



HAL
open science

Zero-free regions of radar ambiguity functions and moments

Philippe Jaming

► **To cite this version:**

| Philippe Jaming. Zero-free regions of radar ambiguity functions and moments. 2006. hal-00080459v2

HAL Id: hal-00080459

<https://hal.science/hal-00080459v2>

Preprint submitted on 13 Dec 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

UNCERTAINTY PRINCIPLES FOR RADAR AMBIGUITY FUNCTIONS AND MOMENTS

PHILIPPE JAMING

ABSTRACT. In this article, we give an estimate of the zero-free region around the origin of the ambiguity function of a one-dimensional signal u in terms of the moments of u . This is done by proving an uncertainty relation between the first zero of the Fourier transform of a non-negative function and the moments of the function. As a corollary, we also give an estimate of how much a function needs to be translated to obtain a function that is orthogonal to the original function. The paper is completed with a proof of the local uncertainty principle for the ambiguity function.

1. INTRODUCTION

Woodward's time-frequency correlation function or radar ambiguity function [Wo, Wi, AT], as defined by

$$A(u)(x, y) = \int_{-\infty}^{+\infty} u\left(t + \frac{x}{2}\right) \overline{u\left(t - \frac{x}{2}\right)} e^{-2i\pi yt} dt$$

plays a central role in evaluating the ability of a transmitted radar waveform $\operatorname{Re}(u(t)e^{i\omega_0 t})$ to distinguish targets that are separated by range delay x and Doppler frequency y . Ideally, one would like to have $A(u)(x, y) = \delta_{(0,0)}(x, y)$, the Dirac mass at $(0, 0)$, but this desideratum is not achievable because of the so-called “*ambiguity uncertainty principle*”, that is the constraint

$$\iint_{\mathbb{R}^2} |A(u)(x, y)|^2 dx dy = A(u)(0, 0)^2 = \left(\int_{\mathbb{R}} |u(t)|^2 dt \right)^2.$$

As $A(u)$ is continuous when $u \in L^2(\mathbb{R})$ is a signal of finite energy, it follows that $A(u)$ can not vanish in a neighborhood of $(0, 0)$ unless $u = 0$. Further limitations are given by various versions of the uncertainty principle for the ambiguity function, see *e.g.* [BDJ, De, Gr, GZ] and the references therein. These state in particular that if one concentrates $A(u)$ in the x -variable, then one loses concentration in the y -variable. Since ideal behavior is not achievable, it becomes important to determine how closely to the ideal situation one can come.

A first attempt in that direction is due to Price and Hofstetter [PH] who considered the quantity

$$V(E) = \iint_E |A(u)(x, y)|^2 dx dy$$

where E is a measurable subset of \mathbb{R}^2 . Under the assumption that $V(E) \rightarrow V_0$ when E shrinks to $\{(0, 0)\}$, they proved that, if A is a convex set symmetric with respect to the origin, then $V(A) \geq \frac{V_0}{4} \operatorname{area}(A)$. It should however be mentioned that, for this result to be significant, one has to go outside

1991 *Mathematics Subject Classification.* 42B10.

Key words and phrases. zero-free region, ambiguity functions, short time Fourier transform, windowed Fourier transform, local uncertainty principle.

Research partially financed by: *European Commission* Harmonic Analysis and Related Problems 2002-2006 IHP Network (Contract Number: HPRN-CT-2001-00273 - HARP) .

the class of signals of finite energy. Indeed, when u has finite energy, from the continuity of $A(u)$, one gets that

$$V(E) \leq \text{area}(E) \sup_{(x,y) \in E} |A(u)(x,y)|^2 \xrightarrow{E \searrow \{(0,0)\}} \text{area}(\{(0,0)\}) |A(u)(0,0)|^2 = 0.$$

So for $u \in L^2(\mathbb{R})$, $V_0 = 0$ and the bound in [PH] is trivial. Nevertheless, it is possible to define $A(u)$ when u is a Schwartz distribution and the results in [PH, Section II] have then to be read under the implicit assumption that $A(u) \in L^2_{\text{loc}}(\mathbb{R}^2 \setminus \{(0,0)\})$ in which case V_0 may not be 0 and their result is then meaningful.

We will here restrict our attention to signals u of *finite energy* and have some L^2 -moments.¹ In this context, we strengthen the above remark by giving an upper estimate of $V(E)$. This is done by a direct computation when E is a large enough ball and by generalizing Faris' local uncertainty principle when E is the complementary of a small enough ball. In particular we prove the following :

Theorem A. *Let $u \in L^2(\mathbb{R})$ such that $tu, \xi \hat{u} \in L^2(\mathbb{R})$ and let $\delta > 0$. Then*

$$\begin{aligned} - \text{ if } \rho \leq \frac{\max(\|tu\|_2, \|t\hat{u}\|_2)}{65\|u\|_2} \delta^2 \text{ then } \iint_{x^2+y^2 \geq \rho^2} |A(u)(x,y)|^2 dx dy &\geq (1 - \delta^2)\|u\|^4; \\ - \text{ if } R \geq \frac{2(\|tu\|_2 + \|t\hat{u}\|_2)}{\delta\|u\|_2} \text{ then } \iint_{x^2+y^2 \leq \rho^2} |A(u)(x,y)|^2 dx dy &\geq (1 - \delta^2)\|u\|^4. \end{aligned}$$

The second aim of this paper is to determine more precisely the neighborhood of $(0,0)$ on which $A(u)$ does not vanish. To do so, we prove a new form of uncertainty principle, showing that there is an exclusion relation between the function u having moments and its Fourier transform being 0 near 0. More precisely, we prove the following:

Theorem B.

