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A MATHEMATICAL MODEL FOR THE EVAPORATION OF A LIQUID FUEL DROPLET, SUBJECT TO NONLINEAR CONSTRAINTS

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We study the mathematical evolution of a liquid fuel droplet inside a vessel. In particular, we give the radius evolution of the droplet on a finite time interval, starting from a hyperbolic system involving the pressure and the velocity of the surrounding gas. Existence of bounded solutions for the mass fraction of the liquid, submitted to nonlinear constraints, is shown. Numerical simulations are given, in agreement with known physical experiments.

Keywords: Hyperbolic system, droplet radius, nonlinear parabolic equation.

AMS Subject Classification: 35Q35, 76T10.

1. INTRODUCTION

The evaporation of a single drop in a gas involves simultaneous heat and mass transfer processes in which the heat for evaporation is transferred to the drop surface by conduction and convection, while vapor is carried by convection and diffusion back into the gas stream. The evaporation rate depends on the pressure, temperature and physical properties of the gas, the temperature, volatility and diameter of the drop in the spray.

In the experimental study of a single drop evaporation performed by the LCSR, Combustion Laboratory of the University of campus from Orleans, France, see also [7], the drop is suspended from a silicate tube. The elliptic shape of the drop is corrected to a sphere of equal volume. Important quantities of interest for these experiments are time evolution of the radius of the drop, as well as classical quantities such as mass fractions or temperatures of the liquid, gas. In the experiments

made above, the so-called D^2 law is used to simplify two-phase fluid models and then propose adequate numerical schemes. This law simply states that the time evolution of the radius behaves as $\frac{1}{t^2}$ in time flow, and is purely phenomenological.

Our purpose in this paper is exactly in the opposite sense. We start from phenomenological fluid (mixtures) PDE modeling the drop evaporation process, compute the time-evolution of the drop radius, and then deduce other quantities of interest such as mass fractions of the liquid and gas. Thus, we shall consider a drop initially represented by a single component mixture (chemical specie 1) while the surrounding gas at time $t = 0$ is pure (chemical specie 2).

During the evaporation process, the vapor of liquid is transferred into the gas, while by condensation at the droplet surface and then by diffusion the specie 2 appears in the drop.

We make the important simplification that the moving interface between the drop and the surrounding gas (i.e. between the two species) is spherical with radius $R = R(t)$ evolving in time.

Let ρ_G (resp. ρ_L) denote the gas density (resp. liquid density). Let v_G (resp. v_L) denote the gas velocity (resp. velocity density). Then one has the classical overall continuity and momentum conservation laws

$$\partial_t \rho_k + \operatorname{div}(\rho_k v_k) = 0 \quad (1.1)$$

$$\rho_k \partial_t v_k + \rho_k v_k \cdot \nabla v_k = -\nabla p \quad (1.2)$$

where k is the gas G or the liquid L depending on whether one considers the gas or liquid and p the state equation of the gas.

Let Y_{L1}, Y_{L2} (resp. Y_{G1}, Y_{G2}) the mass fractions of the liquid (resp. gas) obtained after diffusion of species in the surrounding gas. Therefore for two species, one has

$$Y_{G1} + Y_{G2} = Y_{L1} + Y_{L2} = 1.$$

Along with equation (1.1), we have to add the equation giving the conservation of species. So for the liquid we have

$$\rho_L \partial_t Y_{Lk} + \rho_L v_L \cdot \nabla Y_{Lk} + \operatorname{div}(\rho_L Y_{Lk} v_{Lk}) = -\rho_L f(Y_{Lk}), \quad k = 1, 2 \quad (1.3)$$

Y_{Lk} denoting the mass fraction of the liquid, and f a continuous function associated to a friction or a resistance for the drop.

We assume that the speed of the liquid is small, and thus settled it to 0. Equations (1.1) and (1.3) can then be written under conservative form as

$$\partial_t(\rho \tilde{g}) + \operatorname{div}(\rho \tilde{g} v) = F(\tilde{g}) \quad (1.4)$$

or

$$\partial_t u + \frac{\partial}{\partial x}(f(u)) = F(u) \quad (1.5)$$

in a system of particular coordinates.

