Formalisme thermodynamique et variations de la dimension de Hausdorff des ensembles de Julia des polynômes quadratiques

Guillaume Havard, Michel Zinsmeister

THERMODYNAMIC FORMALISM AND VARIATIONS OF THE HAUSDORFF DIMENSION OF QUADRATIC JULIA SETS

GUILLAUME HAVARD, MICHEL ZINSMEISTER

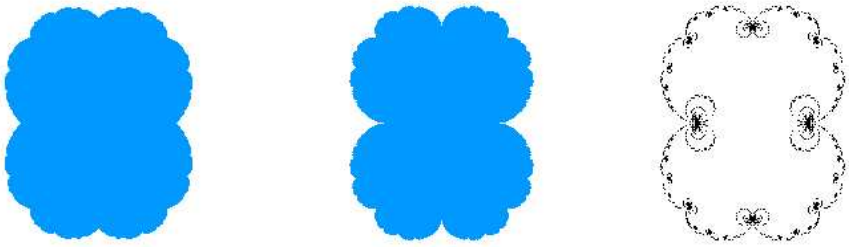
ABSTRACT. Let $d(c)$ denotes the Hausdorff dimension of the Julia set of the polynomial $z \mapsto z^2 + c$. The function d restricted to $[0, +\infty)$ is real analytic in $[0, \frac{1}{4}) \cup (\frac{1}{4}, +\infty)$ ([Ru^2]), is left-continuous at $\frac{1}{4}$ ([Bo,Zi]) but not continuous ([Do,Se,Zi]). We prove that $c \mapsto d'(c)$ tends to $+\infty$ from the left at $\frac{1}{4}$ as $(\frac{1}{4} - c)^{d(\frac{1}{4})-\frac{3}{2}}$. In particular the graph of d has a vertical tangent on the left at $\frac{1}{4}$, result which comforts the numerical experiments.

1. INTRODUCTION AND STATEMENT OF THE RESULT

By quadratic family we mean the family of polynomials

$$P_c : z \longmapsto z^2 + c, \qquad c \in \mathbb{C}.$$

As usual K_c and J_c will denote respectively the filled-in Julia set and the Julia set of P_c. We recall that $J_c = \partial K_c$ and that K_c is the set of complex numbers z for which the sequence obtained by induction by $z_0 = z$, $z_{n+1} = P_c(z_n)$ does not converge to ∞.



$c = 0,21$ $c = 0,25$ $c = 0,251$

FIGURE 1. Filled-in Julia sets for different values of the parameter c (*Pictures made with Arnaud Cheritat's program*).

We are interested in the function $c \mapsto d(c)$ where $d(c)$ denotes Hausdorff dimension of J_c. A well known result of Ruelle ([Ru^2]), based on thermodynamic formalism (Bowen Formula) asserts that this function is real-analytic on the complement of the Mandelbrot set $\mathbb{M} = \{c \in \mathbb{C} \mid K_c \text{ is connected}\}$ as well as on every hyperbolic component of $\overset{o}{\mathbb{M}}$.
So d is real-analytic on $]\frac{-3}{4}, \frac{1}{4}[\cup]\frac{1}{4}, \infty[$, since $]\frac{-3}{4}, \frac{1}{4}[$ is the trace on $\mathbb{R}$ of the main cardioïd of $\mathbb{M}$, the hyperbolic component of values of c such that P_c has an attracting

fixed point, and $]\frac{1}{4}, \infty[\subset \mathbb{C} \setminus \mathbb{M}$.
The point $\frac{1}{4}$ corresponds dynamically to a bifurcation; for $0 < c < \frac{1}{4}$ P_c has two real fixed points, one attracting and one repelling and they both converge to $\frac{1}{2}$, the parabolic fixed point for $P_{\frac{1}{4}}$, as c goes to $\frac{1}{4}$. When c is greater than $\frac{1}{4}$, P_c possesses two repelling conjugated fixed points.
In [Bo,Zi] it has been proved that $d_{|\mathbb{R}}$ is continuous from the left at $\frac{1}{4}$. On the other hand, parabolic implosion produces a topological discontinuity which induces a jump of the function d; more precisely it is shown in [Do,Se,Zi] that

$$d(\frac{1}{4}) < \liminf_{c \to \frac{1}{4}+0} d(c) \leq \limsup_{c \to \frac{1}{4}+0} d(c) < 2.$$

Notice that it is not known if d has a limit from the right at $\frac{1}{4}$.
The purpose of the present article is to precise the speed of convergence of $d(c)$ towards $d(\frac{1}{4})$ from the left. From now on we write $\delta = d(\frac{1}{4})$.
Bowen's formula is relatively well adapted to computation and several numerical experiments have been performed [Ga], [Bo,Zi], [McMu]. They all show the same behaviour before $\frac{1}{4}$, namely a vertical tangent at this point. We prove this rigorously modulo the accepted bound $\delta < \frac{3}{2}$. Indeed all quoted numerical values give $\delta \simeq 1,07 \pm 0,02$ which is very far from $\frac{3}{2}$. We do not feel like going into tedious computations to prove this, even if it can certainly be done.

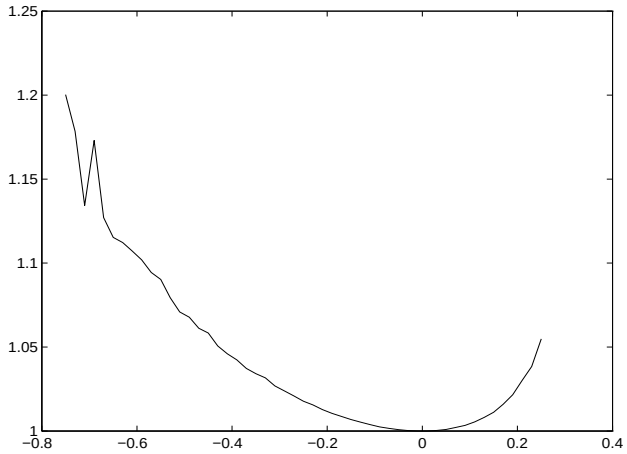


FIGURE 2. The graph of d obtained in [Bo,Zi].

Theorem 1.1. *There exist $c_0 < \frac{1}{4}$ and $K > 1$ such that*

$$\forall c \in [c_0, \frac{1}{4}[\qquad \frac{1}{K} \leq (\frac{1}{4} - c)^{(\frac{3}{2} - \delta)} d'(c) \leq K.$$

Remark 1.2. Notice that it implies

$$\delta - d(c) \sim (\frac{1}{4} - c)^{\delta - \frac{1}{2}}.$$

2. Bowen's formula and thermodynamic formalism.

The Julia set J_0 admits a natural Markov partition corresponding to dyadic development of the argument measured in "numbers of turns"; namely the partition into two half circles $[0, \frac{1}{2}[$ and $[\frac{1}{2}, 1[$ which will be denoted by **0** and **1** respectively. On the other hand the polynomials P_c, $0 \leq c \leq \frac{1}{4}$ admit a Böttcher coordinate, i.e. there exists $\phi_c : \mathbb{C} \setminus \overline{\mathbb{D}} = \mathbb{C} \setminus K_0 \longrightarrow \mathbb{C} \setminus K_c$ holomorphic bijective satisfying $\phi_c(z) = z + \cdots$ at infinity and conjugating P_0 into P_c. The sets J_c being for $0 \leq c \leq \frac{1}{4}$ Jordan curves, the ϕ_c's have homeomorphic extensions from J_0 to J_c, which allow to define a Markov partition for all J_c's. This Markov partition has of course refining at all orders n; more specifically J_c is for every n a union of 2^n cylinders γ two of these cylinders having closure intersecting at at most one point. We denote by $\Gamma_n(c)$ the set of all cylinders of order n for J_c.
If $c < \frac{1}{4}$ then P_c is hyperbolic and Kœbe distortion theorem applies to all cylinders : there exists a constant $c \geq 1$ independent of x or n (but not on c!) such that

$$\frac{1}{K}\text{diam } \gamma_n(z) \leq |(P_c^{\circ n})'(z)|^{-1} \leq K\text{diam } \gamma_n(z) \tag{2.1}$$

where $\gamma_n(z)$ is the cylinder of order n containing z. Writing, for a cylinder γ of order n, if f is a continuous function on $\overline{\gamma}$

$$f(\gamma) = \sup_{z \in \overline{\gamma}} f(z),$$

we define

$$\Pi_n(t) = \sum_{\gamma \in \Gamma_n(c)} |(P_c^n)'|^{-t}(\gamma) \tag{2.2}$$

Then we have (see [Bow], [Wa[1]], [Ru[1]]) that

$$\Pi(t) = \lim_{n \to +\infty} \frac{\log \Pi_n(t)}{n} \tag{2.3}$$

exists and define a convex function on $\mathbb{R}$ strictly decreasing from $+\infty$ to $-\infty$. Bowen's formula [Bow] asserts that $d(c)$ is the unique real t such that $\Pi(t) = 0$.
Using Böttcher coordinate, (2.2) becomes

$$\Pi_n(t) = \sum_{\gamma \in \Gamma_n(0)} e^{S_n(-t \log |2\phi_c|)(\gamma)} \tag{2.4}$$

where

$$S_n(\varphi)(z) = \sum_{k=0}^{n-1} \varphi(z^{2^k}).$$

The function Π then appears to be the pressure of the continuous function $-t \log |2\phi_c|$. For a continuous function φ on J_0 the pressure $P(\varphi)$ is defined (see [Wa[1]]) by

$$P(\varphi) = \lim_{n \to +\infty} \frac{1}{n} \log \left(\sum_{\gamma \in \Gamma_n(0)} e^{S_n(\varphi)(\gamma)} \right). \tag{2.5}$$

