}{2}\int_0^t \partial^{(2)}_{xx}\Psi(s,X_s)(1-f_s)ds, \quad 0 \leq t \leq 1,$$

with

$$Y^{\Psi}_t = \Psi(t,X_t) - \Psi(0,X_0) - \int_0^t \partial_s\Psi_s(s,X_s)ds - \frac{1}{2}\int_0^t \partial^{(2)}_{xx}\Psi(s,X_s)ds.$$

Moreover X is a weak Brownian motion of order 1. This implies $\mathbb{E}\left[Y^{\Psi}_t\right] = 0$, for every $0 \leq t \leq 1$. We derive that

$$\mathbb{E}\left[\int_0^t \psi(s,X_s)d^-X_s + \frac{1}{2}\int_0^t \partial^{(2)}_{xx}\Psi(s,X_s)(f_s-1)ds\right] = \mathbb{E}\left[Y^{\Psi}_t\right] = 0.$$

Since the law of X_t is $N(0,t)$, by Fubini theorem and integration by parts on the real line we obtain

$$\mathbb{E}\left[\int_0^t \partial^{(2)}_{xx}\Psi(s,X_s)(f_s-1)ds\right] = \mathbb{E}\left[\int_0^t \psi(s,X_s)\frac{(1-f_s)X_s}{s}ds\right].$$

This concludes the proof of the second point.

□

From [13] we can extract an example of an $\mathcal{A}$-semimartingale which is not a semimartingale.

Example 4.18. *Suppose that $(B_t, 0 \leq t \leq 1)$ is a Brownian motion on the probability space $(\Omega, \mathbb{G}, P)$, being $\mathbb{G}$ some filtration on $(\Omega, \mathcal{F}, P)$. Set*

$$X_t = \begin{cases} B_t, & 0 \leq t \leq \frac{1}{2} \\ B_{\frac{1}{2}} + (\sqrt{2}-1)B_{t-\frac{1}{2}}, & \frac{1}{2} < t \leq 1. \end{cases}$$

Then X is a continuous weak Brownian motion of order 1, which is not a $\mathbb{G}$-semimartingale. Moreover it is possible to show that $d[X]_t = f_t dt$, with $f = I_{[0,\frac{1}{2}]} + (\sqrt{2}-1)^2 I_{[\frac{1}{2},1]}$. In particular, thanks to point 2. of previous proposition, $X + \int_0^{\cdot} \frac{(1-f_s)X_s}{2s} ds$ is an $\mathcal{A}_X^1$-martingale.

A natural question is the following. Supposing that X is an $\mathcal{A}$-martingale with respect to a probability measure Q equivalent to P, what can we say about the nature of X under P ?. The following proposition provides a partial answer to this problem when $\mathcal{A} = \mathcal{A}_X^1$.

Proposition 4.19. *Let X be as in proposition 4.17, and σ satisfy assumption 4.15. Assume, furthermore, that X is an $\mathcal{A}_X^1$-martingale under a probability measure Q with $P << Q$, Then the law of X_t is absolutely continuous with respect to Lebesgue measure, for every $0 \leq t \leq 1$.*

Proof. Since $P << Q$, for every $0 \leq t \leq 1$, $X_t(P) << X_t(Q)$. Then it is sufficient to observe that by proposition 4.17, for every $0 \leq t \leq 1$, the law of X_t under Q is absolutely continuous with respect to Lebesgue. □

Corollary 4.20. *Let X be as in proposition 4.17, and σ satisfy assumption 4.15. Assume, furthermore, that X is an $\mathcal{A}_X$-martingale under a probability measure Q with $P << Q$, Then the law of X_t is absolutely continuous with respect to Lebesgue measure, for every $0 \leq t \leq 1$.*

Proof. Clearly $\mathcal{A}_X^1$ is contained in $\mathcal{A}_X$. The result is then a consequence of previous proposition. □

Proposition 4.21. *Let $(X_t, 0 \leq t \leq 1)$ be a continuous weak Brownian motion of order 8. Then, for every $\psi : [0,1] \times \mathbb{R} \to \mathbb{R}$, Borel measurable with polynomial growth, the forward integral $\int_0^{\cdot} \psi(t, X_t) d^- X_t$, exists and*

$$\mathbb{E}\left[\int_0^{\cdot} \psi(t, X_t) d^- X_t, \right] = 0.$$

In particular, X is an $\mathcal{A}_X$-martingale.

Proof. Let $\psi : [0,1] \times \mathbb{R} \to \mathbb{R}$ be Borel measurable and t in $0 \leq t \leq 1$ be fixed. Set

$$I_\varepsilon^X(t) = I(\varepsilon, \psi(\cdot, X), X) \quad I_\varepsilon^B(t) = I(\varepsilon, \psi(\cdot, B), B),$$

being B a Brownian motion on a filtered probability space $(\Omega^B, \mathbb{F}^B, P^B)$.

Since X is a weak Brownian motion of order 8, it follows that

$$\mathbb{E}\left[\left|I_\varepsilon^X(t)-I_\delta^X(t)\right|^4\right]=\mathbb{E}^{P^B}\left[\left|I_\varepsilon^B(t)-I_\delta^B(t)\right|^4\right],\quad\forall\,\varepsilon,\delta>0.$$

We show now that $I_\varepsilon^B(t)$ converges in $L^4(\Omega)$. This implies that $I_\varepsilon^X(t)$ is of Cauchy in $L^4(\Omega)$.

In [29], chapter 3.5, it is proved that $I_\varepsilon^B(t)$ converges in probability when ε goes to zero, and the limit equals the Itô integral $\int_0^t\psi(s,B_s)dB_s$. Applying Fubini theorem for Itô integrals, theorem 45 of [22], chapter IV and Burkholder-Davies-Gundy inequality, we can perform the following estimate, for every $p>4$:

$$\begin{aligned}\mathbb{E}^{P^B}\left[\left|I_\varepsilon^B(t)\right|^p\right] &= \mathbb{E}^{P^B}\left[\left|\int_0^t\left(\frac{1}{\varepsilon}\int_{r-\varepsilon}^r\psi(s,B_s)ds\right)dB_r\right|^p\right]\\ &\le c\mathbb{E}^{P^B}\left[\int_0^1\frac{1}{\varepsilon}\int_{r-\varepsilon}^r|\psi(s,B_s)|^p\,dsdr\right]\\ &\le c\sup_{t\in[0,1]}\mathbb{E}^{P^B}\left[|\psi(t,B_t)|^p\right]<+\infty,\end{aligned}$$

for some positive constant c. This implies the uniformly integrability of the family of random variables $\left((I_\varepsilon^B(t))^4\right)_{\varepsilon>0}$ and therefore the convergence in $L^4(\Omega^B,P^B)$ of $\left(I_\varepsilon^B(t)\right)_{\varepsilon>0}$.

Consequently, $\left(I_\varepsilon^X(t)\right)_{\varepsilon>0}$ converges in $L^4(\Omega)$ toward a random variable $I(t)$. It is clear that $\mathbb{E}\left[I(t)\right]=0$, being $I(t)$ the limit in $L^2(\Omega)$ of random variables having zero expectation.

To conclude we show that Kolmogorov lemma applies to find a continuous version of $(I(t),0\le t\le 1)$. Let $0\le s\le t\le 1$. Applying the same arguments used above

$$\begin{aligned}\mathbb{E}\left[|I(t)-I(s)|^4\right] &= \lim_{\varepsilon\to0}\mathbb{E}^{P^B}\left[\left|\int_s^t\left(\frac{1}{\varepsilon}\int_u^{u+\varepsilon}\psi(u,B_u)dB_r\right)du\right|^4\right]\\ &\le c\mathbb{E}^{P^B}\left[\left|\int_s^t\left(\frac{1}{\varepsilon}\int_{r-\varepsilon}^r\psi(u,B_u)du\right)^2dr\right|^2\right]\\ &\le |t-s|\,\mathbb{E}^{P^B}\left[\int_s^t\frac{1}{\varepsilon}\int_{r-\varepsilon}^r|\psi(u,B_u)|^4\,dudr\right]\\ &\le \sup_{u\in[0,1]}\mathbb{E}^{P^B}\left[|\psi(u,B_u)|^4\right]|t-s|^2,\quad c>0.\end{aligned}$$

□

4.3. Optimization problems and $\mathcal{A}$-martingale property

4.3.1. Gâteaux-derivative: recalls

In this part of the paper we recall the notion of Gâteaux differentiability and we list some related properties.

Definition 4.22. *A function $f : \mathcal{A} \to \mathbb{R}$ is said* ***Gâteaux-differentiable*** *at $\pi \in \mathcal{A}$, if there exists $Df_\pi : \mathcal{A} \to \mathbb{R}$ such that*

$$\lim_{\varepsilon \to 0} \frac{f(\pi + \varepsilon\theta) - f(\pi)}{\varepsilon} = Df_\pi(\theta), \quad \forall \theta \in \mathcal{A}.$$

If f is Gâteaux-differentiable at every $\pi \in \mathcal{A}$, then f is said Gâteaux-differentiable on $\mathcal{A}$.

Definition 4.23. *Let $f : \mathcal{A} \to \mathbb{R}$. A process π is said* ***optimal*** *for f in $\mathcal{A}$ if*

$$f(\pi) \geq f(\theta), \quad \forall \theta \in \mathcal{A}.$$

We state this useful lemma omitting its straightforward proof.