For every $q > 0$ there exists $\kappa_q > 0$ such that, if $u \in L^1(\mathbb{R})$ is a non-negative function, $u \neq 0$, then

$$\inf\{\xi > 0 : \hat{u}(\pm\xi) = 0\}^q \inf_{t_0 \in \mathbb{R}} \| |t - t_0|^q u(t) \|_1 \geq \kappa_q \|u\|_1.$$

This result is inspired by a recent result of Luo and Zhang [LZ]. They prove that if a real non-negative function is supported in $[0, +\infty)$ then there is an uncertainty principle of the above form relating its moments and the first zero its Fourier transform.

It turns out that the ambiguity function, when restricted in a given direction, is always the Fourier transform of a non-negative function, but the support condition of Luo and Zhang is not valid. We thus start by removing that condition in their uncertainty principle, which can be done at little expenses apart from some numerical constants. This then allows to obtain a zero-free region for the ambiguity function $A(u)$ when u and its Fourier transforms have L^2 -moments. This region turns out to be a rectangle when one considers dispersions:

Theorem C.

Let $u \in L^2(\mathbb{R})$ be such that $tu(t) \in L^2(\mathbb{R})$ and $\xi \hat{u}(\xi) \in L^2(\mathbb{R})$. Then the ambiguity function $A(u)$ of u has no zero in the convex hull of the four points

$$\left(\pm \frac{\sqrt{2}\|u\|_2}{2\pi \inf_{a \in \mathbb{R}} \| |t - a| u(t) \|_2}, 0 \right) \quad \text{and} \quad \left(0, \pm \frac{\sqrt{2}\|u\|_2}{2\pi \inf_{\omega \in \mathbb{R}} \| |\xi - \omega| \hat{u}(\xi) \|_2} \right).$$

The article is divided in two sections. The first one is devoted to the extension of Luo and Zhang's uncertainty principle. In the second section, after some preliminaries on the fractional Fourier transform and on the ambiguity function, we give a proof of Theorem C. The section is completed with

¹Recall that this is part of the narrow-band assumption of the model in which the radar measures $A(u)$, see e.g. [AT].

the adaptation of the local uncertainty principle to ambiguity function, in particular with the proof of Theorem A.

Notation : In this paper $d \geq 1$ is an integer. For $1 \leq p < \infty$ we define $L^p(\mathbb{R}^d)$ as the space of measurable functions such that

$$\|f\|_p^p := \int_{\mathbb{R}^d} |f(t)|^p dt < +\infty.$$

For $u \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ we define the Fourier transform as

$$\mathcal{F}u(\xi) = \widehat{u}(\xi) = \int_{\mathbb{R}^d} u(t)e^{-2i\pi\xi t} dt, \quad \xi \in \mathbb{R}$$

and then extend it to $L^2(\mathbb{R}^d)$ in the usual way.

2. ZERO-FREE REGIONS OF THE FOURIER TRANSFORM

In this section, we restrict our attention to dimension $d = 1$.

Theorem 2.1.

For every $q > 0$ there exists $\kappa_q > 0$ such that, if $u \in L^1(\mathbb{R})$ is a non-negative function, $u \neq 0$, then

$$\inf\{\xi > 0 : \widehat{u}(\pm\xi) = 0\}^q \inf_{t_0 \in \mathbb{R}} \| |t - t_0|^q u(t) \|_1 \geq \kappa_q \|u\|_1.$$

Remark :

— One may take for the constant $\kappa = \frac{1}{c(2\pi)^q}$ where c is the smallest constant in Equation (2.2) below.

— A similar result has been proved in [LZ] but with the extra assumption that u be supported in $[0, +\infty)$. The constant κ is then explicitly known and better than the one above.

— It is enough to prove that if $u, |t|^q u \in L^1$ and $u \geq 0$, then for $|\xi| < \left(\kappa_q \frac{\|u\|_1}{\||t|^q u\|_1} \right)^{1/q}$, $\widehat{u}(\xi) \neq 0$.

Once this is done, we can apply the result to the translate $u(t - t_0)$ of u .

— Assume the theorem has been proved for a given q and let $q' > q$. Let $u \geq 0$ be in L^1 and such that $|t|^{q'} u \in L^1$. From Hölder's inequality, we get that $\||t|^q u\|_1 \leq \||t|^{q'} u\|_1^{\frac{q}{q'}}$ and $\|u\|_1^{1 - \frac{q}{q'}}$ and $|t|^q u \in L^1$. It further follows that

$$\left(\kappa_{q'} \frac{\|u\|_1}{\||t|^{q'} u\|_1} \right)^{1/q'} \leq \left(\kappa_{q'}^{q/q'} \frac{\|u\|_1}{\||t|^q u\|_1} \right)^{1/q}$$

and we obtain the theorem for q' if we set $\kappa_{q'} = \kappa_q^{q'/q}$. In particular $\kappa_q \geq \kappa_1^q$ for $q \geq 1$.