Notice that ϕ_c (and thus $\log |2\phi_c|$) is a Hölder function on J_0 because $(c, z) \mapsto \phi_c(z)$ is a holomorphic motion and thus for all $c \in [0, \frac{1}{4}[$, ϕ_c has a quasiconformal extension to $\mathbb{C}$. Ruelle theorem follows from the fact that the map $\varphi \mapsto P(\varphi)$ is real-analytic on the spaces of Hölder functions.
The key in Ruelle's result is to interpret the pressure in terms of an operator called

transfer operator or Ruelle operator.
If φ is continuous on J_0, the associated operator is defined as

$$\forall x \in \partial\mathbb{D} \qquad \mathcal{L}_\varphi(f)(x^2) = e^{\varphi(x)} f(x) + e^{\varphi(-x)} f(-x). \tag{2.6}$$

Perron-Frobenius-Ruelle theorem [Ru[1]] asserts that if φ is Hölder then $\beta = e^{P(\varphi)}$ is a single eigenvalue of $\mathcal{L}_\varphi$ associated with an eigenvector $h > 0$ which is Hölder with same exponent.
On the other hand there exists a unique probability measure ω on the circle such that $\mathcal{L}^*_\varphi(\omega) = \beta\omega$. When $\varphi = \varphi_c = -d(c)\log|2\phi_c|$ this measure corresponds on J_c to a measure ω_c which is The Hausdorff measure (normalized to be of mass 1) in dimension $d(c)$ on J_c.
It can be shown that it is the unique conformal measure on J_c, i.e. it is the unique probability measure for which there exists $t > 0$ such that

$$\omega_c(P_c(A)) = \int_A |P_c'|^t d\omega_c \tag{2.7}$$

for every $A \subset J_c$ on which P_c is injective.
For every $0 < c < \frac{1}{4}$, if $\widetilde{h}_c$ denote the eigenvector of $\mathcal{L}_{\varphi_c}$ we define $h_c = \widetilde{h}_c \circ \phi_c^{-1}$. We can choose $\widetilde{h}_c$ so that $\mu_c = h_c\omega_c$ is a probability measure. It is then the unique P_c-invariant measure equivalent to ω_c. In thermodynamic terms, it is an equilibrium state, that is an invariant probability measure maximizing some functional called free energy. There is no other equilibrium state (no phase transition). For $c = \frac{1}{4}$, there are exactly two equilibrium states [Ha[1]]; there is a phase transition similar to the transition water-ice.
This long introduction on thermodynamic formalism had for purpose the following theorem which will allow us to start the computation.

Proposition 2.1. *Let $0 < c < \frac{1}{4}$, we have*

$$d'(c) = -\frac{d(c)}{\int_{\partial\mathbb{D}} \log|2\phi_c| d\widetilde{\mu}_c} \int_{\partial\mathbb{D}} \frac{\partial}{\partial c} \left(\log|2\phi_c|\right) d\widetilde{\mu}_c. \tag{2.8}$$

Proof. Let $\varphi_c = -d(c)\log|2\phi_c|$: because of Bowen's formula, we know that the pressure of φ_c is 0. So $\mathcal{L}_{\varphi_c}$ and $\mathcal{L}^*_{\varphi_c}$ have an eigenvector associated with the eigenvalue 1. Denote by $\widetilde{h}_c$ and $\widetilde{\omega}_c$ those eigenvectors. We choose $\widetilde{\omega}_c$ so that it is a probability measure and $\widetilde{h}_c$ so that $\widetilde{\mu}_c = \widetilde{h}_c\widetilde{\omega}_c$ is a probability measure.
We have

$$\mathcal{L}_{\varphi_c}(\widetilde{h}_c) = \widetilde{h}_c.$$

Differentiating, we obtain

$$\left(\frac{\partial}{\partial c}\mathcal{L}_{\varphi_c}\right)(\widetilde{h}_c) + \mathcal{L}_{\varphi_c}\left(\frac{\partial}{\partial c}\widetilde{h}_c\right) = \frac{\partial}{\partial c}\widetilde{h}_c.$$

Now we integrate with respect to $\widetilde{\omega}_c$. Using the fact that it is an eigenvector for $\mathcal{L}^*_{\varphi_c}$ we get

$$\int_{\partial\mathbb{D}} \left(\frac{\partial}{\partial c}\mathcal{L}_{\varphi_c}\right)(\widetilde{h}_c) d\widetilde{\omega}_c = 0. \tag{2.9}$$

Using (2.6) we compute

$$\forall f \in C(\partial\mathbb{D}) \qquad \left(\frac{\partial}{\partial c}\mathcal{L}_{\varphi_c}\right)(f) = \mathcal{L}_{\varphi_c}\left(\left[-d'(c)\log|2\phi_c| - d(c)\frac{\partial}{\partial c}\left(\log|2\phi_c|\right)\right] f\right),$$

and (2.9) leads to

$$\int_{\partial\mathbb{D}} \left(-d'(c) \log|2\phi_c| - d(c) \frac{\partial}{\partial c} (\log|2\phi_c|) \right) d\widetilde{\mu}_c = 0$$

which is what we wanted.

■

The result of [Bo,Zi] implies that $d(c) \to \delta$ as $c \to \frac{1}{4}$. On the other hand the denominator in (2.8) is the Lyapounov exponent of P_c. In the next paragraph we prove the convergence as $c \to \frac{1}{4}$ of this Lyapounov exponent towards the Lyapounov exponent of $P_{\frac{1}{4}}$ which is positive.
Finally the numerator will be estimated in section 4.

3. Convergence of equilibrium states and Lyapounov exponents

3.1. **Approximate Fatou coordinates.** We will denote $z_c$and ζ_c respectively the repelling and attracting fixed point of P_c :

$$z_c = \frac{1+\sqrt{1-4c}}{2} \qquad \zeta_c = \frac{1-\sqrt{1-4c}}{2}.$$

If $\varepsilon_c = \frac{1}{4} - c$, the approximate Fatou coordinates is defined as

$$Z_c(z) = \frac{1}{2\sqrt{\varepsilon_c}} \log\left(\frac{z-z_c}{z-\zeta_c} \right) \tag{3.1}$$

$$\mathbb{C}\backslash[\zeta_c, z_c] \xrightarrow{\frac{z-z_c}{z-\zeta_c}} \mathbb{C}\backslash]-\infty, 0[\xrightarrow{\frac{1}{2\sqrt{\varepsilon_c}}\log(.)} \{\mathcal{R}e\, z \in \mathbb{R}\} \cap \{\mathcal{I}m\, z \in]-i\frac{\pi}{2\sqrt{\varepsilon_c}}, i\frac{\pi}{2\sqrt{\varepsilon_c}}[\}.$$

Notice that z being fixed we have

$$Z_c \underset{\varepsilon_c \to 0}{\sim} -\frac{1}{z-\frac{1}{2}} :$$

which is the approximate Fatou coordinate for $c = \frac{1}{4}$.

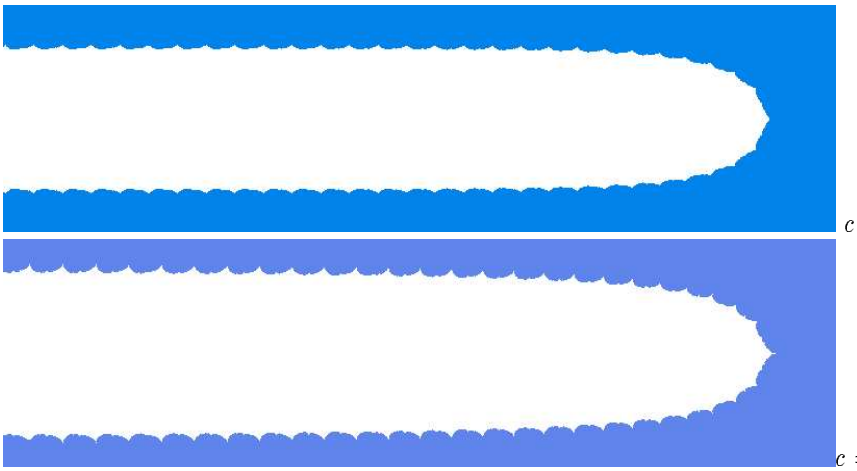


Figure 3. Filled-in Julia sets in Fatou coordinates
(*Pictures made with Arnaud Cheritat's program).*

Under this change of coordinates P_c is conjugated to F_c.
Let $E_c(Z) = e^{2\sqrt{\varepsilon_c}Z}$: one can compute using $c - \zeta_c = \zeta_c^2$ and $c - z_c = z_c^2$,

(3.2)

$$F_c(Z) = Z + \frac{1}{2\sqrt{\varepsilon_c}} \log(1 + 2\sqrt{\varepsilon_c}) + \frac{1}{2\sqrt{\varepsilon_c}} \log\left(1 + \frac{4E_c(Z)\varepsilon_c}{(1-4\varepsilon_c)E_c(Z) - (1+2\sqrt{\varepsilon_c})}\right).$$

Next lemma gives an estimation of how far from translation by one is F_c when $|Z|$ is big.