Lemma 4.24. *Let $f : \mathcal{A} \to \mathbb{R}$. For every π and θ in $\mathcal{A}$ define $f_{\pi,\theta} : \mathbb{R} \longrightarrow \mathbb{R}$ in the following way:*

$$f_{\pi,\theta}(\lambda) = f(\pi + \lambda(\theta - \pi)).$$

Then it holds:

1. *f is Gâteaux-differentiable if and only if for every π and θ in $\mathcal{A}$, $f_{\pi,\theta}$ is differentiable on $\mathbb{R}$. Moreover $f'_{\pi,\theta}(\lambda) = Df_{\pi+\lambda(\theta-\pi)}(\theta - \pi)$.*
2. *f is concave if and only if $f_{\pi,\theta}$ is concave for every π and θ in $\mathcal{A}$.*

Proposition 4.25. *Let $f : \mathcal{A} \to \mathbb{R}$ be Gâteaux-differentiable. Then, if π is optimal for f in $\mathcal{A}$, $Df_\pi = 0$. If f is concave*

$$\pi \text{ is optimal for } f \text{ in } \mathcal{A} \quad \Longleftrightarrow \quad Df_\pi = 0.$$

Proof. It is immediate to prove that π is optimal for f if and only if $\lambda = 0$ is a maximum for $f_{\pi,\theta}$, for every θ in $\mathcal{A}$. By lemma 4.24 $f'_{\pi,\theta}(0) = Df_\pi(\theta)$, for every θ in $\mathcal{A}$. The conclusion follows easily. □

4.3.2. An optimization problem

In this part of the paper F will be supposed to be a measurable function on $(\Omega \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}))$, almost surely in $C^1(\mathbb{R})$, strictly increasing, with F' being the derivative of F with respect to x, bounded on $\mathbb{R}$, uniformly in Ω. In the sequel ξ will be a continuous finite quadratic variation process with $\xi_0 = 0$.

The starting point of our construction is the following hypothesis.

Assumption 4.26. *1. If θ belongs to $\mathcal{A}$, then $\theta I_{[0,t]}$ belongs to $\mathcal{A}$ for every $0 \leq t < 1$.*

2. Every θ in $\mathcal{A}$ ξ-improperly forward integrable, and

$$\mathbb{E}\left[\left|\int_0^1 \theta_t d^- \xi_t\right| + \left|\int_0^1 \theta_t^2 d[\xi]_t\right|\right] < +\infty.$$

Definition 4.27. *Let θ be in $\mathcal{A}$. We denote*

$$L^\theta = \int_0^1 \theta_t d^- \xi_t - \frac{1}{2}\int_0^1 \theta^2 d[\xi]_t, \quad dQ^\theta = \frac{F'(L^\theta)}{\mathbb{E}\left[F'(L^\theta)\right]}$$

and we set $f(\theta) = \mathbb{E}\left[F(L^\theta)\right]$.

We observe that point 2. of assumption 4.26 and the boundedness of F' implies that $\mathbb{E}\left[\left|F(L^\theta)\right|\right] < +\infty$. Therefore f is well defined.

Remark 4.28. *Point 2. of assumption 4.26 implies that $\mathbb{E}\left[|\xi_t| + [\xi]_t\right] \leq +\infty$, for every $0 \leq t \leq 1$. This is due to the fact that $\mathcal{A}$ must contain real constants.*

We are interested in describing a link between the existence of an optimal process for f in $\mathcal{A}$ and the $\mathcal{A}$-semimartingale property for ξ under some probability measure equivalent to P, depending on the optimal process.

Lemma 4.29. *The function f is Gâteaux-differentiable on $\mathcal{A}$. Moreover for every π and θ in $\mathcal{A}$*

$$Df_\pi(\theta) = \mathbb{E}\left[F'(L^\pi)\int_0^1 \theta_t d^-\left(\xi_t - \int_0^t \pi_s d\left[\xi\right]_s\right)\right].$$

If F is concave, then f inherits the property.

Proof. Regarding the concavity of f, we recall that if F is increasing and concave, it is sufficient to verify that, for every θ and π in $\mathcal{A}$, it holds

$$L^{\pi+\lambda(\theta-\pi)} - L^\pi - \lambda\left(L^\theta - L^\pi\right) \geq 0, \quad 0 \leq \lambda \leq 1.$$

A short calculus shows that, for every $0 \leq \lambda \leq 1$,

$$L^{\pi+\lambda(\theta-\pi)} - L^\pi - \lambda\left(L^\theta - L^\pi\right) = \frac{1}{2}\lambda(1-\lambda)\int_0^1 (\theta_t - \pi_t)^2 d\left[\xi\right]_t \geq 0.$$

Using the differentiability of F we can write

$$a_\varepsilon = \frac{1}{\varepsilon}(f(\pi+\varepsilon\theta) - f(\pi)) = \mathbb{E}\left[H^\varepsilon_{\pi,\theta}\int_0^1 F'\left(L^\pi + \mu\varepsilon H^\varepsilon_{\pi,\theta}\right) d\mu\right],$$

with

$$H^\varepsilon_{\pi,\theta} = \int_0^1 \theta_t d^-\xi_t - \frac{1}{2}\int_0^1 (\theta_t^2\varepsilon + 2\theta_t\pi_t) d\left[\xi\right]_t.$$

The conclusion follows by Lebesgue dominated convergence theorem, which applies thanks to the boundedness of F' and point 2. in assumption 4.26. □

Putting together lemma 4.29 and proposition 4.25 we can formulate the following.

Proposition 4.30. *If a process π in $\mathcal{A}$ is optimal for $\theta \mapsto \mathbb{E}\left[F\left(L^{\theta}\right)\right]$, then the process $\xi - \int_0^{\cdot} \pi_t d\left[\xi\right]_t$ is an $\mathcal{A}$-martingale under Q^{π}. If F is concave the converse holds.*

Proof. Thanks to lemma 4.29 and point 1. in assumption 4.26, for every θ in $\mathcal{A}$ and $0 \leq t \leq 1$

$$\begin{aligned} 0 &= Df_{\pi}(\theta I_{[0,t]}) = \mathbb{E}\left[F'(L^{\pi}) \int_0^t \theta_s d^{-}\left(\xi_s - \int_0^s \pi_r d\left[\xi\right]_r\right)\right] \\ &= \mathbb{E}^{Q^{\pi}}\left[\int_0^t \theta_s d^{-}\left(\xi_s - \int_0^s \pi_r d\left[\xi\right]_r\right)\right]. \end{aligned}$$

□

The following proposition describes some sufficient conditions to recover the semimartingale property for ξ with respect to a filtration $\mathbb{G}$ on $(\Omega, \mathcal{F})$, when the set $\mathcal{A}$ is made up of $\mathbb{G}$-adapted processes. It can be proved using proposition 4.7.

Proposition 4.31. *Assume that ξ is adapted with respect to some filtration $\mathbb{G}$ and that $\mathcal{A}$ satisfies the hypothesis $\mathcal{D}$ with respect to $\mathbb{G}$. If a process π in $\mathcal{A}$ is optimal for $\theta \mapsto \mathbb{E}\left[F(L^{\theta})\right]$, then the process $\xi - \int_0^{\cdot} \beta_t d\left[\xi\right]_t$ is a $\mathbb{G}$-martingale under P, where $\beta = \pi + \frac{1}{p^{\pi}} \frac{d[p^{\pi}, \xi]}{d[\xi, \xi]}$, and $p^{\pi} = \mathbb{E}\left[\frac{dP}{dQ^{\pi}} \mid \mathcal{G}_{\cdot}\right]$. If F is concave, then the converse holds.*

Proof. Thanks to point 2. of assumption 4.26, for every $0 \leq t < 1$, the random variable $\xi_t - \int_0^t \pi_t d\left[\xi\right]_t$ is in $L^1(\Omega)$ and so in $L^1(\Omega, Q^{\pi})$ being $\frac{dQ^{\pi}}{dP}$ bounded. Then proposition 4.7 applies to state that $\xi - \int_0^{\cdot} \pi_t d\left[\xi\right]_t$ is a $\mathbb{G}$-martingale under Q^{π}. Using Girsanov theorem, chapter 6 of [22], we get the necessity condition. As far as the converse is concerned, we observe that, thanks to the hypotheses on $\mathcal{A}$, if $\xi - \int_0^{\cdot} \pi_t d\left[\xi\right]_t$ is a $\mathbb{G}$-martingale, then for every θ in $\mathcal{A}$, the process $\int_0^{\cdot} \theta_t d^{-}\left(\xi_t - \int_0^t \pi_s d\left[\xi\right]_s\right)$ is a $\mathbb{G}$-martingale starting at zero with zero expectation. This concludes the proof. □

Proposition 4.32. *Suppose that there exists a measurable process $(\gamma_t, 0 \leq t \leq 1)$ such that the process $\xi - \int_0^{\cdot} \gamma_t d\left[\xi\right]_t$ is an $\mathcal{A}$-martingale. Assume, furthermore, the existence of a sequence of processes $\left(\theta^n\right)_{n \in \mathbb{N}} \subset \mathcal{A}$ with*

$$\lim_{n \to +\infty} \mathbb{E}\left[\int_0^1 \left|\theta_t^n - \gamma_t\right|^2 d\left[\xi\right]_t\right] = 0.$$

If γ belongs to $\mathcal{A}$ then γ is optimal for $\theta \mapsto \mathbb{E}\left[L^{\theta}\right]$. Moreover if there exists an optimal process π, then $d\left|[\xi]\right|\left\{t \in [0,1), \gamma_t \neq \pi_t\right\} = 0$, almost surely.

Proof. Using proposition 4.30 we deduce that a process π is optimal for f if and only if the process $\int_0^{\cdot}(\gamma_t - \pi_t)d\,[\xi]_t$ is an $\mathcal{A}$-martingale under P. Then π is optimal if and only if for every θ is in $\mathcal{A}$ it holds: $\mathbb{E}\left[\int_0^{\cdot}\theta_t(\gamma_t - \pi_t)d\,[\xi]_t\right] = 0$. This permits to achieve immediately the end of the proof. □

4.3.3. An example of $\mathcal{A}$-martingale and a related optimization problem

We illustrate a setting where proposition 4.32 applies. It will be deduced by [18]. In that paper the authors study a particular case of the optimization problem considered in proposition 4.32. As process ξ they take a Brownian motion W, and they find sufficient conditions in order to have existence of a process γ such that $W - \int_0^{\cdot}\gamma_t dt$ is an $\mathcal{A}$-martingale, being $\mathcal{A}$ some specific set we shall clarify later. To get their goal, they consider an anticipating setting and combine Malliavin calculus with substitution formulae, the anticipation being generated by a random variable possibly depending on the whole trajectory of W.

We work into the specific framework of subsection 3.2.