— Taking $u = 1$ on $[-a, a]$ and $u = 0$ elsewhere gives the bound $\kappa_q \leq \frac{1}{(q+1)2^q}$.

We will give two proofs of this result. The first one gives an explicit constant and works easily in all dimensions, but is restricted to $q \geq 1$.

First proof of Theorem 2.1. From elementary calculus, we get that $|\widehat{u}(\xi) - \widehat{u}(0)| \leq |\xi| \sup_{\eta \in [0, \xi]} |\nabla \widehat{u}(\eta)|$ so that

$$(2.1) \quad |\widehat{u}(\xi)| \geq \widehat{u}(0) - |\xi| \sup_{\eta \in [0, \xi]} |\nabla \widehat{u}(\eta)|.$$

But $\nabla \widehat{u}(\eta) = \nabla \int_{\mathbb{R}} u(t) e^{-2i\pi t \eta} dt = -2i\pi \int_{\mathbb{R}} t u(t) e^{-2i\pi t \eta} dt$ as long as $\int_{\mathbb{R}} |t| |u(t)| dt < \infty$. It follows that $\sup_{\eta \in [0, \xi]} |\nabla \widehat{u}(\eta)| \leq 2\pi \int_{\mathbb{R}} |t| |u(t)| dt$. On the other hand $\widehat{u}(0) = \int_{\mathbb{R}} u(t) dt$ so that (2.1) gives

$$|\widehat{u}(\xi)| \geq \int_{\mathbb{R}} u(t) dt - 2\pi |\xi| \int_{\mathbb{R}} |t| |u(t)| dt > 0$$

as long as $|\xi| < \frac{1}{2\pi} \frac{\|u\|_1}{\| |t| u \|_1}$. From the above remarks, we get the desired result with $\kappa_q = \frac{1}{(2\pi)^q}$. \square

Second proof of Theorem 2.1. The proof is very similar to that of Luo and Zhang. The idea is that $\operatorname{Re} e^{ix} = \cos x \geq 1 - x^2/2$ for all x . Thus $\operatorname{Re} \widehat{u} \geq \int_{\mathbb{R}} (1 - (2\pi t \xi)^2/2) u(t) dt$. It follows that if $\widehat{u}(\xi) = 0$, then $\operatorname{Re} \widehat{u}(\xi) = 0$ thus

$$\xi^2 \|t^2 u(t)\|_1 \geq \frac{1}{2\pi^2} \|u(t)\|_1.$$

The constant $\frac{1}{2\pi^2}$ can be improved when u is supported in $[0, +\infty)$ using a refined version of the inequality $\cos x \geq 1 - x^2/2$ (see [LZ, Proposition 1.2]) and can further be extended by replacing $\|t^2 u\|$ by other moments $\|t^q u\|$. Our substitute to [LZ, Proposition 1.2] is the following :

Fact. For every $q > 0$, there exists a, c such that, for all $x \in \mathbb{R}$,

$$(2.2) \quad a \cos x \geq 1 - c|x|^q.$$

It then follows that, if ξ is such that $\widehat{u}(\xi) = 0$, then

$$\begin{aligned} 0 &= a \operatorname{Re} \widehat{u}(\xi) = \int_{\mathbb{R}} u(t) a \cos 2\pi t \xi dt \geq \int_{\mathbb{R}} u(t) (1 - c(2\pi |\xi|)^q |t|^q) dt \\ &= \|u\|_1 - c(2\pi |\xi|)^q \| |t|^q u \|_1. \end{aligned}$$

Now, as \widehat{u} is continuous, we may take $\xi = \pm\tau$, and get that

$$|\tau|^q \| |t|^q u \|_1 \geq \frac{1}{c(2\pi)^q} \|u\|_1$$

as claimed. \square

Proof of (2.2). The fact is trivial as long as one does not look for best constants. The simplest way is to take $a = 2$ and c such that $1 - c|\frac{\pi}{3}|^q = -2$ i.e. $c = 3(\frac{3}{\pi})^q$. In this case, for $0 \leq |x| \leq \frac{\pi}{3}$, $1 - c|x|^q \leq 1 \leq 2 \cos x$ and for $|x| > \frac{\pi}{3}$, $1 - c|x|^q \leq -2 \leq 2 \cos x$. \square

Remark :

— One can slightly refine this argument, taking $a = 1 + \eta$, $x_\eta = \arccos \frac{1}{1 + \eta}$ and c such that $1 - c|x_\eta|^q = -(1 + \eta)$ that is $c = \frac{2 + \eta}{(\arccos(1 + \eta)^{-1})^q}$. Then for $x < x_\eta$, $1 - c|x|^q \leq 1 \leq (1 + \eta) \cos x$ and for $x \geq x_\eta$, $1 - c|x|^q \leq -(1 + \eta) \leq (1 + \eta) \cos x$. One may then minimize over $\eta > 0$ and some values are given in the following table :

q	3	4	5	6
a	3.26	3.94	4.61	5.27
c	2.134	1.656	1.241	0.908

— The constant c may be slightly improved by a further refinement. It is of course enough to establish the inequality for $0 \leq x \leq \pi$. Now that we know that for each $\eta > 0$, there exists a constant c such that $1 - c|x|^q \leq (1 + \eta) \cos x$ for all x , it is enough to take the biggest such constant c_η . The curves $y = 1 - c_\eta|x|^q$ and $y = (1 + \eta) \cos x$ then have a contact point at which they are tangent. Thus there exists x_η such that $(1 + \eta) \cos x_\eta = 1 - c_\eta x_\eta^q$ and $(1 + \eta) \sin x_\eta = c_\eta q x_\eta^{q-1}$. This implies that $\cos x_\eta + \frac{1}{q} x_\eta \sin x_\eta = \frac{1}{1 + \eta}$. It is then enough to show that this equation has a unique solution x_η in $[0, \pi]$ which then gives $c_\eta = \frac{(1 + \eta) \sin x_\eta}{q x_\eta^{q-1}}$.

