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## Homotopes and Conformal Deformations of Symmetric spaces

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**Abstract.** *Homotopy* is an important feature of associative and Jordan algebraic structures: such structures always come in families whose members need not be isomorphic among other, but still share many important properties. One may regard homotopy as a special kind of *deformation* of a given algebraic structure. In this work, we investigate the global counterpart of this phenomenon on the geometric level of the associated symmetric spaces – on this level, homotopy gives rise to *conformal deformations of symmetric spaces*. These results are valid in arbitrary dimension and over general base fields and -rings.

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## 0. Introduction

**0.1. Associative algebras and general linear groups.** The example of general linear groups is well-suited to illustrate what we mean by “conformal deformation”, and by the concept of “homotopy”. If  $G$  is any group with unit element  $\mathbf{1}$ , then we can define new group laws on  $G$  by choosing an arbitrary element  $a \in G$  and defining

$$x \cdot_a y := xay.$$

The unit element of  $(G, \cdot_a)$  is then  $a^{-1}$ . Of course, all these group laws are isomorphic among each other: left translation  $l_a : G \rightarrow G$  is an isomorphism from  $(G, \cdot_a)$  onto  $(G, \cdot)$  (one may also use right translation), and hence this construction seems not to be very interesting. However, things would start to become more interesting, if one could take a “limit” for  $a$  converging to some element in the “boundary of  $G$ ”; then  $(G, \cdot_a)$  should converge to some deformed group which is no longer isomorphic to  $G$  (a “homotope” of  $G$ ).

In order to make this idea more precise, we need some additional structure. For instance, assume that  $G = A^\times$  is the unit group of an associative unital algebra  $A$  over a ring or field  $\mathbb{K}$ . As explained above,  $G := A^\times$  is a group with product  $\cdot_a$ , for any  $a \in A$ . We want to allow  $a$  to “converge” to some non-invertible element of  $A$  – the problem being that, although the *homotope algebra*  $(A, \cdot_a)$  with product  $x \cdot_a y = xay$  is an associative algebra for all  $a \in A$ , it will not have a unit element as soon as  $a$  is non-invertible. In order to circumvent this problem, we translate, for invertible  $a$ , the unit element  $a^{-1}$  by the translation  $\tau = \tau_{-a^{-1}}$  to the origin  $0 \in A$ , that is, for invertible  $a$  we consider the group law

$$\begin{aligned} x \diamond_a y &= \tau(\tau^{-1}(x) \cdot_a \tau^{-1}(y)) \\ &= (x + a^{-1})a(y + a^{-1}) - a^{-1} \\ &= xay + y + x \end{aligned} \tag{0.1}$$

which is defined on the set

$$\begin{aligned} G_a &= \{x \in A \mid a^{-1} + x \in A^\times\} \\ &= \{x \in A \mid \mathbf{1} + ax \in A^\times\}. \end{aligned} \tag{0.2}$$

Now it is easy to take the “limit as  $a$  becomes non-invertible”: one checks that the last expressions of (0.1) and (0.2) define a group  $(G_a, \diamond_a)$  for any element  $a \in A$ . (The inverse of  $x \in G_a$  is  $y = -x(\mathbf{1} + ax)^{-1}$ , as is seen by direct calculation.) For  $a \in A^\times$  these groups are, by construction, all isomorphic to  $G$ , but for non-invertible  $a \in A$  they are not: they are “homotopes”, but not isotopes of  $G$ . In particular, for  $a = 0$ , we get the translation group  $(A, +)$  as a deformation of  $A$ . Also, it is clear that, if  $a, b \in A$  belong to the same  $A^\times$ -orbit (under left, or under right action), then  $G_a$  and  $G_b$  are isomorphic. Therefore, if we can classify the orbits of the action of  $A^\times \times A^\times$  on  $A$ , then we can also classify the various groups  $G_a$  obtained by the preceding construction. For instance, this is possible in the case where  $A = M(n, n; \mathbb{K})$  is the algebra of square matrices over a field  $\mathbb{K}$ , and then a complete description of the groups  $G_a$  is not difficult (see Section 5.1). Finally, one remarks that, on the level of Lie algebras, the corresponding situation is very simple: whenever the notion of a “Lie algebra of  $G_a$ ” makes sense, then this is simply  $A$  with the  $a$ -deformed Lie bracket

$$[x, y]_a = xay - yax. \tag{0.3}$$

**0.2. Jordan algebraic structures and symmetric spaces.** The concept of *homotopy* has its natural setting in Jordan theory (the theory of Jordan algebras, -triple systems and -pairs),

and our aim is to generalize the preceding example in this framework. The basic algebraic facts on homotopy of Jordan structures are summarized in Chapter 1. The main point is that all such structures come actually in families which are parametrized by a “continuous” set of parameters and hence allow for deformations as the parameter tends to a “singular” value. These facts being well-known, we want to describe the corresponding deformation theory on the geometric level of *symmetric spaces*. We use the term “Conformal Deformations” in order to distinguish our approach from other, more formal, theories on deformations of, say, Lie groups and Lie algebras; this terminology is motivated by the fact that our families of deformed spaces have, in some sense, a common underlying *generalized conformal structure*.

Recall that a *symmetric space* is a manifold  $M$  equipped with a family of symmetries  $\sigma_x$ ,  $x \in M$ , such that the product map  $\mu : M \times M \rightarrow M$ ,  $(x, y) \mapsto \mu(x, y) := \sigma_x(y)$  satisfies some natural identities; when fixing a base point  $o \in M$ , it is useful to consider the modified product map  $(x, y) \mapsto \mathcal{Q}(x) := \sigma_x \sigma_o$ , and then the data  $(M, \mathcal{Q}, o)$  have a strong formal similarity with the data  $(G, \cdot, e)$  of a group (see Appendix A, Proposition A.2). The manifold structure will not play an important rôle since symmetric spaces associated to Jordan structures are algebraic in nature; however, in view of applications in harmonic analysis, we will state also the “smooth versions” of the main results.

The notions of homotope that we consider in the three main Jordan theoretic categories are as follows:

**UJA** (Unital Jordan algebras.) If  $(J, \bullet)$  is a Jordan algebra, then for any element  $a \in J$  the new product  $\bullet_a$  on  $J$

$$x \bullet_a y := (x \bullet a) \bullet y + x \bullet (a \bullet y) - a \bullet (x \bullet y).$$

defines again a Jordan algebra  $J_a = (J, \bullet_a)$ , called the *a-homotope* of  $(J, \bullet)$ . It is unital if, and only if, the element  $a$  is invertible in  $J$ , and then  $a^{-1}$  is the unit element of  $V_a$ .

**JP** (Jordan pairs.) If  $(V^+, V^-)$  is Jordan pair over  $\mathbb{K}$  with trilinear maps  $T^\pm$ , then, for any  $a \in V^-$ ,

$$x \bullet_a y := T^+(x, a, y) \tag{0.4}$$

defines a Jordan algebra structure on  $V^+$ , called the *a-homotope algebra*. In general, this algebra has no unit element.

**JTS** (Jordan triple systems.) If  $V$  is a JTS over  $\mathbb{K}$  with trilinear map  $T$ , a linear map  $\alpha : V \rightarrow V$  is called *symmetric* or *T-selfadjoint* if, for all  $x, y, z \in V$ ,

$$\alpha T(x, \alpha y, z) = T(\alpha x, y, \alpha z). \tag{0.5}$$

Then (Lemma 1.13)  $T_\alpha(x, y, z) := T(x, \alpha y, z)$  defines a new Jordan triple product on  $V$ , called the *α-homotope* of  $(V, T)$ .

The last notion is the most general one since, taking for  $\alpha$  the quadratic operator  $Q(a)$  of a JTS or of a Jordan pair, one recovers the preceding two notions. We call *structure variety*  $\text{Svar}(T)$  of the JTS  $(V, T)$  the set of all  $T$ -selfadjoint maps; by what we have just said, this space of deformation parameters contains the quadratic operators  $Q(a)$ , but it also contains all symmetric Bergmann operators  $B(a, a)$ , as well as all involutive automorphisms of  $T$ . Thus the corresponding deformation theory is very rich. In the last two chapters we give some examples and comments on possible applications in Harmonic Analysis, which are far from being exhaustive. In the following, we describe in more detail how symmetric spaces and their deformations associated to the three mentioned categories are constructed.

**0.3. Jordan algebras and quadratic prehomogeneous symmetric spaces.** In the category UJA, the construction is very similar to the one for unital associative algebras described in Section 0.1: the unit group  $A^\times$  of an associative algebra is now replaced by the set  $J^\times$  of invertible

elements of the unital Jordan algebra  $J$ , which carries a natural symmetric space structure given by

$$\mu(a, y) = U_a U_y^{-1} y$$

where  $U_x(y) = 2x(xy) - x^2y$  is the quadratic operator. With respect to the base point 1, the quadratic map  $\mathcal{Q}(x)$  of the symmetric space and the quadratic operator  $U_x$  coincide; for this reason symmetric spaces of the type  $J^\times$  have been called *quadratic prehomogeneous spaces* in [Be00, Ch. II]. Now, as already mentioned, for invertible  $a$  the  $a$ -homotope  $(J, \bullet_a)$  corresponds to the base point  $a^{-1}$ . Therefore, in order to define a deformed symmetric space as “limit when  $a$  becomes non-invertible”, we may use the same trick as in the associative case: we translate the symmetric space with base point  $(J^\times, a^{-1})$  to the origin  $0 \in J$  via the translation  $\tau = \tau_{-a^{-1}}$  and derive an explicit formula for the resulting structure (cf. Prop. 2.2). Then it turns out that this formula defines in fact a symmetric space for any  $a \in J$ . Moreover, this formula makes sense for any Jordan pair  $(V^+, V^-)$  instead of a Jordan algebra  $J$ , and hence we will state and prove it in this more general context (Theorem 2.3). This result, generalizing (0.1) and (0.2), says that, for an arbitrary element  $a \in V^-$  the set  $\mathcal{U}_a$  of elements  $x \in V^+$  such that  $(x, -a)$  is quasi-invertible is stable under the symmetry

$$\sigma_x := \tau_{2x+Q(x)a} \circ (-B(x, -a)) \circ \tilde{\tau}_{-a},$$

(where  $\tau_v$  is translation by  $v \in V^+$ ,  $\tilde{\tau}_w(x) = x^w$  the *quasi-inverse* of the pair  $(x, w)$  and  $B(x, y)$  the *Bergmann operator*), and the product

$$\mu : \mathcal{U}_a \times \mathcal{U}_a \rightarrow \mathcal{U}_a, \quad (x, y) \mapsto \mu(x, y) := \sigma_x(y)$$

defines on  $\mathcal{U}_a$  the structure of a symmetric space. As in the associative case, for  $a = 0$  we get the most degenerate case: the flat space  $V^+$  with product  $\mu(x, y) = 2x - y$ ; on the other hand, the most non-degenerate case arises if  $a$  has maximal rank. For instance, in the Jordan pair  $(V^+, V^-) = (M(p, q; \mathbb{K}), M(q, p; \mathbb{K}))$  of rectangular matrices over a field  $\mathbb{K}$  with  $p \leq q$ , this means that the matrix  $a \in V^-$  has maximal rank  $p$ . Then the symmetric space  $\mathcal{U}_a$  is isomorphic to a semidirect product of  $\mathrm{GL}(p, \mathbb{K})$  with  $M(q - p, p; \mathbb{K})$  (see Section 5.2). The case  $p = q$  plays a special rôle: only in this case we get a reductive symmetric space; we are back in the case of a unital Jordan algebra discussed above. In the following, we give a geometric version of the construction of the symmetric space  $\mathcal{U}_a$ .

**0.4. Generalized projective and polar geometries.** The concept of a *generalized projective geometry*  $(\mathcal{X}^+, \mathcal{X}^-)$  is the geometric analog of the one of a Jordan pair  $(V^+, V^-)$  (see [Be02]). Roughly, this is a pair of sets such that every element  $a \in \mathcal{X}^-$  determines on some part  $\mathfrak{v}_a \subset \mathcal{X}$  the structure of an affine space over  $\mathbb{K}$ , and vice versa, such that certain natural properties are satisfied. These properties are the geometric analogs of the identities characterizing Jordan pairs (cf. Appendix B). Now fix two elements  $a, b \in \mathcal{X}^-$  and assume that the intersection  $\mathcal{U}_{a,b} := \mathfrak{v}_b \cap \mathfrak{v}_a$  of the corresponding affine parts is non-empty. From a dual point of view, this means that, for every element  $x \in \mathcal{U}_{a,b}$ , the points  $a$  and  $b$  belong to the affine part  $\mathfrak{v}_x \subset \mathcal{X}^-$  determined by  $x$ . Let  $p(x) \in \mathfrak{v}_x \subset \mathcal{X}^-$  be the midpoint of  $a$  and  $b$  in the affine space  $\mathfrak{v}_x$ . The *midpoint map*  $p : \mathcal{X}^+ \supset \mathcal{U}_{a,b} \rightarrow \mathcal{X}^-$  thus defined shares many properties the polarities known from projective geometry; but in general it is far from behaving like a bijection (take, e.g., an ordinary projective space  $\mathcal{X}^+ = \mathbb{P}(V)$  together with its dual  $\mathcal{X}^- = \mathbb{P}(V^*)$ ; then the image of the midpoint map is essentially the ordinary projective line joining  $b$  and  $c$ ). Nevertheless, the construction from [Be02, Theorem 4.1] associating to each (generalized) polar geometry a symmetric space still works: for any  $x \in \mathcal{U}_{a,b}$ , the element  $p(x)$  belongs to  $\mathfrak{v}_x$ ; hence  $x$  can be chosen as origin in  $\mathfrak{v}_x$ ; let  $(-1)_{x,p(x)}$  be the negative of the identity in the vector space  $(\mathfrak{v}_x, x)$ ; then (Theorem 3.1) the set  $\mathcal{U}_{a,b}$  becomes a symmetric space when equipped with the

multiplication map  $\mu(x, y) = (-1)_{x, p(x)}(y)$ . This construction is indeed the geometric version of the one previously described: we fix one of two points  $b, c$  to be the origin in  $\mathcal{X}^-$ , so we are working in a fixed affine space  $V^+$ , and the other point corresponds to the “deformation parameter”  $a$  from the preceding section. Then we recover the symmetric space described there (Theorem 3.3). In case  $a = b$ , the midpoint map is constant  $= a$ , and we get the affine space  $\mathfrak{v}_a$  with  $\mu(x, y) = (-1)_x(y) = 2x - y$ ; on the other hand, the case of unital Jordan algebras corresponds to the case where, for a “generic choice” of  $a$  and  $b$ , the midpoint map  $p$  does indeed extend to a bijection which is then called an *inner polarity* (cf. [Be03]).

**0.5. General deformations of polar geometries.** Finally, let us explain the most general result concerning conformal deformations, which corresponds to deformations  $T_\alpha$  of a Jordan triple system  $T$  by  $T$ -symmetric maps  $\alpha$ . Jordan triple systems are nothing but Jordan pairs with involution, so they correspond to generalized projective geometries  $(\mathcal{X}^+, \mathcal{X}^-)$  with involution  $p: \mathcal{X}^+ \rightarrow \mathcal{X}^-$ ; to be precise,  $p$  has to be a *polarity*. The data  $(\mathcal{X}^+, \mathcal{X}^-; p)$  are called a *generalized polar geometry*. Then (cf. [Be02, Theorem 4.1]) the *set of non-isotropic points* (with respect to  $p$ ),

$$M^{(p)} := \{x \in \mathcal{X}^+ \mid x \in \mathfrak{v}_{p(x)}\}, \quad (0.6)$$

with product map  $\mu(x, y) := (-1)_{x, p(x)}(y)$  is a symmetric space. On the other hand, for every  $T$ -selfadjoint map  $\alpha$ , we can do the same thing with respect to the  $\alpha$ -homotope  $T_\alpha$  of the Jordan triple system  $T$ , and then get a symmetric space  $M_\alpha$ . But, *a priori*, for various  $\alpha$ , these symmetric spaces are constructed independently of each other. If we want to understand them as deformations of the given symmetric space  $M = M^{(p)}$ , we need a common realization on one underlying manifold – this is precisely the content of Theorem 4.3: all symmetric spaces  $M_\alpha$  have a canonical realization inside  $\mathcal{X}^+$  (and, in the topological case, these are open submanifolds), such that, on the (open) part  $M_\alpha \cap \mathfrak{v}_{p(o)}$ , the symmetric space structure (and other differential geometric data) can be described by explicit Jordan theoretic formulae in terms of the  $\alpha$ -homotope  $T_\alpha$ .