Lemma 3.1. *There exist $K > 0$ and $\varepsilon_0 > 0$ such that for $\mathcal{R}e\ Z \leq -K$ and $\varepsilon_c \leq \varepsilon_0$,*

$$|F_c(Z) - (Z+1)| \leq \frac{1}{4}.$$

Proof. By formula (3.2) this boils down to proving that

$$\left|\frac{1}{2\sqrt{\varepsilon_c}} \log\left(1 + \frac{4E_c(Z)\varepsilon_c}{(1-4\varepsilon_c)E_c(Z) - (1+2\sqrt{\varepsilon_c})}\right)\right| \leq \frac{1}{10},$$

for $\mathcal{R}e\ Z \leq -K$ and $\varepsilon_c \leq \varepsilon_0$.
Put $K \geq 100$, then if $\mathcal{R}e\ Z \leq -K$,

$$\begin{aligned} |(1+2\sqrt{\varepsilon_c}) - (1-4\varepsilon_c)E_c(Z)| &\geq (1+2\sqrt{\varepsilon_c}) - (1-4\varepsilon_c)e^{-100\sqrt{\varepsilon_c}} \\ &\geq 50\sqrt{\varepsilon_c} \qquad \text{if } \varepsilon_c \leq \varepsilon_0. \end{aligned}$$

For $\mathcal{R}e\ Z \leq -K < -100$ and $\varepsilon_c \leq \varepsilon_0$ this leads to

$$\left|\frac{4E_c(Z)\varepsilon_c}{(1-4\varepsilon_c)E_c(Z) - (1+2\sqrt{\varepsilon_c})}\right| \leq \frac{4\varepsilon_c e^{-K\sqrt{\varepsilon_c}}}{50\sqrt{\varepsilon_c}}.$$

So we can conclude that

$$\left|\frac{1}{2\sqrt{\varepsilon_c}} \log\left(1 + \frac{4E_c(Z)\varepsilon_c}{(1-4\varepsilon_c)E_c(Z) - (1+2\sqrt{\varepsilon_c})}\right)\right| \leq \frac{1}{10}.$$

■

We want now to estimate the size of the cylinders near z_c. For $n \geq 1$ let $C_n(c)$ be the set of points of J_c with external angle belonging to $[2^{-(n+1)}, 2^{-n}]$. In symbolic dynamics it corresponds to the cylinder

$$C_n(c) = \underbrace{\mathbf{1} \cdots \mathbf{1}}_{n}\ \mathbf{0}.$$

Lemma 3.2. *There exist $c_0 < \frac{1}{4}$ and $K > 0$ such that*

$$\forall c \in [c_0, \frac{1}{4}] \qquad \forall z_n \in C_n(c) \qquad \sum_{n \geq 0} |\mathcal{I}m\ z_n| \leq K.$$

Proof. The proof will be done by showing relations between $\mathcal{I}m\ z_n = y_n$ and $\mathcal{I}m\ Z_c(z_n) = \mathcal{I}m\ Z_n = Y_n$.
By construction,

$$z_n = \frac{1}{2} + \sqrt{\varepsilon_c} + 2\sqrt{\varepsilon_c} \frac{e^{2\sqrt{\varepsilon_c}Z_n}}{1 - e^{2\sqrt{\varepsilon_c}Z_n}}.$$

This leads to

$$y_n = \frac{\sqrt{\varepsilon_c}}{2} \frac{\sin(\sqrt{\varepsilon_c}Y_n)}{(\sinh(\sqrt{\varepsilon_c}X_n))^2 + (\sin(\sqrt{\varepsilon_c}Y_n))^2}.$$

This can also be written

$$y_n = \frac{\sin(\sqrt{\varepsilon_c}Y_n)}{2\sqrt{\varepsilon_c}} \frac{1}{\left(\frac{\sinh(\sqrt{\varepsilon_c}X_n)}{\sqrt{\varepsilon_c}X_n}\right)^2 X_n^2 + \left(\frac{\sin(\sqrt{\varepsilon_c}Y_n)}{\sqrt{\varepsilon_c}Y_n}\right)^2 Y_n^2}. \tag{3.3}$$

Using this relation and the fact that $\sqrt{\varepsilon_c}Y_n \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we easily get

$$|y_n| \leq \frac{\pi^2}{8}\frac{|Y_n|}{|Z_n|^2}. \tag{3.4}$$

With the lemma 3.1 this leads to

$$|y_n| \leq \lambda \frac{|Y_n|}{n^2} \leq \frac{\lambda}{n}. \tag{3.5}$$

On the other hand, $|Y_n| \leq |Y_0| + \sum_{k=0}^{n-1} |Y_{k+1} - Y_k|$, and

$$|Y_{k+1} - Y_k| = \frac{1}{2\sqrt{\varepsilon_c}} \left| \mathcal{I}m \ \log\left(\frac{z_{k+1} - z_c}{z_k - z_c}\frac{z_k - \zeta_c}{z_{k+1} - \zeta_c}\right)\right|.$$

Since $\frac{z_{k+1}-z_c}{z_k - z_c} = \frac{z_{k+1}-z_c}{P_c(z_{k+1})-P_c(z_c)} = \frac{1}{z_{k+1}+z_c}$, we have

$$\begin{aligned} |Y_{k+1} - Y_k| &\leq \frac{1}{2\sqrt{\varepsilon_c}} \left| \mathcal{I}m \ \log\left(\frac{z_{k+1}+\zeta_c}{z_{k+1}+z_c}\right)\right| \\ &\leq \frac{1}{2\sqrt{\varepsilon_c}} \left| \mathcal{A}rg \ \left(1 - \frac{2\sqrt{\varepsilon_c}(x_{k+1}+z_c)}{|z_{k+1}+z_c|^2} + i\frac{2\sqrt{\varepsilon_c}y_k}{|z_{k+1}+z_c|^2}\right)\right|. \end{aligned}$$

We can conclude that there is $\lambda > 0$ such that

$$|Y_{k+1} - Y_k| \leq \lambda |y_k| \qquad \text{and also } |Y_n| \leq |Y_0| + \lambda \sum_{k=0}^{n-1} |y_k|. \tag{3.6}$$

We use (3.5) and obtain

$$|y_n| \leq \frac{\lambda |Y_0|}{n^2} + \frac{\lambda}{n^2}\sum_{k=1}^{n}\frac{1}{k} \leq \lambda \frac{\log n}{n^2}. \tag{3.7}$$

The proof of the lemma is finished since $\sum \frac{\log n}{n^2} < +\infty$.

■

Using (3.6) and the lemma 3.2 one obtains

Corollary 3.3. *There exists $c_0 < \frac{1}{4}$ and $K > 0$ such that*

$$\forall c \in [c_0, \frac{1}{4}] \qquad Z_c(J_c) \subset \{\mathcal{I}m \ Z \in [-K, K]\}.$$

We are now in position to prove the main technical result of this section.

Lemma 3.4. *There exist $\alpha > 0$, $c_0 < \frac{1}{4}$ and $K > 0$ such that for all $c \in [c_0, \frac{1}{4}]$, all $n \in \mathbb{N}$ and all $Z_n = X_n + iY_n \in C_n(c)$ we have*

$$n\sqrt{\varepsilon_c} < \alpha \Longrightarrow \quad \frac{1}{Kn^2} \leq \textit{diam } (C_n(c)) \leq \frac{K}{n^2}$$

$$n\sqrt{\varepsilon_c} \geq \alpha \Longrightarrow \quad \frac{\varepsilon_c e^{2X_n\sqrt{\varepsilon_c}}}{K} \leq \textit{diam } (C_n(c)) \leq K\varepsilon_c e^{2X_n\sqrt{\varepsilon_c}}.$$

Remark 3.5. There is K such that inequalities

$$\frac{\varepsilon_c e^{-Kn\sqrt{\varepsilon_c}}}{K} \leq \text{diam } (C_n(c)) \leq \frac{K}{n^2}$$

are true for all integer n and all $c \in [c_0, \frac{1}{4}]$.