Assumption 4.33. *We suppose the existence of a random variable L in $\mathbb{D}^{1,2}$, satisfying the following assumption:*

1. $\int_{\mathbb{R}}\mathbb{E}\left[|L|^2 I_{\{0\le x\le L\}\cup\{0\ge x\ge L\}}\right]dx < +\infty$;
2. *for a.a. t in $[0,1]$ the process*

$$I(\cdot,t,L) := I_{[t,1]}(\cdot)I_{\left\{\int_t^1 (D_sL)^2 ds>0\right\}}\left(\int_t^1 (D_sL)^2 ds\right)^{-1}(D_tL)(D_{\cdot}L)$$

belongs to $Dom\delta$ and there exists a $\mathcal{P}(\mathbb{F})\times\mathcal{B}(\mathbb{R})$-measurable random field $(h(t,x), 0\le t\le 1, x\in\mathbb{R})$ such that $h(\cdot,L)$ belongs to $L^2(\Omega\times[0,1])$ and

$$\mathbb{E}\left[\int_0^1 I(u,t,L)dW_u \mid \mathcal{F}_t\vee\sigma(L)\right] = h(t,L), \quad 0\le t\le 1.$$

Let $\Theta(L)$ be the set of processes $(\theta_t, 0\le t<1)$ such that there exists a random field $(u(t,x), 0\le t\le 1, x\in\mathbb{R})$ with $\theta_t = u(t,L)$, $0\le t<1$ and

$$\begin{cases} u(t,\cdot)\in C^1(\mathbb{R})\ \forall\ 0\le t\le 1. \\ \int_{-n}^{n}\int_0^1(\partial_x u(t,x))^2 dtdx < +\infty, \forall n\in\mathbb{N}\ a.s.. \\ \mathbb{E}\left[\int_{\mathbb{R}}\left(\int_0^1(\partial_x u(t,x))^2 dt\right)^2 dx + \int_0^1(u(t,0))^2 dt\right] < +\infty. \\ \mathbb{E}\left[\int_0^1(\partial_x u(t,L))^2(D_tL)^2 dt + \left(\int_0^1(\partial_x u(t,L))^2 dt\right)\left(\int_0^1(D_tL)^2 dt\right)\right] < +\infty. \end{cases}$$

Suppose that $\mathcal{A}$ equals $\Theta(L)$. With the specifications above we have the following.

Corollary 4.34. *Let b be a process in $L^2(\Omega\times[0,1])$, such that $h(\cdot,L)+b$ belongs to the closure of $\mathcal{A}$ in $L^2(\Omega\times[0,1])$. There exists an optimal process π in $\mathcal{A}$ for the function*

$$\theta \mapsto \mathbb{E}\left[\int_0^1 \theta_t d^-\left(W_t+\int_0^t b_s ds\right)-\frac{1}{2}\int_0^1 \theta_t^2 dt\right]$$

if and only if $h(\cdot,L)+b$ belongs to $\mathcal{A}$ and $h(\cdot,L)+b=\pi$.

Proof. It is clear that $\mathcal{A}$ is a real linear set of measurable and with bounded paths processes verifying condition 1. of assumption 4.26. Proposition 2.8 of [18] shows that every θ in $\mathcal{A}$ is in $L^2(\Omega\times[0,1])$, that it is W-improperly forward integrable and that the improper integral belongs to $L^2(\Omega)$. In particular, condition 2. of assumption 4.26 is verified. Furthermore, the proof of theorem 3.2 of [18] implicitly shows that the process $W-\int_0^{\cdot} h(t,L)dt$, is a $\mathcal{A}$-martingale. This implies that $W+\int_0^{\cdot} b_t dt-\int_0^{\cdot}\gamma_t dt$, with $\gamma=h(\cdot,L)+b$, is an $\mathcal{A}$-martingale. The end of the proof follows then by proposition 4.32. □

5. The market model

We consider a market offering two investing possibilities in the time interval $[0,1]$. Prices of the two traded assets follow the evolution of two stochastic processes $\left(S_t^0, 0\le t\le 1\right)$ and $\left(S_t, 0\le t\le 1\right)$. We assume that

$$S^0=\left(\exp(V_t), 0\le t\le 1\right),$$

where $(V_t, 0\le t\le 1)$ is a positive process starting at zero with bounded variation, and S is a continuous strictly positive process, with finite quadratic variation.

Remark 5.1. *If $V=\int_0^{\cdot} r_s ds$, being $(r_t, 0\le t\le 1)$ the short interest rate, S^0 represents the price process of the so called money market account. Here we do not assume that V is a riskless asset, being that assumption not necessary to develop our calculus. We only suppose that S^0 is less risky then S.*

Assuming that S has a finite quadratic variation is not restrictive at least for two reasons.

Consider a market model involving an inside trader: that means an investor having additional informations with respect to the honest agent. Let $\mathbb{F}$ and $\mathbb{G}$ be the filtrations representing the information flow of the honest and the inside investor, respectively. Then it could be worthwhile to demand the absence of free lunches with vanishing risk (FLVR) among all simple $\mathbb{F}$-predictable strategies. Under the hypothesis of absence of (FLVR), by theorem 7.2, page 504 of [7], S is a semimartingale on the underlying probability space $(\Omega,P,\mathbb{F})$. On the other hand S could fail to be a $\mathbb{G}$-semimartingale, since (FLVR) possibly exist for the insider. Nevertheless, the inside investor is still allowed to suppose that S has finite quadratic variation thanks to proposition 2.7.

Secondly, as already specified in the introduction, if we want to include S as a self-financing-portfolio, we have to require that $\int_0^{\cdot} Sd^-S$ exists. This is equivalent to assume that S has finite quadratic variation, see proposition 4.1 of [28].

5.1. Portfolio strategies

We assume the point of view of an investor whose flow of information is modeled by a filtration $\mathbb{G}= (\mathcal{G}_t)_{t\in[0,1]}$ of $\mathcal{F}$, which satisfies the usual assumptions.

We denote with $C_b^-([0,1))$ the set of processes which have paths being left continuous and bounded on each compact set of $[0,1)$.

Definition 5.2. *A **portfolio strategy** is a couple of $\mathbb{G}$-adapted processes $\phi = \left(\left(h_t^0, h_t\right), 0 \le t < 1\right)$. The market value X of the portfolio strategy ϕ is the so called **wealth process** $X = h^0S^0 + hS$.*

We stress that there is no point in defining the portfolio strategy at the end of the trading period, that is for $t = 1$. Indeed, at time 1, the agent has to liquidate his portfolio.

Definition 5.3. *A portfolio strategy $\phi = \left(h^0, h\right)$ is **self-financing** if both h^0 and h belong to $C_b^-([0,1))$, the process h is locally S-forward integrable and its wealth process X verifies*

$$X = X_0 + \int_0^{\cdot} h_t^0 dS_t^0 + \int_0^{\cdot} h_t d^- S_t. \tag{13}$$

The interpretation of the first two items is straightforward: h^0 and h represent, respectively, the number of shares of S^0 and S held in the portfolio; X is its market value. The self-financing condition (13) seems to be an appropriate formalization of the intuitive idea of trading strategy not involving exogenous sources of money. Among its justifications we can include the following ones.

As already explained in the introduction, the discrete time version of condition (13) reads as the classical self-financing condition. Furthermore, if S is a $\mathbb{G}$-semimartingale, forward integrals of $\mathbb{G}$-adapted processes with left continuous and bounded paths, agree with classical Itô integrals, see proposition 2.8 and 2.7.

In the sequel we will choose as *numéraire* the positive process S^0. That means that prices will be expressed in terms of S^0. We will denote with $\widetilde{Y}$ the value of a stochastic process $(Y_t, 0 \le t \le 1)$ *discounted* with respect to S^0 : $\widetilde{Y}_t = Y_t(S_t^0)^{-1}$, for every $0 \le t \le 1$.

The following lemma shows that, as well as in a semimartingale model, a portfolio strategy which is self-financing is uniquely determined by its initial value and the process representing the number of shares of S held in the portfolio. We remark that previous definitions and considerations can be made without supposing that the investor is able to observe prices of S and S^0. However, we need to make this hypothesis for the following characterization of self-financing portfolio strategies.

Assumption 5.4. *From now on we suppose that S and S^0 are $\mathbb{G}$-adapted processes.*

Lemma 5.5. *Let $\phi = \left(\left(h_t^0, h_t\right), 0 \leq t < 1\right)$ be a couple of $\mathbb{G}$-adapted processes in $C_b^-([0,1))$. Suppose that h is locally S-forward integrable. Then the portfolio strategy ϕ is self-financing if and only if its discounted wealth process $\widetilde{X}$ verifies*

$$\widetilde{X} = X_0 - \int_0^{\cdot} e^{-V_t} h_t S_t dV_t + \int_0^{\cdot} e^{-V_t} d^- \int_0^t h_s d^- S_s. \tag{14}$$

On the other hand, let $(h_t, 0 \leq t < 1)$ be a $\mathbb{G}$-adapted process in $C_b^-([0,1))$, which is locally S-forward integrable, and X_0 be a $\mathcal{G}_0$-random variable. Then the couple

$$\phi = \left(\left((X_t - h_t S_t)(S_t^0)^{-1}, h_t\right), 0 \leq t < 1\right),$$

with X defined as in equality (14), is a self-financing portfolio strategy with wealth process X.

Proof. Regarding the first part of the statement we observe that corollary 2.9 and equality $X = h^0 B + hS$ imply the equivalence between equalities (14) and (13).

Let h, X_0 and X be as in the second part of the statement. It is clear that $h^0 = \left((X_t - h_t S_t)\,(S_t^0)^{-1}, 0 \leq t < 1\right)$ is $\mathbb{G}$-adapted and belongs to $C_b^-([0,1))$. By construction, the wealth process corresponding to the strategy $\phi = (h^0, h)$ is equal to X. The conclusion follows by the first part of the statement. □

Remark 5.6. *Suppose that (h^0, h) is a self-financing portfolio strategy with h locally forward integrable with respect to $\widetilde{S}$. Corollary 2.9 and previous lemma imply that its discounted wealth process $\widetilde{X}$ can be also be rewritten in the following way*

$$\widetilde{X} = X_0 + \int_0^{\cdot} h_t d^- \widetilde{S}_t + R,$$

with

$$R = \int_0^{\cdot} e^{-V_t} d^- \int_0^t h_s d^- S_s - \int_0^{\cdot} h_t d^- \int_0^t e^{-V_s} d^- S_s.$$

Lemma 5.5 leads to conceive the following definition.