**0.6. Globalization of structural maps.** The result from Theorem 4.3 is closely related to a rather classical categorical problem in projective geometry. The question is simply: *what shall be the morphisms in the “category of projective spaces (over a given field or ring  $\mathbb{K}$ )”* ? – shall we admit only maps induced by injective linear maps (so we get globally defined maps of projective spaces) or admit maps induced by arbitrary non-zero linear maps (so our maps are not everywhere defined)? The answer is of course that both definitions make sense and thus we get two different categories. Here the theory of Jordan pairs provides a very satisfactory way to understand this situation: Jordan pairs can be turned into a category in two different ways which correspond exactly to the possibilities mentioned above. The first one, the “usual” homomorphisms, extend to globally defined morphisms of associated geometries, whereas the second one, called *structural transformations*, do not always extend. Then a natural question is: what is the “biggest” or “natural” domain in the geometry to which structural transformations extend? This topic is treated in Appendix C, which may be of independent interest, and it is the key ingredient for the proof of Theorem 4.3.

**0.7. Applications and problems.** The point of view of deformations and contractions of Lie groups leads to interesting applications in harmonic analysis, see, e. g., [DR85]. We think that the setting presented here is very suitable for applications of this kind since the dependence of many relevant structures of the symmetric spaces  $M_\alpha$  on the deformation parameter  $\alpha$  can be described by explicit (algebraic or at least analytic) formulae – see Sections 4.7, together with examples given in Chapter 5 and further comments on problems related to the concept of *group contractions* discussed in Section 5.5.

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**Notation.** Throughout, we work over a commutative base ring  $\mathbb{K}$  in which 2 and 3 are invertible and hence use the concepts *linear* Jordan pairs, algebras and triple systems.

## 1. Homotopes of Lie and Jordan algebraic structures

**1.1. Scalar homotopes and c-duality.** If  $V$  is an algebra over  $\mathbb{K}$  (Lie or other), with product  $m(x, y)$ , then for any scalar  $\lambda \in \mathbb{K}$ , we may define a new product  $m_\lambda(x, y) := \lambda m(x, y)$ , called a *scalar homotope of  $(V, m)$* . Then  $\varphi := \lambda \text{id}_V$  is a homomorphism from  $(V, m_\lambda)$  to  $(V, m)$ . In particular, the algebras  $(V, m_\lambda)$  for invertible  $\lambda$  are all isomorphic among each other. Thus, if  $\mathbb{K}$  is a field, this construction gives us nothing interesting.

Now let  $(V, R)$  be a triple system (i.e.,  $R : V^3 \rightarrow V$  is a trilinear map). Then  $R_\lambda(X, Y, Z) := \lambda R(X, Y, Z)$  defines a new triple product on  $V$ , again called a *scalar homotope*. But now  $\lambda \text{id}_V$  is a homomorphism from  $(V, R_{\lambda^2})$  to  $(V, R)$ , and hence we can only conclude that all homotopes  $(V, R_\mu)$ , with  $\mu$  a square of an invertible element in  $\mathbb{K}$ , are isomorphic to  $(V, R)$ . For instance, if  $\mathbb{K} = \mathbb{R}$ , we get a new triple system  $(V, R_{-1})$  which is called the *c-dual of  $R$* . For a general field, we say that the  $(V, R_\lambda)$  with invertible  $\lambda$  are *scalar isotopes of  $(V, R)$* ; they need not be isomorphic to  $(V, R)$ .

The aim of this chapter is to recall some basic results showing that for Jordan algebraic structures there are interesting generalizations of the notion of scalar homotopy, whereas for Lie algebraic structures this is not the case.

**1.2. Graded Lie algebras.** Let  $(\Gamma, +)$  be an abelian group. A Lie algebra  $\mathfrak{g}$  over  $\mathbb{K}$  is called  $\Gamma$ -graded if it is of the form  $\mathfrak{g} = \bigoplus_{n \in \Gamma} \mathfrak{g}_n$  with  $[\mathfrak{g}_m, \mathfrak{g}_n] \subset \mathfrak{g}_{n+m}$ . The  $\mathbb{Z}/(2)$ -graded Lie algebras correspond bijectively to *symmetric Lie algebras*, i.e., to Lie algebras together with an automorphism of order 2. A Lie algebra is called *3-graded* if it is  $\mathbb{Z}$ -graded, with  $\mathfrak{g}_n = 0$  if  $n \neq -1, 0, 1$ . Note that then  $\mathfrak{g} = \mathfrak{g}_0 \oplus (\mathfrak{g}_1 \oplus \mathfrak{g}_{-1})$  gives rise to a  $\mathbb{Z}/(2)$ -grading, so  $\mathfrak{g}$  is then also a symmetric Lie algebra (of a rather special type).

**1.3. Lie triple systems.** A Lie triple system (LTS) is a  $\mathbb{K}$ -module  $\mathfrak{q}$  together with a trilinear map

$$R : \mathfrak{q} \times \mathfrak{q} \times \mathfrak{q} \rightarrow \mathfrak{q}, \quad (X, Y, Z) \mapsto R(X, Y, Z) =: -[X, Y, Z]$$

such that

$$(LT1) \quad R(X, X) = 0$$

$$(LT2) \quad R(X, Y, Z) + R(Y, Z, X) + R(Z, X, Y) = 0 \quad (\text{the Jacobi identity})$$

$$(LT3) \quad \text{For } D := R(X, Y) \text{ we have } DR(U, V, W) = R(DU, V, W) + R(U, DV, W) + R(U, V, DW).$$

If  $\mathfrak{g}$  is a  $\mathbb{Z}/(2)$ -graded Lie algebra, then  $\mathfrak{g}_{-1}$  with  $[X, Y, Z] := [[X, Y], Z]$  becomes an LTS, and every LTS arises in this way since one may reconstruct a Lie algebra from an LTS via the *standard imbedding* (see [Lo69]).

**1.4. Linear Jordan pairs.** If  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$  is a 3-graded Lie algebra, then we let  $V^\pm := \mathfrak{g}_{\pm 1}$  and define trilinear maps by

$$T_\pm : V_\pm \times V_\mp \times V_\pm \rightarrow V_\pm, \quad (x, y, z) \mapsto [[x, y], z].$$

The maps  $T_\pm$  satisfy the following identities:

$$(LJP1) \quad T^\pm(x, y, z) = T^\pm(z, y, x)$$

$$(LJP2) \quad T^\pm(u, v, T^\pm(x, y, z)) = T^\pm(T^\pm(u, v, x), y, z) - T^\pm(x, T^\mp(v, u, y), z) + T^\pm(x, y, T^\pm(u, v, z))$$

In general, a pair of  $\mathbb{K}$ -modules  $(V_+, V_-)$  with trilinear maps  $T^\pm : V_\pm \times V_\mp \times V_\pm \rightarrow V_\pm$  is called a *linear Jordan pair* if (LJP1) and (LJP2) hold. Every linear Jordan pair arises by the construction just described. The reconstruction of the Lie algebra  $\mathfrak{g}$  from a Jordan pair  $(V^+, V^-)$  is essentially the standard imbedding in disguise since  $\mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$  is a Lie triple system.

**1.5. Jordan triple systems.** An *involution of a 3-graded Lie algebra* is an automorphism  $\theta$  of order 2 such that  $\theta(\mathfrak{g}_\pm) = \mathfrak{g}_\mp$  and  $\theta(\mathfrak{g}_0) = \mathfrak{g}_0$ . Then we let  $V := \mathfrak{g}_+$  and define

$$T : V \times V \times V \rightarrow V, \quad (X, Y, Z) \mapsto [[X, \theta Y], Z].$$

Then  $T$  satisfies the identities (LJT1) and (LJT2) obtained from (LJP1) and (LJP2) by omitting the indices  $\pm$ . A  $\mathbb{K}$ -module together with a trilinear map satisfying these identities is called a (*linear*) *Jordan triple system* (JTS). Every linear Jordan triple system arises by the construction just described. One may also define the notion of an *involution of a Jordan pair*; then Jordan triple systems are the same as Jordan pairs with involution (see [Lo75]). For any JTS  $(V, T)$ , the Jordan pair  $(V^+, V^-) = (V, V)$  with  $T^\pm = T$  is called the *underlying Jordan pair*.

**1.6. The Jordan-Lie functor.** If  $T$  is a JTS on  $V$ , then

$$[X, Y, Z] = T(X, Y, Z) - T(Y, X, Z)$$

defines a LTS on  $V$ . This defines a functor from the category of JTS to the category of LTS over  $\mathbb{K}$ , which we call the *Jordan-Lie functor*. It is neither injective nor surjective (cf. [Be00]). The LTS  $(V, R)$  belongs to the subalgebra  $\mathfrak{g}^\theta$  of  $\mathfrak{g}$  whose involution is induced by the canonical involution of the 3-graded Lie algebra  $\mathfrak{g}$ .

**1.7. Jordan algebras.** A *Jordan algebra* is a  $\mathbb{K}$ -module  $J$  with a bilinear and commutative product  $x \bullet y$  such that the identity

$$(J2) \quad x \bullet (x^2 \bullet y) = x^2 \bullet (x \bullet y)$$

holds (cf. [McC04]; the reader not familiar with Jordan theory may throughout think of the example  $J = A^\tau$ , the set of fixed points of an anti-automorphism  $\tau$  of an associative algebra  $A$ , with bilinear product  $x \bullet y = \frac{xy + yx}{2}$ .) Jordan algebras always come in families: for any Jordan algebra  $J$  and element  $a \in J$ , the *homotope*  $J_a$  is defined as follows. Writing  $L(x)$  for the left multiplication operator  $L(x)y = x \bullet y$ , we can define a new product on  $J$  by

$$x \bullet_a y := (L(xa) + [L(x), L(a)])y = (x \bullet a) \bullet y + x \bullet (a \bullet y) - a \bullet (x \bullet y).$$

One proves that  $J_a = (J, \bullet_a)$  is again a Jordan algebra (see [McC04]), called the *a-homotope of  $(J, \bullet)$* , and that  $T(x, a, y) := x \bullet_a y$  defines a Jordan triple product on  $J$  (see [Be00, Ch. II] for a geometric proof of this fact). Thus  $(J, J)$  is a Jordan pair, but not all Jordan pairs are of this form.

**1.8. Jordan pairs with invertible elements.** For any Jordan pair  $(V^+, V^-)$  and every  $a \in V^-, V^+$  with the bilinear product

$$x \cdot_a y := T^+(x, a, y)$$

is a Jordan algebra, again called the *a-homotope algebra*. This algebra is unital if, and only if, the element  $a$  is *invertible in  $(V^+, V^-)$* , which means that the operator

$$Q(a) := Q^-(a) := \frac{1}{2}T^-(a, \cdot, a) : V_+ \rightarrow V_-$$

is invertible; then  $a^{-1} := (Q(a))^{-1}(a)$  is the unit element of  $J_a$ . Every unital Jordan algebra arises in this way from a Jordan pair having invertible elements, after a choice of such an element.

**1.9. Invertibility.** In a Jordan algebra  $(J, \bullet)$ , the quadratic operator  $U_x$  is defined by

$$U_x y = (2L_x^2 - L_{x^2})y = 2x \bullet (x \bullet y) - (x \bullet x) \bullet y.$$

An element  $x \in J$  is called *invertible* if  $U_x$  is invertible; let

$$J^\times = \{x \in J \mid U_x \in \text{GL}(J)\},$$

be the set of invertible elements. For such  $x$ , we define the *inverse*  $x^{-1} = U_x^{-1}x$  and call the map  $j : J^\times \rightarrow J$ ,  $x \mapsto x^{-1}$  the *Jordan inverse*. Then the following identities hold:

(SB1)  $U_{U_x y} = U_x U_y U_x$  (the “fundamental formula”)

(SB2)  $U_{x^{-1}} = (U_x)^{-1}$

(SB3)  $x^{-1} = U_x^{-1}x$

(cf. [McC04, p. 200 and 201]). In Section 2.1 we give a geometric interpretation of these identities.

**1.10. Quasi-invertibility.** In general, a Jordan pair  $(V^+, V^-)$  does not have invertible elements. The important notion is the one of *quasi-inverse*: let

$$\begin{aligned} B(x, y) &:= \text{id}_V - T(x, y) + Q(x)Q(y) \\ \tilde{\tau}_y(x) &:= x^y := B(x, y)^{-1}(x - Q(x)y) \end{aligned}$$

the *Bergman operator*, resp. the *quasi-inverse* (the latter provided that  $(x, y)$  is *quasi-invertible*, which means that  $B(x, y)$  is invertible). If  $x$  is invertible and  $(x, y)$  quasi-invertible, then we have  $\tilde{\tau}_y(x) = (x^{-1} + y)^{-1}$ .

In case  $(V^+, V^-) = (J, J)$  comes from a unital Jordan algebra  $V$ , we can also write  $\tilde{\tau}_y = j \circ \tau_{-y} \circ j$  where  $j$  is the Jordan inverse. For  $y = 1$  (the unit element), the identities  $\tilde{\tau}_1 = j \circ \tau_{-1} \circ j$  and  $\tau_{-1} \circ j \circ \tau_1 = \tilde{\tau}_1 \circ (-\text{id})$ , where  $j$  is the Jordan inverse, can be interpreted via an action of the group  $\text{Sl}(2, \mathbb{K})$  on  $J$  by fractional linear maps via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . x = (ax + b)(cx + d)^{-1}.$$

Then the maps  $\tau_1$ ,  $\tilde{\tau}_1$ ,  $j$  and  $-\text{id}$  correspond to the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and one checks the corresponding equations by a direct matrix computation.

**1.11. Symmetric maps of Jordan triple systems and associated homotopes.** For Jordan triple systems, there is a much richer supply of homotopes than for Lie triple systems: let  $(V, T)$  be a Jordan triple system. A *symmetric* or *self-adjoint map* of  $(V, T)$  is an endomorphism  $\alpha$  of  $V$  with the property

$$\forall x, y, z \in V : \quad T(\alpha x, y, \alpha z) = \alpha T(x, \alpha y, z). \quad (1.1)$$

**Lemma 1.12.** *If  $\alpha$  is  $T$ -selfadjoint, then the formula*

$$T^{(\alpha)}(x, y, z) := T(x, \alpha y, z) \quad (1.2)$$

*defines a Jordan triple product  $T_\alpha$  on  $V$ , called the  $\alpha$ -homotope of  $T$ . The quadratic map and the Bergmann operator of  $T_\alpha$  are given by*

$$Q_\alpha(x) = Q(x) \circ \alpha, \quad B_\alpha(x, y) = B(x, \alpha y).$$

**Proof.** It is clear that  $T_\alpha$  satisfies (LJT1). For the proof of (LJT2) take the identity (LJT2) for  $T$  and replace the middle elements  $v$  and  $y$  by  $\alpha v$  and  $\alpha y$  and get

$$T(u, \alpha v, T(x, \alpha y, z)) = T(T(u, \alpha v, x), \alpha y, z) - T^\pm(x, T(\alpha v, u, \alpha y), z) + T^\pm(x, \alpha y, T^\pm(u, \alpha v, z))$$

Then apply the defining relation of  $\alpha$  to the second term on the right hand side, and one gets the identity (LJT2) for  $T_\alpha$ . The quadratic operator for this JTS clearly is  $Q_\alpha(x) = Q(x) \circ \alpha$ , and it follows that

$$\begin{aligned} B_\alpha(x, y) &= \text{id} - T_\alpha(x, y) + Q_\alpha(x)Q_\alpha(y) = \text{id} - T(x, \alpha y) + Q(x) \circ \alpha Q(y) \circ \alpha \\ &= \text{id} - T(x, \alpha y) + Q(x)Q(\alpha y) = B(x, \alpha y). \end{aligned} \quad \blacksquare$$

The set of symmetric maps of  $(V, T)$ ,

$$\text{Svar}(T) := \{\alpha \in \text{End}(V) \mid \forall x, y, z \in V : T(\alpha x, y, \alpha z) = \alpha T(x, y, z)\}$$

is called the *structure variety of the Jordan triple system*  $T$ . Note that, by the fundamental formula  $Q(Q(x)z) = Q(x)Q(z)Q(x)$ , all quadratic maps  $Q(x)$  belong to  $\text{Svar}(T)$ , and so do the Bergmann operators of the form  $B(x, x)$  for all  $x \in V$  (by Bergmann structurality  $Q(B(x, y)z) = B(x, y)Q(z)B(y, x)$ , cf. [Lo75, JP26]). Also,  $\text{Svar}(T)$  contains all involutive automorphisms of  $T$ , together with their negatives. More generally,  $\text{Svar}(T)$  is a cone, i.e., invariant under taking scalar multiples. See [Be00] for a classification of the connected components of  $\text{Svar}(T) \cap \text{GL}(V)$  for  $\mathbb{K} = \mathbb{R}$  and  $(V, T)$  a simple finite-dimensional JTS of classical type.