Proof. Let Z_n be a point in $Z_c(C_n(c))$, by lemma 3.1

$$|Z_n| \geq A + Bn.$$

Then by bounded distortion

$$\text{diam } Z_c(C_n(c)) \leq K|Z_n - Z_{n+1}| \leq K,$$

and

$$\frac{1}{K}|(Z_c^{-1})'(Z_n)| \leq \text{diam } C_n(c) \leq K|(Z_c^{-1})'(Z_n)|.$$

But

$$Z_c^{-1}(Z) = \frac{\zeta_c E_c(Z) - z_c}{E_c(Z) - 1} \qquad \text{and} \qquad (Z_c^{-1})'(Z) = \frac{4\varepsilon_c E_c(Z)}{(E_c(Z)-1)^2} = \frac{\varepsilon_c}{(\sinh(\sqrt{\varepsilon_c}Z))^2}.$$

So we have

$$\begin{aligned} |(Z_c^{-1})'(Z)| &= \frac{\varepsilon_c}{(\sinh(\sqrt{\varepsilon_c}X))^2 + (\sin(\sqrt{\varepsilon_c}Y))^2} \\ &= \frac{1}{\left(\frac{\sinh(\sqrt{\varepsilon_c}X)}{\sqrt{\varepsilon_c}X}\right)^2 X^2 + \left(\frac{\sin(\sqrt{\varepsilon_c}Y)}{\sqrt{\varepsilon_c}Y}\right)^2 Y^2}. \end{aligned}$$

But $\sqrt{\varepsilon_c}Y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ thus $\frac{4}{\pi^2} \leq \left(\frac{\sinh(\sqrt{\varepsilon_c}Y)}{\sqrt{\varepsilon_c}Y}\right)^2 \leq 1$. And if $-\alpha \leq \sqrt{\varepsilon_c}X \leq 2$ then $1 \leq \left(\frac{\sin(\sqrt{\varepsilon_c}X)}{\sqrt{\varepsilon_c}X}\right)^2 \leq \left(\frac{\sinh(\alpha)}{\alpha}\right)^2$. So if $Z \in Z_c(J_c)$ and if $-\alpha \leq \sqrt{\varepsilon_c}X \leq 2$ then

$$\left(\frac{\alpha}{\sinh(\alpha)}\right)^2 \frac{1}{|Z|^2} \leq |(Z_c^{-1})'(Z)| \leq \frac{\pi^2}{4}\frac{1}{|Z|^2}. \tag{3.8}$$

On the other hand if Z lies in $Z_c(J_c)$, corollary 3.3 implies $|\mathcal{I}m\ Z| \leq K$. If $\sqrt{\varepsilon_c}X \leq -\alpha$ and if ε_c is small enough then

$$\left(\frac{1 - e^{-2\alpha}}{2}\right)^2 e^{-2\sqrt{\varepsilon_c}X} \leq (\sinh(\sqrt{\varepsilon_c}X))^2 + (\sin(\sqrt{\varepsilon_c}Y))^2 \leq e^{-2\sqrt{\varepsilon_c}X}.$$

We can conclude that for $Z \in Z_c(J_c)$ and $\sqrt{\varepsilon_c}X \leq -\alpha$ we have

$$\varepsilon_c e^{2\sqrt{\varepsilon_c}X} \leq |(Z_c^{-1})'(Z)| \leq \frac{4}{(1 - e^{-2\alpha})^2}\varepsilon_c e^{2\sqrt{\varepsilon_c}X} \tag{3.9}$$

Applying estimations (3.8) and (3.9) with $Z = Z_n$ gives the result.

■

3.2. **Equicontinuity of the ϕ_c's.** The function $c \mapsto \phi_c(1)$ converges to $\phi_{\frac{1}{4}}(1)$ when c grows to $\frac{1}{4}$. It follows that ϕ_c converges to $\phi_{\frac{1}{4}}$ on the dense set of points of the unit circle with dyadic argument. To show that ϕ_c converges uniformly on $\partial\mathbb{D}$ to $\phi_{\frac{1}{4}}$ it thus suffices to show that the family $\{\phi_c\}$ is equicontinuous.
In order to prove this it suffices to find a sequence ε_n decreasing to 0 such that

$$\forall \gamma_c \in \Gamma_n(c) \qquad \text{diam } \gamma_c < \varepsilon_n \tag{3.10}$$

Among the cylinders $\Gamma_n(c)$ we will distinguish the "good" cylinders as the ones for which the symbolic writing ends with $\mathbf{1\ 0}$ or $\mathbf{0\ 1}$. These cylinders are the cylinders for which the diameter may be computed "uniformly" by Kœbe distortion theorem. More precisely there exists $K > 0$ independent of c such that if $\gamma_n(c)$ is a good cylinders of order n then

$$\frac{1}{K}\frac{1}{|(P_c^{\circ n})'(\gamma_n(c))|} \leq \text{diam } \gamma_n(c) \leq \frac{K}{|(P_c^{\circ n})'(\gamma_n(c))|} \tag{3.11}$$

Lemma 3.6. *There exist $c_0 \in]0, \frac{1}{4}[$, $n_0 > 0$ and a constant $K > 0$ such that if $\gamma_n(c)$ is a good cylinder of order $n \geq n_0$ and $c \in [c_0, \frac{1}{4}[$ then*

$$|(P_c^{\circ n})'(\gamma_n(c))| \geq \lambda n^2 \tag{3.12}$$

Proof. First of all let us consider a point in the cylinder $\mathbf{0\ 0\ 1}$. Its image by P_c is in the cylinder $\mathbf{0\ 1}$ which is easily seen to be included in $E = \{|z| \geq \frac{1}{2}\} \cap \{\mathcal{R}e\ z \leq 0\} \cap \{\mathcal{I}m\ z \geq 0\}$. So $z \in \mathbf{0\ 0\ 1}$ can be written $\sqrt{\zeta - c}$ with $\zeta \in E$. It follows that

$$|z| = \sqrt{|\zeta - c|} \geq (\frac{1}{4} + c^2)^{\frac{1}{4}} = m_c.$$

Since m_c converges to $\frac{5^{\frac{1}{4}}}{2}$ as c converges to $\frac{1}{4}$ we conclude that there exists $c_0 \in [0, \frac{1}{4}[$ and $\varepsilon_0 > 0$ such that

$$\forall c \geq c_0 \qquad \forall z \in \mathbf{0\ 0\ 1} \qquad |2z| \geq 1 + \varepsilon_0.$$

By symmetry it is also true for $\mathbf{0\ 1\ 0}$, $\mathbf{1\ 0\ 1}$ and $\mathbf{1\ 1\ 0}$.
In other words if z belongs to a good cylinder of order 3 then $|2z| \geq 1 + \varepsilon_0$ if $c \in [c_0, \frac{1}{4}[$.
Let now $\gamma_n(c)$ be any good cylinder of order $n \geq 3$, $\gamma_n(c) = x_1 \cdots x_n$ ($x_i \in \{\mathbf{0}, \mathbf{1}\}$). Let p be the number of indices $i \leq n-3$ such that $x_i x_{i+2} x_{i+2}$ is a good cylinder and let $k_1, \cdots, k_q$ be the lengths of the long (≥ 2) sequences of $\mathbf{0}$ or $\mathbf{1}$ that appear in $\gamma_n(c)$. Clearly $p \geq q$ and by the result of the preceding paragraph

$$|(P_c^{\circ n})'(M_n\ \mathbf{0})| \geq \max(\lambda n^2, 1)$$

where M_n denotes the cylinder of length n with n $\mathbf{1}$. The above estimation implies

$$|(P_c^{\circ n})'(\gamma_n(c))| \geq (1+\varepsilon_0)^p (\max(\lambda k_1^2, 1)) \cdots (\max(\lambda k_q^2, 1)) \tag{3.13}$$

If $(1+\varepsilon_0)^p \geq n^2$ there is nothing to prove. If $(1+\varepsilon_0)^p \leq n^2$ then $p \leq K \log n$. On the other hand the largest k_j is greater than $\frac{k_1 + \cdots + k_q}{q}$. It follows that

$$|(P_c^{\circ n})'(\gamma_n(c))| \geq \lambda (1+\varepsilon_0)^p \left(\frac{k_1 + \cdots + k_q}{q}\right)^2.$$

Since $p \leq \log n$, necessarily $k_1 + \cdots + k_q \geq \frac{n}{2}$ if n is greater than some n_0, and

$$|(P_c^{\circ n})'(\gamma_n(c))| \geq \frac{\lambda}{4}\frac{(1+\varepsilon_0)^p}{q^2} n^2 \geq K n^2,$$

since $\frac{(1+\varepsilon_0)^p}{q^2} \geq \frac{(1+\varepsilon_0)^p}{p^2} \geq K$.