Definition 5.7.

1. *A **self-financing portfolio** is a couple (X_0, h) of a $\mathcal{G}_0$-measurable random variable X_0, and a process h in $C_b^-([0,1))$ which is $\mathbb{G}$-adapted and locally S-forward integrable.*
2. *The discounted wealth process $\widetilde{X}$ of the self financing portfolio (X_0, h), and the number of shares h^0 of S^0 held in that portfolio are given by*

$$\begin{cases} \widetilde{X} = X_0 - \int_0^{\cdot} e^{-V_t} h_t S_t dV_t + \int_0^{\cdot} e^{-V_t} d^- \int_0^t h_s d^- S_s \\ h^0 = (X - hS)(S^0)^{-1}. \end{cases}$$

3. *In the sequel we let us employ the term **portfolio** to denote the process h, in a self-financing portfolio, representing the number of shares of S held. Without further specifications the initial wealth of an investor will be assumed to be equal to zero.*

Lemma 5.5 and remark 5.6 immediately imply the following.

Corollary 5.8. *Let (X_0, h) be a self-financing portfolio such that h is locally $\widetilde{S}$-forward integrable and $\int_0^\cdot e^{-V_t} d^- \int_0^t h_s d^- S_s = \int_0^\cdot h_t d^- \int_0^t e^{-V_s} d^- S_s$. Then its discounted value $\widetilde{X}$ verifies the equality $\widetilde{X} = X_0 + \int_0^\cdot h d^- \widetilde{S}$.*

Remark 5.9. *If S is a $\mathbb{G}$-semimartingale, the hypothesis required on h in previous remark is always verified. Indeed, forward integrals coincide with classical Itô integrals for which the associative property holds true, see proposition 2.7.*

Some conditions to insure the existence of chain-rule formulae, when the semimartingale property of the integrator process fails to hold, can be found in section 3. For more informations about this topic we also refer to [10] and [9].

Assumption 5.10. *We assume the existence of a real linear space of portfolios $\mathcal{A}$, that is of $\mathbb{G}$-adapted processes h belonging to $C_b^-([0,1))$, which are locally S-forward integrable. The set $\mathcal{A}$ will represent the set of all **admissible strategies** for the investor.*

We proceed furnishing examples of sets behaving as the set $\mathcal{A}$ in assumption 5.10. They correspond to the examples discussed in section 3.

5.1.1. Admissible strategies via Itô fields

We refer the reader to subsection 3.1 for notations and definitions.

The following proposition is a straightforward consequence of proposition 3.7.

Proposition 5.11. *Let $\mathcal{A}$ be the set of processes $(h_t, 0 \leq t < 1)$ such that for every $0 \leq t < 1$ the process in $hI_{[0,t]}$ belongs to $\mathcal{S}(\mathcal{C}_S^1(\mathbb{G}))$. Then $\mathcal{A}$ is a real linear space satisfying the hypotheses of assumption 5.10.*

5.1.2. Admissible strategies via Malliavin calculus

For this example we refer to subsection 3.2. We recall that there, W was a real valued Wiener process defined on the canonical probability space $(\Omega, \mathbb{F}, \mathcal{F}, P)$. Regarding the price of S we make the following assumption.

Assumption 5.12. *We suppose that $S = S_0 \exp\left(\int_0^\cdot \sigma_t dW_t + \int_0^\cdot \left(\mu_t - \frac{1}{2}\sigma_t^2\right) dt\right)$, where μ and σ are $\mathbb{F}$-adapted, μ belongs to $L^{1,q}$ for some $q > 4$, σ has bounded and left continuous paths, it belongs $L_-^{1,2} \cap L^{2,2}$ and the random variable*

$$\sup_{t\in[0,1]} \left(|\sigma_t| + \sup_{s\in[0,1]} |D_s\sigma_t| \sup_{s,u\in[0,1]} |D_s D_u \sigma_t| \right)$$

is bounded.

Remark 5.13. *By remark of page 32, section 1.2 of [19] σ is in $L_-^{1,2}$ and $D^-\sigma = 0$.*

Using remarks 3.9 and 3.12, lemma 3.15 and lemma 3.16, it is not difficult to prove that the process $\log(S)$ belongs to $L_-^{1,q}$.

Proposition 5.14. *Let $\mathcal{A}$ be the set of all $\mathbb{G}$-adapted processes h in $C_b^-([0,1))$, such that for every $0 \leq t < 1$, the process $hI_{[0,t]}$ belongs to $L_-^{1,p}$, for some $p > 4$. Then $\mathcal{A}$ is a real linear space satisfying the hypotheses of assumption 5.10.*

Proof. Let h be in $\mathcal{A}$. We set $A = \log(S) - \log(S_0) + \frac{1}{2}\int_0^{\cdot} \sigma_t^2 dt = \int_0^{\cdot} \sigma_t dW_t + \int_0^{\cdot} \mu_t dt$. We recall that, thanks to lemma 2.12, for every $0 \leq t < 1$, $hI_{[0,t]}$ is S-forward integrable if and only if $hI_{[0,t]}S$ is forward integrable with respect to A. Let $0 \leq t < 1$, be fixed. Each component of the vector process $u = \big(hI_{[0,t]}, \log(S)\big)$ belongs to $L_-^{1,p}$ for some $p > 4$ and it has left continuous and bounded paths. We can thus apply proposition 3.21 to state that $hI_{[0,t]}S$ is forward integrable with respect to $\int_0^{\cdot} \sigma_t dW_t$. This implies that $hI_{[0,t]}S$ is A-forward integrable. Letting t vary in $[0,1)$ we find that h is S-improperly integrable and we get the end of the proof. □

5.1.3. Admissible strategies via substitution

We consider the setting of subsection 3.3. More precisely, we assume the existence of a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$ on $(\Omega, \mathcal{F}, P)$, with $\mathcal{F}_1 = \mathcal{F}$, and of an $\mathcal{F}$-measurable random variable L with values in $\mathbb{R}^d$ such that $\mathcal{G}_t = (\mathcal{F}_t \vee \sigma(L))$, for every $0 \leq t \leq 1$. We suppose that $\mathbb{G}$ is right continuous. We assume that S and S^0 are $\mathbb{F}$-adapted, and that S is an $\mathbb{F}$-semimartingale.

We observe that this situation arises when the investor trades as an ***insider***, that is having an extra information about prices, at time 0, represented by the random variable L.

Proposition 5.15. *Let $\mathcal{A}$ be the set of processes h such that, for every $0 \leq t < 1$, the process $hI_{[0,t]}$ belongs to $\mathcal{S}(\mathcal{A}^{p,\gamma}(L))$, for some $p > 1$ and $\gamma > d$. Then $\mathcal{A}$ satisfies the hypotheses of assumption 5.10.*

Proof. Processes in $\mathcal{A}$ are clearly $\mathbb{G}$-adapted and in $C_b^-([0,1))$. The end of the proof is a consequence of proposition 3.37 and remark 3.28. □

The following lemma shows that is not so reductive to restrict the class of possible portfolio strategies to the collection of sets $(\mathcal{S}(\mathcal{A}^{p,\gamma}(L)), p > 1, \gamma > d)$.

Lemma 5.16. *Let $(\pi_t, 0 \leq t < 1)$ be a bounded $\mathcal{P}^{\mathbb{G}}$-measurable process. Then there exists a $\mathcal{P}^{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R})$-measurable function $\big(h(t,x), 0 \leq t < 1, x \in \mathbb{R}^d\big)$, such that $\pi = h(\cdot, L)$, almost surely.*

Proof. Define $L^{0,\mathcal{P}^{\mathbb{F}}}$ as the set of all functions $(h(t,x), 0 \leq t < 1, x \in \mathbb{R}^d)$ which are $\mathcal{P}^{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. Set

$$\mathcal{M} = \Big\{ u : \Omega \times [0,1) \to \mathbb{R}, \mid \exists h \in L^{0,\mathcal{P}^{\mathbb{F}}}, \text{ s.t. } h(\cdot, L) = u \text{ a.s.} \Big\}.$$

The set $\mathcal{M}$ is a *monotone vector space*, see definition in chapter 1 of [22]. Indeed, it is a linear vector space of bounded real functions containing all constants and, if $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence of positive random elements in $\mathcal{M}$, with $u = \sup_{n\in\mathbb{N}} u_n$ bounded, then u belongs to $\mathcal{M}$. In fact $h = sup_n h_n$ is still in $L^{0,\mathcal{P}^{\mathbb{F}}}$ and $u = h(\cdot, L)$. Consider the set $\mathcal{S}^{\mathbb{G}}$ of all $\mathcal{P}^{\mathbb{G}}$-measurable processes of the form $u = I_{\{0\}}h_0(L)f_0 + \sum_{i=0}^{k-2} I_{(t_i,t_{i+1}]}h_i(L)f_i + I_{(t_{k-1},1)}h_{k-1}(L)f_{k-1}$, where $0 = t_0 < t_1 < ... < t_k = 1$, and h_i is $\mathcal{B}(\mathbb{R})$-measurable and bounded, f_i is $\mathcal{F}_{t_i}$-measurable and bounded, for every $i = 0, ..., k$. It is clear that $\mathcal{S}^{\mathbb{G}}$ is stable with respect to multiplication. Moreover $\sigma\left(\mathcal{S}^{\mathbb{G}}\right)$ contains the σ-algebra generated by all bounded and $\mathcal{P}^{\mathbb{G}} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable function. We can thus apply theorem 8 of [22] to get the result. □

5.2. Completeness and arbitrage: $\mathcal{A}$-martingale measures

Definition 5.17. *Let h be a self financing portfolio in $\mathcal{A}$, which is S-improperly forward integrable and X its wealth process. Then h is an $\mathcal{A}$-**arbitrage** if $X_1 = \lim_{t\to 1} X_t$ exists almost surely, $P(\{X_1 \geq 0\}) = 1$ and $P(\{X_1 > 0\}) > 0$.*

Definition 5.18. *If there are no $\mathcal{A}$-arbitrages we say that the market is $\mathcal{A}$-**arbitrage free**.*

Definition 5.19. *A probability measure $Q \sim P$ is said $\mathcal{A}$-**martingale measure** if under Q the process $\widetilde{S}$ is an $\mathcal{A}$-martingale according to definition 4.1.*

We will need the following assumption.