One may define the notion of  $T$ -selfadjoint maps also for Jordan pairs to be a linear map  $\alpha : V^+ \rightarrow V^-$  such that  $(\alpha, \alpha)$  is a structural transformation between the Jordan pair  $(V^+, V^-)$  and its opposite pair  $(V^-, V^+)$ , cf. Appendix C. Then  $T^+(x, \alpha y, z)$  defines a JTS on  $V^+$ .

## 2. Deformations of quadratic prehomogeneous symmetric spaces

**2.1. The symmetric space of invertible elements in a Jordan algebra.** Assume  $J$  is a Jordan algebra over a ring  $\mathbb{K}$  with Jordan product  $x \bullet y$  and unit element  $\mathbf{1}$ . The set  $J^\times$  of invertible elements of  $J$  is stable under the maps  $j(x) = x^{-1}$  (the Jordan inverse) and  $U_x$  for every  $x \in J^\times$ . Let  $\mathcal{Q}(x) := U_x$ . Then the identities (SB1) – (SB3) from Section 1.9 are exactly the identities (SB1) – (SB3) from Appendix A.1, and therefore Prop. A.2 implies that  $J^\times$  carries a natural structure of a reflection space given by

$$\mu(x, y) = \sigma_x(y) = U_x U_y^{-1} y.$$

The differential at  $\mathbf{1}$  of the symmetry  $\sigma_{\mathbf{1}} = j$  can be calculated algebraically by using scalar extension by dual numbers  $\mathbb{K}[\varepsilon] = \mathbb{K} \oplus \varepsilon \mathbb{K}$ ,  $\varepsilon^2 = 0$ . It is easily checked that, in  $J \otimes_{\mathbb{K}} \mathbb{K}[\varepsilon] = J \oplus \varepsilon J$ , we have  $(\mathbf{1} + \varepsilon v)^{-1} = \mathbf{1} - \varepsilon v$ , and it follows that  $T_{\mathbf{1}} \sigma_{\mathbf{1}} = -\text{id}_J$ . If  $a \in V^\times$  is another base point, the maps  $\mathcal{Q}$  and  $j$  are replaced by the quadratic map, resp. Jordan inverse in the homotope algebra  $(V, \bullet_{a^{-1}})$  (which has  $a$  as unit element), and hence we have also  $T_a \sigma_a = -\text{id}_J$ . Thus also property (S4) defining a symmetric space is satisfied (in an algebraic sense), and hence  $V^\times$  is a symmetric space. We call such spaces *quadratic prehomogeneous symmetric spaces* (cf. [Be00, Chapter II]). Now we describe what happens when we translate the space  $V^\times$  such that a base point  $a$  is translated into the origin  $0 \in J$ . Recall from Section 1.10 the definition of the Bergmann-operator and the quasi-inverse.

**Proposition 2.2.** *We equip  $J^\times$  with its symmetric space structure  $\mu(x, y) = \sigma_x(y) = U_x U_y^{-1} y$ , let  $a \in J^\times$  and  $\tau = \tau_{-a^{-1}} : J \rightarrow J$  be translation by  $-a^{-1}$ . Then the translated symmetric space  $\tau(J^\times)$  is given by*

$$\tau(J^\times) = \{x \in J \mid B(x, -a) \in \text{GL}(J)\}$$

with product map

$$\tau(\mu(\tau^{-1}(x), \tau^{-1}(y))) = 2x + U_x a + B(x, -a) \tilde{\tau}_a(-y).$$

**Proof.** First of all,  $x \in \tau(J^\times)$  if and only if  $a^{-1} + x$  is invertible, if and only if  $U_{a^{-1}+x}$  is invertible; but since  $U_a$  is assumed to be invertible, we have the identity  $B(x, a) = U_a U_{a^{-1}-x}$  (see [Lo75, I.2.12]) and hence  $x$  belongs to  $\tau(J^\times)$  if and only if  $B(x, -a)$  is invertible, i.e., iff  $(x, -a)$  is a quasi-invertible pair.

Now let us calculate the translated product map, or, which amounts to the same, via Prop. A.2, the new quadratic map and inversion with respect to a base point. With respect to the old base point  $a^{-1} \in J^\times$ , the quadratic map was given by  $\mathcal{Q}(x)y = U_x U_a y$  and the symmetry being the Jordan inverse  $j_a$  in the Jordan algebra  $J_a$  with unit element  $a^{-1}$ . We have to show that transformation by  $\tau$  leads to the new quadratic map and new inverse

$$\begin{aligned} \tau\left(\mathcal{Q}(\tau^{-1}(x))\tau^{-1}(y)\right) &= 2x + U_x a + B(x, -a)y, \\ \tau(\sigma_a(\tau^{-1}(x))) &= \tilde{\tau}_a(-x). \end{aligned} \tag{2.2}$$

The first equality in (2.2) follows by using several times the fundamental formula and once the identity  $T(a, y)Q_a = Q_a T(y, a)$  ([Lo75, Identity (JP1)]; here  $Q_a = U_a$ ):

$$\begin{aligned} U_{x+a^{-1}} U_a (y + a^{-1}) - a^{-1} &= \frac{1}{2}(T(x + a^{-1}, U_a y + a, x + a^{-1}) - a^{-1}) \\ &= 2x + y + U_x U_a y + U_x a + T(x, a, y) \\ &= 2x + U_x a + B(x, -a)y \end{aligned}$$

If  $a = \mathbf{1}$ , then the second equality from (2.2) can be written

$$(\mathbf{1} + x)^{-1} - \mathbf{1} = \tilde{\tau}_1(-x).$$

This identity can in fact be used to *define* the quasi-inverse (cf. [McC04, 20.3] or [Lo75, I.3]; strictly speaking, in loc. cit. the unital hull  $\hat{J}$  of  $J$  is used [see below]; but since we work in a unital Jordan algebra, this identity holds already in  $J$ ). Now, since the quasi-inverse is in fact a Jordan pair notion, we may as well work in the homotope algebra  $J_a$  and then get the equality for any  $a \in J^\times$ .  $\blacksquare$

Next we will show that the formulae from the preceding proposition define a symmetric space for any  $a \in J$ , invertible or not. Note that for  $a = 0$  this is trivial: in this case we get the flat symmetric space structure on  $J$ . The proof becomes more transparent in the more general framework of *Jordan pairs* – in this context, the following equivalent version of the formulae from Prop. 2.2 holds:

**Theorem 2.3.** *Assume  $(V^+, V^-)$  is a linear Jordan pair over  $\mathbb{K}$  and fix an element  $a \in V^-$ . Let  $\mathcal{U}_a := \{x \in V^+ \mid B(x, -a) \in \text{GL}(V^+)\}$  be the set of elements  $x \in V^+$  such that  $(x, -a)$  is quasi-invertible. Then, for all  $x \in \mathcal{U}_a$ , the set  $\mathcal{U}_a$  is stable under the transformations*

$$\begin{aligned} \mathcal{Q}(x) &:= \tau_{2x+Q(x)a} \circ B(x, -a), \\ \sigma_0 &:= -\tilde{\tau}_{-a} = \tilde{\tau}_a \circ (-\text{id}), \end{aligned}$$

and  $(\mathcal{U}_a, \mathcal{Q}, \sigma_0)$  is a reflection space with base point 0. Moreover, the algebraic differential of the symmetry  $\sigma_x$  at  $x$ , defined by  $\sigma_x(x + \varepsilon v) = \sigma_x(x) + \varepsilon D\sigma_x(x)v$ , equals  $-\text{id}_J$ , and thus  $\mathcal{U}_a$  is a symmetric space.

**Proof.** We use the homotope  $J = V_a$  with product  $x \bullet_a y = \frac{1}{2}T(x, a, y)$  and adjoin a unit element  $\hat{J} = \mathbb{K}\mathbf{1} \oplus J$  (if  $J$  happens to be unital, we forget the unit element of  $J$ ). For  $x, y \in J$ , we calculate in  $\hat{J}$ ,

$$\begin{aligned} U_x y &= (2L_x^2 - L_{x^2})y = 2x \bullet_a (x \bullet_a y) - (x \bullet_a x) \bullet_a y = Q^+(x)Q^-(a)y \\ U_{\mathbf{1} \oplus x}(y) &= y + 2x \bullet_a y + U_x y = (\text{id} + T(x, a) + Q(x)Q(a))y = B(x, -a)y, \\ U_{\mathbf{1} \oplus x}(\mathbf{1} \oplus y) &= U_{\mathbf{1} \oplus x}(\mathbf{1}) + U_{\mathbf{1} \oplus x}(y) = (\mathbf{1} + x)^2 + U_{\mathbf{1} \oplus x}y = \mathbf{1} \oplus (2x + x^2 + U_{\mathbf{1} \oplus x}y) \\ &= \mathbf{1} \oplus (2x + Q(x)a + B(x, -a)y). \end{aligned} \tag{2.3}$$

The second equality shows that  $B(x, -a)$  is the restriction of  $U_{\mathbf{1} \oplus x}$  to  $J$ , and hence  $(x, -a)$  is quasi-invertible in  $(V^+, V^-)$  if and only if  $\mathbf{1} + x$  is invertible in  $\hat{J}$ , i.e.,

$$\mathbf{1} + \mathcal{U}_a = \hat{J}^\times \cap (\mathbf{1} + J).$$

This set is a symmetric subspace of  $\hat{V}^\times$ . In fact, the third equation of (2.3) shows that it is stable under the quadratic map  $\mathcal{Q}$  (with respect to the origin 1), and it is also stable under inversion: if  $(x, -a)$  is quasi-invertible, then the inverse of  $\mathbf{1} + x$  in  $\hat{J}$  is given by (see [Lo75, I.3])

$$(\mathbf{1} + x)^{-1} = \mathbf{1} + \tilde{\tau}_{-a}(-x) = \mathbf{1} - \tilde{\tau}_a(x).$$

Transferring the symmetric space structure from  $\mathbf{1} + \mathcal{U}_a$  to  $\mathcal{U}_a$  via the map  $\mathbf{1} \oplus x \mapsto x$ , we obtain the formula from the claim.

Algebraically, the differential  $D\sigma_0(0) : V^+ \rightarrow V^+$  may be defined by scalar extension with dual numbers  $\mathbb{K}[\varepsilon] = \mathbb{K} \oplus \varepsilon\mathbb{K}$ ,  $\varepsilon^2 = 0$ :

$$\sigma_0(0 + \varepsilon v) = -B(\varepsilon x, -a)^{-1}(\varepsilon x + Q(\varepsilon x)a) = -(\text{id} - \varepsilon T(x, a))\varepsilon x = -\varepsilon x,$$

whence  $D\sigma_0(0) = -\text{id}_{V^+}$ . Since  $\sigma_x = \mathcal{Q}(x) \circ \sigma_0$ , it follows that  $D\sigma_x(x) = -\text{id}_{V^+}$ .  $\blacksquare$

**Remark.** By the preceding calculations, the fundamental formula  $U_{\mathbf{1}-x}U_{\mathbf{1}-y}U_{\mathbf{1}-x} = U_{U_{\mathbf{1}-x}(\mathbf{1}-y)}$  in  $\mathbf{1} - \mathcal{U}_a$  translates to the identity

$$B(x, a)B(y, a)B(x, a) = B(2x - Q(x)a + B(x, a)y, a),$$

which in turn can be interpreted by saying that, for every  $a \in V^-$ , the map  $x \mapsto B(x, a)$  is a homomorphism of the reflection space  $\mathcal{U}_a$  into the structure group, seen as a reflection space.

**2.4. The topological case.** Assume  $\mathbb{K}$  is a topological ring with dense unit group. A *topological Jordan pair* is a pair of Hausdorff topological  $\mathbb{K}$ -modules  $V^\pm$  such that the trilinear maps  $T^\pm$  are continuous. Following [BeNe05], we say that  $V^\pm$  satisfies the condition (C2) if the following holds:

(C2) for all  $a \in V^\pm$ , the sets  $\mathcal{U}_a$  are open and the maps  $\mathcal{U}_a \times V \rightarrow V$ ,  $(x, z) \mapsto B(x, -a)^{-1}z$  are continuous.

**Theorem 2.5.** *Assume  $(V^+, V^-)$  is a topological Jordan pair satisfying the condition (C2).*

- (1) *The symmetric space  $(\mathcal{U}_a, \mu)$  from Theorem 2.3 is smooth, i.e., it is a symmetric space in the category of manifolds over  $\mathbb{K}$ .*
- (2) *With respect to the base point 0, the Lie triple system of the symmetric space  $(\mathcal{U}_a, \mathcal{Q}, \sigma_0)$  is given by antisymmetrizing the Jordan triple product  $S(u, v, w) := T^+(u, Q(a)v, w)$  in the first two variables:*

$$[x, y, z] = S(x, y, z) - S(y, x, z).$$

**Proof.** (1) Smoothness of  $\mu$  follows from [BeNe05, Prop. 5.2], and the Property (S4) of a symmetric space follows from the calculation of  $D\sigma_0(0)$  given above (the differential in the sense of differential calculus is the same as the algebraic differential, cf. [Be06]).

(2) Once again, the easiest proof is by reduction to unital Jordan algebras: for a unital Jordan algebra  $(J, e)$ , the Lie triple system of the pointed symmetric space  $(J^\times, e)$  is given by

$$[x, y, z] = T(x, y, z) - T(y, x, z) = [L_x, L_y]z = x \bullet (y \bullet z) - y \bullet (x \bullet z) \quad (2.4)$$

(see [BeNe05, Theorem 3.4] and its proof which is algebraic in nature). For invertible  $a$ , the Lie triple system of  $(J^\times, a)$  is then given by the same formula, but replacing  $\bullet$  by  $\bullet_a$  and  $T$  by  $T_a(x, y, z) = T(x, Q(a)y, z) = S(x, y, z)$ .

Starting now with a Jordan pair  $(V^+, V^-)$  and arbitrary  $a \in V^-$ , we apply the preceding remarks to the unital Jordan algebra  $\widehat{J} = \mathbb{K}\mathbf{1} \oplus J$  with  $J = V_a$ , as in the proof of Theorem 2.3. Then the Lie triple system of the pointed symmetric space  $(\mathbf{1} + \mathcal{U}_a, \mathbf{1}) = \widehat{J}^\times \cap (\mathbf{1} + J)$  is given by restricting the triple product (2.4) (with  $\bullet = \bullet_a$ ) to  $J$ . ■

The preceding result applies to all finite-dimensional and to all Banach Jordan pairs over  $\mathbb{R}$  or  $\mathbb{C}$ , but also to many other not so classical cases. Note that the (C2)-condition assures in particular that the sets  $\mathcal{U}_a$  are not reduced to  $\{0\}$ . As to Part (2) of the theorem, it is a general fact that antisymmetrizing a Jordan triple system in the first two variables gives a Lie triple system (the ‘‘Jordan-Lie functor’’, see Section 1.6). The preceding construction works also in other contexts where some sort of differential calculus can be implemented (cf. Remark 10.1 in [BeNe05]), and it can also be applied if we start with any non-unital Jordan algebra instead of  $J_a$ ; then  $\mathcal{U}_a$  corresponds to the set of quasi-invertible elements in this algebra.

### 3. Deformations of generalized projective geometries

The geometric or ‘‘integrated’’ version of a Jordan pair  $(V^+, V^-)$  is a *generalized projective geometry*  $(\mathcal{X}^+, \mathcal{X}^-)$ , see Appendix B. We give an analog of Theorem 2.3 in this context; it generalizes [Be03, Section 6], where only the case of Jordan algebras has been treated (cf. last part of the claim).