Indeed, set $\varphi(x) = \cos x + \frac{x \sin x}{q}$ then $\varphi(0) = 1 \geq \frac{1}{1 + \eta}$ and $\varphi(\pi) = -1$ so that $\varphi(x) = \frac{1}{1 + \eta}$ has at least one solution in $[0, \pi]$. Moreover $\varphi'(x) = -\sin x + \frac{1}{q}(x \cos(x) + \sin(x))$. Elementary calculus and the study of the comparative behaviour of $\sin x$ and $\frac{1}{q}(x \cos(x) + \sin(x))$ shows that, if $q \geq 2$ then φ is strictly decreasing over $[0, \pi]$ and if $q < 2$, there exists x_1 such that φ is increasing over $[0, x_1]$ and strictly decreasing over $[x_1, \pi]$. In both cases, this implies that $\varphi(x) = 1/(1 + \eta)$ has a unique solution x_η in $[0, \pi]$.

To obtain the optimal constant is then difficult as one has to minimize c_η over η but this is not explicitly known. It should be noted that $\eta \rightarrow 0$ may not be optimal. For instance, a computer plot will convince the reader that

$$1.02 \cos x \geq 1 - 0.52x^{3/2} \quad , \quad 1.1 \cos x \geq 1 - 0.42x^2.$$

Notation

For $q > 0$, we write

$$\Delta_q(u) = \inf_{t_0 \in \mathbb{R}^d} \| |t - t_0|^q f \|_2^2.$$

To conclude this section, let us give a first application of this result. We ask whether a translate $f_a(t) = f(t - a)$ of f can be orthogonal to f . But

$$0 = \int_{\mathbb{R}} f(t) \overline{f(t - a)} dt = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 e^{2ia\xi} d\xi = \mathcal{F}[|\widehat{f}|^2](-a).$$

Similarly, if the modulation $f^{(\omega)}(t) = e^{2i\pi\omega t} f(t)$ of f is orthogonal to f then $\mathcal{F}[|\widehat{f}|^2](\omega) = 0$. From Theorem 2.1, we get the following:

Corollary 2.2.

Let $q > 0$ and κ_q be the constant of Theorem 2.1. Let $f \in L^2(\mathbb{R})$.

— Assume that $(1 + |\xi|)^{q/2} \widehat{f} \in L^2$. Then for f and its translate f_a to be orthogonal, it is necessary that $|a|^q \Delta_{q/2}(\widehat{f}) \geq \kappa_q \|f\|_2^2$.

— Assume that $(1 + |t|)^{q/2} f \in L^2$. Then for f and its modulation $f^{(\omega)}$ to be orthogonal, it is necessary that $|\omega|^q \Delta_{q/2}(f) \geq \kappa_q \|f\|_2^2$.

3. AMBIGUITY FUNCTIONS AND MOMENTS

3.1. Preliminaries on the fractional Fourier transform.

Here again, we restrict our attention to dimension $d = 1$. For $\alpha \in \mathbb{R} \setminus \pi\mathbb{Z}$, let $c_\alpha = \frac{\exp \frac{i}{2}(\alpha - \frac{\pi}{2})}{\sqrt{|\sin \alpha|}}$ be

a square root of $1 - i \cot \alpha$. For $f \in L^1(\mathbb{R})$ and $\alpha \notin \pi\mathbb{Z}$, define

$$\mathcal{F}_\alpha f(\xi) = c_\alpha e^{-i\pi\xi^2 \cot \alpha} \int_{\mathbb{R}} f(t) e^{-i\pi t^2 \cot \alpha} e^{-2i\pi t \xi / \sin \alpha} dt = c_\alpha e^{-i\pi\xi^2 \cot \alpha} \mathcal{F}[f(t) e^{-i\pi t^2 \cot \alpha}](\xi / \sin \alpha)$$

while for $k \in \mathbb{Z}$, $\mathcal{F}_{2k\pi}f = f$ and $\mathcal{F}_{(2k+1)\pi}f(\xi) = f(-\xi)$. This transformation has the following properties :