Assumption 5.20. *Suppose that for all h in $\mathcal{A}$ the following conditions hold.*

1. *The process $e^V h$ belongs to $\mathcal{A}$.*
2. *h is $\widetilde{S}$-improperly forward integrable and*

$$\int_0^{\cdot} e^{-V_t} d^- \int_0^t h_s d^- S_s = \int_0^{\cdot} h_t e^{-V_t} d^- S_t = \int_0^{\cdot} h_t d^- \int_0^t e^{-V_s} d^- S_s. \quad (15)$$

For the following proposition the reader should keep in mind the notation in equality (8). We omit its proof which is a direct application of corollary 4.20.

Proposition 5.21. *Let $\mathcal{A} = \mathcal{A}_S$. Suppose that $d[S]_t = \sigma(t, S_t)^2 dt$, where σ satisfies assumption 4.15. If there exists a $\mathcal{A}$-martingale measure then the law of $\widetilde{S}_t$ is absolutely continuous with respect to Lebesgue measure, for every $0 \leq t \leq 1$.*

Proposition 5.22. *Under assumption 5.20, if there exists an $\mathcal{A}$-martingale measure Q, the market is $\mathcal{A}$-arbitrage free.*

Proof. Suppose that h is an $\mathcal{A}$-arbitrage. Since $\widetilde{S}$ is an $\mathcal{A}$-martingale under Q, using corollary 5.8 we find $\mathbb{E}^Q[\widetilde{X}_1] = \mathbb{E}^Q[\int_0^1 h_t d^- \widetilde{S}_t] = 0$. This contradicts the arbitrage condition $Q(\{X_1 > 0\}) > 0$. □

The proposition which follows characterizes the set of all $\mathcal{A}$-martingale measures.

Proposition 5.23. *Under assumption 5.20 the process $\widetilde{S}$ is an $\mathcal{A}$-martingale under Q, if and only if the process $S - \int_0^{\cdot} S_t dV_t$ is an $\mathcal{A}$-martingale under Q.*

Proof. If h is in $\mathcal{A}$ by assumption 5.20 we have

$$\begin{aligned}
\mathbb{E}^Q\left[\int_0^{\cdot} h_t d^-\left(S_t - \int_0^t S_s dV_s\right)\right] &= \mathbb{E}^Q\left[\int_0^{\cdot} (h_t e^{V_t}) e^{-V_t} d^-\left(S - \int_0^t S_s dV_s\right)\right] \\
&= \mathbb{E}^Q\left[\int_0^{\cdot} (h_t e^{V_t}) d^- \int_0^t e^{-V_s} d^- S_s\right] \\
&+ \mathbb{E}^Q\left[\int_0^{\cdot} (h_t e^{V_t}) d^- \int_0^t S_s de^{-V_s}\right] \\
&= \mathbb{E}^Q\left[\int_0^{\cdot} (h_t e^{V_t}) d^- \widetilde{S}_t\right] = 0.
\end{aligned}$$

□

We proceed discussing *completeness*.

Definition 5.24. *A **contingent claim** is an $\mathcal{F}$-measurable random variable. $\mathcal{L}$ will be a set of $\mathcal{F}$-measurable random variables; it will represent all the contingent claims the investor is interested in.*

Definition 5.25.
1. *A contingent claim C is said **$\mathcal{A}$-attainable** if there exists a self financing portfolio (X_0, h) with h in $\mathcal{A}$, which is S-improperly forward integrable, such that the corresponding wealth process X verifies $\lim_{t\to 1} X_t = C$, almost surely. The portfolio h is said the **replicating** or **hedging** portfolio for C, X_0 is said the **replication price** for C.*
2. *The market is said to be $(\mathcal{A}, \mathcal{L})$**-complete** if every contingent claim in $\mathcal{L}$ is attainable trough a portfolio in $\mathcal{A}$.*

Assumption 5.26. *For every $\mathcal{G}_0$-measurable random variable η, and h in $\mathcal{A}$ the process $u = h\eta$, belongs to $\mathcal{A}$.*

Proposition 5.27. *Suppose that the market is $\mathcal{A}$-arbitrage free, and that assumption* 5.26 *is realized. Then the replication price of an attainable contingent claim is unique.*

Proof. Let (X_0, h) and (Y_0, k) be two replicating portfolios for a contingent claim C, with h and k in $\mathcal{A}$, and wealth processes X and Y, respectively. We have to prove that

$$P\left(\{X_0 - Y_0 \neq 0\}\right) = 0.$$

Suppose, for instance, that $P\left(X_0 - Y_0 > 0\right) \neq 0$. We set $A = \{X_0 - Y_0 > 0\}$. By assumption 5.26, $I_A(k-h)$ is a portfolio in $\mathcal{A}$ with wealth process $I_A(Y_t - X_t)$. Since both (X_0, h) and (Y_0, k) replicate C, $\lim_{t\to 1} I_A(Y_t - X_t) = I_A(X_0 - Y_0)$, with $P(\{I_A(X_0 - Y_0 > 0)\}) > 0$. Then $I_A(k-h)$ is an $\mathcal{A}$-arbitrage and this contradicts the hypotheses. □

Proposition 5.28. *Suppose that there exists an $\mathcal{A}$-martingale measure Q. Then the following statements are true.*

1. *Under assumptions 5.20 and 5.26, the replication price of an $\mathcal{A}$-attainable contingent claim C is unique and equal to* $\mathbb{E}^Q\left[\widetilde{C} \mid \mathcal{G}_0\right]$.
2. *Let $\mathcal{G}_0$ be trivial. If Q and Q_1 are two $\mathcal{A}$-martingale measures, then $\mathbb{E}^Q[\widetilde{C}] = \mathbb{E}^{Q_1}[\widetilde{C}]$, for every $\mathcal{A}$-attainable contingent claim C. In particular, if the market is $(\mathcal{A}, \mathcal{L})$-complete and $\mathcal{L}$ is an algebra, all $\mathcal{A}$-martingale measures coincide on the σ-algebra generated by all bounded discounted contingent claims in $\mathcal{L}$.*

Proof. Let (X_0, h) be a replicating $\mathcal{A}$-portfolio for C. By corollary 5.8

$$\mathbb{E}^Q\left[\widetilde{C} \mid \mathcal{G}_0\right] = X_0 + \mathbb{E}^Q\left[\int_0^1 h_t d^- \widetilde{S}_t \mid \mathcal{G}_0\right].$$

We observe that $\mathbb{E}^Q\left[\int_0^1 h_t d^- \widetilde{S}_t \mid \mathcal{G}_0\right] = 0$. In fact, if η is a $\mathcal{G}_0$-measurable random variable, then, thanks to assumption 5.26, ηh belongs to $\mathcal{A}$, so as to have $\mathbb{E}^Q\left[\left(\int_0^1 h_t d^- \widetilde{S}_t\right)\eta\right] = \mathbb{E}^Q\left[\int_0^1 \eta h_t d^- \widetilde{S}_t\right] = 0$. This implies point 1.

If $\mathcal{G}_0$ is trivial, we deduce that, if Q and Q_1 are two $\mathcal{A}$-martingale measures, $\mathbb{E}^Q[\widetilde{C}] = \mathbb{E}^{Q_1}[\widetilde{C}]$, for every $\mathcal{A}$-attainable contingent claim. The proof of the last point is then an application of theorem 8, chapter 1 of [22]. □

5.3. Hedging

In this part of the paper we price contingent claims via partial differential equations. In particular we show robustness of Black-Scholes formula for *European* and *Asian* contingent claims within a non-semimartingale model.

The following proposition generalizes a result obtained in a slight different form in [33], when the process S is supposed to be the sum of a Wiener process and a continuous process with zero quadratic variation.

We suppose here that $d[S]_t = \sigma(t, S_t)^2 S_t^2 dt$ and $dV_t = rdt$, with $r > 0$ and $\sigma : [0,1] \times (0, +\infty) \to \mathbb{R}$.

Proposition 5.29. *Let ψ be a function in $C^0(\mathbb{R})$. Suppose that there exists $(v(t,x), 0 \le t \le 1, x \in \mathbb{R})$ of class $C^{1,2}([0,1) \times \mathbb{R}) \cap C^0([0,1] \times \mathbb{R})$, which is a solution of the following Cauchy problem*

$$\begin{cases} \partial_t v(t,y) + \frac{1}{2}(\widetilde{\sigma}(t,y))^2 y^2 \partial^{(2)}_{yy} v(t,y) &= 0 \quad \text{on } [0,1) \times \mathbb{R} \\ v(1,y) &= \widetilde{\psi}(y), \end{cases} \tag{16}$$

where

$$\begin{cases} \widetilde{\sigma}(t,y) = \sigma(t, ye^{rt}) & \forall (t,y) \in [0,1] \times \mathbb{R}, \\ \widetilde{\psi}(y) = \psi(ye^r)e^{-r} & \forall y \in \mathbb{R}. \end{cases}$$

Set

$$h_t = \partial_y v(t, \widetilde{S}_t), \quad 0 \le t < 1, \quad X_0 = v(0, S_0).$$

Then (X_0, h) is a self-financing portfolio replicating the contingent claim $\psi(S_1)$.