**Theorem 3.1.** *Let  $(\mathcal{X}^+, \mathcal{X}^-)$  be the generalized projective geometry associated to a Jordan pair  $(V^+, V^-)$ , fix  $a, b \in \mathcal{X}^-$  such that  $\mathcal{U}_{a,b} := \mathfrak{v}_a \cap \mathfrak{v}_b$  is non-empty, and let  $p : \mathcal{U}_{a,b} \rightarrow \mathcal{X}^-$ ,  $x \mapsto \mathbf{\Pi}_{\frac{1}{2}}(a, x, b)$  be the midpoint map assigning to  $x$  the geometric midpoint of  $a$  and  $b$  in the affine space  $\mathfrak{v}_x$ . Then the set  $\mathcal{U}_{a,b}$  becomes a symmetric space when equipped with the multiplication map*

$$\mu(x, y) = (-1)_{x, p(x)}(y) = \mathbf{\Pi}_{-1}(x, \mathbf{\Pi}_{\frac{1}{2}}(a, x, b), y).$$

*In case  $(V^+, V^-) = (J, J)$  comes from a unital Jordan algebra  $J$ , then  $p$  extends to a bijection denoted by  $p : \mathcal{X}^- \rightarrow \mathcal{X}^+$ ; in this case the symmetric space  $\mathcal{U}_{a,b}$  is the symmetric space associated to a polar geometry (see Chapter 3), and it is isomorphic to the quadratic prehomogeneous symmetric space  $J^\times$ .*

**Proof.** Let us show first that the set  $\mathcal{U}_{a,b}$  is indeed stable under the symmetries  $\sigma_x = (-1)_{x,p(x)}$  for  $x \in \mathcal{U}_{a,b}$ . To this end, note that, since  $(-1) = (-1)^{-1}$ , the identity (PG1) implies that, for  $x \in \mathcal{U}_{c,b}$ , the pair

$$(g^+, g^-) := (\sigma_x^+, \sigma_x^-) = ((-1)_{x,p(x)}, (-1)_{p(x),x})$$

acts as an automorphism on  $(\mathcal{X}^+, \mathcal{X}^-)$ . By definition,  $\sigma_x = \sigma_x^+$  is the first component of this automorphism. Then, using that  $\mathbf{\Pi}_r(u, c, v) = \mathbf{\Pi}_{1-r}(v, c, u)$  and that  $r_{v,c}^{-1} = (r^{-1})_{v,c}$ , we see that

$$\sigma_x^-(c) = \mathbf{\Pi}_{-1}(\mathbf{\Pi}_{\frac{1}{2}}(a, x, b), x, a) = \mathbf{\Pi}_2(a, x, \mathbf{\Pi}_{\frac{1}{2}}(a, x, b)) = b,$$

and similarly  $\sigma_x^-(b) = a$ . Therefore  $\sigma_x^+(\mathfrak{v}_b) = \mathfrak{v}_{\sigma^-(b)} = \mathfrak{v}_a$  and  $\sigma_x(\mathfrak{v}_a) = \mathfrak{v}_b$  and hence  $\sigma_x$  preserves  $\mathcal{U}_{a,b}$ . Moreover, it follows that the group  $G(\mathcal{U}_{a,b})$  generated by all products of two symmetries preserves both  $a$  and  $b$  and hence acts affinely both on  $\mathfrak{v}_a$  and on  $\mathfrak{v}_b$ .

The defining identities (S1) – (S4) of a symmetric space are now all easily checked:  $x$  is fixed by  $\sigma_x$  since  $(-1)_{x,c}(x) = x$  for all  $c \top x$ ;  $\sigma_x^2 = \text{id}$  since  $(-1)_{x,c}^2 = 1_{x,c} = \text{id}$ ; the tangent map of  $\sigma_x$  at  $x$  is  $-\text{id}_{T_x M}$  since in the chart  $\mathfrak{v}_{p(x)}$ , centered at  $x$ , this map is just the negative of the identity; finally, in order to prove (S3) note first that, because of the fundamental identity (PG2), the pair  $(p, p)$  is self-adjoint, and thus for all scalars  $r$ ,

$$p \circ r_{x,p(x)}(y) = p \mathbf{\Pi}_r(x, p(x), y) = \mathbf{\Pi}_r(p(x), x, p(y)) = r_{p(x),x} \circ p(y)$$

In particular, for  $r = -1$ , we have  $p \circ (-1)_{x,p(x)} = (-1)_{p(x),x} \circ p$ , i.e.,  $p \circ \sigma_x = \sigma'_x \circ p$ . Using this, we obtain

$$\begin{aligned} \sigma_x \mu(y, z) &= \sigma_x \mathbf{\Pi}_{-1}(y, p(y), z) = \mathbf{\Pi}_{-1}(\sigma_x y, \sigma_x^- p(y), \sigma_x z) \\ &= \mathbf{\Pi}_{-1}(\sigma_x y, p(\sigma_x y), \sigma_x z) = \mu(\sigma_x y, \sigma_x z), \end{aligned}$$

which is precisely the property (S3).

Finally, assume that  $(V^+, V^-)$  comes from a unital Jordan algebra. In this case, there is a canonical identification of  $\mathcal{X}^+$  and  $\mathcal{X}^-$ , denoted by  $n : \mathcal{X}^+ \rightarrow \mathcal{X}^-$  (see [BeNe04], [Be03]), and then the property that  $\mathfrak{v}_a \cap \mathfrak{v}_b$  be non-empty is equivalent to saying that the pair  $(a, n(b))$  is quasi-invertible. If this is the case, the map  $p$  is, via  $n$ , identified with  $(-1)_{a,n(b)}$  ([Be03, 2.1]), and hence extends canonically to a bijection  $\mathcal{X}^- \rightarrow \mathcal{X}^+$  (called an *inner polarity*). In [Be03, Section 6] it is shown that the symmetric space associated to an inner polarity is, in a suitable chart, the same as the quadratic prehomogeneous space  $J^\times$ . ■

**3.2. The quasi-inverse revisited.** In order to show that the symmetric spaces from Theorem 3.1 and from Theorem 2.3 coincide, we need the global interpretation of the quasi-inverse via translation groups: in any module  $\mathfrak{m}$  over a ring  $\mathbb{K}$  define for  $r \in \mathbb{K}$  the proper dilations  $r_x(y) = (1 - r)x + ry$ ; then if 2 is invertible in  $\mathbb{K}$ , vector addition can be recovered from the dilations via  $x + y = 2 \frac{x+y}{2} = 2_0 2_x^{-1}(y)$ , i.e., translations are given by  $\tau_x = 2_0 2_x^{-1}$ . In a generalized projective geometry, and with respect to a fixed base point  $(o, o')$ , the identity (PG1) then implies that the pair

$$(\tau_x, \tilde{\tau}_x) = (2_{o,o'} 2_{x,o'}^{-1}, 2_{o',o}^{-1} 2_{o',x})$$

is an automorphism. Dually,  $(\tilde{\tau}_a, \tau_a)$  for  $a \in V^-$  is then defined. One can show that, in the affine chart  $(V^+, V^-)$  corresponding to the base point  $(o, o')$ ,  $\tilde{\tau}_a$  is indeed the quasi-inverse from Jordan theory. Similarly, we have

$$(\tau_{-x}, \tilde{\tau}_{-x}) = (2_{x,o'} 2_{o,o'}^{-1}, 2_{o',x}^{-1} 2_{o',o}).$$

**Theorem 3.3.** *Let notation be as in Theorem 3.1 and fix  $o' := b$  as base point in  $\mathcal{X}^-$  and choose an arbitrary base point  $o \in V_a \cap V_b$ . We write  $(V^+, V^-)$  for the Jordan pair associated to the base point  $(o, o')$ . Then  $\mathcal{U}_{a,b} = \mathcal{U}_{a,o'}$  is naturally identified with  $\mathcal{U}_a$ , and in the global chart  $V^+$  of  $\mathcal{U}_{a,b}$ , the symmetric space structure from Theorem 3.1 is given by the explicit formula from Theorem 2.3.*

**Proof.** We calculate in the Jordan pair  $(V^+, V^-) = (\mathfrak{v}_{o'}, \mathfrak{v}'_o)$ . For  $a \in V^-$  and  $x \in V^+$ , the following are equivalent:

$$\begin{aligned} x \top a &\Leftrightarrow x \in \mathfrak{v}_a = \tilde{\tau}_a(\mathfrak{v}_{o'}) = \tilde{\tau}_a(V^+) \\ &\Leftrightarrow \tilde{\tau}_{-a}(x) \in V^+ \\ &\Leftrightarrow (x, -a) \text{ quasi-invertible.} \end{aligned}$$

Hence  $\mathfrak{v}_a \cap \mathfrak{v}_{o'} = \tilde{\tau}_a(V^+) \cap V^+ = \mathcal{U}_a$ .

Now let us show that the symmetric space structure from Theorem 3.1 is given by the explicit formulae

$$\begin{aligned} \sigma_o &= -\tilde{\tau}_{-a} \\ \sigma_x \sigma_o &= \mathcal{Q}(x) = \tau_{2x+Q(x)a} \circ B(x, -a). \end{aligned} \tag{3.1}$$

The proof is in several steps. Let  $d := \frac{1}{2}a$ , i.e.,  $a = 2d = 2_{o,o'}d$ . Then

$$\begin{aligned} \sigma_x(y) &= \mathbf{\Pi}_{-1}(x, \mathbf{\Pi}_{\frac{1}{2}}(o', x, 2d), y) = \mathbf{\Pi}_{-1}(x, 2_{o',x}^{-1} 2_{o',o}d, y) \\ &= \mathbf{\Pi}_{-1}(x, \tilde{\tau}_{-x}d, y) \\ &= \tau_{-x} \mathbf{\Pi}_{-1}(2x, d, y+x) \\ &= \tau_{-x} \circ (-1)_{2x,d} \circ \tau_x(y), \end{aligned} \tag{3.2}$$

and for  $x = o$  this gives

$$\sigma_o = (-1)_{o,d} = (-1)_{\tau_d \circ o, \tau_d \circ o'} = \tilde{\tau}_d(-1)_{o,o'} \tilde{\tau}_{-d} = -\tilde{\tau}_{-2d} = -\tilde{\tau}_{-a},$$

proving the first relation from (3.1). Now we are going to prove the second relation from (3.1). Recall from [Be02, Cor. 5.8] or [Be00, X.3.2] that one can express the dilations  $r_{z,b}$  for a scalar  $r \in \mathbb{K}$  and a quasi-invertible pair  $(z, b)$  via translations, quasi-inverses and the dilation  $r \text{id}_{V^+} = r_{o,o'}$ . For the scalar  $r = -1$ , the formula reads:

$$(-1)_{z,b} = \tau_x \circ \tilde{\tau}_{2\tau_{-z}(b)} \circ \tau_z \circ (-1) = (-1) \circ \tau_{-z} \circ \tilde{\tau}_{-2\tau_{-z}(b)} \circ \tau_{-z} \tag{3.3}$$

where  $(-1) = (-1)_{o,o'} = -\text{id}$ . Using this, we get from (3.2) along with  $\sigma_o = (-1)_{o,d}$ ,

$$\begin{aligned} \sigma_x \sigma_o &= \tau_{-x} \circ (-1)_{2x,d} \circ \tau_x \circ (-1)_{o,d} \\ &= \tau_{-x} \circ (-1) \circ \tau_{-2x} \circ \tilde{\tau}_{-2\tau_{-2x}(d)} \circ \tau_{-2x} \circ \tau_x \circ (-1) \circ \tilde{\tau}_{-2d} \\ &= \tau_x \circ \tilde{\tau}_{2\tau_{-2x}(d)} \circ \tau_x \circ \tilde{\tau}_{-2d} \end{aligned} \tag{3.4}$$

From the proof of Theorem 3.1, we know already that  $\sigma_x \sigma_o$  acts as an affine transformation on  $\mathcal{U}_a = V^+ \cap \mathfrak{v}_a$ . Thus, in order to prove the second relation from (3.1), it suffices to determine its linear part and its value at  $o$ . The value at  $o$  of this transformation is

$$\begin{aligned} \sigma_x \sigma_o(o) &= x + \tilde{\tau}_{2\tau_{-2x}(d)}(x) \\ &= x + \frac{1}{2} \tilde{\tau}_{\tau_{-2x}(d)}(2x) \\ &= x + \frac{1}{2} (2x + Q(2x)d) = 2x + 2Q(x)d = 2x + Q(x)a \end{aligned}$$

where we passed to the last line by using the covariance property of the quasi-inverse map [Lo75, Th. I.3.7] which implies, in the notation of [Lo75],

$$\begin{aligned}\tilde{\tau}_{\tau-z}^{-1}(z) &= z^{(b-z)} = B(-z, b)((z-z)^b - (-z)^b) \\ &= -B(-z, b)B(-z, b)^{-1}(-z - Q(-z)b) = z + Q(z)b.\end{aligned}$$

Let us determine the linear part of  $\sigma_x \sigma_o$ . For a general automorphism  $g$  such that  $g(o) \in V^+$ , the linear part in the triple decomposition  $g \in \tau_{V^+} \text{Str}(V) \tilde{\tau}_{V^-}$  is the inverse of the *denominator*  $d_g(o)$  (see [BeNe04, Theorem 2.10 and its proof]). Now, the denominator  $d_g(o)$  of the transformation  $g = \tilde{\tau}_w \circ \tau_v$  is  $d_g(o) = B(v, w)$  (see [BeNe04]), and hence the linear part of  $\sigma_x \sigma_o$  is

$$\begin{aligned}(d_{\sigma_x \sigma_o}(o))^{-1} &= (d_{\tilde{\tau}_{2\tau-2x}(d)} \circ \tau_x(o))^{-1} \\ &= B(x, 2\tilde{\tau}_{-2x}(d))^{-1} = B(2x, \tilde{\tau}_{-2x}(d))^{-1} \\ &= B(2x, -d) = B(x, -2d) = B(x, -a)\end{aligned}$$

where we used JP35 in passing to the last line. Summing up,  $\sigma_x \sigma_o = \tau_{2x+Q(x)a} \circ B(x, -a)$ , and (3.1) is proved.  $\blacksquare$

Note that combining Theorem 3.1 and Theorem 3.3 gives another (in fact, independent) proof of Theorem 2.3. Such a proof does not involve the technique (which is artificial from a geometric point of view) of adjoining a unit element to the Jordan algebras  $V_a$ .

## 4. Conformal deformations of polar geometries

**4.1. The symmetric space of a generalized polar geometry.** Recall from Appendix B.3 that Jordan triple systems  $(V, T)$  correspond to *generalized polar geometries*  $(\mathcal{X}^+, \mathcal{X}^-; p)$ , i.e., to generalized projective geometries with polarity  $p: \mathcal{X}^+ \rightarrow \mathcal{X}^-$ . We denote by

$$M^{(p)} := \{x \in \mathcal{X}^+ \mid x \top p(x)\} \quad (4.1)$$

the set of non-isotropic points of the polarity  $p$ . This set becomes a symmetric space when equipped with the product

$$\mu(x, y) := \sigma_x(y) := (-1)_{x, p(x)}(y) = \mathbf{II}_{-1}(x, p(x), y) \quad (4.2)$$

(see [Be02, th. 4.1]; the proof is similar to the one of Theorem 3.1). Fixing a point  $o \in M^{(p)}$  and choosing the transversal pair  $(o, o') = (o, p(o))$  as base point in  $(\mathcal{X}^+, \mathcal{X}^-)$ , we get as affine image of  $M^{(p)}$  the analog of the well-known “bounded realization” of a Hermitian symmetric space. Explicitly, when we identify  $\mathcal{X}^+$  and  $\mathcal{X}^-$  and  $V^+$  and  $V^-$  via  $p$ , then  $(x, p(x))$  is a transversal pair is transversal iff  $(x, -x)$  is a quasi-invertible pair (see Chapter 3), i.e., iff the Bergman operator  $B(x, -x)$  is invertible, and hence<sup>1</sup>

$$M^{(p)} \cap V = \{x \in V \mid B(x, -x) \text{ is invertible}\}. \quad (4.3)$$

---

<sup>1</sup> One should note that the sign in the correspondence between spaces and triple systems is a matter of convention. Geometers look at Hermitian symmetric spaces of non-compact type as having “negative” curvature, and on the other hand, one is used to associate them to “positive” Hermitian JTS, see [Lo77]. In order to check signs, one may use that for the JTS  $\mathbb{R}$  with  $T(x, y, z) = 2xyz$ , the positive Bergmann polynomial  $B(x, -x) = (1 + x^2)^2$  must belong to the compact space  $\mathbb{R}\mathbb{P}^1$  and the “opposite” Bergmann polynomial  $B(x, x) = (1 - x^2)^2$  to its non-compact dual.