■

If $\gamma_n(c)$ is a good cylinder, the lemma 3.6 and (3.11) immediately imply

$$\text{diam } \gamma_n(c) \leq \frac{K}{n^2} \tag{3.14}$$

We recall that M_n is the cylinder of length n with n **1**, and let $\overline{M_n}$ be the cylinder of length n with n **0**.
M_n can be written as $\cup_{k\geq n} M_k$ **0**, thus

$$\text{diam } M_n \leq \sum_{k\geq n} \text{diam } M_k \text{ } \mathbf{0}.$$

But M_k **0** ends with a good cylinder so that

$$\text{diam } M_n \leq \frac{K}{n}. \tag{3.15}$$

One can now conclude. Let X be any cylinder of order $n \geq n_0$. If it is good we already know that diam $X \leq \frac{K}{n^2}$. If not then we may assume that $X = YM_k$ (or $X = Y\overline{M_k}$) with $k \geq 2$ and Y good. If $k \geq \frac{n}{2}$ then diam $X \leq K$diam $M_k \leq \frac{K}{n}$. If $k < \frac{n}{2}$ then Y is a good cylinder of order $n - k \geq \frac{n}{2}$ and

$$\text{diam } X \leq \text{diam } Y \leq \frac{K}{n^2}.$$

In any case we have proven that there exists $K > 0$ such that

$$\forall n \geq n_0 \qquad \forall \gamma_n(c) \in \Gamma_n(c) \qquad \text{diam } \gamma_n(c) \leq \frac{K}{n} \tag{3.16}$$

3.3. **Weak convergence of equilibrium states.** As we already know that the ϕ_c's converge uniformly, and in order to prove that the Lyapounov exponents converge to a non zero limit, it suffices to prove that the equilibrium states weakly converge to a measure $\widetilde{\mu}$ such that

$$\int_{\partial\mathbb{D}} \log|2\phi_{\frac{1}{4}}| d\widetilde{\mu} > 0. \tag{3.17}$$

The first prove of this fact is in [Ha^2]. The strategy there is as follow : we write $\widetilde{\mu}_c = \widetilde{h}_c\widetilde{\omega}_c$ and we use the fact that $\widetilde{\omega}_c$ converges weakly to $\widetilde{\omega}_{\frac{1}{4}}$, the normalized Hausdorff measure on $J_{\frac{1}{4}}$ ([Bo,Zi]). Unfortunately $(\widetilde{h}_c)_{c\in[c_0,\frac{1}{4}[}$ is not an equicontinuous family because $\widetilde{h}_c(\phi_c(1))$ tends to $+\infty$ as c grows to $\frac{1}{4}$. The difficulty is overcome by showing that $(\frac{1}{\widetilde{h}_c})_{c\in[c_0,\frac{1}{4}[}$ actually is equicontinuous. Every weak limit of $\widetilde{\mu}_{c_k}$ is then seen to be absolutely continuous with respect to $\widetilde{\omega}_{\frac{1}{4}}$. The author then uses Aaronson-Denker-Urbański theorem [Aa,De,Ur] saying that there is only one measure and that it satisfies (3.17).
In the purpose of being self contained we give another proof. It uses the same idea as in [Do,Se,Zi]; we use a renormalization of P_c which will appear to be uniformly hyperbolic.
We denote by C_0 the set of points in J_0 with external angles between $\frac{1}{3}$ and $\frac{2}{3}$ and by C_j, $j \geq 1$ the successive inverse images of C_0 that are in the upper-half plane. The associated Markov partition is then $\{C_j,\ j \geq 0\} \cup \{\overline{C_j},\ j \geq 1\}$. On each C_j or $\overline{C_j}$ we replace P_c by $P_c^{\circ j}$ sending C_j into C_0. The advantage is that this new

dynamical system is uniformly hyperbolic, the drawback being that the "alphabet" becomes infinite. But the results of Walters or Mauldin-Urbański [Wa^2], [Ma,Ur] show that usual features go through in this case. In particular for every $c \in [0, \frac{1}{4}]$ there exists a unique invariant measure μ_c^r (r for renormalized) for the new dynamics which is equivalent to the conformal measure ω_c, which is independent of the renormalization. Moreover, we claim that $(\mu_c^r)_c \in [0, \frac{1}{4}[$ weakly converges to $\mu_{\frac{1}{4}}^r$. Indeed $\mu_c^r = h_c^r \omega_c$ and in this case, because of thermodynamic formalism, the family $(h_c^r)_c \in [0, \frac{1}{4}]$ is equicontinuous.
It remains to elucidate the link between μ_c^r and μ_c. We first observe that, if continuous on $C(\partial\mathbb{D})$, the linear form

$$\varphi \longmapsto \int_{\partial\mathbb{D}} S_{N(x)}(\varphi)(x) d\widetilde{\mu_c^r}(x),$$

where $N(x) = j$ if $x \in C_j \cup \overline{C_j}$, defines a P_0-invariant measure. This form is indeed continuous since

$$\begin{aligned}
\int_{\partial\mathbb{D}} |S_{N(x),c}(\varphi)(x)| d\widetilde{\mu_c^r}(x) &\leq ||\varphi||_\infty \textstyle\int_{\partial\mathbb{D}} N(x) d\widetilde{\mu_c^r}(x) \\
&\leq K||\varphi||_\infty \left(\sum_{j \geq 0} (1+j)\widetilde{\omega_c}(C_j) \right) \\
&\leq K||\varphi||_\infty \left(\sum_{j \geq 0} \frac{1}{j^{2d(c)-1}} \right) \\
&\leq K||\varphi||_\infty.
\end{aligned}$$

Since $d(c) \geq 1 + \alpha$ for some $\alpha > 0$ on $[\frac{1}{8}, \frac{1}{4}]$.
One can now conclude. If $\varphi \in C(\partial\mathbb{D})$ then

$$\begin{aligned}
\int_{\partial\mathbb{D}} \varphi d\widetilde{\mu_c} &= \textstyle\int_{\partial\mathbb{D}} S_{N(x)}(\varphi)(x) d\widetilde{\mu_c^r}(x) \\
&\underset{c \to \frac{1}{4}}{\longrightarrow} \textstyle\int_{\partial\mathbb{D}} S_{N(x)}(\varphi)(x) d\widetilde{\mu_c^r}(x) = \int_{\partial\mathbb{D}} \varphi d\widetilde{\mu}_{\frac{1}{4}}.
\end{aligned}$$

This approach also allows us to have estimates on $\mu_c(C_j)$ which will be used later. What precedes indeed also proves that μ_c is uniformly equivalent to ω_c on C_0 and

Lemma 3.7. *There exist $c_0 < \frac{1}{4}$, $\lambda > 1$ such that for all $c \in [c_0, \frac{1}{4}]$ and all $n \geq 1$,*

$$\frac{1}{\lambda} \varepsilon_c^{d(c)-\frac{1}{2}} e^{-\lambda n \sqrt{\varepsilon_c}} \leq \widetilde{\mu_c}(C_n(c)) \leq \frac{\lambda}{n^{2d(c)-1}}.$$

Notice that this approach gives a new proof in this particular case of the theorem in [Aa,De,Ur].

Proof. By construction,

$$\widetilde{\mu_c}(C_n(c)) = \sum_{k=n}^{+\infty} \widetilde{\mu_c^r}(C_n(c)).$$

But $\frac{1}{\lambda}\widetilde{\mu_c^r} \leq \widetilde{\omega_c} \leq \lambda \widetilde{\mu_c^r}$. On the other hand ω_c is normalized Hausdorff measure and bounded distortion theorem implies

$$\frac{1}{\lambda}\widetilde{\omega_c}(C_n(c)) \leq \text{diam } (C_n(c))^{d(c)} \leq \lambda \widetilde{\omega_c}(C_n(c)).$$

Using lemma 3.4 and remark 3.5 we get

$$\frac{\varepsilon_c^{d(c)}}{\lambda}\sum_{k=n}^{+\infty} e^{-\lambda k\sqrt{\varepsilon_c}} \leq \widetilde{\mu_c}(C_n(c)) \leq \lambda \sum_{k=n}^{+\infty} \frac{1}{k^{2d(c)}}.$$

And this leads to

$$\frac{1}{\lambda}\varepsilon_c^{d(c)-\frac{1}{2}} e^{-\lambda n\sqrt{\varepsilon_c}} \leq \widetilde{\mu_c}(C_n(c)) \leq \frac{\lambda}{n^{2d(c)-1}}.$$

■

4. Proof of the main theorem

First of all if we want to give an estimation of $d'(c)$ we need (see 2.8) an explicit expression of $\frac{\partial}{\partial c}(\log|2\phi_c|)$.
Differentiating the following functional equation

$$\phi_c(z^2) = \phi_c(z)^2 + c \tag{4.1}$$

we obtain, denoting $\dot{\phi_c}$ for $\frac{\partial}{\partial c}(\phi_c)$,

$$\dot{\phi_c}\,(z^2) = 2\phi_c(z)\;\dot{\phi_c}\,(z) + 1. \tag{4.2}$$

This can also be written

$$\dot{\phi_c}\,(z) = -\frac{1}{2\phi_c(z)} - \frac{1}{2\phi_c(z)}\;\dot{\phi_c}\,(z^2). \tag{4.3}$$

Reinjecting this formula one obtains

(4.4)

$$\dot{\phi_c}\,(z) = -\sum_{k=0}^{n-1} \frac{1}{2\phi_c(z)2\phi_c(z^2)\cdots 2\phi_c(z^{2^k})} - \frac{1}{2\phi_c(z)2\phi_c(z^2)\cdots 2\phi_c(z^{2^{n-1}})}\;\dot{\phi_c}\,(z^{2^n})$$