Proof. Assumption 5.4 tells us that h is a $\mathbb{G}$-adapted process in $C_b^-([0,1))$. By proposition 2.11, h is locally $\widetilde{S}$-forward integrable. Combining lemma 2.12 and proposition 2.11, it is possible to prove that

$$\int_0^{\cdot} e^{-V_t} d^- \int_0^t h_s d^- S_s = \int_0^{\cdot} h_t e^{-V_t} d^- S_t = \int_0^{\cdot} h_t d^- \int_0^t e^{-V_s} d^- S_s.$$

Similar arguments were used in [10], corollary 23. Corollary 5.8 implies then that its discounted wealth process verifies

$$\widetilde{X} = X_0 + \int_0^{\cdot} h d^- \widetilde{S}. \tag{17}$$

On the other hand by point 2. of proposition 3.7

$$[\tilde{S}] = \int_0^{\cdot} \widetilde{S}_s^2 \widetilde{\sigma}(s, \widetilde{S}_s)^2 ds. \tag{18}$$

Applying proposition 2.11, recalling equation (16), equalities (17) and (18) we find that

$$\widetilde{X}_t = v(t, \widetilde{S}_t), \quad \forall 0 \le t < 1.$$

In particular $X_0 + \lim_{t\to 1} \int_0^t h_s d^- \widetilde{S}_s$ exists finite and coincides with $v(1, \widetilde{S}_1) = \widetilde{\psi}(\widetilde{S_1}) = \psi(S_1)e^{-r}$. □

Remark 5.30. *In particular, under some minimal regularity assumptions on σ and no degeneracy, the market is $(\mathcal{A}_S, \mathcal{L})$-complete, if $\mathcal{L}$ equals the set of all contingent claims of type $\psi(S_1)$ with ψ in $C^0(\mathbb{R})$ with linear growth.*

The result of proposition 5.29 can also be adapted to hedge Asian contingent claims, that is contingent claims C depending on the mean of S over the traded period: $C = \psi\left(\frac{1}{S_1}\left(\int_0^1 S_t dt\right)\right)$, for some ψ in $C^0(\mathbb{R})$.

Proposition 5.31. *Suppose that $\sigma(t,x) = \sigma$, for every (t,x) in $[0,1]\times\mathbb{R}$, for some $\sigma > 0$. Let ψ be a function in $C^0(\mathbb{R})$ and $v(t,y)$ a continuous solution of class $C^{1,2}([0,1)\times\mathbb{R}) \cap C^0([0,1]\times\mathbb{R})$ of the following Cauchy problem*

$$\begin{cases} \frac{1}{2}\sigma^2 y^2 \partial_{yy}^{(2)} v(t,y) + (1-ry)\partial_y v(t,y) + \partial_t v(t,y) &= 0, \quad \text{on } [0,1)\times\mathbb{R} \\ v(1,y) &= \psi(y). \end{cases}$$

Set $Z_t = \int_0^t S_s ds - K$, for some $K > 0$, $X_0 = v(0, \frac{K}{S_0})S_0$ and $h_t = v(t, \frac{Z_t}{S_t}) - \partial_y v(t, \frac{Z_t}{S_t})\frac{Z_t}{S_t}$, for all $0 \le t \le 1$. Then (X_0, h) is a self-financing portfolio which replicates the contingent claim $\psi\left(\frac{1}{S_1}\left(\int_0^1 S_t dt - K\right)\right) S_1$.

Proof. We set $\xi_t = \frac{Z_t}{S_t}, 0 \le t \le 1$. Applying proposition 2.11 to the function $u(t,z,s) = v(t, \frac{z}{s}e^{-rt})s$ and using the equation fulfilled by v we can expand the process $(e^{-rt}v(t,\xi_t)S_t, 0 \le t < 1)$ as follows:

$$u(t, Z_t, \widetilde{S}_t) = v\left(t, \xi_t\right) \tilde{S}_t = v\left(0, \xi_0\right) S_0 + \int_0^t h_t d^- \widetilde{S}_t. \tag{19}$$

By arguments which are similar to those used in the proof of previous proposition, it is possible to show that h is a self-financing portfolio and that (19) implies that $u(t, Z_t, \widetilde{S}_t) = \widetilde{X}_t$ for every $0 \leq t < 1$. Therefore $\lim_{t\to 1} \widetilde{X}_t$ is finite and equal to $\psi(\xi_1) S_1 e^{-r}$. This concludes the proof. □

5.4. Utility maximization

5.4.1. Formulation of the problem

We consider the problem of maximization of expected utility from terminal wealth starting from initial capital $X_0 > 0$, being X_0 a $\mathcal{G}_0$-measurable random variable. We define the function $U(x)$ modeling the utility of an agent with wealth x at the end of the trading period. The function U is supposed to be of class $C^2((0, +\infty))$, strictly increasing, with $U'(x)x$ bounded.

We will need the following assumption.

Assumption 5.32. *The utility function U verifies $\frac{U''(x)x}{U'(x)} \leq -1, \quad \forall x > 0$.*

A typical example of function U verifying assumption 5.32 is $U(x) = \log(x)$.

We will focus on portfolios with strictly positive value. As a consequence of this, before starting analyzing the problem of maximization, we show how it is possible to construct portfolio strategies when only positive wealth is allowed.

Definition 5.33. *For simplicity of calculation we introduce the process*

$$A = \log(S) - \log(S_0) + \frac{1}{2}\int_0^{\cdot} \frac{1}{S_t^2} d\left[S\right]_t .$$

Lemma 5.34. *Let $\theta = (\theta_t, 0 \leq t < 1)$ be a $\mathbb{G}$-adapted process in $C_b^-([0,1))$ such that*

1. *θ is A-improperly forward integrable.*
2. *The process $A^\theta = \int_0^{\cdot} \theta_s d^- A_s$ has finite quadratic variation.*
3. *If X^θ is the process defined by*

$$X^\theta = X_0 \exp\left(\int_0^{\cdot} \theta_t d^- A_t + \int_0^{\cdot} (1-\theta_t)\, dV_t - \frac{1}{2}\left[A^\theta\right]\right),$$

then $\int_0^{\cdot} X_t^\theta \theta_t d^- A_t$ and $\int_0^{\cdot} X_t^\theta d^- \int_0^t \theta_s d^- A_s$ improperly exist and

$$\int_0^{\cdot} X_t^\theta \theta_t d^- A_t = \int_0^{\cdot} X_t^\theta d^- \int_0^t \theta_s d^- A_s. \tag{20}$$

Then the couple (X_0, h), with $h_t = \frac{\theta X_t^\theta}{S_t}$, $0 \leq t < 1$, is a self-financing portfolio with strictly positive wealth X^θ. In particular, $\lim_{t\to 1} X_t^\theta = X_1^\theta$ exists and it is strictly positive.

Proof. Thanks to lemma 2.12 h is locally S-forward integrable and $\int_0^\cdot h_t d^- S_t = \int_0^\cdot \theta_t X_t^\theta d^- A_t$. Applying corollary 2.9, proposition 2.11, and using hypothesis 3., $\widetilde{X^\theta}$ can be rewritten in the following way:

$$\begin{aligned} \widetilde{X_t^\theta} &= X_0 + \int_0^\cdot \widetilde{X_t^\theta} d^- \int_0^t \theta_s d^- A_s - \int_0^\cdot \widetilde{X_t^\theta} \theta_t dV_t \\ &= X_0 + \int_0^\cdot e^{-V_t} d^- \int_0^t h_s d^- S_s - \int_0^\cdot e^{-V_t} h_t S_t dV_t. \end{aligned} \tag{21}$$

Remark 5.6 tells us that X^θ is the wealth of the self-financing portfolio (X_0, h). □

Remark 5.35. *The process θ in previous lemma represents the **proportion** of wealth invested in S.*

Remark 5.36. *Let θ be as in lemma 5.34. Then, for every $0 \leq t < 1$, X is, indeed, the unique solution, on $[0,t]$, of equation*

$$X^\theta = X_0 + \int_0^\cdot X_t^\theta d^- \left(\int_0^t \theta_s d^- A_s + \int_0^t (1-\theta_s) dV_s - \frac{1}{2} \left[A^\theta \right]_t \right).$$

In fact, uniqueness is insured by corollary 5.5 of [28]. It is important to highlight that, without the assumption on θ regarding the chain rule in equality (20), we cannot conclude that X^θ solves equation (21). However we need to require that X^θ solves the latter equation to interpret it as the value of a portfolio whose proportion invested in S is constituted by θ. In the sequel we will construct, in some specific settings, classes of processes defining proportions of wealth as in lemma 5.34. We will consider, in particular, two cases already contemplated in [3] and [18]. Our definitions of those sets will result more complicated than the ones defined in the above cited papers. This happens because, in those works, the chain rule problem arising when the forward integral replaces the classical Itô integral is not clarified.

Assumption 5.37. *We assume the existence of a real linear space $\mathcal{A}^+$ of $\mathbb{G}$-adapted processes $(\theta_t, 0 \leq t < 1)$ in $C_b^-([0,1))$, such that*

1. *θ verifies condition 1., 2. and 3. of lemma 5.34, and $\left[A^\theta\right] = \int_0^\cdot \theta_t^2 d\left[A\right]_t$.*
2. *$\theta I_{[0,t]}$ belongs to $\mathcal{A}^+$ for every $0 \leq t < 1$.*

For every θ in $\mathcal{A}^+$ we denote with Q^θ the probability measure defined by:

$$\frac{dQ^\theta}{dP} = \frac{U'(X_1^\theta) X_1^\theta}{\mathbb{E}\left[U'(X_1^\theta) X_1^\theta\right]}.$$

The utility maximization problem consists in finding a process π in $\mathcal{A}^+$ maximizing expected utility from terminal wealth, i.e.:

$$\pi = \arg\max_{\theta \in \mathcal{A}^+} \mathbb{E}\left[U(X_1^\theta)\right]. \tag{22}$$

Problem (22) is not trivial because of the uncertain nature of the processes A and V and the non zero quadratic variation of A. Indeed, let us suppose that $[A] = 0$ and that both A and V are deterministic. Then, it is sufficient to consider

$$\sup_{\lambda \in \mathbb{R}} \mathbb{E}\left[U(X_1^\lambda)\right] = \lim_{x \to +\infty} U(x),$$

and remind that U is strictly increasing, to see that a maximum can not be realized. The problem is less clear when the term $-\frac{1}{2}\int_0^\cdot \theta_t^2 d\,[A]_t$ and a source of randomness are added.