**4.2. The symmetric space associated to the  $\alpha$ -homotope.** Now consider the Jordan triple systems  $V_\alpha = (V, T_\alpha)$ , the  $\alpha$ -homotopes of  $V$  with respect to symmetric maps  $\alpha : V \rightarrow V$  (cf. Lemma 1.12). Since  $V_\alpha$  is a JTS in its own right, the preceding construction associates to  $V_\alpha$  a generalized polar geometry  $(\mathcal{X}_\alpha^+, \mathcal{X}_\alpha^-, p_\alpha)$  with symmetric space  $M_\alpha$ . The explicit formula for the affine image (analog of the “bounded realization”) is of the same type as (4.3), where the underlying  $\mathbb{K}$ -modules is the same, but  $T$  is replaced by  $T_\alpha$  and  $B$  by  $B_\alpha(u, v) = B(u, \alpha v)$ . However, *a priori*, the global spaces  $\mathcal{X}_\alpha^+$  for various  $\alpha$ , are different spaces. In order to compare the geometries  $\mathcal{X}_\alpha$  and the symmetric spaces  $M_\alpha$  with each other and to realize them as deformations of  $M^{(p)}$ , we need a common realization, which is given by the following result:

**Theorem 4.3.** *Let  $(\mathcal{X}^+, \mathcal{X}^-; p)$  be the generalized polar geometry of a Jordan triple system  $(V, T)$ , and let  $\alpha : V \rightarrow V$  be a  $T$ -symmetric map, i.e.,  $\alpha \in \text{Svar}(T)$ . Let  $(V, T_\alpha)$  be the  $\alpha$ -homotope of  $(V, T)$  and let  $(\mathcal{X}_\alpha^+, \mathcal{X}_\alpha^-; p_\alpha)$  be the generalized polar geometry associated to  $(V, T_\alpha)$ , with symmetric space denoted by  $M_\alpha \subset \mathcal{X}_\alpha^+$ . Then there exists a canonical embedding  $\Phi_\alpha : \mathcal{X}_\alpha^+ \rightarrow \mathcal{X}^+$  having the following properties:*

- (1) *Identifying  $V_\alpha$  and  $V$  as affine spaces, the affine image of  $M_\alpha$  under this imbedding coincides with the description of  $M_\alpha$  in the chart  $V_\alpha$ , i.e.*

$$\Phi_\alpha(M_\alpha) \cap V = \{x \in V \mid B(x, -\alpha x) \in \text{GL}(V)\}$$

- (2) *The tangent map of  $\Phi_\alpha$  at the origin can be algebraically defined, and it is the identity map  $V_\alpha \rightarrow V$ .*
- (3) *If  $\alpha$  is invertible, then the image of this imbedding is precisely the symmetric space  $M^{(p\alpha)}$  associated to the polarity  $p \circ \alpha : \mathcal{X}^+ \rightarrow \mathcal{X}^-$ .*

**Proof.** First we describe the construction of  $\Phi_\alpha$ . Saying that  $\alpha : V \rightarrow V$  is  $T$ -selfadjoint is equivalent to saying that the map of Jordan pairs

$$(\alpha, \alpha) : (V, V) \rightarrow (V, V)$$

is a structural transformation (see Appendix C for definition). We apply Lemma C.3, with  $(W^+, W^-) = (V^+, V^-)$  and  $(f, g) = (\alpha, \alpha)$ ; then the Jordan pair  $(V^+, W^-)$  from Lemma C.3 is  $(V_\alpha, V_\alpha)$ , with the JTS  $V_\alpha$  defined by Lemma 1.12. The second point of Lemma C.3 says that

$$\varphi = (\text{id}, \alpha) : (V_\alpha, V_\alpha) \rightarrow (V, V)$$

is a homomorphism of Jordan pairs. As in Section C.4, the homomorphism  $(\text{id}, \alpha)$  induces a homomorphism

$$(\Phi^+, \Phi^-) : (\mathcal{X}^+(V_\alpha), \mathcal{X}^-(V_\alpha)) \rightarrow (\mathcal{X}^+(V), \mathcal{X}^-(V))$$

with injective first component  $\Phi^+ : \mathcal{X}^+(V_\alpha) \rightarrow \mathcal{X}^+(V)$ .

We define  $\Phi_\alpha := \Phi^+$  and show that this imbedding has the desired properties. Since  $(\Phi^+, \Phi^-)$  is a homomorphism of generalized projective geometries, its restriction to affine parts with respect to the base points is given by the pair of linear maps which induces it, namely by  $(\text{id}, \alpha)$ , and hence the algebraic tangent map can be defined and is equal to the pair  $(\text{id}, \alpha)$  (cf. [Be02]). Thus the tangent map of  $\Phi^+$  at the base point  $o$  is the identity map  $V_\alpha \rightarrow V$ . This proves (2).

We prove (1). Let  $x \in V^+ = V$ . Recall that the element  $p(x) \in \mathcal{X}^-$  is identified with  $x \in V^- = V$ . By definition,  $x$  belongs to the symmetric space  $M^{(p)}$  if and only if  $x$  and  $p(x)$  are transversal, if and only if the pair  $(x, -p(x)) = (x, -x)$  is transversal (see proof of 3.1), if and only if  $B(x, -x)$  is invertible. The same argument, applied to the JTS  $T_\alpha$ , shows that  $x$

belongs to  $M_\alpha$  if and only if  $B_\alpha(x, -x)$  is invertible. But  $B_\alpha(x, y) = B(x, \alpha y)$  (Lemma 1.12), and hence  $x$  belongs to  $M_\alpha$  if and only if  $B(x, -\alpha x)$  is invertible.

We prove (3). Assume  $\alpha$  is  $T$ -selfadjoint and invertible. Then  $(\text{id}, \alpha) : (V_\alpha, V_\alpha) \rightarrow (V, V)$  is a Jordan pair isomorphism, and the induced map  $(\Phi^+, \Phi^-)$  is an isomorphism of geometries. Under this isomorphism, the polarity of  $(\mathcal{X}^+(V_\alpha), \mathcal{X}^-(V_\alpha))$  defining the symmetric space  $M_\alpha$  (the exchange automorphism) corresponds to the map  $p \circ \alpha$  which is therefore the polarity defining the symmetric space  $\Phi_\alpha(M_\alpha)$ . This had to be shown. ■

**Remark 4.4.** (Realisation as a graph.) Assume that  $\alpha$  is a  $T$ -symmetric map. Then it is easily checked that the graph of  $\alpha$ ,  $\Gamma_\alpha = \{(v, \alpha v) \mid v \in V\}$ , is a sub-JTS of the polarized JTS  $(V \oplus V, \tilde{T})$  with

$$\tilde{T}((x, a), (y, b), (z, c)) = (T(x, b, z), T(a, y, c)),$$

and that  $\Gamma_\alpha$  is isomorphic to  $V_\alpha$  as JTS. By the preceding construction, this situation globalizes as follows: the polarized JTS  $(V \oplus V, \tilde{T})$  corresponds to the polar geometry  $(\mathcal{X}^+ \times \mathcal{X}^-, \mathcal{X}^- \times \mathcal{X}^+)$  with the exchange automorphism (see [Be02, Prop. 3.6]) and with associated symmetric space  $M = \{(x, \alpha) \in \mathcal{X}^+ \times \mathcal{X}^- \mid x \top \alpha\}$ . Then  $M_\alpha$  is the intersection of  $M$  with the graph of the map

$$\Phi^+(\mathcal{X}^+) \rightarrow \mathcal{X}^-, \quad \Phi^+(x) \mapsto \Phi^-(x).$$

**Remark 4.5.** (Alternative proof in the real finite-dimensional case.) In the finite-dimensional real or complex case, the main statements of Theorem 4.3 can be proved in a different way: if  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$  with involution  $\Theta$  is the involutive 3-graded Lie algebra associated to the JTS  $T$ , then

$$\mathfrak{q}_\alpha := \{v + \Theta(\alpha v) \mid v \in \mathfrak{g}_1\}$$

is a sub-Lie triple system of  $\mathfrak{g}$  (a “defining plane” in the terminology of [Ri70]). Let  $\mathfrak{h}_\alpha = [\mathfrak{q}_\alpha, \mathfrak{q}_\alpha]$  and define the subalgebra  $\mathfrak{g}_\alpha := \mathfrak{h}_\alpha \oplus \mathfrak{q}_\alpha$  of  $\mathfrak{g}$ . Let  $G_\alpha$  and  $H_\alpha$  be the analytic subgroups of  $G = \text{Aut}(\mathfrak{g})$  with Lie algebra  $\mathfrak{g}_\alpha$ , resp.  $\mathfrak{h}_\alpha$ . Then the orbit  $M_\alpha := G_\alpha \cdot o$  of the base point  $o$  in  $\mathcal{X}^+$  is open, and it is a homogeneous symmetric space isomorphic to  $G_\alpha/H_\alpha$  coinciding (possibly up to topological connected components) with the space  $M_\alpha$  from Theorem 4.3. This construction can already be found in the work [Ri70] by A.A. Rivillis. Since it uses the exponential map as essential ingredient, it gives rather results of local nature, and it does not work for general base fields.

**Theorem 4.6.** *Assume that  $\mathbb{K}$  is a topological ring with dense unit group, and that  $T$  is a topological Jordan pair over  $\mathbb{K}$  such that*

(C1) *The set  $(V \times V)^\times = \{(x, y) \in V \times V \mid B(x, y) \in \text{GL}(V)\}$  of quasi-invertible pairs is open in  $V \times V$ , and the map*

$$(V \times V)^\times \times V \rightarrow V, \quad (x, y, z) \mapsto B(x, y)^{-1}z$$

*is continuous.*

*Assume moreover that  $\alpha$  is a continuous  $T$ -symmetric map. Then  $M_\alpha$  is a smooth symmetric space over  $\mathbb{K}$ , and it is an open submanifold of  $\mathcal{X} = \mathcal{X}^+$ . The Lie triple system of the pointed symmetric space  $(M_\alpha, o)$  is  $V$  equipped with the Lie triple product*

$$[X, Y, Z]_\alpha := T_\alpha(X, Y, Z) - T_\alpha(Y, X, Z) = T(X, \alpha Y, Z) - T(Y, \alpha X, Z).$$

**Proof.** This follows directly from [BeNe05, Theorem 6.3] since the JTS  $T_\alpha$  again satisfies the property (C1). ■

In particular, the LTS  $[\cdot, \cdot, \cdot]_\alpha$  and  $[\cdot, \cdot, \cdot]_{-\alpha}$  are negatives of each other, which means that the symmetric spaces  $M_\alpha$  and  $M_{-\alpha}$  are  $c$ -duals of each other.

## 5. Behavior of geometric structures under deformation

For applications in geometry and harmonic analysis one not only needs to know how the affine image  $M_\alpha \cap V$  of the symmetric space  $M_\alpha$  as a set, but also how its geometric structure depends on the deformation parameter  $\alpha$ . We give some answers to this question. As a rule, the dependence on the parameter  $\alpha$  can be made explicit if we simply mind the rules: in the following formulae, replace  $M$  by  $M_\alpha$ ,  $T$  by  $T_\alpha$ ,  $B(x, y)$  by  $B(x, \alpha y)$  and  $Q(x)$  by  $Q(x) \circ \alpha$ .

**5.1. The product map.** The product map of the symmetric space  $M$  is given by Formula (3.3), by letting  $x = z$ ,  $b = p(x)$ :

$$\begin{aligned} \sigma_x(y) &= (-1)_{x,p(x)}(y) = \tau_x \tilde{\tau}_{\tilde{\tau}_x(x)} \circ (-1)_{o,o'} \circ \tilde{\tau}_{-\tilde{\tau}_x(x)} \tau_{-x}(y) \\ &= \tau_x \tilde{\tau}_{2\tilde{\tau}_x(x)} \tau_x (-1)_{o,o'}(y) \\ &= \tau_{2\tilde{\tau}_x(x)} B(2\tilde{\tau}_x(x), -x)^{-1} \tilde{\tau}_{2\tilde{\tau}_x(x)}(y) \end{aligned} \quad (5.1)$$

In particular, the “square”  $x^2 := \sigma_x(o)$  in the symmetric space  $M$  is

$$\sigma_x(o) = 2\tilde{\tau}_x(x) = 2(\text{id}_V - Q(x))^{-1}x \quad (5.2)$$

(the last equality is proved as in [Be00, p. 195]).

**5.2. Second derivatives and second order tangent bundles.** The tangent bundle  $TM$  of  $M$ , which is again a symmetric space, is simply obtained by scalar extension by dual numbers, as already used in Section 2.1 (cf. [BeNe05, Th. 5.5]). Hence the geometrically important second tangent bundle  $TTM$  is given by scalar extension by the second order tangent ring  $TT\mathbb{K} = \mathbb{K}[\varepsilon_1, \varepsilon_2]$ . The same statement holds for the geometry  $(\mathcal{X}^+, \mathcal{X}^-)$  itself. For instance, by a straightforward calculation using that  $\varepsilon_1^2 = 0 = \varepsilon_2^2$ , we get from the definition of the quasi-inverse and from (5.1)

$$\begin{aligned} \tilde{\tau}_a(\varepsilon_1 v_1 + \varepsilon_2 v_2) &= B(\varepsilon_1 v_1 + \varepsilon_2 v_2, a)^{-1}(\varepsilon_1 v_1 + \varepsilon_2 v_2 - Q(\varepsilon_1 v_1 + \varepsilon_2 v_2)a) \\ &= \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_1 \varepsilon_2 T(v_1, a, v_2), \\ \sigma_x(x + \varepsilon_1 v_1 + \varepsilon_2 v_2) &= x - \varepsilon_1 v_1 - \varepsilon_2 v_2 + \varepsilon_1 \varepsilon_2 T(v_1, \tilde{\tau}_x(x), v_2). \end{aligned}$$

On the other hand, ordinary second differentials and second order tangent maps of twice continuously differentiable maps  $f$  are related via

$$f(x + \varepsilon_1 v_1 + \varepsilon_2 v_2) = f(x) + \varepsilon_1 df(x)v_1 + \varepsilon_2 df(x)v_2 + \varepsilon_1 \varepsilon_2 d^2 f(x)(v_1, v_2)$$

(see [Be06, Eqn. (7.18)]), so that from the preceding calculation we get the second differential of  $\tilde{\tau}_a$  at the origin and the one of  $\sigma_x$  at  $x$ .

**5.3. The canonical affine connection.** It is possible to define the canonical affine connection of the symmetric space in a purely algebraic way. In the situation of Theorem 4.6 it coincides with any of the various differential geometric definitions of a connection (cf. [Be06, Chapter 26]). More precisely, in [Be06, Chapter 26] it is shown that the canonical connection  $\nabla$  of a symmetric space  $M$  can be defined in terms of the second order tangent bundle  $TTM$ , and we have the following chart description: let  $x \in M$ ,  $V$  a chart around  $x$  inducing the flat connection  $\nabla^0$  on some neighborhood of  $x$ ; then the difference  $C := \nabla - \nabla^0$  (i.e., the Christoffel tensor of the canonical connection with respect to the chart) at the point  $x$  is simply the second derivative of

the symmetry  $\sigma_x$  at  $x$ :  $C_x = d^2\sigma_x(x)$ . Since we have calculated this second differential above, we get

$$C_x(u, v) = T(v_1, \tilde{\tau}_{-x}(x), v_2) = T(v_1, (\text{id}_V - Q(x))^{-1}x, v_2) \quad (5.3)$$

(for the last equality, see (5.2)). In the finite-dimensional real case this result is already given in [BeHi01, Theorem 2.3]; however, the proof given there does not carry over to the general situation considered here. As explained in [BeHi01], (5.3) implies that the canonical connections of the symmetric spaces  $M_\alpha$  are *conformally equivalent* among each other. Note also that, for  $\alpha = 0$ , we get  $C_x = 0$  for all  $x$ , and hence  $\nabla = \nabla_0$ , as expected.

**5.4. Trace form and pseudo-metrics.** It is clear that semi-simplicity is not preserved under deformations. If  $\mathbb{K}$  is a field and the JTS  $V$  is finite-dimensional over  $\mathbb{K}$ , then we say that the JTS  $V$  is *non-degenerate* if the *trace form*  $g_0(u, v) := \text{tr}(T(u, v, \cdot))$  is non-degenerate. If this is the case, then  $g_0$  extends to a pseudo-Riemannian metric tensor (non-degenerate symmetric two fold covariant tensor field) with affine picture given by

$$g_x(u, v) := g_0(u, B(x, -x)^{-1}v), \quad (u, v \in V, x \in M) \quad (5.4)$$

(see [Be00, X.6]). On the other hand, the dual two fold contravariant tensor field with affine picture  $\gamma_x(\varphi, \psi) = g_0(\varphi, B(x, -x)\psi)$  can always be defined and exists also for the deformed spaces.

**5.5. Invariant measures: group case.** Let  $\mathbb{K} = \mathbb{R}$  and  $A$  a finite dimensional unital  $\mathbb{R}$ -algebra. In general, the Lie group  $G = A^\times$  and its deformations  $G_\alpha$  will not be unimodular (take, e. g., the algebra of upper triangular square matrices). However, if  $G = A^\times$  happens to be unimodular (e.g, the case of the general linear groups), then also its deformations  $G_\alpha$  remain unimodular: the modular function is  $g \mapsto \det(\text{Ad}(g))$ ; if this function is trivial for all deformation parameters in some interval, it remains trivial for all deformation parameters in the closure of this interval.