For $c < \frac{1}{4}$ the rest in (4.4) tends to 0 as n grows to infinity. So $\dot{\phi_c}$ can be written as an infinite sum

$$\dot{\phi_c}\,(z) = -\sum_{k=0}^{+\infty} \frac{1}{2\phi_c(z)2\phi_c(z^2)\cdots 2\phi_c(z^{2^k})}. \tag{4.5}$$

In particular we will use later the following functional equality

$$\dot{\phi_c}\,(z) = \frac{1}{(P_c^{\circ n})'(\phi_c(z))}\;\dot{\phi_c}\,(z^{2^n}) + \sum_{k=1}^{n} \frac{1}{(P_c^{\circ k})'(\phi_c(z))}. \tag{4.6}$$

Notice finally that

$$\frac{\partial}{\partial c}(\log|2\phi_c|) = \mathcal{R}e\;\left(\frac{\dot{\phi_c}}{\phi_c}\right). \tag{4.7}$$

In order to prove that the main contribution in $\int_{\partial\mathbb{D}} \mathcal{R}e\;\left(\frac{\dot{\phi_c}}{\phi_c}\right) d\widetilde{\mu}_c$ comes from the integral near z_c, we must show that as long as we keep far from z_c, the integral of the modulus is bounded by a constant independent of $c_0 < c < \frac{1}{4}$.
Let N be an integer which will be chosen later. We define $\mathcal{B}_N$ as $\mathbb{D} \setminus \mathcal{M}_N$, where

$\mathcal{M}_N = M_N \cup \overline{M_N}$. Because of bounded distortion theorem and our estimation in section 3.3 we know that there is $\lambda(N) > 0$ independent of $c_0 < c < \frac{1}{4}$ such that

$$\forall z \in \partial\mathbb{D} \qquad P_c^{\circ n}(\phi_c(z)) \in \mathcal{B}_N \Longrightarrow \frac{1}{|(P_c^{\circ n})'(\phi_c(z))|} \leq \frac{\lambda(N)}{n^2} \tag{4.8}$$

and also

$$\forall A \subset \mathcal{B}_N \qquad \widetilde{\mu}_c(A) \leq \lambda(N)\widetilde{\omega}_c(A) \tag{4.9}$$

We prove

Proposition 4.1. *For all $N \in \mathbb{N}$ there exists $\lambda(N) > 0$ such that*

$$\forall c \in [c_0, \frac{1}{4}[\qquad \int_{\mathcal{B}_N} |\frac{\dot{\phi_c}}{\phi_c}| d\widetilde{\mu}_c \leq \lambda(N).$$

Proof. Notice that $|\phi_c| \geq \frac{1}{2}$ so we just have to bound $\int_{\mathcal{B}_N} |\dot{\phi_c}| d\widetilde{\mu}_c$.
Let U_n $(n \geq 1)$ denotes the set of points in $\partial\mathbb{D}$ that come back or arrived for the first time in $\mathcal{B}_N$ after exactly n iterations of $T : z \mapsto z^2$.
Let N_0 be an integer and set $A_{N_0,n} = T^{-N_0}(U_n) \cap \mathcal{B}_N$. The set $\{A_{N_0,n}\}_{n\geq 1}$ is a $\widetilde{\mu}_c$-partition of $\mathcal{B}_N$. So we can write

$$\int_{\mathcal{B}_N} |\dot{\phi_c}| d\widetilde{\mu}_c = \sum_{n=1}^{+\infty} \int_{A_{N_0,n}} |\dot{\phi_c}| d\widetilde{\mu}_c. \tag{4.10}$$

Now we study $\int_{A_{N_0,n}} |\dot{\phi_c}| d\widetilde{\mu}_c$ and using (4.6) with $n + N_0$ we get

$$\int_{A_{N_0,n}} |\dot{\phi_c}| d\widetilde{\mu}_c \leq \int_{A_{N_0,n}} \frac{1}{|(P_c^{\circ n+N_0})'(\phi_c)|} |\dot{\phi_c} \circ T^{\circ n+N_0}| d\widetilde{\mu}_c + (n+N_0)\widetilde{\mu}_c(A_{N_0,n})$$

But if $z \in A_{N_0,n}$ then $P_c^{\circ(n+N_0)}(\phi_c(z)) \in \mathcal{B}_N$ so using (4.8) we obtain

$$\int_{A_{N_0,n}} |\dot{\phi_c}| d\widetilde{\mu}_c \leq \frac{\lambda(N)}{(n+N_0)^2} \int_{A_{N_0,n}} |\dot{\phi_c} \circ T^{\circ n+N_0}| d\widetilde{\mu}_c + (n+N_0)\widetilde{\mu}_c(A_{N_0,n}).$$

We recall that $\widetilde{\mu}_c$ is T-invariant, and also that $\mathbb{1}_{A_{N_0,n}} \leq \mathbb{1}_{\mathcal{B}_N} \circ T^{\circ n+N_0}$ so we have

$$\int_{A_{N_0,n}} |\dot{\phi_c}| d\widetilde{\mu}_c \leq \frac{\lambda(N)}{(n+N_0)^2} \int_{\mathcal{B}_N} |\dot{\phi_c}| d\widetilde{\mu}_c + (n+N_0)\widetilde{\mu}_c(A_{N_0,n}). \tag{4.11}$$

We need an estimation of $\widetilde{\mu}_c(A_{N_0,n})$. But $A_{N_0,n}$ can be written as

$$A_{N_0,n} = \bigcup_{i=2}^{N_0} \bigcup_{X \in \mathbb{B}_i} XC_{n+N+N_0-i},$$

where $\mathbb{B}_i$ is the set of good cylinders of length i (see section 1). We use (4.9) in order to obtain

$$\widetilde{\mu}_c(A_{N_0,n}) \leq \lambda(N) \sum_{i=2}^{N_0} \sum_{X \in \mathbb{B}_i} \widetilde{\omega}_c(XC_{n+N+N_0-i}). \tag{4.12}$$

We know that $\widetilde{\omega}_c$ is a kind of quasi-Bernoulli measure on good cylinders. More precisely there exists $\lambda > 1$ independent of $c \in [c_0, \frac{1}{4}]$ such that :

$$\frac{1}{\lambda}\widetilde{\omega_c}(X)\widetilde{\omega_c}(Y) \leq \widetilde{\omega_c}(XY) \leq \lambda\widetilde{\omega_c}(X)\widetilde{\omega_c}(Y)$$

for all good cylinders X. This leads to

$$\widetilde{\mu}_c(A_{N_0,n}) \leq \lambda(N) \sum_{i=2}^{N_0} \widetilde{\omega}_c(C_{n+N+N_0-i}) \sum_{X \in \mathbb{B}_i} \widetilde{\omega}_c(X).$$

But $\widetilde{\omega}_c(C_{n+N+N_0-i}) \leq \frac{\lambda}{(n+N+N_0-i)^{2d(c)}}$, and $\sum_{X\in\mathbb{B}_i} \widetilde{\omega}_c(X) \leq 1$. So we have shown that

$$\widetilde{\mu}_c(A_{N_0,n}) \leq \lambda(N) \frac{N_0}{(n+N)^{2d(c)}}.$$

This estimation in (4.11) gives

$$\int_{A_{N_0,n}} |\dot{\phi}_c| d\widetilde{\mu}_c \leq \frac{\lambda(N)}{(n+N_0)^2} \int_{\mathcal{B}_N} |\dot{\phi}_c| d\widetilde{\mu}_c + \lambda(N) N_0 \frac{n+N_0}{(n+N)^{2d(c)}}.$$

And with (4.10) we get

$$\begin{aligned} \int_{\mathcal{B}_N} |\dot{\phi}_c| d\widetilde{\mu}_c &\leq \lambda(N) \left(\sum_{n=N_0}^{+\infty} \frac{1}{n^2} \right) \int_{\mathcal{B}_N} |\dot{\phi}_c| d\widetilde{\mu}_c + N_0 \lambda(N) \sum_{n=N}^{+\infty} \frac{1}{n^{2d(c)-1}} + N_0^2 \lambda(N) \sum_{n=N}^{+\infty} \frac{1}{n^{2d(c)}} \\ &\leq \frac{\lambda(N)}{N_0} \int_{\mathcal{B}_N} |\dot{\phi}_c| d\widetilde{\mu}_c + \frac{\lambda(N) N_0}{N^{2d(c)-2}} + \frac{\lambda(N) N_0^2}{N^{2d(c)-1}}. \end{aligned}$$

For $\frac{\lambda(N)}{2} < N_0 \leq \frac{\lambda(N)}{2} + 1$ we get

$$\int_{\mathcal{B}_N} |\dot{\phi}_c| d\widetilde{\mu}_c \leq \frac{\lambda(N)}{N^{2d(c)-1}}.$$

∎

We are now in position to give a proof theorem 1.1.