In the sequel, we will always assume the following.

Assumption 5.38. *For every θ in $\mathcal{A}^+$,*

$$\mathbb{E}\left[\left|\int_0^1 \theta_t d^-(A_t - V_t)\right| + \frac{1}{2}\int_0^1 \theta_t^2\,[A]_t\right] < +\infty.$$

Definition 5.39. *A process π is said **optimal portfolio** in $\mathcal{A}^+$, if it is optimal for the function $\theta \mapsto \mathbb{E}\left[U(X_1^\theta)\right]$ in $\mathcal{A}^+$, according to definition 4.23.*

Remark 5.40. *Set $\xi = A - V$, $\mathcal{A} = \mathcal{A}^+$, and*

$$F(\omega, x) = U\left(X_0(\omega)e^{x + V_1(\omega)}\right), \quad (\omega, x) \in \Omega \times \mathbb{R}.$$

According to definitions of section 4.3.2, $\mathcal{A}$ satisfies assumption 4.26, the function F is measurable, almost surely in $C^1(\mathbb{R})$, strictly increasing and with bounded first derivative. If U satisfies assumption 5.32 then F is also concave. Moreover $F(L^\theta) = U(X_1^\theta)$ for every θ in $\mathcal{A}^+$.

Before stating some results about the existence of an optimal portfolio, we provide examples of sets of admissible strategies with positive wealth.

5.4.2. Admissible strategies via Itô fields

For this example the reader should keep in mind subsection 3.1.

Proposition 5.41. *Let $\mathcal{A}^+$ be the set of all processes $(\theta_t, 0 \le t < 1)$ such that θ is the restriction to $[0,1)$ of a process h belonging to $\mathcal{S}(\mathcal{C}_A^1(\mathbb{G}))$. Then $\mathcal{A}^+$ satisfies the hypotheses of assumption 5.37.*

Proof. Let h be in $\mathcal{S}(\mathcal{C}_A^1(\mathbb{G}))$ and θ its restriction on $[0,1)$. It is clear that θ is in $C_b^-([0,1))$ and $\mathbb{G}$-adapted. Thanks to proposition 3.7, h is A-forward integrable, $\int_0^\cdot h_t d^- A_t$ belongs to $\mathcal{S}(\mathcal{C}_A^2(\mathbb{G}))$ and the process $\int_0^\cdot h_t d^- A_t$ has finite quadratic variation equal to $\int_0^\cdot h_t^2 d\,[A]_t$. By remark 2.5, $\int_0^\cdot h_t d^- A_t = \int_0^\cdot \theta_t d^- A_t$, and conditions 1. and 2. of lemma 5.34 are thus satisfied. Remark 3.6 implies that the process

$$\exp\left(\int_0^\cdot \theta_t d^- A_t + \int_0^\cdot (1 - \theta_s) dV_s - \frac{1}{2}\int_0^\cdot \theta_t^2 d\,[A]_t\right)$$

belongs to $\mathcal{S}(\mathcal{C}_A^1(\mathbb{G}))$. Then, by proposition 3.7, again, θ fulfills also condition 3. of lemma 5.34. By construction, $\theta I_{[0,t]}$ is an element of $\mathcal{A}^+$ for every $0 \leq t < 1$ and this concludes the proof. □

5.4.3. Admissible strategies via Malliavin calculus

We restrict ourselves to the setting of section 5.1.2. We recall that in that case $A = \int_0^\cdot \sigma_t dW_t + \int_0^\cdot \mu_t dt$. We make the following additional assumption:

$$S^0 = e^{\int_0^\cdot r_t dt},$$

with r in $L^{1,z}$ for some $z > 4$ and $\mathbb{F}$-adapted.

Proposition 5.42. *Let $\mathcal{A}^+$ be the set of all $\mathbb{G}$-adapted processes in $C_b^-([0,1))$ being the restriction on $[0,1)$ of processes h in $L_-^{1,2} \cap L^{2,2}$, such that D^-h is in $L_-^{1,2}$, and the random variable*

$$\sup_{t\in[0,1]} \left(|h_t| + \sup_{s\in[0,1]} |D_s h_t| + \sup_{s,u\in[0,1]} |D_s D_u h_t| \right)$$

is bounded.

Then $\mathcal{A}^+$ satisfies the hypotheses of assumption 5.37.

Proof. Let h be as in the hypotheses. Proposition 3.18 applies to to state that h is A-forward integrable and

$$\begin{aligned} \int_0^\cdot h_t d^- A_t &= \int_0^\cdot h_t \sigma_t d^- W_t + \int_0^\cdot h_t \mu_t dt \\ &= \int_0^\cdot h_t \sigma_t \delta W_t + \int_0^\cdot \left(h_t \mu_t + \sigma_t D_t^- h_t \right) dt. \end{aligned}$$

On the other hand, proposition 3.17 applies to obtain

$$\left[\int_0^\cdot h_t d^- A_t \right] = \left[\int_0^\cdot h_t \sigma_t \delta W_t \right] = \int_0^\cdot h_t^2 \sigma_t^2 dt.$$

In particular, if θ is the restriction of h on $[0,1)$, then θ fulfills point 1. and 2. of lemma 5.34.

Consider the vector process $\left(\int_0^\cdot h_t d^- A_t, \int_0^\cdot (1-h_t) dV_t, \int_0^\cdot h_t^2 d[A] \right)_t$. We affirm that each of its components belongs to $L_-^{1,v}$ for some $v > 4$. In fact, the first component is equal to the sum of $\int_0^\cdot h_t \sigma_t \delta W_t$ and $\int_0^\cdot \left(\sigma_t D_t^- h_t + h_t \mu_t \right) dt$; the first term of the sum belongs to $L_-^{1,p}$ by lemma 3.16, which applies thanks to remark 3.9; the second term is in $L_-^{1,q\wedge p}$ thanks to lemma 3.15; remark 3.9 and lemma 3.15 again imply that both $\int_0^\cdot (1-h_t) r_t dt$ and $\int_0^\cdot h_t^2 \sigma_t^2 dt$ belong to $L_-^{1,z}$. We can thus apply proposition 3.21 to find that

$$\int_0^\cdot X_t^h d^- \int_0^t h_s \sigma_s dW_s = \int_0^\cdot X_t^h h_t d^- \int_0^t \sigma_s dW_s,$$

with $X^h = \exp\left(\int_0^\cdot h_t d^- A_t - \int_0^\cdot (1-h_s)dV_s - \frac{1}{2}\int_0^\cdot h^2 d\,[A]_t\right)$. This permits to conclude the proof. □

5.4.4. Admissible strategies via substitution

We return here to the framework of subsection 5.1.3.

Proposition 5.43. *Let $\mathcal{A}^+$ be the set of all processes which are the restriction to $[0,1)$ of processes in $\mathcal{S}(\mathcal{A}^{p,\gamma}(L))$ for some $p > 3$ and $\gamma > 3d$. Then $\mathcal{A}^+$ satisfies the hypotheses of assumption 5.37.*

Proof. Let h be in $\mathcal{S}(\mathcal{A}^{p,\gamma}(L))$ for some $p > 3$ and $\gamma > 3d$. Proposition 3.37 insures that h is A-forward integrable, and that $\int_0^\cdot h_t d^- A_t$ has finite quadratic variation equal to $\int_0^\cdot h_t^2 d\,[A]_t$. The process

$$X^h = \exp\left(\int_0^\cdot h_t d^- A_t - \int_0^\cdot (1-h_t)dV_t - \frac{1}{2}\int_0^\cdot h_t^2 d\,[A]_t\right)$$

has bounded paths. Then, thanks to point 1. of remark 2.3, to prove that

$$\int_0^\cdot X_t^h d^- \int_0^t h_s d^- A_s = \int_0^\cdot X_t^h h_t d^- A_t, \tag{23}$$

we are allowed to replace X^h by $\psi(\log(X^h))$, being ψ a function of class $C^\infty(\mathbb{R})$ with bounded derivative. Using lemma 3.27 it is possible to show that the process $\psi\left(\log(X^h)\right)$ belongs to $\mathcal{S}(\mathcal{A}^{\frac{p}{2},\frac{\gamma}{2}}(L))$. Proposition 3.37 again let us get equality (23). □

5.4.5. Optimal portfolios and $\mathcal{A}^+$-martingale property

Adapting results contained in section 4.3.2 to the utility maximization problem, we can formulate the following propositions. We omit their proofs, being particular cases of the ones contained in that section.

Proposition 5.44. *If a process π in $\mathcal{A}^+$ is an optimal portfolio, then the process $A - V - \int_0^\cdot \pi_t d\,[A]_t$ is an $\mathcal{A}^+$-martingale under Q^π. If U fulfills assumption 5.32, then the converse holds.*

Proposition 5.45. *Suppose that $\mathcal{A}^+$ satisfies assumption $\mathcal{D}$ with respect to $\mathbb{G}$. If a process π in $\mathcal{A}^+$ is an optimal portfolio, then the process $A - V - \int_0^\cdot \beta_t d\,[A]_t$ is a $\mathbb{G}$-martingale under P, with*

$$\beta = \pi + \frac{1}{p^\pi}\frac{d\,[p^\pi, A]}{d\,[A]}, \quad \text{and} \quad p^\pi = \mathbb{E}^{Q^\pi}\left[\frac{dP}{dQ^\pi} \mid \mathcal{G}_\cdot\right].$$

If U fulfills assumption 5.32, then the converse holds.