**5.6. Invariant measures on symmetric spaces.** Let  $\mathbb{K} = \mathbb{R}$  and  $V$  a finite-dimensional JTS over  $\mathbb{R}$ . Then in general there is no measure on  $M$  which is invariant under the transvection group  $G(M)$ . Such a measure exists if  $V$  is non-degenerate: in this case, the corresponding invariant integral is given by the explicit formula

$$I(f) = \int_{M \cap V} f(x) |\det B(x, -x)|^{-\frac{1}{2}} dx, \quad (5.5)$$

where  $dx$  is the Lebesgue measure on  $V$  (see [Be00, X.6.2]). However, as above, it is true that, if  $I(f)$  is an invariant integral for the symmetric space  $M_{\text{id}_V}$ , then the same formula defines an invariant integral for the deformed symmetric spaces  $M_\alpha$ . The proof uses essentially the same argument as the one for unimodular Lie groups: by the invariance properties of the Bergmann operator  $B(x, x)$  (see [Be00, VIII.2.8]), the change of variables formula shows that the integral  $I$  is invariant under a diffeomorphism  $g$  if we have

$$\forall y \in M \cap V : \quad |\det d(\tilde{\tau}_{-y}g)(0)| = |\det d(g^{-1}\tau_y)|.$$

In the non-degenerate case this is true for all symmetries  $g = \sigma_x$ . The dependence of both sides on the deformation parameter  $\alpha$  is continuous, and the set of parameters  $\alpha$  corresponding to non-degenerate homotopes  $T_\alpha$  is open and dense in the structure variety; therefore we have equality for all  $\alpha$  in the structure variety. In particular, for  $\alpha \rightarrow 0$ , the invariant measure tends to the Lebesgue measure of  $V$ .

**5.7. Further results and topics.** Further results on geometric structures on the spaces  $M_\alpha$  in the real, non-degenerate case can be found in the paper [Ma79] by B.O. Makarevich, and it

would certainly be interesting to generalize some of these results. Finally, one would like to study the dependence of harmonic analysis and representation theory on the symmetric spaces  $M_\alpha$  on the deformation parameter  $\alpha$ . Although the situation here is different from the one considered, e.g., in [DR85], it seems possible to transfer ideas developed in the context of contractions of semisimple Lie groups to the set-up just described. Since all relevant geometric data of  $M_\alpha$  depend algebraically or at least analytically on  $\alpha$ , one may expect that something similar is true on the level of data of harmonic analysis. See, for instance [FaPe04] and [Pe02] for examples of harmonic analysis on symmetric spaces of the type  $M_\alpha$ . The parameter  $\alpha$  is fixed in these works, so that the dependence on  $\alpha$  is not visible. When varying  $\alpha$ , not only the aspect of deformations of  $\alpha$  to singular values is of interest, but also the one of *analytic continuation* between isotopic, but non isomorphic values of  $\alpha$ . For instance, the duality between *compactly causal* and *non-compactly causal symmetric spaces* is reflected by the isotopy between  $\alpha$  and  $-\alpha$  (cf. [Be00, XI.3]).

## 6. Some examples

**6.1. Quadratic matrices and associative algebras.** Let us return to the setting of Section 0.1 and consider the associative algebra  $A = M(n, n; \mathbb{K})$  of square matrices over a field  $\mathbb{K}$ . Then the  $A^\times \times A^\times$ -orbits are represented by the idempotent matrices

$$e := e_r := \begin{pmatrix} \mathbf{1}_r & 0 \\ 0 & 0 \end{pmatrix}, \quad r = 0, \dots, n.$$

The group  $G_e = \{x \in A \mid \mathbf{1} + ex \in A^\times\}$  is explicitly given by

$$G_e = \left\{ x = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \det \begin{pmatrix} \mathbf{1}_r + \alpha & \beta \\ 0 & \mathbf{1}_{n-r} \end{pmatrix} \neq 0 \right\} = \left\{ \begin{pmatrix} \mathbf{1}_r + \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \det(\mathbf{1}_r + \alpha) \neq 0 \right\}$$

(where  $\beta, \gamma, \delta$  are arbitrary), with product  $x \diamond_e y = xey + x + y$ , i.e.

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \diamond_e \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & \alpha\beta' \\ \gamma\alpha' & \gamma\beta' \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

This group is isomorphic to a semidirect product of  $\mathrm{GL}(r, \mathbb{K})$  and a Heisenberg type group: for  $\beta, \gamma, \delta$  zero, we get a subgroup isomorphic to  $\mathrm{GL}(r, \mathbb{K})$ , and for  $\alpha = 0$ , we obtain a subgroup with product

$$\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} \diamond_e \begin{pmatrix} 0 & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma\beta' \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} + \begin{pmatrix} 0 & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

It is easily checked that  $G_e$  is the semidirect product of these two subgroups. The Lie algebra of  $G_e$  is given by the bracket  $[x, y]_e = xey - yex$ , which yields explicitly

$$\left[ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \right]_e = \begin{pmatrix} \alpha\alpha' - \alpha'\alpha & \alpha\beta' - \alpha'\beta \\ \gamma\alpha' - \gamma'\alpha & \gamma\beta' - \gamma'\beta \end{pmatrix}$$

so that for  $\alpha = 0 = \alpha'$  we get the 2-step nilpotent algebra

$$\left[ \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} 0 & \beta' \\ \gamma' & \delta' \end{pmatrix} \right]_e = \begin{pmatrix} 0 & 0 \\ 0 & \gamma\beta' - \gamma'\beta \end{pmatrix}.$$

If  $A$  is a general associative algebra and  $e$  an idempotent in  $A$ , then the preceding calculations carry over, where matrices have to be understood with respect to the *Peirce-decomposition* of  $A$ , where  $c := \mathbf{1} - e$ :

$$A = cA \oplus eA = cAc \oplus cAe \oplus eAc \oplus eAe.$$

Then  $G_e$  is isomorphic to a semidirect product of  $(eAe)^\times$  with a Heisenberg type group  $cAe \oplus eAc \oplus cAc$ . However, in general not every element of  $A$  is similar to an idempotent, and hence other types of deformations can appear.

**6.2. Rectangular matrices.** We consider the Jordan pair  $(V^+, V^-) = (M(p, q; \mathbb{K}), M(q, p; \mathbb{K}))$  of rectangular matrices over a ring  $\mathbb{K}$ , with trilinear maps  $T^\pm(x, y, z) = xyz + zyx$ . The automorphism group of this Jordan pair contains  $\mathrm{GL}(p, \mathbb{K}) \times \mathrm{GL}(q, \mathbb{K})$ , acting in the usual way from the right and from the left, and every matrix  $a \in V^-$  is conjugate under this action to a matrix of the form  $e = e_r$  with  $e_{ii} = 1$  for  $i = 1, \dots, r$  and 0 else. Thus we may assume  $a = e$ , and then the symmetric space structure on  $\mathcal{U}_e$  from Theorem 2.3 is calculated in a similar way as above. Recall that a pair  $(x, y) \in (M(p, q; \mathbb{K}), M(q, p; \mathbb{K}))$  is quasi-invertible if and only if the matrix  $\mathbf{1}_p - xy$  is invertible in  $M(p, p; \mathbb{K})$ ; therefore  $\mathcal{U}_e$  is the set of matrices  $x \in M(p, q; \mathbb{K})$  such that  $\mathbf{1}_p + xe$  is invertible, i.e.

$$\mathcal{U}_e = \left\{ \begin{pmatrix} \mathbf{1}_r + \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \det(\mathbf{1}_r + \alpha) \neq 0, \quad \beta, \gamma, \delta \text{ arbitrary} \right\}.$$

Using that  $Q(x)e = xex$  and  $B(x, -e)y = (\mathbf{1} + xe)y(\mathbf{1} + ex)$ , we get the quadratic map of the symmetric space  $\mathcal{U}_e$

$$Q(x)y = 2x + Q(x)e + B(x, -e)y = 2x + y + xex + xey + yex + xeyex.$$

Again one finds that  $\mathcal{U}_e$  has a bundle structure: the basis is given by letting  $\beta, \gamma, \delta$  equal to zero; it is the space of  $r \times r$ -Matrices, with the symmetric space structure

$$Q(\alpha)\alpha' = 2\alpha + \alpha' + \alpha^2 + \alpha\alpha' + \alpha'\alpha + \alpha\alpha'$$

which is isomorphic to the group  $\mathrm{GL}(r, \mathbb{K})$ , seen as a symmetric space (group case, cf. Appendix A.3). The fiber is given by letting  $\alpha = 0$ , and we get explicitly

$$Q\left(\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}\right)\begin{pmatrix} 0 & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} 0 & 2\beta + \beta' \\ 2\gamma + \gamma' & 2\delta + \delta' \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma\beta + \gamma'\beta' + \gamma'\beta \end{pmatrix}.$$

Knowing that the quasi-inverse in the Jordan pair of rectangular matrices is given by  $\tilde{\tau}_a(x) = (\mathbf{1} - ax)^{-1}x(\mathbf{1} - xa)^{-1}$ , we find that the symmetry with respect to the origin in the fiber is

$$\sigma_0\left(\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}\right) = \begin{pmatrix} 0 & -\beta \\ -\gamma & -\delta + \gamma\beta \end{pmatrix},$$

so that the product map of the fiber is finally

$$\mu\left(\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} 0 & \beta' \\ \gamma' & \delta' \end{pmatrix}\right) = \begin{pmatrix} 0 & 2\beta - \beta' \\ 2\gamma - \gamma' & 2\delta - \delta' \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (\gamma - \gamma')(\beta - \beta') \end{pmatrix}.$$

In contrast to the preceding section, where the fiber was a two-step nilpotent group, for the symmetric space structure the fiber is now simply a flat a symmetric space. This is not surprising since the triple Lie bracket on a two-step nilpotent group is zero. Explicitly, one can check by direct calculation that the map

$$F : \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} 0 & \beta \\ \gamma & \delta + \gamma\beta \end{pmatrix}$$

is an isomorphism from the fiber onto the same set, but equipped with the flat symmetric space structure  $2x - y$ . (The situation is completely similar to the one considered in [Be06], where the symmetric space structure on the second order tangent bundle  $TTM$  of a symmetric space  $M$  is calculated, see loc. cit., Chapters 26 and BA.)

**6.3. Symmetric spaces associated to idempotents.** The preceding calculations and remarks can be generalized for an arbitrary Jordan pair  $(V^+, V^-)$ , as follows: we assume that  $e := e^- \in V^-$  is an element which can be completed to an idempotent, i.e., there exists  $e^+ \in V^+$  such that  $Q^+(e^+)e^- = e^+$  and  $Q^-(e^-)e^+ = e^+$ . (In other words,  $e$  is *von Neumann regular*, cf. [Lo75].) Associated to an idempotent  $(e^+, e^-)$ , there is a *Peirce-decomposition*  $V^+ = V_2 \oplus V_1 \oplus V_0$  which is the eigenspace decomposition of the operator  $T(e^+, e^-)$ . The space  $V_1 \oplus V_0$  depends only on  $e^-$  since  $V_1 \oplus V_0 = \ker Q(e^-)$ , and  $V_2$  depends only on  $e^+$  since  $V_2 = Q(e^+)V^-$ . Moreover,  $V_2$  is a unital Jordan algebra with unit element  $e^+$  (see [Lo75] for all this).

**Theorem 6.4.** (Bundle structure of  $\mathcal{U}_e$ .) *Assume  $e = e^- \in V^-$  as a von Neumann regular element and choose  $e^+ \in V^+$  such that  $(e^+, e^-)$  is an idempotent.*

- (1)  $\mathcal{U}_e = \{x = x_0 + x_1 + x_2 \mid x_0 \in V_0, x_1 \in V_1, x_2 + e^+ \in V_2^\times\}$
- (2) *The space  $\mathcal{U}_e \cap V_2 = e^+ - V_2^\times$  is a symmetric subspace of  $\mathcal{U}_e$  (called the base), isomorphic to the quadratic prehomogeneous symmetric space  $V_2^\times$ .*
- (3) *The space  $V_1 \oplus V_0$  is a symmetric subspace of  $\mathcal{U}_e$  (called the fiber), and its symmetric space structure is isomorphic to the flat symmetric structure  $2x - y$  on this set.*

**Proof.** (1) We use the identity JP24 of [Lo75] which gives

$$B(x, e)B(x, -e) = B(x, Q(e)x) = B(Q(x)e, e).$$

Assume  $x \in V_0 \oplus V_1$ , i.e.,  $Q(e)x = 0$ . Hence  $B(x, e)B(x, -e) = B(x, 0) = \text{id}$ . In the same way,  $B(x, -e)B(x, e) = \text{id}$ , and so  $B(x, -e)$  is always invertible, with inverse  $B(x, e)$ . This shows that  $V_0 \oplus V_1 \subset \mathcal{U}_e$ . Moreover, then  $\tilde{\tau}_x(e) = e$  since  $B(e, x)^{-1}(e - Q(e)x) = B(e, -x)(e - Q(e)x) = B(e, -x)e = e$ .

Assume  $x \in V_2$ . Then  $B(x, -e)$  is invertible in  $V^+$  if and only if its restriction to  $V_2$  is invertible, and hence, by the results on the Jordan algebra case (Prop. 2.2), we see that this is equivalent to saying that  $e^+ + x$  is invertible in  $V_2$ .

Now let  $z \in V^+$  and decompose  $z = x + y$  with  $y \in V_0 \oplus V_1$  and  $y \in V_2$ . Then  $(y, e)$  is quasi-invertible with  $\tilde{\tau}_y(e) = e$ , as we have seen above, and hence we may apply the identity JP34 of [Lo75] in order to obtain  $B(x + y, e) = B(x, \tilde{\tau}_y(e))B(y, e) = B(x, e)B(y, e)$ . This operator is invertible if and only if  $x$  and  $y$  satisfy the previously described conditions, and this proves the claim.

(2) This follows directly from (1) since the formula for the product  $\mu(x, y)$  with  $x, y \in (V_2^\times + e^+)$  uses only data which depend on the Jordan algebra  $V_2$ .

(3) We first calculate  $\sigma_0(y)$  for an element  $y \in \ker Q(e)$ : using that  $B(y, -e)^{-1} = B(y, e)$ , we get

$$\begin{aligned} \sigma_0(y) &= \tilde{\tau}_e(-y) = B(-y, e)^{-1}(-y - Q(y)e) = -B(y, e)y - B(y, e)Q(y)e \\ &= -y + T(y, e, y) - Q(y)e + T(y, e, Q(y)e) = -y + Q(y)e \end{aligned}$$

since, by JP1,  $T(y, e, Q(y)e) = Q(y)T(e, y, e) = 0$ . Next we calculate the map  $Q(x)y = 2x + Q(x)e + B(x, -e)(y)$ . Since  $B(x, -e)y = y + T(x, e, y) + Q(x)Q(e)y = y + T(x, e, y)$ , we get

$$Q(x)y = 2x + y + Q(x)e + T(x, e, y) = 2x + y + \frac{1}{2}T(x, e, x) + T(x, e, y).$$

As in the matrix case (Section 6.2), it is checked by a direct calculation that the map

$$F : V_0 \oplus V_1 \rightarrow V_0 \oplus V_1, \quad z \mapsto z + Q(z)e$$

is an isomorphism from the symmetric space  $V_0 \oplus V_1$  with the usual flat structure  $(x, y) \mapsto 2x - y$  onto the symmetric space  $V_0 \oplus V_1$  equipped with the symmetric space structure that we have just described.  $\blacksquare$

The preceding proof was purely Jordan theoretic. It would be interesting to state and prove Theorem 6.4 in a geometric way, i.e., in the context of Theorem 3.1. The starting point should be a suitable geometric interpretation of von Neumann regularity. This is indeed possible and will be taken up elsewhere.