Proof. By (2.8) and (4.7) we know that

$$d'(c) = \frac{d(c)}{\int_{\partial\mathbb{D}} \log|2\phi_c| d\widetilde{\mu}_c} \int_{\partial\mathbb{D}} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c. \tag{4.13}$$

Let N be an integer, we have

$$\int_{\partial\mathbb{D}} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c \geq \int_{\mathcal{M}_N} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c - \int_{\mathcal{B}_N} |\frac{\dot{\phi}_c}{\phi_c}| d\widetilde{\mu}_c.$$

So using proposition 4.1 we get

$$\int_{\partial\mathbb{D}} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c \geq \int_{\mathcal{M}_N} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c - \lambda(N).$$

We study $\int_{\mathcal{M}_N} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c$. We know that

$$\int_{\mathcal{M}_N} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c = \sum_{n=N}^{+\infty} \int_{C_n} \mathcal{R}e \left(\sum_{k=1}^{+\infty} \frac{1}{(P_c^k)'(\phi_c(z))} \frac{1}{\phi_c(z)} \right) d\widetilde{\mu}_c(z).$$

This leads to

$$\int_{\mathcal{M}_N} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c \geq \sum_{n=N}^{+\infty} \sum_{k=1}^{n} \int_{C_n} \mathcal{R}e \left(\frac{1}{(P_c^k)'(\phi_c(z))} \right) d\widetilde{\mu}_c(z) -$$

$$\sum_{n=N}^{+\infty}\int_{C_n}\left|\sum_{k=n+1}^{+\infty}\frac{1}{(P_c^k)'(\phi_c(z))}\right|d\widetilde{\mu}_c(z).$$

But

$$\begin{aligned}\mathbb{1}_{C_n}(z)\left|\sum_{k=n+1}^{+\infty}\frac{1}{(P_c^k)'(\phi_c(z))}\right| &= \mathbb{1}_{C_n}(z)\left|\frac{1}{(P_c^n)'(\phi_c(z))}\right|\left|\dot{\phi}_c\circ T^n(z)\right| \\ &\leq \tfrac{\lambda(N)}{n^2}\mathbb{1}_{\mathcal{B}_N}\circ T^{\circ n}(z)|\ \dot{\phi}_c\circ T^n(z)|.\end{aligned}$$

Using the T-invariance we get

$$\sum_{n=N}^{+\infty}\int_{C_n}\left|\sum_{k=n+1}^{+\infty}\frac{1}{(P_c^k)'(\phi_c(z))}\right|d\widetilde{\mu}_c(z)\leq\lambda(N)\int_{\mathcal{B}_N}|\ \dot{\phi}_c\ |d\widetilde{\mu}_c\sum_{n=N}^{+\infty}\frac{1}{n^2}.$$

With proposition 4.1 this leads to

$$\sum_{n=N}^{+\infty}\int_{C_n}\left|\sum_{k=n+1}^{+\infty}\frac{1}{(P_c^k)'(\phi_c(z))}\right|d\widetilde{\mu}_c(z)\leq\lambda(N). \tag{4.14}$$

Concerning $\int_{\mathcal{M}_N}\mathcal{R}e\ \left(-\frac{\dot{\phi}_c}{\phi_c}\right)d\widetilde{\mu}_c$ we have

$$\int_{\mathcal{M}_N}\mathcal{R}e\ \left(-\frac{\dot{\phi}_c}{\phi_c}\right)d\widetilde{\mu}_c\geq\sum_{n=N}^{+\infty}\sum_{k=1}^{n}\int_{C_n}\mathcal{R}e\ \left(\frac{1}{(P_c^k)'(\phi_c(z))}\right)d\widetilde{\mu}_c(z)-\lambda(N).$$

Now we fix N large enough so that $|\mathcal{A}rg\ (P_c^k)'(\phi_c(z))|\leq\frac{\pi}{6}$ for all $z\in C_n$ with $n\geq N$ and $k\leq n$. Thanks to lemma 3.2 we know that it is possible and we obtain

$$\forall n\geq N\qquad\forall k\geq n\qquad\forall z\in C_n\qquad\mathcal{R}e\ \left(\frac{1}{(P_c^k)'(\phi_c(z))}\right)\geq\frac{1}{\sqrt{3}}\frac{1}{|(P_c^k)'(\phi_c(z))|}.$$

So we have shown that

$$\int_{\partial\mathbb{D}}\mathcal{R}e\ \left(-\frac{\dot{\phi}_c}{\phi_c}\right)d\widetilde{\mu}_c\geq\frac{1}{\sqrt{3}}\sum_{n=N}^{+\infty}\sum_{k=1}^{n}\int_{C_n}\frac{1}{|(P_c^k)'(\phi_c(z))|}d\widetilde{\mu}_c(z)-\lambda(N).$$

The estimation

$$\forall z\in C_n\qquad|(P_c^k)'(\phi_c(z))|\leq\lambda\frac{n^2}{(n-k)^2}, \tag{4.15}$$

which is a consequence of lemma 3.4, is true for $n\sqrt{\varepsilon_c}<\alpha$, ($\varepsilon_c=\frac{1}{4}-c$).
Using (4.15) we write

$$\int_{\partial\mathbb{D}}\mathcal{R}e\ \left(-\frac{\dot{\phi}_c}{\phi_c}\right)d\widetilde{\mu}_c\geq\lambda\sum_{n=0}^{\frac{\alpha}{\sqrt{\varepsilon_c}}}\sum_{k=1}^{n}\widetilde{\mu_c}(C_n)\left(\frac{n-k}{n}\right)^2-\lambda(N).$$

We know (see lemma 3.7) that $\widetilde{\mu}_c(C_n)\geq\lambda\varepsilon_c^{d(c)-\frac{1}{2}}e^{-\lambda n\sqrt{\varepsilon_c}}$. This leads to

$$\begin{aligned}\int_{\partial\mathbb{D}}\mathcal{R}e\ \left(-\frac{\dot{\phi}_c}{\phi_c}\right)d\widetilde{\mu}_c &\geq \frac{1}{\lambda}\varepsilon_c^{d(c)-\frac{1}{2}}\sum_{n=N}^{\frac{\alpha}{\sqrt{\varepsilon_c}}}\frac{e^{-\lambda n\sqrt{\varepsilon_c}}}{n^2}\sum_{k=1}^{n}k^2-\lambda(N) \\ &\geq \frac{1}{\lambda}\varepsilon_c^{d(c)-\frac{1}{2}}\sum_{n=N}^{\frac{\alpha}{\sqrt{\varepsilon_c}}}ne^{-\lambda n\sqrt{\varepsilon_c}}-\lambda(N) \\ &\geq \frac{1}{\lambda}\varepsilon_c^{d(c)-\frac{3}{2}}-\lambda(N).\end{aligned}$$

Finally we obtain

$$\int_{\partial\mathbb{D}} \mathcal{R}e\ \left(-\frac{\dot{\phi}_c}{\phi_c}\right) d\widetilde{\mu}_c \geq \frac{\lambda}{(\frac{1}{4}-c)^{\frac{3}{2}-d(c)}} - \lambda(N).$$

In particular $d'(c) \to +\infty$ as $c \to \frac{1}{4}$ and so if $c \in [c_0, \frac{1}{4}]$, $d(c) < \delta$ and

$$\forall c \in [c_0, \frac{1}{4}] \qquad d'(c) \geq \frac{K}{(\frac{1}{4}-c)^{\frac{3}{2}-\delta}}.$$

In order to get the opposite inequality we note that

$$\left|\int_{\partial\mathbb{D}} \mathcal{R}e\ \left(-\frac{\dot{\phi}_c}{\phi_c}\right) d\widetilde{\mu}_c\right| \leq \int_{\mathcal{M}_N} \left|\frac{\dot{\phi}_c}{\phi_c}\right| d\widetilde{\mu}_c + \int_{\mathcal{B}_N} \left|\frac{\dot{\phi}_c}{\phi_c}\right| d\widetilde{\mu}_c. \tag{4.16}$$

Since the proposition 4.1 asserts that $|\int_{\mathcal{B}_N}|$ is bounded by a constant $\lambda(N)$ which only depends on N, we have

$$\left|\int_{\partial\mathbb{D}} \mathcal{R}e\ \left(-\frac{\dot{\phi}_c}{\phi_c}\right) d\widetilde{\mu}_c\right| \leq \int_{\mathcal{M}_N} \left|\frac{\dot{\phi}_c}{\phi_c}\right| d\widetilde{\mu}_c + \lambda(N). \tag{4.17}$$

For $\int_{\mathcal{M}_N}$ we write

$$\int_{\mathcal{M}_N} \left|\frac{\dot{\phi}_c}{\phi_c}\right| d\widetilde{\mu}_c \leq \sum_{n=N}^{+\infty}\sum_{k=1}^{n}\int_{C_n} \frac{1}{|(P_c^{\circ k})'(\phi_c)|} d\widetilde{\mu}_c + \sum_{n=N}^{+\infty}\int_{C_n}\left|\sum_{k=n+1}^{+\infty}\frac{1}{(P_c^{\circ k})'(\phi_c)}\right| d\widetilde{\mu}_c.$$

