



HAL
open science

FINITE DIFFERENCE SCHEMES AS A MATRIX EQUATION

Claire David, Pierre Sagaut

► **To cite this version:**

Claire David, Pierre Sagaut. FINITE DIFFERENCE SCHEMES AS A MATRIX EQUATION. 2006.
hal-00081010v1

HAL Id: hal-00081010

<https://hal.science/hal-00081010v1>

Preprint submitted on 21 Jun 2006 (v1), last revised 3 Aug 2006 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

FINITE DIFFERENCE SCHEMES AS A MATRIX EQUATION

CLAIRE DAVID AND PIERRE SAGAUT*

Abstract.

Finite difference schemes are here solved by means of a linear matrix equation. The theoretical study of the related algebraic system is exposed.

Key words. Finite difference schemes, Sylvester equation

AMS subject classifications. 15A15, 15A09, 15A23

1. Introduction: Scheme classes. Finite difference schemes used to approximate linear differential equations are usually solved by means of a recursive calculus, expensive in time and not of practical handling. We here propose a completely different approach, which consists in solving an equivalent matrix equation.

Consider the transport equation:

$$(1.1) \quad \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0, \quad x \in [0, L], \quad t \in [0, T]$$

with the initial condition $u(x, t = 0) = u_0(x)$.

PROPOSITION 1.1.

A finite difference scheme for this equation can be written under the form:

$$(1.2) \quad \alpha u_i^{n+1} + \beta u_i^n + \gamma u_i^{n-1} + \delta u_{i+1}^n + \varepsilon u_{i-1}^n + \zeta u_{i+1}^{n+1} + \eta u_{i-1}^{n-1} + \theta u_{i-1}^{n+1} + \vartheta u_{i+1}^{n-1} = 0$$

where:

$$(1.3) \quad u_l^m = u(l dx, m dt)$$

$l \in \{i-1, i, i+1\}$, $m \in \{n-1, n, n+1\}$, $j = 0, \dots, n_x$, $n = 0, \dots, n_t$, dx , dt denoting respectively the mesh size and time step ($L = n_x dx$, $T = n_t dt$).

The Courant-Friedrichs-Lewy number (*cfl*) is defined as $\sigma = cdt/dx$.

A numerical scheme is specified by selecting appropriate values of the coefficients α , β , γ , δ , ε , ζ , η , θ and ϑ in equation (1.2). Values corresponding to numerical schemes retained for the present works are given in Table 1.1.

The number of time steps will be denoted n_t , the number of space steps, n_x . In general, $n_x \gg n_t$.

The paper is organized as follows. The equivalent matrix equation is exposed in section 2. Specific properties of the involved matrices are exposed in section 2.3.

*Université Pierre et Marie Curie-Paris 6, Laboratoire de Modélisation en Mécanique, UMR CNRS 7607, Boîte courrier n°162, 4 place Jussieu, 75252 Paris cedex 05, France (david@lmm.jussieu.fr).

TABLE 1.1
Numerical scheme coefficient.

Name	α	β	γ	δ	ϵ	ζ	η	θ	ϑ
Leapfrog	$\frac{1}{2\tau}$	0	$\frac{-1}{2\tau}$	$\frac{c}{2h}$	$\frac{-c}{2h}$	0	0	0	0
Lax	$\frac{1}{\tau}$	0	0	$\frac{-1}{2\tau} + \frac{c}{2h}$	$\frac{-1}{2\tau} - \frac{c}{2h}$	0	0	0	0
Lax-Wendroff	$\frac{1}{\tau}$	$\frac{-1}{\tau} + \frac{c^2\tau}{h^2}$	0	$\frac{(1-\sigma)c}{2h}$	$\frac{-(1+\sigma)c}{2h}$	0	0	0	0
Crank-Nicolson	$\frac{1}{\tau} + \frac{c}{h^2}$	$\frac{-1}{\tau} + \frac{c}{h^2}$	0	$\frac{-c}{h^2}$	$\frac{c}{h^2}$	0	$\frac{-c}{h^2}$	$\frac{-c}{h^2}$	0