- (1) $\int_{\mathbb{R}} \mathcal{F}_\alpha f(\xi) \overline{\mathcal{F}_\alpha g(\xi)} d\xi = \int_{\mathbb{R}} f(t) \overline{g(t)} dt$ which allows to extend \mathcal{F}_α from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ as a unitary operator on $L^2(\mathbb{R})$;
- (2) $\mathcal{F}_\alpha \mathcal{F}_\beta = \mathcal{F}_{\alpha+\beta}$;
- (3) if $f_{a,\omega}(t) = e^{-2i\pi\omega t} f(t-a)$ then

$$\mathcal{F}_\alpha f_{a,0}(\xi) = \mathcal{F}_\alpha f(\xi + a \cos \alpha) e^{-i\pi a^2 \cos \alpha \sin \alpha - 2i\pi a \xi \sin \alpha},$$

and

$$\mathcal{F}_\alpha f_{0,\omega}(\xi) = \mathcal{F}_\alpha f(\xi + \omega \sin \alpha) e^{i\pi\omega^2 \cos \alpha \sin \alpha + 2i\pi\omega \xi \cos \alpha},$$

- (4) if $f \in L^2$ is such that $tf \in L^2$ then

$$\mathcal{F}_\alpha [tf](\xi) = \xi \mathcal{F}_\alpha f(\xi) \cos \alpha + i[\mathcal{F}_\alpha f]'(\xi) \sin \alpha.$$

3.2. Preliminaries on the ambiguity function.

Let $d \geq 1$ be an integer. Let us recall that the ambiguity function of $u, v \in L^2(\mathbb{R}^d)$ is defined by

$$A(u, v)(x, y) = \int_{\mathbb{R}^d} u\left(t + \frac{x}{2}\right) \overline{v\left(t - \frac{x}{2}\right)} e^{-2i\pi\langle t, y \rangle} dt.$$

Closely related transforms are the short-time Fourier transform also known as the windowed Fourier transform, defined by $S_v u(x, y) = e^{-i\pi\langle x, y \rangle} A(u, v)(x, y)$ and the Wigner transform

$$W(u, v)(\eta, \xi) = \int_{\mathbb{R}^d} u\left(\xi + \frac{t}{2}\right) \overline{v\left(\xi - \frac{t}{2}\right)} e^{2i\pi\langle t, \eta \rangle} dt$$

which is the inverse Fourier transform of $A(u, v)$ in \mathbb{R}^{2d} . Further

$$W(u, v)(\eta, \xi) = 2^d A(u, \check{v})(2\xi, 2\eta)$$

where $\check{v}(t) = v(-t)$.

We will now list the properties that we need. they are all well-known *e.g.* [Al, AT, Wi]:

- (1) $A(u, v), W(u, v) \in L^2(\mathbb{R}^2)$ with $\|A(u, v)\|_{L^2(\mathbb{R}^2)} = \|W(u, v)\|_{L^2(\mathbb{R}^2)} = \|u\|_2 \|v\|_2$ and are continuous;
- (2) if we denote $A(u) = A(u, u)$ then $A(u)(0, 0) = \|u\|_2^2$ where it is maximal;
- (3) for $a, \omega \in \mathbb{R}^d$ let $u_{a,\omega}(t) = e^{2i\pi\langle \omega, t \rangle} u(t-a)$, then

$$A(u_{a,\omega}, v_{b,\eta})(x, y) = e^{i\pi((\omega+\eta)x + (a+b)(y-\omega+\eta))} A(u, v)(x-a+b, y-\omega+\eta);$$

- (4) in dimension $d = 1$, $A(\mathcal{F}_\alpha u, \mathcal{F}_\alpha v)(x, y) = A(u, v)(x \cos \alpha + y \sin \alpha, x \sin \alpha - y \cos \alpha)$;
- (5) $W(\hat{u}, \hat{v})(\eta, \xi) = W(u, v)(\xi, -\eta)$.

Property 4 was proved in [Wi] when the fractional Fourier transform is defined in terms of Hermite polynomials and in [Al] with the above definition of the fractional Fourier transform.

3.3. Zero free region around the origin of $A(u)$ in dimension $d = 1$.

In this section, we restrict our attention to the one-dimensional situation. Noticing that $A(u)$ is a Fourier transform and in particular that $A(u)(0, y) = \mathcal{F}[|u|^2](y)$, we get from Property 4 of the ambiguity function that

$$A(u)(y \sin \alpha, -y \cos \alpha) = \mathcal{F}[|\mathcal{F}_\alpha u|^2](y).$$

Definition.

Let us define, for $\theta \in]0, \pi[$

$$\tau_\theta = \inf\{t > 0 : A(u)(t \cos \theta, t \sin \theta) = 0 \text{ or } A(u)(-t \cos \theta, -t \sin \theta) = 0\}.$$

Let $q > 0$ and κ_q be given by Theorem 2.1. Applying this theorem, we obtain

$$(3.3) \quad \tau_\theta^q \inf_{t_0 \in \mathbb{R}} \| |t - t_0|^q |\mathcal{F}_{\theta - \pi/2} u|^2 \|_1 \geq \kappa_q \| |\mathcal{F}_{\theta - \pi/2} u|^2 \|_1 = \kappa_q \|u\|_2^2.$$

Let us now show that in the specific case $q = 2$ a more precise result can be obtained: if $u \in L^2$ is such that $tu \in L^2$ and $t\hat{u} \in L^2$ then from Property 4 of the fractional Fourier transform

$$\begin{aligned} \|t\mathcal{F}_\alpha u\|_2 &= \|tu(t) \cos(-\alpha) - iu'(t) \sin \alpha\|_2 \leq \|tu(t)\|_2 |\cos \alpha| + \|u'\|_2 |\sin \alpha| \\ &= \|tu(t)\|_2 |\cos \alpha| + \|\xi\hat{u}(\xi)\|_2 |\sin \alpha|. \end{aligned}$$