We use (4.14) and we obtain

$$\int_{\mathcal{M}_N} \left|\frac{\dot{\phi}_c}{\phi_c}\right| d\widetilde{\mu}_c \leq \sum_{n=N}^{+\infty}\sum_{k=1}^{n}\int_{C_n} \frac{1}{|(P_c^{\circ k})'(\phi_c)|} d\widetilde{\mu}_c + \lambda(N). \tag{4.18}$$

Note that bounded distortion theorem implies

$$\frac{1}{|(P_c^{\circ k})'(\phi_c(C_n))|} \sim \frac{|(P_c^{\circ(n-k)})'(\phi_c(C_{n-k}))|}{|(P_c^{\circ n})'(\phi_c(C_n))|} \leq \lambda \frac{\text{diam } C_n(c)}{\text{diam } C_{n-k}(c)}.$$

Using this inequality, the lemma 3.7 and inequality (4.18), (4.17) becomes

$$\left|\int_{\partial\mathbb{D}} \mathcal{R}e\ \left(-\frac{\dot{\phi}_c}{\phi_c}\right) d\widetilde{\mu}_c\right| \leq \lambda(N) + \lambda \sum_{n=N}^{+\infty} \frac{1}{n^{2d(c)-1}} \sum_{k=1}^{n} \frac{\text{diam } C_n(c)}{\text{diam } C_k(c)}.$$

Let $S_1 = \sum_{n=N}^{\frac{\alpha}{\sqrt{\varepsilon_c}}}\sum_{k=1}^{n}$, $S_2 = \sum_{\frac{\alpha}{\sqrt{\varepsilon_c}}}^{+\infty}\sum_{k=1}^{\frac{\alpha}{\sqrt{\varepsilon_c}}}$ and $S_3 = \sum_{\frac{\alpha}{\sqrt{\varepsilon_c}}}^{+\infty}\sum_{k=\frac{\alpha}{\sqrt{\varepsilon_c}}}^{n}$, we study those three pieces showing that they are all bounded by $\lambda(\frac{1}{4}-c)^{d(c)-\frac{3}{2}}$.

For $n \leq \frac{\alpha}{\sqrt{\varepsilon_c}}$ and $k \leq n$, lemma 3.4 implies

$$\frac{\text{diam } C_n(c)}{\text{diam } C_k(c)} \leq \lambda \left(\frac{k}{n}\right)^2.$$

So we have

$$\begin{aligned} |S_1| &\leq \lambda \sum_{n=N}^{\frac{\alpha}{\sqrt{\varepsilon_c}}} \frac{1}{n^{2d(c)+1}} \sum_{k=1}^{n} k^2 \\ &\leq \lambda \sum_{n=N}^{\frac{\alpha}{\sqrt{\varepsilon_c}}} \frac{1}{n^{2d(c)-2}} \\ &\leq \lambda \varepsilon_c^{d(c)-\frac{3}{2}}. \end{aligned}$$

For $n \geq \frac{\alpha}{\sqrt{\varepsilon_c}}$ and $1 \leq k \leq \frac{\alpha}{\sqrt{\varepsilon_c}}$ lemma 3.4 implies

$$\frac{\text{diam } C_n(c)}{\text{diam } C_k(c)} \leq \lambda k^2 \varepsilon_c e^{-\lambda n \sqrt{\varepsilon_c}}.$$

So

$$\begin{aligned} |S_2| &\leq \lambda \varepsilon_c \sum_{n=\frac{\alpha}{\sqrt{\varepsilon_c}}}^{+\infty} \frac{e^{-\lambda n \sqrt{\varepsilon_c}}}{n^{2d(c)-1}} \sum_{k=1}^{\frac{\alpha}{\sqrt{\varepsilon_c}}} k^2 \\ &\leq \lambda \frac{1}{\sqrt{\varepsilon_c}} \sum_{n=\frac{\alpha}{\sqrt{\varepsilon_c}}}^{+\infty} \frac{1}{n^{2d(c)-1}} \\ &\leq \lambda \varepsilon_c^{d(c)-\frac{3}{2}}. \end{aligned}$$

And for $n \geq \frac{\alpha}{\sqrt{\varepsilon_c}}$ and $\frac{\alpha}{\sqrt{\varepsilon_c}} \leq k \leq n$, we have

$$\frac{\text{diam } C_n(c)}{\text{diam } C_k(c)} \leq \lambda e^{2(X_n - X_k)\sqrt{\varepsilon_c}}$$

where X_n and X_k are real part respectively of $Z_n \in C_n(c)$ and $Z_k \in C_k(c)$. Bounded distortion theorem and lemma 3.1 imply

$$(X_n - X_k) \leq -\lambda(n-k)$$

and we obtain

$$\begin{aligned} |S_3| &\leq \lambda \sum_{n=\frac{\alpha}{\sqrt{\varepsilon_c}}}^{+\infty} \frac{1}{n^{2d(c)-1}} \sum_{k=\frac{\alpha}{\sqrt{\varepsilon_c}}}^{n} e^{-\lambda(n-k)\sqrt{\varepsilon_c}} \\ &\leq \lambda \sum_{n=\frac{\alpha}{\sqrt{\varepsilon_c}}}^{+\infty} \frac{1}{n^{2d(c)-1}} \sum_{k=0}^{n-\frac{\alpha}{\sqrt{\varepsilon_c}}} e^{-\lambda k \sqrt{\varepsilon_c}} \\ &\leq \lambda \frac{1}{\sqrt{\varepsilon_c}} \sum_{n=\frac{\alpha}{\sqrt{\varepsilon_c}}}^{+\infty} \frac{1}{n^{2d(c)-1}} \\ &\leq \lambda \varepsilon_c^{d(c)-\frac{3}{2}} \end{aligned}$$

.

We have shown

$$d'(c) \leq \lambda (\frac{1}{4} - c)^{d(c)-\frac{3}{2}}. \tag{4.19}$$

To finish the proof it remains to see why inequality (4.19) is actually equivalent with the same one with $d(c)$ replaced by δ. We would like to thank Pierrette Sentenac for showing us the following proof.
Putting $t = \frac{1}{4} - c$ and $y(t) = \delta - d(\frac{1}{4} - t)$, the question boils down to proving that $t \mapsto t^{-y(t)}$ is bounded at the origin. But we know that $y(t)$ converges to 0 at 0, so

$y(t) \leq \frac{1}{2}$ for t less than some $\eta > 0$. Integrating the analogue of (4.19) for y we then see that $|y(t)| \leq Kt^{\delta-1}$, $t < \eta$, and the result follows.

■

References

[Aa,De,Ur] J.Aaronson, M.Denker, M.Urbański.
Ergodic theory for Markov fibered systems and parabolic rational maps.
Trans.Amer.math.soc. (1993), pp. 495-548

[Bo,Zi] O.Bodart, M.Zinsmeister.
Quelques résultats sur la dimension de Hausdorff des ensembles de Julia des polynômes quadratiques.
Fund.Math.(1996), pp. 121-137

[Bow] R.Bowen.
Equilibrium states and the ergodic theory of Anosov diffeomorphisms.
Lecture Notes in Math. Vol 470, Springer-Verlag, (1975)

[De,Ur^1] M.Denker, M.Urbański.
Hausdorff and conformal measures on Julia sets with a rationally indifferent fixed point.
J.London Math. Soc. (1991), pp.107-118.

[De,Ur^2] M.Denker, M.Urbański.
Absolutely continuous invariant measures on Julia sets with rationally indifferent fixed points.
Forum Math. (1991), pp.561-579.

[Do,Se,Zi] A.Douady, P.Sentenac, M.Zinsmeister
Implosion parabolique et dimension de Hausdorff.
CRAS, (1997).

[Ga] L.Garnett.
A computer algorithm for determining the Hausdorff dimension of certains fractals.
Math.comp.51(1988), pp.291-300.

[Ha^1] G.Havard.
Mesures invariantes pour les fractions rationnelles géométriquement finie.
Fund.Math.(A paraître courant 1999).

[Ha^2] G.Havard.
Thèse de l'université d'Orléans.

[McMu] C.McMullen.
Hausdorff dimension and conformal dynamics 3 : Computation of dimension.
Preprint. (1997).

[Ma,Ur] R.D.Mauldin, M.Urbański.
Dimensions and measures in infinite iterated function systems.
Proc.London Math.soc.(3)73(1996), pp.105-154.

[Ru^1] D.Ruelle.
Thermodynamic Formalism. Addison-Wesley (1978)

[Ru^2] D.Ruelle.
Repellers for real analytic maps.
Ergodic Theory Dynamical Systems. Vol 2, (1982), pp.99-108.

[Wa^1] P.Walters
A variationnal principle for the pressure of continuous transformations.
Amer.J.Math., 97 (1976), pp.937 -971

[Wa^2] P.Walters
Invariant measures and equilibrium states for some mappings which expand distances.
T.A.M.S. (1978).