2. The Sylvester equation.

2.1. Matricial form of the finite differences problem. Let us introduce the rectangular matrix defined by:

$$(2.1) \quad U = [u_i^n]_{1 \leq i \leq n_x - 1, 1 \leq n \leq n_t}$$

THEOREM 2.1. *The problem (1.2) can be written under the following matricial form:*

$$(2.2) \quad M_1 U + U M_2 + \mathcal{L}(U) = M_0$$

where M_1 and M_2 are square matrices respectively $n_x - 1$ by $n_x - 1$, n_t by n_t , given by:

$$(2.3) \quad M_1 = \begin{pmatrix} \beta & \delta & 0 & \dots & 0 \\ \varepsilon & \beta & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \beta & \delta \\ 0 & \dots & 0 & \varepsilon & \beta \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & \gamma & 0 & \dots & 0 \\ \alpha & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \gamma \\ 0 & \dots & 0 & \alpha & 0 \end{pmatrix}$$

the matrix M_0 being given by:

$$(2.4) \quad M_0 = \begin{pmatrix} -\gamma u_1^0 - \varepsilon u_0^1 - \eta u_0^0 - \theta u_0^1 & -\varepsilon u_0^2 - \eta u_0^0 - \theta u_0^2 & \dots & \dots & -\varepsilon u_0^{n_t} - \eta u_0^{n_t-1} - \theta u_0^{n_t} \\ -\gamma u_2^0 - \eta u_1^0 - \vartheta u_2^0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\gamma u_{n_x-2}^0 & 0 & \dots & \dots & 0 \\ -\gamma u_{n_x-1}^0 - \delta u_{n_x}^1 - \eta u_{n_x-1}^0 - \zeta u_{n_x}^2 - \vartheta u_{n_x}^0 & -\delta u_{n_x}^2 - \vartheta u_{n_x}^1 - \zeta u_{n_x}^3 & \dots & \dots & -\delta u_{n_x}^{n_t} - \eta u_{n_x-1}^{n_t} - \vartheta u_{n_x}^{n_t-1} \end{pmatrix}$$

and where \mathcal{L} is a linear matricial operator which can be written as:

$$(2.5) \quad \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$$

where \mathcal{L}_1 , \mathcal{L}_2 , \mathcal{L}_3 and \mathcal{L}_4 are given by:

$$(2.6) \quad \mathcal{L}_1(U) = \zeta \begin{pmatrix} u_2^2 & u_2^3 & \dots & u_2^{n_t} & 0 \\ u_3^2 & u_3^3 & \dots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{n_x-1}^2 & u_{n_x-1}^3 & \dots & u_{n_x-1}^{n_t} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad \mathcal{L}_2(U) = \eta \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & u_1^1 & u_1^2 & \dots & u_1^{n_t-1} \\ 0 & u_1^0 & u_1^1 & \dots & u_2^{n_t-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & u_{n_x-2}^1 & u_{n_x-2}^2 & \dots & u_{n_x-2}^{n_t-1} \end{pmatrix}$$

$$(2.7) \quad \mathcal{L}_3(U) = \theta \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ u_1^2 & u_1^3 & \dots & u_1^{n_t} & 0 \\ u_2^2 & u_2^3 & \dots & u_2^{n_t} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{n_x-2}^2 & u_{n_x-2}^3 & \dots & u_{n_x-2}^{n_t} & 0 \end{pmatrix} \quad \mathcal{L}_4(U) = \vartheta \begin{pmatrix} 0 & u_2^1 & u_2^2 & \dots & u_2^{n_t-1} \\ 0 & u_3^1 & u_3^2 & \dots & u_3^{n_t-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & u_{n_x-1}^1 & \dots & \dots & u_{n_x-1}^{n_t-1} \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

PROPOSITION 2.2. *The second member matrix M_0 bears the initial conditions, given for the specific value $n = 0$, which correspond to the initialization process when computing loops, and the boundary conditions, given for the specific values $i = 0$, $i = n_x$.*

Denote u_{exact} the exact solution of (1.1).
The corresponding matrix U_{exact} will be:

$$(2.8) \quad U_{exact} = [U_{exact_i}^n]_{1 \leq i \leq n_x, 1 \leq n \leq n_t}$$

where:

$$(2.9) \quad U_{exact_i}^n = U_{exact}(x_i, t_n)$$

with $x_i = i \, dx$, $t_n = n \, dt$.