In particular, the ambiguity function $A(u)$ of u has no zero in the region

$$\left\{ (t \sin \alpha, -t \cos \alpha) : 0 < \alpha < \pi, |t| \leq \frac{\sqrt{2}\|u\|_2}{2\pi(\|tu(t)\|_2 |\cos \alpha| + \|\xi\hat{u}(\xi)\|_2 |\sin \alpha|)} \right\}.$$

This region is a rombus with endpoints

$$\left(\pm \frac{\sqrt{2}\|u\|_2}{2\pi\|tu(t)\|_2}, 0 \right) \quad \text{and} \quad \left(0, \pm \frac{\sqrt{2}\|u\|_2}{2\pi\|\xi\hat{u}(t)\|_2} \right).$$

Further, changing $u(t)$ into $u(t-a)e^{i\omega t}$ leaves the modulus of $A(u)$ unchanged, and so are the zero-free regions of $A(u)$. We have thus proved

Theorem 3.1.

Let $u \in L^2(\mathbb{R})$ be such that $tu(t) \in L^2(\mathbb{R})$ and $\xi\hat{u}(\xi) \in L^2(\mathbb{R})$. Then the ambiguity function $A(u)$ of u has no zero in the convex hull of the four points

$$\left(\pm \frac{\sqrt{2}\|u\|_2}{2\pi\Delta_1(u)^{1/2}}, 0 \right) \quad \text{and} \quad \left(0, \pm \frac{\sqrt{2}\|u\|_2}{2\pi\Delta_1(\hat{u})^{1/2}} \right).$$

The area of that rombus is

$$\frac{\frac{1}{\pi}\|u\|_2^2}{\Delta_1(u)^{1/2}\Delta_1(\hat{u})^{1/2}} \leq 4$$

according to Heisenberg's uncertainty principle.

Note also that the numerical constant $\frac{\sqrt{2}}{2\pi}$ can be improved to 0.248 using the inequality $1.1 \cos x \geq 1 - 0.41x^2$ instead of $\cos x \geq 1 - x^2/2$.

3.4. The local uncertainty principle.

In this section, we switch back to the higher dimensional situation $d \geq 1$.

The first result we will prove may be well-known, but we do not know an reference for it. Let $\mu(u) = \frac{1}{\|u\|_2} \int_{\mathbb{R}^d} t|u(t)|^2 dt$. Writing $h_x(t) = u\left(t + \frac{x}{2}\right)v\left(t - \frac{x}{2}\right)$ we get as for [BDJ, Lemma 5.2]:

$$\begin{aligned} \int_{\mathbb{R}^{2d}} |x|^2 |A(u)(x, y)|^2 dx dy &= \int_{\mathbb{R}^d} |x|^2 \int_{\mathbb{R}} |\widehat{h_x}(y)|^2 dy dx = \int_{\mathbb{R}^d} |x|^2 \int_{\mathbb{R}} |\widehat{h_x}(t)|^2 dt dx \\ &= \int_{\mathbb{R}^{2d}} |r - \mu(u) + \mu(v) - s|^2 |u(r)|^2 |v(s)|^2 dr ds \\ &= \| |r - \mu(u)|u\|_2^2 \|v\|_2^2 + \|u\|_2^2 \| |r - \mu(v)|v\|_2^2 \\ &= \Delta_1(u) \|v\|_2^2 + \|u\|_2^2 \Delta_1(v). \end{aligned}$$

Using Property 4 of the ambiguity function, we get that

$$\int_{\mathbb{R}^2} |x|^2 |A(u)(x, y)|^2 dx dy = \Delta_1(\hat{u}) \|v\|_2^2 + \|u\|_2^2 \Delta_1(\hat{v}).$$

It follows that

$$\begin{aligned} \iint_{x^2+y^2 \geq R^2} |A(u)(x, y)|^2 dx dy &\leq \frac{1}{R^2} \iint_{\mathbb{R}^2} (|x|^2 + |y|^2) |A(u)(x, y)|^2 dx dy \\ &\leq \frac{1}{R^2} \left((\Delta_1(u) + \Delta_1(\hat{u})) \|v\|_2^2 + \|u\|_2^2 (\Delta_1(v) + \Delta_1(\hat{v})) \right). \end{aligned}$$

As $\|A(u)\|_2^2 = \|u\|_2^2 \|v\|_2^2$ we get the following :

Proposition 3.2.