**6.5. Grassmannians.** Next we give some examples illustrating Theorem 4.3. Assume  $W = \mathbb{K}^{p+q}$  is a vector space over a field  $\mathbb{K}$  of characteristic different from 2 and let  $(\mathcal{X}^+, \mathcal{X}^-) = (\mathcal{G}ras_p(W), \mathcal{G}ras_q(W))$  the Grassmannian geometry of  $p$ , resp.  $q$ -spaces in  $W$ . Then  $(\mathcal{X}^+, \mathcal{X}^-)$  is a generalized projective geometry; with respect to the base point  $(o^+, o^-) = (\mathbb{K}^p \oplus 0, 0 \oplus \mathbb{K}^q)$ , it corresponds to the Jordan pair  $(M(p, q; \mathbb{K}), M(q, p; \mathbb{K}))$ . We define a family of symmetric space structures associated to  $(\mathcal{X}^+, \mathcal{X}^-)$ : assume  $\beta : W \times W \rightarrow \mathbb{K}$  is a symmetric or skew-symmetric bilinear form on  $W$  such that  $\beta$  is non-degenerate on the first summand  $o^+ = \mathbb{K}^p$ . If  $\beta$  is non-degenerate on all of  $W$ , then  $p : \mathcal{X}^+ \rightarrow \mathcal{X}^-$ ,  $E \mapsto E^\beta$ , where  $E^\beta$  is the orthogonal complement of  $E$  with respect to  $\beta$ , is a globally defined polarity, but if  $\beta$  is degenerate, then  $p$  is only defined on the (open dense) set  $\{E \in \mathcal{X}^+ \mid \dim(E^\beta) = q\}$ . In any case, the space of non-degenerate linear subspaces of  $W$ ,

$$M^\beta := \{E \subset W \mid W = E \oplus E^\beta\}$$

is a symmetric space with product map

$$\mu(E, F) = \sigma_E(F) = (-1)_{E, E^\beta}(F),$$

where, for any scalar  $r \in \mathbb{K}$  and decomposition  $W = U \oplus V$ , we let  $r_{U, V}$  be the linear map which is 1 on  $U$  and  $r$  on  $V$ . Let us study how the symmetric space structure and the Jordan triple system of  $M^\beta$  depends on  $\beta$ . In order to do this, we have to assume that  $p(o^+) = o^-$ , i.e., that  $\beta(o^+, o^-) = 0$ . In other words,  $\beta$  is of the form

$$\beta = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad \text{i.e.} \quad \beta((x, y), (x', y')) = \beta_1(x, x') + \beta_2(y, y') = x^t B_1 x' + y^t B_2 y'$$

with  $x, x' \in \mathbb{K}^p$ ,  $y, y' \in \mathbb{K}^q$  and (skew-) symmetric matrices  $B_1$  (size  $p \times p$ ) and  $B_2$  (size  $q \times q$ ). The property that  $p(o^+) = o^-$  assures that the affine picture of  $p$  is a linear map from  $M(p, q; \mathbb{K})$  to  $M(q, p; \mathbb{K})$ . We calculate this map explicitly: let  $X \in M(p, q; \mathbb{K})$ ; this element corresponds to the graph

$$\Gamma_X = \{Xx \mid x \in \mathbb{K}^p\} \in \mathcal{X}^+.$$

Using that  $B_1$  is invertible, we find that the orthogonal complement of  $\Gamma_X$  is given by

$$\begin{aligned} (\Gamma_X)^\beta &= \{(u, v) \in \mathbb{K}^p \times \mathbb{K}^q \mid \forall x \in \mathbb{K}^p : \beta((u, v), (x, Xx)) = 0\} \\ &= \{(u, v) \in \mathbb{K}^p \times \mathbb{K}^q \mid \forall x \in \mathbb{K}^p : u^t B_1 x + v^t B_2 Xx = 0\} \\ &= \{(u, v) \in \mathbb{K}^p \times \mathbb{K}^q \mid \forall x \in \mathbb{K}^p : u^t x + v^t B_2 X B_1^{-1} x = 0\} \\ &= \{(u, v) \in \mathbb{K}^p \times \mathbb{K}^q \mid \forall x \in \mathbb{K}^p : u^t x = -(B_2 X B_1^{-1} v)^t x\} \\ &= \{( -(B_2 X B_1^{-1} v)^t, u) \mid u \in \mathbb{K}^q\} = \Gamma_{-(B_2 X B_1^{-1})^t}. \end{aligned}$$

The linear picture of the map  $p$  is therefore

$$M(p, q; \mathbb{K}) \rightarrow M(q, p; \mathbb{K}), \quad X \mapsto -(B_2 X B_1^{-1})^t = -(\alpha X)^t$$

with  $\alpha(X) = B_2 X B_1^{-1}$ , and the JTS of the symmetric space  $M^\beta$  is

$$T_\alpha(X, Y, Z) = X(\alpha Y)^t Z + Z(\alpha Y)^t X = X(B_2 Y B_1^{-1})^t Z + Z(B_2 Y B_1^{-1})^t X,$$

which is the  $\alpha$ -homotope of  $T(X, Y, Z) = XY^t Z + ZY^t X$ .

Let us consider some special choices of  $B_1$  and  $B_2$ . Recall that  $B_1$  has to be invertible, and  $B_1$  and  $B_2$  are either both symmetric or both skew-symmetric. In case  $\mathbb{K} = \mathbb{R}$  and  $\mathbb{K} = \mathbb{C}$ , the symmetric spaces we obtain are essentially homogeneous and of the type  $M^\beta \cong O(B_1 \oplus B_2)/(O(B_1) \times O(B_2))$ . The isomorphism classes of symmetric spaces obtained for invertible matrices  $B_2$  are listed in [Be00, Tables XII.4]:

- Symmetric case: by choosing for  $B_1$  and  $B_2$  invertible diagonal matrices having coefficients  $b_{ii} \in \{\pm 1\}$ , we get all symmetric spaces of the type  $O(l+j, k+i)/(O(l, k) \times O(i, j))$ .
- Skew-symmetric case: by choosing the usual matrices of symplectic forms (the dimensions have to be pair), we get all symmetric spaces of the type  $\text{Sp}(l+j, \mathbb{K})/(\text{Sp}(l, \mathbb{K}) \times \text{Sp}(j, \mathbb{K}))$ .

Standard examples of deformations to degenerate cases are obtained by taking some of the diagonal coefficients of  $B_2$  equal to zero. Since orthogonal groups of degenerate forms can be described as semidirect products of vector groups, general linear groups and orthogonal groups of non-degenerate forms, it is possible to give explicit descriptions of such symmetric spaces. Note that in case  $p$  even and  $q$  odd a new series of symmetric spaces arises which does not arise as deformation of a semisimple series: choose  $B_1$  to be a symplectic form and  $B_2$  skew-symmetric and non-zero.

**6.6. Other examples, and some final comments.** Similar examples of deformations of semi-simple symmetric spaces arise also in all other classical series of symmetric spaces (see the classification in [Be00, Chapter XII]) and also for about half of the simple exceptional symmetric spaces (cf. comments in [Be00, Section 0.6]). In order to keep this work in reasonable bounds, we cannot go here into further details. Let us just mention that all classical groups arise as group cases by our construction (the general linear groups are related to associative algebras, see Section 0.1, and all orthogonal, symplectic and unitary groups are related to certain Jordan triple systems); it is then an interesting problem to determine the deformations of these symmetric spaces that are again of groupe type. In other words, we raise the problem: *which deformations of groupe type symmetric spaces come from deformations of the overlying group structure* (as was the case for the general linear group in Section 0.1)? This question should not be confused with another, also very natural problem: *if  $(M_\alpha)_{\alpha \in \text{Svar}(T)}$  is a deformation of the symmetric space  $M = M_{\text{id}}$ , in which sense are then the transvection groups  $G_\alpha := G(M_\alpha)$  deformations of the transvection group  $G = G(M)$ ?* They will in general not be “contractions of  $G$ ” in the sense of [DR85] since the dimension of  $G(M_\alpha)$  will not be constant as a function of  $\alpha$ . Replacing transvection groups by automorphism groups leads to a variant of this problem. It seems that contractions of  $\text{Aut}(M)$  in the sense of [DR85] can only be obtained by using some additional parameter which in some sense memorizes the structure of the stabilizer group of a point.

## Appendix A. Symmetric spaces

**A.1. Reflection spaces with base point: a group-like concept.** A *reflection space* is a set  $M$  such that, for all  $x \in M$ , there is a *symmetry*  $\sigma_x : M \rightarrow M$  satisfying

$$(S1) \quad \sigma_x(x) = x$$

$$(S2) \quad \sigma_x \circ \sigma_x = \text{id}_M$$

$$(S3) \quad \text{for all } x, y \in M, \sigma_x \sigma_y \sigma_x = \sigma_{\sigma_x(y)}.$$

The map

$$\mu : M \times M \rightarrow M, \quad (x, y) \mapsto \mu(x, y) := \sigma_x(y)$$

is called the *multiplication map*. The concept of a reflection space is “affine”, in the sense that it does not refer to a distinguished base point. On the other hand, for some purposes it is necessary to choose a base point  $o \in M$ . Following Loos [Lo69], we then define the *quadratic map* and the *square*, for  $x \in M$ ,

$$\mathcal{Q}(x) := \mathcal{Q}_o(x) := \sigma_x \circ \sigma_o : M \rightarrow M, \quad x^2 := \mathcal{Q}(x)o = \sigma_x(o).$$

The symmetry  $\sigma_o$  is called *inversion* and is sometimes denoted by  $x^{-1} := \sigma_o(x)$ . Note that  $\mu(x, y)$  can be recovered from these data via

$$\mu(x, y) = \sigma_x(y) = \sigma_x \sigma_o \sigma_o(y) = \mathcal{Q}(x)j(y) = \mathcal{Q}(x)\mathcal{Q}(y)^{-1}y.$$

The following relations are easily checked (cf. [Lo69] or [Be00, Chapter I]): for all  $x, y \in M$ ,

$$(SB1) \quad \mathcal{Q}(\mathcal{Q}(x)y) = \mathcal{Q}(x)\mathcal{Q}(y)\mathcal{Q}(x)$$

$$(SB2) \quad \mathcal{Q}(x^{-1}) = \mathcal{Q}(x)^{-1}$$

$$(SB3) \quad x^{-1} = \mathcal{Q}(x)^{-1}x$$

**Proposition A.2.** *There is an equivalence of categories between the category of reflection spaces with base point and the category of pointed sets  $(M, o)$  equipped with two maps  $\mathcal{Q} : M \times M \rightarrow M$ ,  $(x, y) \mapsto \mathcal{Q}(x)y$  and  $\sigma_o : M \rightarrow M$ ,  $x \mapsto x^{-1}$  such that (SB1) – (SB3) hold.*

**Proof.** Given  $(M, o; \mathcal{Q}, \sigma_o)$ , we define

$$\mu(x, y) := \sigma_x(y) := \mathcal{Q}(x)y^{-1} = \mathcal{Q}(x)\mathcal{Q}(y)^{-1}y,$$

and (S1) – (S3) are easily proved from (SB1) – (SB3). (For (S3), start by proving that  $\sigma_o \mathcal{Q}(x) \sigma_o = \mathcal{Q}(x)^{-1}$ .) This construction clearly is inverse to the one described above, and it is also clear that we get the same morphisms. ■

The various spaces  $(M, a, \mathcal{Q}_a, \sigma_a)$  for  $a \in M$  are in general not isomorphic among each other (the map  $\mathcal{Q}(a)$  is an isomorphism from  $(M, o)$  onto  $(M, a^2)$  and not onto  $(M, a)$ ). In order to change base points, we have the “isotopy formula”:

$$\mathcal{Q}_a(x) = \sigma_x \sigma_a = \sigma_x \sigma_o (\sigma_a \sigma_o)^{-1} = \mathcal{Q}(x)\mathcal{Q}(a)^{-1}.$$

**A.3. Symmetric spaces.** A *symmetric space* is a reflection space such that

(S4) the tangent map  $T_x \sigma_x$  of the symmetry  $\sigma_x$  at the fixed point  $x$  is equal to the negative of the identity of the tangent space  $T_x M$ .

Of course, this makes only sense when the notion of tangent map is defined, e.g., if  $M$  is a smooth manifold and  $\mu : M \times M \rightarrow M$  is a smooth map. But in some interesting cases tangent maps can also be defined in a purely algebraic sense. Typically, there will be some natural chart around  $x$  such that  $\sigma_x$  is, in this chart, represented by the negative of the identity; then (S4) holds.

If  $(M, o)$  is a symmetric space with base point, in the category of smooth manifolds, one may define its *Lie triple system* in a similar way as the Lie algebra of a Lie group; this is most naturally done by using the quadratic maps  $\mathcal{Q}(x)$ , cf. [Lo69] or [Be06, Chapters 5 and 27].

Important examples of symmetric spaces arise from the *group cases*: a group  $G$  with the new product  $\mu(g, h) = \sigma_g(h) = gh^{-1}g$  satisfies (S1), (S2), (S3), and if  $G$  is, e.g., a Lie group, then also (S4) holds. Subspaces of  $(G, \mu)$  are given by *spaces of symmetric elements*: if  $\tau$  is an involution of a group  $G$  (anti-automorphism of order 2), then  $M := \{g \in G \mid \tau(g) = g^{-1}\}$  is stable under  $\mu$ , i.e., it is a subspace.

## Appendix B: Generalized projective geometries

The geometric counterpart of a Jordan pair is a *generalized projective geometry*, and the one of a JTS a *generalized polar geometry*. In this appendix we recall the basic definitions and refer to [Be02] and [BeNe04] for further details.

**B.1. Affine pair geometries.** A *pair geometry* is given by a pair of sets, denoted by  $(\mathcal{X}^+, \mathcal{X}^-)$ , and a binary *transversality relation*  $(\mathcal{X}^+ \times \mathcal{X}^-)^\top \subset \mathcal{X}^+ \times \mathcal{X}^-$ ; we write  $x \top \alpha$  or  $\alpha \top x$  if  $(x, \alpha)$  belongs to  $(\mathcal{X}^+ \times \mathcal{X}^-)^\top$ , and we require that for all  $\alpha \in \mathcal{X}^\mp$  there exists  $x \in \mathcal{X}^\pm$  such that  $x \top \alpha$ . In other words,  $\mathcal{X}^\pm$  is covered by the sets

$$\mathfrak{v}_\alpha := \mathfrak{v}_\alpha^\pm := \alpha^\top := \{x \in \mathcal{X}^\mp \mid x \top \alpha\} \quad (\text{B.1})$$

as  $\alpha$  runs through  $\mathcal{X}^\mp$ . An *affine pair geometry over a commutative ring*  $\mathbb{K}$  is a pair geometry such that, for every  $\alpha \in \mathcal{X}^\mp$ , the set  $\mathfrak{v}_\alpha$  is equipped with the structure of an affine space over  $\mathbb{K}$ ; then the sets  $\mathfrak{v}_\alpha$  will be called *affine parts* or *affine charts* of  $\mathcal{X}^\pm$ . Given an affine pair geometry, we let, for  $(x, a), (y, a) \in (\mathcal{X}^+ \times \mathcal{X}^-)^\top$  and  $r \in \mathbb{K}$ ,  $r_{x,a}(y) := ry$  be the product  $r \cdot y$  in the  $\mathbb{K}$ -module  $\mathfrak{v}_a$  with zero vector  $x$ , and we define the *structure maps* by

$$\mathbf{\Pi}_r := \mathbf{\Pi}_r^+ : (\mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+)^\top \rightarrow \mathcal{X}^+, \quad (x, a, y) \mapsto \mathbf{\Pi}_r(x, a, y) := r_{x,a}(y), \quad (\text{B.2})$$

where

$$(\mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+)^\top := \{(x, a, y) \in \mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+ \mid x \top a, y \top a\}, \quad (\text{B.3})$$

and dually the maps  $\mathbf{\Pi}_r^-$  are defined. Similarly, we get structure maps defined by vector addition:

$$\mathbf{\Sigma} := \mathbf{\Sigma}^+ : (\mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+ \times \mathcal{X}^+)^\top \rightarrow \mathcal{X}^+, \quad (x, a, y, z) \mapsto \mathbf{\Sigma}(x, a, y, z) := y +_{x,a} z, \quad (\text{B.4})$$

where the sum is taken in the  $\mathbb{K}$ -module  $(\mathfrak{v}_a, x)$  and the set  $(\mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+ \times \mathcal{X}^+)^\top$  is defined by a similar condition as in (B.3). Dually,  $\mathbf{\Sigma}^-$  is defined.

**B.2. Categorical notions.** Assume  $(\mathcal{X}^+, \mathcal{X}^-)$  and  $(\mathcal{Y}^+, \mathcal{Y}^-)$  are affine pair geometries over  $\mathbb{K}$ .

1. **Duality.** All axioms of an affine pair geometry appear together with their dual version: thus  $(\mathcal{X}^-, \mathcal{X}^+; \top; \mathbf{\Pi}^-, \mathbf{\Pi}^+, \mathbf{\Sigma}^-, \mathbf{\Sigma}^+)$  is again an affine pair geometry, called the *dual geometry* of  $(\mathcal{X}^+, \mathcal{X}^-; \top; \mathbf{\Pi}^+, \mathbf{\Pi}^-, \mathbf{\Sigma}^+, \mathbf{\Sigma}^-)$ .
2. **Homomorphisms.** These are pairs of maps  $(g^+ : \mathcal{X}^+ \rightarrow \mathcal{Y}^+, g^- : \mathcal{X}^- \rightarrow \mathcal{Y}^-)$  which are compatible with the transversality relations and with the structure maps, in the obvious sense.
3. **Antiautomorphisms.** These are isomorphisms  $(g^+, g^-) : (\mathcal{X}^+, \mathcal{X}^-) \rightarrow (\mathcal{X}^-, \mathcal{X}^+)$  onto the dual geometry. If  $g^- = (g^+)^{-1}$ , then  $(g^+, g^-)$  is a *correlation*, and if, moreover, there exists a *non-isotropic point* (i.e., there is  $x \in \mathcal{X}^+$  with  $x \top g^+(x)$ ), then  $(g^+, g^-)$  is called a *polarity*.
4. **Adjoint (or: structural) pairs of morphisms.** These are pairs of maps  $(f : \mathcal{X}^+ \rightarrow \mathcal{Y}^+, h : \mathcal{Y}^- \rightarrow \mathcal{X}^-)$  which are compatible with transversality and with structure maps in the sense that  $x \top h(a)$  iff  $f(x) \top a$  and

$$f \cdot \mathbf{\Pi}_s(u, h.v, w) = \mathbf{\Pi}_s(f.u, v, f.w), \quad f \cdot \mathbf{\Sigma}(u, h.v, w, h.z) = \mathbf{\Sigma}(f.u, v, h.w, z).$$

It is necessary to allow here also maps that are not everywhere defined (see Appendix C). A map  $f : \mathcal{X}^+ \rightarrow \mathcal{X}^-$  is called *selfadjoint* if  $(f, f)$  is an adjoint pair of morphisms into the dual geometry (compare with Section 1.12).