DEFINITION 2.3. *We will call error matrix the matrix defined by:*

$$(2.10) \quad E = U - U_{exact}$$

Let us consider the matrix F defined by:

$$(2.11) \quad F = M_1 U_{exact} + U_{exact} M_2 + \mathcal{L}(U_{exact}) - M_0$$

PROPOSITION 2.4. *The error matrix E satisfies:*

$$(2.12) \quad M_1 E + E M_2 + \mathcal{L}(E) = F$$

2.2. The matrix equation.

2.2.1. Theoretical formulation. THEOREM 2.5. *Minimizing the error due to the approximation induced by the numerical scheme is equivalent to minimizing the norm of the matrices E satisfying (2.12).*

Note: Since the linear matricial operator \mathcal{L} appears only in the Crank-Nicolson scheme, we will restrain our study to the case $\mathcal{L} = 0$. The generalization to the case $\mathcal{L} \neq 0$ can be easily deduced.

PROPOSITION 2.6.

The problem is then the determination of the minimum norm solution of:

$$(2.13) \quad M_1 E + E M_2 = F$$

which is a specific form of the generalized Sylvester equation:

$$(2.14) \quad AX + BXC = D$$

where A and B are m by m matrices, C and F n by n matrices, and X , a m by n matrix. C has to be a Jordan matrix.

PROPOSITION 2.7. *Since the coefficient α never vanishes, the matrix M_2 is invertible.*

The solving of the Sylvester equation is generally based on Schur decomposition: for a given square n by n matrix A , n being an even number of the form $n = 2p$, there exists a unitary matrix U and a upper triangular block matrix T such that:

$$(2.15) \quad A = U^* T U$$

where U^* denotes the (complex) conjugate matrix of the transposed matrix $^T U$. The diagonal blocks of the matrix T correspond to the complex eigenvalues λ_i of A :

$$(2.16) \quad T = \begin{pmatrix} T_1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & T_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & T_p \end{pmatrix}$$

where the block matrices T_i , $i = 1, \dots, p$ are given by:

$$(2.17) \quad \begin{pmatrix} \operatorname{Re}[\lambda_i] & \operatorname{Im}[\lambda_i] \\ -\operatorname{Im}[\lambda_i] & \operatorname{Re}[\lambda_i] \end{pmatrix}$$

Re being the real part of a complex number, and Im the imaginary one.

Due to this decomposition, the Sylvester equation require, to be solved, that the

dimensions of the matrices be even numbers. We will therefore, in the following,

restrain our study to $n_x - 1$ and n_t being even numbers. So far, it is interesting to note that the Schur decomposition being more stable for higher order matrices, it perfectly fits finite differences problems.

Complete parametric solutions of the generalized Sylvester equation (2.13) is given in [2], [3].

As for the determination of the solution Sylvester equation, it is a major topic in control theory, and has been the subject of numerous works (see [1], [6], [8], [9], [10], [11], [12]).

In [1], the method is based on the reduction of the observable pair (A, C) to an observer-Hessenberg pair (H, D) , H being a block upper Hessenberg matrix. The reduction to the observer-Hessenberg form (H, D) is achieved by means of the staircase algorithm (see [4], ...).

In [9], in the specific case of B being a companion form matrix, the authors propose a very neat general complete parametric solution, which is expressed in terms of the controllability of the matrix pair (A, B) , a symmetric matrix operator, and a parametric matrix in the Hankel form.