Let $u, v \in L^2(\mathbb{R}^d)$ and let $0 < \delta < 1$. Then for

$$R \geq \max \left(\frac{(\Delta_1(u) + \Delta_1(\hat{u}))^{1/2}}{\|u\|}, \frac{(\Delta_1(v) + \Delta_1(\hat{v}))^{1/2}}{\|v\|} \right) \frac{\sqrt{2}}{\delta},$$

then

$$\iint_{x^2+y^2 \leq R^2} |A(u, v)(x, y)|^2 dx dy \geq (1 - \delta^2) \|u\|_2^2 \|v\|_2^2.$$

We will now prove a similar result about the L^2 -mass of the ambiguity function outside *small sets* thanks to the local uncertainty principle. Let us recall that Faris [Fa] and Price [Pr1] proved that for $0 < \alpha < d/2$, $E \subset \mathbb{R}^d$ of finite measure and $f \in L^2(\mathbb{R}^d)$,

$$(3.4) \quad \int_E |\widehat{f}(\xi)|^2 d\xi \leq K_{\alpha, d} |E|^{2\alpha/d} \| |x|^\alpha f \|_2^2,$$

where

$$K_{\alpha, d} = \frac{(d + 2\alpha)^2}{(2\alpha)^{4\alpha/d}} (d - 2\alpha)^{2\alpha/d - 2}.$$

On the other hand Price [Pr2] proved that for $\alpha > d/2$, $E \subset \mathbb{R}^d$ of finite measure and $f \in L^2(\mathbb{R}^d)$,

$$(3.5) \quad \int_E |\widehat{f}(\xi)|^2 d\xi \leq C_{\alpha, d} |E| \| |x|^\alpha f \|_2^{d/\alpha} \|f\|_2^{2-d/\alpha},$$

where

$$C_{\alpha, d} = \frac{\pi^{d/2} \left(\frac{2\alpha}{d} - 1\right)^{d/2\alpha}}{\alpha \Gamma(d/2) \left(1 - \frac{d}{2\alpha}\right)} \Gamma\left(\frac{d}{2\alpha}\right) \Gamma\left(1 - \frac{d}{2\alpha}\right)$$

and this constant is optimal.

Corollary 3.3.

Let $u, v \in L^2(\mathbb{R}^d)$ and $E \subset \mathbb{R}^{2d}$ be a set of finite measure. Then for $\alpha < d$,

$$(3.6) \quad \int_E |A(u, v)(x, y)|^2 dx dy \leq 2^{2\alpha} K_{\alpha, 2d} |E|^{\alpha/d} \left((\Delta_\alpha(u) + \Delta_\alpha(\hat{u})) \|v\|_2^2 + \|u\|_2^2 (\Delta_\alpha(v) + \Delta_\alpha(\hat{v})) \right)$$

while for $\alpha > d$,

$$(3.7) \quad \begin{aligned} \int_E |A(u, v)(x, y)|^2 dx dy &\leq 2^{2d} C_{\alpha, 2d} |E| \left((\Delta_\alpha(u) + \Delta_\alpha(\hat{u})) \|v\|_2^2 + \|u\|_2^2 (\Delta_\alpha(v) + \Delta_\alpha(\hat{v})) \right)^{2d/\alpha} \\ &\quad \times \left(\|u\|_2 \|v\|_2 \right)^{2-2d/\alpha} \end{aligned}$$

Proof. Using Property 3 of the ambiguity function, it is enough to prove (3.6)-(3.7) with $\| |x|^\alpha \cdot \|$ replacing $\Delta_\alpha(\cdot)$. Indeed, once this is done, it is enough to apply it to $u_{a,\omega}$, $v_{b,\eta}$ and $E_{a,b,\omega,\eta} = E + (a - b, \omega - \eta)$.

Now recall that $|a + b|^t \leq 2^t(|a|^t + |b|^t)$. We will use the following computation which further generalizes [BDJ, Lemma 5.2]: Let $g_x(t) = u(x + \frac{t}{2}) \overline{v(x - \frac{t}{2})}$. Then

$$\begin{aligned}
\iint_{\mathbb{R}^{2d}} |x|^{2\alpha} |W(u, v)(x, y)|^2 dx dy &= \int_{\mathbb{R}^d} |x|^{2\alpha} \int_{\mathbb{R}^d} |\widehat{g_x}(y)|^2 dy dx = \int_{\mathbb{R}^d} |x|^{2\alpha} \int_{\mathbb{R}^d} |g_x(t)|^2 dt dx \\
&= \iint_{\mathbb{R}^{2d}} |x|^{2\alpha} \left| u\left(x + \frac{t}{2}\right) \overline{v\left(x - \frac{t}{2}\right)} \right|^2 dt dx \\
&= 2^{-2\alpha} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |r + s|^{2\alpha} |u(r)|^2 |v(s)|^2 dr ds \\
(3.8) \qquad \qquad \qquad &\leq \| |r|^\alpha u \|_2^2 \|v\|_2^2 + \|u\|_2^2 \| |s|^\alpha v \|_2^2
\end{aligned}$$

Similarly, using the symmetry $W(\hat{u}, \hat{v})(x, y) = W(u, v)(-y, x)$ and Parseval's identity, we get

$$(3.9) \qquad \iint_{\mathbb{R}^{2d}} |y|^{2\alpha} |W(u, v)(x, y)|^2 dx dy \leq \| |r|^\alpha \hat{u} \|_2^2 \|v\|_2^2 + \|u\|_2^2 \| |s|^\alpha \hat{v} \|_2^2$$

Now if $\alpha < d$, from Faris' Inequality (3.4),

$$\iint_E |A(u, v)|^2 dx dy = K_{\alpha, 2d} |E|^{\alpha/d} \| |(x, y)|^\alpha W(u, v) \|_{L^2(\mathbb{R}^{2d})}^2.$$