5. Stability. A pair geometry  $(\mathcal{X}^+, \mathcal{X}^-, \top)$  is called *stable* if any two points  $x, y \in \mathcal{X}^\pm$  are on a common affine chart, i.e., there exists  $a \in \mathcal{X}^\mp$  such that  $x, y \in \mathfrak{v}_a$ .

**B.3. The affine pair geometry associated to a Jordan pair.** Assume that  $V = (V^+, V^-)$  is a linear Jordan pair over  $\mathbb{K}$ , which we obtain in the form  $(V^+, V^-) = (\mathfrak{g}_1, \mathfrak{g}_{-1})$  for a 3-graded Lie algebra  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$  with  $T^\pm(x, y, z) = [[x, y], z]$  (see Section 1.4). Let  $G := \text{PE}(V^+, V^-) = \langle U^+, U^- \rangle$  be the *projective elementary group* associated to this 3-graded Lie algebra, i.e., the group generated by the vector groups  $U^\pm = \exp(\mathfrak{g}_{\pm 1})$ , let  $H$  be the intersection of  $G$  with the automorphism group of the 3-grading (called the *inner structure group*) and let  $P^\pm := U^\pm H$ . Since  $U^\pm$  are vector groups, it is easily seen that

$$(\mathcal{X}^+, \mathcal{X}^-) := (\mathcal{X}^+(V), \mathcal{X}^-(V)) := (G/P^-, G/P^+)$$

is an affine pair geometry. The base point  $(eP^-, eP^+)$  is often denoted by  $(o^+, o^-)$ , and the affine parts  $(\mathfrak{v}_{o^-}, \mathfrak{v}_{o^+})$  corresponding to this base point are identified with  $(V^+, V^-) = (\exp(\mathfrak{g}^+).o^+, \exp(\mathfrak{g}^-).o^-)$ . (In [BeNe04] a realization of this geometry and of its transversality relation in terms of *Lie algebra filtrations* is given.)

The construction is functorial: a Jordan pair homomorphism induces a homomorphism of affine pair geometries ([Be02, Th. 10.1]). In particular, an involution of a Jordan pair induces a polarity of  $(\mathcal{X}^+, \mathcal{X}^-)$ . Since Jordan triple systems are nothing but Jordan pairs with involution, we thus associate a polar geometry to any JTS.

The geometry associated to a Jordan pair  $(V^+, V^-)$  is stable if and only if the maps

$$V^\mp \times V^\pm \rightarrow \mathcal{X}^\pm, \quad (a, x) \mapsto \exp(a) \exp(x).o^\pm$$

are surjective; then  $\mathcal{X}^\pm$  can be seen as  $V^\pm \times V^\mp$  modulo the equivalence relation given by fibers of this map (called *projective equivalence*; cf. [Lo94]). Another equivalent version of stability is that the projective group  $G$  admits the ‘‘Harish-Chandra decomposition’’  $G = \exp(V^+) \exp(V^-) H \exp(V^+)$  and  $G = \exp(V^-) \exp(V^+) H \exp(V^-)$ , where  $H$  is the structure group.

**B.4. Generalized projective geometries** are affine pair geometries  $(\mathcal{X}^+, \mathcal{X}^-)$  such that the structure maps satisfy some algebraic identities (PG1) and (PG2) which should be seen as global analogs of certain Jordan algebraic identities. Recall from B.1 that  $r_{x,a}(y) = \mathbf{\Pi}_r(x, a, y)$ ,  $r_{a,x}(b) = \mathbf{\Pi}_r^-(a, x, b)$  (‘‘left multiplication operators’’), and denote by  $M_{x,y}^{(r)}(a) := \mathbf{\Pi}_r(x, a, y)$ ,  $M_{a,b}^{(r)}(x) := \mathbf{\Pi}_r^-(a, x, b)$  the ‘‘middle multiplication operators’’. Then the geometry  $(\mathcal{X}^+, \mathcal{X}^-)$  associated to a Jordan pair  $(V^+, V^-)$  satisfies the following properties ([Be02, Th. 10.1]):

- (PG1) For any invertible scalar  $r \in \mathbb{K}$  and any transversal pair  $(x, a)$ , the pair of maps  $(g, g') = (r_{x,a}, r_{a,x}^{-1})$  is an automorphism of the affine pair geometry  $(\mathcal{X}^+, \mathcal{X}^-)$ .
- (PG2) For any  $r \in \mathbb{K}$  and  $a, b \in \mathcal{X}^-$  with  $\mathfrak{v}_a \cap \mathfrak{v}_b \neq \emptyset$ , the pair of middle multiplications  $(f, h) = (M_{a,b}^{(r)}, M_{b,a}^{(r)})$  is an adjoint pair of (locally defined) morphisms from  $(\mathcal{X}^+, \mathcal{X}^-)$  to  $(\mathcal{X}^-, \mathcal{X}^+)$ , and dually.

Applied to elements of  $\mathcal{X}^+$ , resp.  $\mathcal{X}^-$ , (PG1) and (PG2) are indeed algebraic identities for the structure maps. One should think of (PG1) as a global analog of the defining identity (LJP2) (Section 1.4) of a Jordan pair, and of (PG2) as the global analog of the fundamental formula.

## Appendix C: Globalization of structural transformations

**C.1. Homomorphisms and structural transformations.** There are two ways to turn Jordan pairs over  $\mathbb{K}$  into a category: one defines *homomorphisms* in the usual way as pairs of linear maps  $(h^-, h^+) : (V^+, V^-) \rightarrow (W^+, W^-)$  which are compatible with the maps  $T^\pm$ , and one defines a *structural transformation* to be a pair of linear maps  $f : V^+ \rightarrow W^+$ ,  $g : W^- \rightarrow V^-$  such that

$$T_W^+(f(x), y, f(z)) = f(T_V^+(x, g(y), z)), \quad T_V^-(g(u), v, g(w)) = g(T_W^-(u, f(v), w))$$

(cf. [Lo95]). In both cases we get a category; these categories are different, but have essentially the same isomorphisms.

**C.2. Globalization: example of projective spaces.** A homomorphism of Jordan pairs  $(\varphi^+, \varphi^-) : (V^+, V^-) \rightarrow (W^+, W^-)$  always lifts to a homomorphism of associated geometries ([Be02, Th. 10.1]), whereas for structural transformations the situation is more difficult. In order to show that structural transformations do not always extend globally, consider the example of ordinary projective geometry  $\mathcal{X}^+ = \mathbb{K}\mathbb{P}^n$  over a field. Its Jordan pair is  $(W, W^*)$ , with  $W = \mathbb{K}^n$ , and

$$T^+(x, \lambda, y) = x\lambda(y) + y\lambda(x), \quad T^-(\lambda, x, \mu) = \lambda(x)\mu + \mu(x)\lambda.$$

This Jordan pair is simple, and hence homomorphisms  $(W, W^*) \rightarrow (V^+, V^-)$  are either injective or trivial. Clearly, every *injective* linear map lifts to a map  $\mathbb{K}\mathbb{P}^n \rightarrow \mathbb{K}\mathbb{P}^m$  of associated projective spaces, which shows that homomorphisms globalize, as stated above. On the other hand, every linear map  $f : V \rightarrow W$  together with its dual map defines a structural transformation  $(f, g) = (f : V \rightarrow W, f^* : W^* \rightarrow V^*)$ . If  $f$  is not injective, then  $f$  does not lift to a globally defined map of projective spaces. The best we can do is to extend  $f$  to a not everywhere defined map  $\tilde{f}$  from  $\mathbb{K}\mathbb{P}^n$  to  $\mathbb{K}\mathbb{P}^m$ . The “natural domain of definition” of  $\tilde{f}$  is

$$D(\tilde{f}) := \left\{ \left[ \begin{pmatrix} v \\ r \end{pmatrix} \right] \mid v \in V, r \in \mathbb{K} : \begin{pmatrix} fv \\ r \end{pmatrix} \neq 0 \right\},$$

and then  $\tilde{f}(\left[ \begin{pmatrix} v \\ r \end{pmatrix} \right]) = \left[ \begin{pmatrix} fv \\ r \end{pmatrix} \right]$ . The set  $D(\tilde{f})$  is strictly bigger than the affine part  $V$  (except for the trivial case  $f = 0$ ). The purpose of this appendix is to define such a natural domain of definition for any structural transformation of Jordan pairs.

**Lemma C.3.** *If  $(f, g) = (f : V^+ \rightarrow W^+, g : W^- \rightarrow V^-)$  is a structural transformation between Jordan pairs, then the pair  $(V^+, W^-)$  with*

$$\begin{aligned} S^+ : V^+ \times W^- \times V^+ &\rightarrow V^-, & (a, b, c) &\mapsto T_V^+(a, g(b), c), \\ S^- : W^- \times V^+ \times W^- &\rightarrow W^-, & (u, v, w) &\mapsto T_W^-(u, f(v), w) \end{aligned}$$

*is a linear Jordan pair, and the maps*

$$\begin{aligned} \varphi &:= (\varphi^+, \varphi^-) := (f, \text{id}) : (V^+, W^-) \rightarrow (W^+, W^-), & (x, y) &\mapsto (f(x), y) \\ \psi &:= (\psi^+, \psi^-) := (\text{id}, g) : (V^+, W^-) \rightarrow (V^+, V^-), & (x, y) &\mapsto (x, g(y)) \end{aligned}$$

*are homomorphisms of Jordan pairs.*

**Proof.** The first claim can be checked by a direct computation which is completely analogous to the proof of Lemma 1.12. We show that  $\varphi$  is a homomorphism:

$$\begin{aligned} \varphi^+ S^+(a, b, c) &= f T^+(a, g b, c) = T^+(f a, b, f c) = T^+(\varphi^+ a, \varphi^- b, \varphi^+ c), \\ \varphi^- S^-(x, y, z) &= T^-(x, f y, z) = T^-(\varphi^- x, \varphi^+ y, \varphi^- z). \end{aligned}$$

Similarly, we see that  $\psi$  is a homomorphism. ■

**C.4. The globalization of a structural transformation.** Assume  $(f, g)$  is a structural transformation as above, and let  $\varphi$  and  $\psi$  the homomorphisms defined in the lemma. By functoriality, we have induced homomorphisms on the level of geometries:

$$\begin{array}{ccccc}
 & (V^+, W^-) & & \mathcal{X}(V^+, W^-) & \\
 (f, \text{id}) \swarrow & & \searrow (\text{id}, g) & & \searrow (\Psi^+, \Psi^-) \\
 (W^+, W^-) & & (V^+, V^-) & \mathcal{X}(W) & \mathcal{X}(V)
 \end{array}$$

In general, if the first (resp. second) component of a Jordan pair homomorphism is injective, then so is the first (resp. second) component of the induced map of geometries<sup>1</sup>, and hence the first component  $\Psi^+$  and the second component  $\Phi^-$  are injective. The following maps together with their domains of definition are called the *globalization of  $(f, g)$* :

$$\begin{aligned}
 \tilde{f} : D(\tilde{f}) &\rightarrow \mathcal{X}^+(W), & D(\tilde{f}) &:= \Psi^+(\mathcal{X}^+(V^+, W^-)), & \tilde{f}(\Psi^+(x)) &:= \Phi^+(x), \\
 \tilde{g} : D(\tilde{g}) &\rightarrow \mathcal{X}^-(V), & D(\tilde{g}) &:= \Phi^-(\mathcal{X}^-(V^+, W^-)), & \tilde{g}(\Phi^-(a)) &:= \Psi^-(a).
 \end{aligned}$$

The following result will not be needed in this work and its proof is therefore omitted:

**Theorem C.5.** *The pair  $(\tilde{f}, \tilde{g})$  is an adjoint pair of morphisms between  $\mathcal{X}(V)$  and  $\mathcal{X}(W)$  whose domain of definition  $(D(\tilde{f}), D(\tilde{g}))$  is complete in the following sense: whenever  $x \in D(\tilde{f})$ , then  $\tilde{f}(x)^\top \subset D(\tilde{g})$ , and whenever  $a \in D(\tilde{g})$ , then  $\tilde{g}(a)^\top \subset D(\tilde{f})$ . Moreover,  $(D(\tilde{f}), D(\tilde{g}))$  is a generalized projective geometry, isomorphic to  $\mathcal{X}(V^+, W^-)$ , with structure maps given by*

$$\mathbf{\Pi}_r^+(x, a, y) = \mathbf{\Pi}_r(x, \tilde{g}(a), y), \quad \mathbf{\Pi}_r^-(a, y, b) = \mathbf{\Pi}_r^-(a, \tilde{f}(y), b), \quad \blacksquare$$

## References

- [Be00] Bertram, W., *The Geometry of Jordan and Lie Structures*, Springer Lecture Notes in Mathematics vol. **1754**, Springer-Verlag, Berlin 2000.
- [Be02] Bertram, W., “Generalized projective geometries: General theory and equivalence with Jordan structures.” *Advances in Geometry* 2 (2002), 329-369.
- [Be03] Bertram, W., “The geometry of null systems, Jordan algebras and von Staudt’s Theorem.” *Ann. Inst. Fourier* 53 (2003) fasc. 1, 193-225.
- [Be06] Bertram, W., *Differential Geometry, Lie Groups and Symmetric Spaces over General Base Fields and Rings*. *Memoirs of the AMS* (to appear); arXiv: math.DG/0502168.
- [BeHi01] Bertram, W. and J. Hilgert, “Characterization of the Kantor-Koecher-Tits algebra by a generalized Ahlfors operator”, *J. of Lie Theory* 11 (2001), 415–426.
- [BeNe04] Bertram, W. and K.-H. Neeb, “Projective completions of Jordan pairs. I.” *J. Algebra* **277**(2) (2004), 193 – 225.
- [BeNe05] Bertram, W. and K.-H. Neeb, “Projective completions of Jordan pairs. II” *Geom. Dedicata* **112** (2005), 73 – 113.
- [DR85] Dooley, A.H. and J.W. Rice, “On contractions of semisimple Lie groups”, *Trans. A.M.S.* **289** vol.1 (1985), 185 – 202.

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<sup>1</sup> cf. [Lo94, 1.3 (3)] for the case of a stable geometry; the proof of the corresponding fact for the general non-stable case is similar but rather lengthy and is therefore omitted here.

- [FaPe04] Faraut, J and M. Pevzner, “Berezin kernels and Analysis on Makarevic spaces”, preprint; arXiv: math.RT/0411294.
- [Lo69] Loos, O., *Symmetric Spaces I*, Benjamin, New York 1969.
- [Lo75] Loos, O., *Jordan Pairs*, Springer LNM 460, Springer, New York 1975.
- [Lo77] Loos, O., *Bounded symmetric domains and Jordan pairs*, Lecture Notes, Irvine 1977.
- [Lo94] Loos, O., “Decomposition of projective spaces defined by unit-regular Jordan pairs”, *Comm. Alg.* **22** (10) (1994), 3925 – 3964.
- [Ma73] Makarevic, B.O., “Open symmetric orbits of reductive groups in symmetric  $R$ -spaces”, *Math. USSR Sbornik* 20 (1973), no. 3, 406 – 418.
- [Ma79] Makarevic, B.O., “Jordan algebras and orbits in symmetric  $R$ -spaces”, *Trans. Moscow Math. Soc.* **39** (1979) (english translation, AMS, *Trans. Moscow Math. Soc.* 1981, issue 1, 169 – 193).
- [McC04] McCrimmon, K., *A Taste of Jordan Algebras*, Springer 2004.
- [Pe02] Pevzner, M., “Analyse conforme sur les algèbres de Jordan.” [Conformal analysis on Jordan algebras] *J. Aust. Math. Soc.* 73 (2002), no. 2, 279–299.
- [Ri70] Rivillis, A.A., “Homogeneous locally symmetric regions in homogeneous spaces associated with semi-simple Jordan algebras”, *Math. USSR Sbornik* 11 (1970), no. 3.

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