We recall that a companion form, or Frobenius matrix is one of the following kind:

$$(2.18) \quad B = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 & -b_0 \\ 1 & 0 & \dots & \dots & 0 & -b_1 \\ 0 & 1 & 0 & \dots & \vdots & \vdots \\ \vdots & 0 & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & -b_{p-1} \end{pmatrix}$$

These results can be generalized through matrix block decomposition to a block companion form matrix:

$$(2.19) \quad M_2 = \begin{pmatrix} M_2^{B^1} & 0 & \dots & \dots & 0 \\ 0 & M_2^{B^2} & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & M_2^{B^k} \end{pmatrix}$$

the $M_2^{B^p}$, $1 \leq p \leq k$ being companion form matrices.

Another method is presented in [14], where the determination of the minimum-norm solution of a Sylvester equation is specifically developed.

The accuracy and computational stability of the solutions is examined in [5].

2.2.2. Existence condition of the solution . THEOREM 2.8. *In the specific cases of the Lax and Lax-Wendroff schemes, (2.14) has a unique solution.*

Proof. (2.14) has a unique solution if and only if A and B have no common eigenvalues.

The characteristic polynomials P_{M_1}, P_{M_2} of M_1 and M_2 , can be respectively calculated as the determinants of respectively $\frac{n_x-1}{2}$ by $\frac{n_x-1}{2}$, $\frac{n_t}{2}$ by $\frac{n_t}{2}$ diagonal block matrices:

$$(2.20) \quad P_{M_1}(\lambda) = ((\lambda - \beta)^2 - \delta \varepsilon)^{\frac{n_x-1}{2}}, \quad P_{M_2}(\lambda) = (\lambda^2 - \alpha \gamma)^{\frac{n_t}{2}}$$

In the specific cases of the Lax and Lax-Wendroff schemes: $\alpha \gamma = 0 \neq \pm \beta + \sqrt{\delta \varepsilon}$. Hence, (2.12) has a unique solution, which accounts for the consistency of the given problem.

□

COROLLARY 2.9. *In the specific case of the Leapfrog scheme, (2.14) has a unique solution if and only if $\tau \neq \frac{h}{c}$.*

2.3. Specific properties of the matrices M_1 and M_2 .

2.3.1. Inversibility of the matrix M_1 . **THEOREM 2.10.** *In the specific case of the Lax and Leapfrog schemes, M_1 is invertible.*

THEOREM 2.11. *In the specific case of the Lax-Wendroff scheme, M_1 is invertible if and only if*

$$(2.21) \quad \left(\frac{-1}{\tau} + \frac{c^2 \tau}{h^2}\right)^2 \neq \frac{(1 - \sigma^2)c^2}{4h^2}$$

Proof. The determinant D_1 of M_1 can be calculated as the determinant of a $\frac{n_t}{2}$ by $\frac{n_t}{2}$ diagonal block matrix:

$$(2.22) \quad D_1 = (\beta^2 - \delta \varepsilon)^{\frac{n_t}{2}}$$

□

2.3.2. Nilpotent components of the matrix M_2 . **PROPOSITION 2.12.** *M_2 can be written as:*

$$(2.23) \quad M_2 = \alpha {}^T N + \gamma N$$

where N is the nilpotent matrix, ${}^T N$ denoting the corresponding transposed matrix:

$$(2.24) \quad N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \dots & 0 & 1 \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

PROPOSITION 2.13.

For either $\alpha = 0$, or $\gamma = 0$, M_2 will thus be nilpotent, of order n_t .

PROPOSITION 2.14.

For $\gamma = 0$, if M_1 is invertible, the solution at $t = n_t dt$ can then be immediately determined.

Proof.