Remark 5.46. *1. We emphasize that if $U(x) = \log(x)$, then the probability measure Q^π appearing in propositions 5.44 and 5.45 is equal to P.*

2. *In [2] it is proved that if the maximum of expected logarithmic utility over all simple admissible strategies is finite, then S is a semimartingale with respect $\mathbb{G}$. This result does not imply proposition 5.45. Indeed, we do not need to assume that our set of portfolio strategies contains the set of simple predictable admissible ones. On the contrary, we want to point out that, as soon as the class of admissible strategies is not large enough, the semimartingale property of price processes could fail, even under finite expected utility.*

Proposition 5.47. *Suppose that $U(x) = \log(x)$, x in $(0, +\infty)$. Assume that there exists a measurable process γ such that $A - V - \int_0^{\cdot} \gamma_t d[A]_t$ is an $\mathcal{A}^+$-martingale and there exists a sequence $(\theta^n)_{n\in\mathbb{N}} \subset \mathcal{A}^+$ such that*

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^1 |\theta_t^n - \gamma_t|^2 \, d[A]_t\right] = 0.$$

Then if γ belongs to $\mathcal{A}^+$ it is an optimal portfolio. On the contrary, if an optimal portfolio π exists, then $d|[A]|\{t \in [0,1), \pi_t \neq \gamma_t\} = 0$ almost surely.

5.4.6. Example

We adopt the setting of section 5.4.3 and we further assume that σ is a strictly positive real

Proposition 5.48. *If a process π is an optimal portfolio in $\mathcal{A}^+$, then the process $W - \int_0^{\cdot} \left(\frac{r_t-\mu_t}{\sigma} + \pi_t\sigma\right) dt$ is an $\mathcal{A}^+$-martingale under Q^π. If U fulfills assumption 5.32, then the converse holds.*

Proof. First of all we observe that it is not difficult to prove that $\mathcal{A}^+$ satisfies assumption 5.38. If a process π is an optimal portfolio in $\mathcal{A}^+$ then proposition 5.44 implies that the process M^π, with $M^\pi = \sigma\left(W - \int_0^{\cdot} \left(\frac{r_t-\mu_t}{\sigma} - \pi_t\sigma\right) dt\right)$, is an $\mathcal{A}^+$-martingale under Q^π. We observe that $\sigma^{-1}\mathcal{A}^+ = \mathcal{A}^+$. Therefore, $\sigma^{-1}M^\pi = W - \int_0^{\cdot} \left(\frac{r_t-\mu_t}{\sigma} + \pi_t\sigma\right) dt$ is an $\mathcal{A}^+$-martingale.

Similarly, if U satisfies assumption 5.32, the converse follows by proposition 5.44. □

Corollary 5.49. *Let $\mathcal{A}^+$ satisfy assumption $\mathcal{D}$ with respect to $\mathbb{G}$. If a process π in $\mathcal{A}^+$ is an optimal portfolio then the process $\widetilde{W} = W - \int_0^{\cdot} \alpha_t dt$ with*

$$\alpha = \pi\sigma + \frac{r-\mu}{\sigma} + \frac{1}{p^\pi}\frac{d[p^\pi, W]}{d[W]}, \quad \text{and} \quad p^\pi = \mathbb{E}^{Q^\pi}\left[\frac{dP}{dQ^\pi} \mid \mathcal{G}_\cdot\right],$$

is a $\mathbb{G}$-Brownian motion under P. If U satisfies assumption 5.32, then the converse holds.

Proof. Let π be an optimal portfolio. By proposition 4.31, the process $\widetilde{W}$ is a $\mathbb{G}$-martingale and so a $\mathbb{G}$-Brownian motion under P. □

The results concerning the example above were proved in [3]. We generalize them in two directions: we replace the geometric Brownian motion A by a finite quadratic variation process and we let the set of possible strategies vary in sets which can, a priori, exclude some simple predictable processes.

5.4.7. Example

We consider the example treated in section 4.3.3. We suppose, for simplicity, that

$$S_t = S_0 e^{\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)t}, \quad S_t^0 = e^{rt} \quad 0 \leq t \leq 1,$$

being σ, μ and r positive constants. This implies $A_t = \sigma W_t + \mu t$, and $V_t = rt$ for $0 \leq t \leq 1$. We set $\mathcal{A}^+ = \Theta(L)$.

Proposition 5.50. *Suppose that $U(x) = \log(x)$, x in $(0, +\infty)$. Suppose that $h(\cdot, L)$ belongs to the closure of $\Theta(L)$ in $L^2(\Omega \times [0,1])$. Then an optimal portfolio π exists if and only if the process $h(\cdot, L) + \int_0^{\cdot} \frac{\mu - r}{\sigma} dt$ belongs to $\Theta(L)$ and $\pi = h(\cdot, L) + \frac{\mu - r}{\sigma}$.*

Proof. The result follows from corollary 4.34. □

Sufficiency for the proposition above was shown, with more general σ, r and μ in theorem 3.2 of [18]. Nevertheless, in this paper we go further in the analysis of utility maximization problem. Indeed, besides observing that the converse of that theorem holds true, we find that the existence of an optimal strategy is strictly connected, even for different choices of the utility function, to the $\mathcal{A}^+$-semimartingale property of W. To be more precise, in that paper the authors show that an optimal process exists, under the given hypotheses, handling directly the expression of the expected utility, which has, in the logarithmic case, a nice expression. Here we reinterpret their techniques at a higher level which permits us to partially generalize those results.

References

[1] Jürgen Amendinger, Peter Imkeller, and Martin Schweizer. Additional logarithmic utility of an insider. *Stochastic Process. Appl.*, 75(2):263–286, 1998.

[2] Stefan Ankirchner and Peter Imkeller. Finite utility on financial markets with asymmetric information and structure properties of the price dynamics. *Ann. Inst. H. Poincaré Probab. Statist.*, 41(3):479–503, 2005.

[3] Francesca Biagini and Bernt Øksendal. A general stochastic calculus approach to insider trading. *Appl. Math. Optim.*, 52(2):167–181, 2005.

[4] Tomas Björk. Interest rate theory. In *Financial mathematics (Bressanone, 1996)*, volume 1656 of *Lecture Notes in Math.*, pages 53–122. Springer, Berlin.

[5] Patrick Cheridito. Arbitrage in fractional Brownian motion models. *Finance Stoch.*, 7(4):533–553, 2003.

[6] Rosanna Coviello and Francesco Russo. Non-semimartingales: stochastic differential equations and weak Dirichlet processes. *To appear. Annals of Probability*, 2005.
[7] Freddy Delbaen and Walter Schachermayer. A general version of the fundamental theorem of asset pricing. *Math. Ann.*, 300(3):463–520, 1994.
[8] Freddy Delbaen and Walter Schachermayer. The existence of absolutely continuous local martingale measures. *Ann. Appl. Probab.*, 5(4):926–945, 1995.
[9] Mohammed Errami and Francesco Russo. n-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes. *Stochastic Process. Appl.*, 104(2), 2003.
[10] Franco Flandoli and Francesco Russo. Generalized integration and stochastic ODEs. *Ann. Probab.*, 30(1):270–292, 2002.
[11] Hans Föllmer. Calcul d'Itô sans probabilités. In *Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French)*, volume 850 of *Lecture Notes in Math.*, pages 143–150. Springer, Berlin.
[12] Hans Föllmer, Philip Protter, and Albert N. Shiryayev. Quadratic covariation and an extension of Itô's formula. *Bernoulli*, 1(1-2):149–169, 1995.
[13] Hans Föllmer, Ching-Tang Wu, and Marc Yor. On weak Brownian motions of arbitrary order. *Ann. Inst. H. Poincaré Probab. Statist.*, 36(4):447–487, 2000.
[14] Axel Grorud and Monique Pontier. Comment détecter le délit d'initiés? *C. R. Acad. Sci. Paris Sér. I Math.*, 324(10):1137–1142, 1997.
[15] Paolo Guasoni. No arbitrage with transaction costs with fractional Brownian motion and beyond. *Preprint 2005.*
[16] Jean Jacod. Grossissement initial, hypothèse (H') et théorème de Girsanov. In Th. Jeulin and M. Yor, editors, *Grossissements de filtrations: exemples et applications*, volume 1118 of *Lecture Notes in Mathematics*. Springer-Verlag, Harlow, 1993.
[17] Jacod Jean. *Calcul stochastique et problèmes de martingales*, volume 714 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.
[18] Jorge A. León, Reyla Navarro, and David Nualart. An anticipating calculus approach to the utility maximization of an insider. *Math. Finance*, 13(1):171–185, 2003. Conference on Applications of Malliavin Calculus in Finance (Rocquencourt, 2001).
[19] David Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, New York, 1995.
[20] David Nualart and Étienne Pardoux. Stochastic calculus with anticipating integrands. *Probab. Theory Related Fields*, 78(4):535–581, 1988.
[21] Igor Pikovsky and Ioannis Karatzas. Anticipative portfolio optimization. *Adv. in Appl. Probab.*, 28(4):1095–1122, 1996.
[22] Philip Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1990. A new approach.
[23] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin,

second edition, 1994.

[24] L. C. G. Rogers. Arbitrage with fractional Brownian motion. *Math. Finance*, 7(1):95–105, 1997.

[25] Francesco Russo and Pierre Vallois. Forward, backward and symmetric stochastic integration. *Probab. Theory Related Fields*, 97(3):403–421, 1993.

[26] Francesco Russo and Pierre Vallois. The generalized covariation process and Itô formula. *Stochastic Process. Appl.*, 59(1):81–104, 1995.

[27] Francesco Russo and Pierre Vallois. Itô formula for C^1-functions of semimartingales. *Probab. Theory Related Fields*, 104(1):27–41, 1996.

[28] Francesco Russo and Pierre Vallois. Stochastic calculus with respect to continuous finite quadratic variation processes. *Stochastics Stochastics Rep.*, 70(1-2):1–40, 2000.

[29] Francesco Russo and Pierre Vallois. Elements of stochastic calculus via regularization. *Preprint LAGA-Paris13 2004-28, Revised version: 2005. To appear: Séminaire de probabilités*, 2005.

[30] Walter Schachermayer. Utility maximisation in incomplete markets. In *Stochastic methods in finance*, volume 1856 of *Lecture Notes in Math.*, pages 255–293. Springer, Berlin, 2004.

[31] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[32] Jeannette H. C. Woerner. Estimation of integrated volatility in stochastic volatility models. *Appl. Stoch. Models Bus. Ind.*, 21(1):27–44, 2005.

[33] Martina Zähle. Long range dependence, no arbitrage and the Black-Scholes formula. *Stoch. Dyn.*, 2(2):265–280, 2002.