But $|x, y|^{2\alpha} = (|x|^2 + |y|^2)^\alpha \leq 2^\alpha (|x|^{2\alpha} + |y|^{2\alpha})$. Thus

$$\begin{aligned}
\| |(x, y)|^\alpha W(u, v) \|_{L^2(\mathbb{R}^{2d})}^2 &= \iint_{\mathbb{R}^{2d}} |(x, y)|^{2\alpha} |W(u, v)(x, y)|^2 dx dy \\
&\leq 2^{2\alpha} \left(\iint_{\mathbb{R}^{2d}} |x|^{2\alpha} |W(u, v)(x, y)|^2 dx dy \right. \\
&\qquad \qquad \qquad \left. + \iint_{\mathbb{R}^{2d}} |y|^{2\alpha} |W(u, v)(x, y)|^2 dx dy \right)
\end{aligned}$$

and (3.8)-(3.9) allows to conclude.

On the other hand, for $\alpha > d$, Price's Inequality (3.4) gives

$$\iint_E |A(u, v)|^2 dx dy = C_{\alpha, 2d} |E| \| |(x, y)|^\alpha W(u, v) \|_{L^2(\mathbb{R}^{2d})}^{2d/\alpha} \|W(u, v)\|_{L^2(\mathbb{R}^{2d})}^{2-2d/\alpha}.$$

Now $\|W(u, v)\|_{L^2(\mathbb{R}^{2d})}^{2-2d/\alpha} = (\|u\|_2 \|v\|_2)^{2-2d/\alpha}$ and the other term has been estimated above. \square

Example.

For $d = 1$, $\alpha = 1/2$ (thus $K_{\alpha, 2d} = 9$) and $u = v$, we first get from Cauchy-Schwarz that $\Delta_{1/2}(u) \leq \Delta_1(u)^{1/2} \|u\|_2$ so that

$$\iint_E |A(u)(x, y)|^2 dx dy \leq 18 |E|^{1/2} \left((\Delta_1(u) + \Delta_1(\hat{u}))^{1/2} \|u\|_2 \|v\|_2^2 + (\Delta_1(v) + \Delta_1(\hat{v}))^{1/2} \|u\|_2^2 \|v\|_2 \right)$$

so that if $\rho \leq \max \left(\frac{(\Delta_1(u) + \Delta_1(\hat{u}))^{1/2}}{\|u\|_2}, \frac{(\Delta_1(v) + \Delta_1(\hat{v}))^{1/2}}{\|v\|_2} \right) \frac{\delta^2}{36\sqrt{\pi}}$ then

$$\iint_{x^2+y^2 \geq \rho^2} |A(u, v)(x, y)|^2 dx dy \geq (1 - \delta^2) \|u\|_2^2 \|v\|_2^2.$$

REFERENCES

- [Al] L. B. ALMEIDA *The fractional Fourier transform and time-frequency representations*. IEEE Trans. Sign. Proc. **42** (1994), 3084–3091.
- [AT] L. AUSLANDER & R. TOLIMIERI *Radar ambiguity functions and group theory*. SIAM J. Math Anal, **16** (1985), 577–601.
- [BDJ] A. BONAMI, B. DEMANGE & PH. JAMING *Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms*. Rev. Mat. Iberoamericana, **19** (2003), 23–55.
- [De] B. DEMANGE *Uncertainty principles for the ambiguity function*. J. London Math. Soc. (2), **72** (2005), 717–730.
- [Fa] W. G. FARIS *Inequalities and uncertainty principles*. J. Mathematical Phys. **19** (1978) 461–466.
- [Gr] K. GRÖCHENIG *Uncertainty principles for time-frequency representations*. Advances in Gabor analysis, 11–30, Appl. Numer. Harmon. Anal., Birkhuser Boston, Boston, MA, 2003.
- [GZ] K. GRÖCHENIG & G. ZIMMERMANN *Hardy’s theorem and the short-time Fourier transform of Schwartz functions*. J. London Math. Soc. (2) **63** (2001), 205–214.
- [LZ] S. LUO & Z. ZHANG, *Estimating the first zero of a characteristic function*. C. R. Acad. Sci. Paris, Ser. I **338** (2004) 203–206.
- [Pr1] J. F. PRICE *Inequalities and local uncertainty principles*. J. Math. Phys. **24** (1983) 1711–1714.
- [Pr2] J. F. PRICE *Sharp local uncertainty inequalities*. Studia Math. **85** (1986) 37–45.
- [PH] R. PRICE & E. M. HOFSTETTER *Bounds on the volume and height distribution of the ambiguity function*. IEEE Trans. Info. Theory (1965), 207–214.
- [Wi] C. H. WILCOX *The synthesis problem for radar ambiguity functions*. MRC Tech. Summary Report **157** (1960), republished in Radar and Sonar part I (eds. R. Blahut, W. Miller and C. Wilcox), I.M.A. vol in Math. and its Appl. **32**, 229–260, Springer, New York, 1991.
- [Wo] P. M. WOODWARD *Probability and Information Theory with Applications to RADAR* Pergamon, 1953.

UNIVERSITÉ D’ORLÉANS, FACULTÉ DES SCIENCES, MAPMO - FÉDÉRATION DENIS POISSON, BP 6759, F 45067 ORLÉANS CEDEX 2, FRANCE

E-mail address: Philippe.Jaming@univ-orleans.fr