In such a case, multiplying (2.2) on the right side by $M_2^{n_t-1}$ leads to:

$$(2.25) \quad M_1 U M_2^{n_t-1} + U M_2 M_2^{n_t} = M_0 M_2^{n_t-1}$$

i. e.:

$$(2.26) \quad M_1 U M_2^{n_t-1} = M_0 M_2^{n_t-1}$$

which leads to:

$$(2.27) \quad U M_2^{n_t-1} = M_1^{-1} M_0 M_2^{n_t-1}$$

Due to:

$$(2.28) \quad M_2^{n_t-1} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

we have:

$$(2.29) \quad U M_2^{n_t-1} = \begin{pmatrix} u_{1n_t} & 0 & 0 & \dots & 0 \\ u_{2n_t} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ u_{n_x n_t} & 0 & \dots & 0 & 0 \end{pmatrix}$$

The solution at $t = n_t dt$ can then be immediately determined.

□

3. Conclusion. Thanks to the above results, we here propose:

1. to solve finite difference problems either through the corresponding Sylvester equation, with the aid of a mathematical software such as Mathematica or Maple;
2. to study the intrinsic properties of various schemes through those of the related matrices M_1 and M_2 ;
3. to optimize finite difference problems through minimization of the symbolic expression of the error as a function of the scheme parameters.

REFERENCES

- [1] P. VAN DOOREN, *Reduced order observers: A new algorithm and proof*, Systems Control Lett., 4(1984), pp. 243-251.
- [2] A. BERMAN AND R. J. PLEMMONS, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia, PA (1994).
- [3] H. R. GAIL, S.L. HANTLER AND B. A. TAYLOR, *Spectral Analysis of M/G/1 and G/M/1 type Markov chains*, Adv. Appl. Probab., 28(1996), pp. 114-165.

- [4] D. L. BOLEY, *Computing the Controllability algorithm / Observability Decomposition of a Linear Time-Invariant Dynamic System, A Numerical Approach*, PhD. thesis, Report STAN-CS-81-860, Dept. Comp. i, Sci., Stanford University, 1981.
- [5] A. S. DEIF, N. P. SEIF, AND S. A. HUSSEIN, *Sylvester's equation: accuracy and computational stability*, Journal of Computational and Applied Mathematics, 61(1995), pp. 1-11.
- [6] J. Z. HEARON, *Nonsingular solutions of $TA - BT = C$* , Linear Algebra and its applications, 16(1977), pp. 57-63.
- [7] C. H. HUO, *Efficient methods for solving a nonsymmetric algebraic equation arising in stochastic fluid models*, Journal of Computational and Applied Mathematics, pp. 1-21 (2004).
- [8] C. C. TSUI, *A complete analytical solution to the equation $TA - FT = LC$ and its applications*, IEEE Trans. Automat. Control AC, 32(1987), pp. 742-744.
- [9] B. ZHOU AND G. R. DUAN, *An explicit solution to the matrix equation $AX - XF = BY$* , Linear Algebra and its applications, 402(2005), pp. 345-366.
- [10] G. R. DUAN, *Solution to matrix equation $AV + BW = EVF$ and eigenstructure assignment for descriptor systems*, Automatica, 28(1992), pp. 639-643.
- [11] G. R. DUAN, *On the solution to Sylvester matrix equation $AV + BW = EVF$ and eigenstructure assignment for descriptor systems*, IEEE Trans. Automat. Control AC, 41(1996), pp. 276-280.
- [12] P. KIRRINNIS, *Fast algorithms for the Sylvester equation $AX - XB^T = C$* , Theoretical Computer Science, 259(2000), pp. 623-638.
- [13] M. KONSTANTINOV, V. MEHRMANN, V. AND P. PETKOV, *On properties of Sylvester and Lyapunov operators*, Linear Algebra and its applications, 312(2000), pp. 35-71.
- [14] A. VARGA, *TA numerically reliable approach to robust pole assignment for descriptor systems*, Future Generation Computer Systems, 19(2003), pp. 1221-1230.
- [15] G. B. WITHAM, *Linear and Nonlinear Wave*, Wiley-Interscience (1974).
- [16] S. WOLFRAM, *The Mathematica book*, Cambridge University Press (1999).