If Γ is a curve of discontinuity of u , then one has

$$[f(u)] = [u] \frac{dx}{dt} \quad (1.6)$$

where $[.]$ denotes the jump of the inner quantity, $s = \frac{dx}{dt}$ is the speed of discontinuity along Γ . The jump Relation (1.6) is known as Rankine-Hugoniot condition. It merely means that discontinuities cannot be completely arbitrary. These classical facts are detailed for instance in [4].

In the case of the drop, to find interface condition at the surface of the drop i.e. for $r = R(t)$, it is sufficient to use (1.5) and (1.4) in polar coordinates, getting

$$[\rho \tilde{g}v] = [\rho \tilde{g}] \frac{dR}{dt}. \quad (1.7)$$

Thus taking $\tilde{g} = 1$, one has

$$(\rho_G - \rho_L) \frac{dR}{dt} = \rho_G v_G - \rho_L v_L$$

that is also with $v = 0$

$$\rho_G \left(v_G - \frac{dR}{dt} \right) = -\rho_L \frac{dR}{dt}. \quad (1.8)$$

Taking $\tilde{g} = Y$ in (1.7), Y denoting the mass fraction of the liquid or the gas after diffusing, we get

$$(\rho_G Y_{Gk} - \rho_L Y_{Lk}) \frac{dR}{dt} = \rho_G Y_{Gk} (v_G + v_{Gk}) - \rho_L Y_{Lk} v_{Lk}$$

and this is equivalent to

$$\rho_G Y_{Gk} (v_G - R') + \rho_G Y_{Gk} v_{Gk} = -\rho_L Y_{Lk} R' + \rho_L Y_{Lk} v_{Lk}. \quad (1.9)$$

Above v_{Gk} (resp. v_{Lk}) is the speed of the specie Gk (resp. Lk), $k = 1, 2$.

Combining relation (1.9) with Fick's law, see [3,4], that is

$$Y_{G1} v_{G1} = -D_{12} \nabla Y_{G1}, \quad Y_{G2} v_{G2} = -D_{21} \nabla Y_{G2}$$

where D_{12} and D_{21} are diffusion coefficients, and with equations relating the thermodynamic state at the interface $r = R(t)$

$$Y_{Gk} = K_k Y_{Lk}, \quad k = 1, 2$$

we obtain for the mass fraction of the liquid Y_{L1} the boundary condition

$$\partial_r Y_{L1} + \frac{R'(t)(K_1 - 1)}{K_2 \rho_G(R(t), t) - K_3} Y_{L1} = 0, \text{ at } r = R(t) \quad (1.10)$$

using polar coordinates.

In [2] we have made huge mathematical and physical simplifications taking the state equation of the gas p as constant in (1.2) and considering the velocity of the gas v_G as a given function of the time t . Thus in our previous work, the system (1.1), (1.2) was reduced to the equation (1.1) with $v_G(t)$ given.

In this paper, we first analyze the hyperbolic system (1.1), (1.2) with an auxiliary state equation of the gas $p_1 = \rho^\gamma$. Once $\rho_G(r, t)$ and $v_G(r, t)$ determined, we compute the radius $R(t)$ of the drop suspended in the gas through the ordinary differential equation (1.8). $R(t)$ being known, we shall then study the mass fraction Y_{L1} of the liquid after the evaporation process through the PDE (1.3) along with the boundary condition (1.10), with a function f submitted to suitable assumptions.

Let us note that spherical symmetry reduces the problem to a transient process of 1-dimensional nature. Within the framework of weighted Sobolev spaces on initial data and for some continuous function f subject to increasing condition, we shall provide an unique local solution for the mass fraction Y_{L1} of the liquid. In addition, we shall show that if the initial condition is bounded, then so is our solution.

2. HYPERBOLIC SYSTEM

The velocity $v_G(r, t)$ of the gas and its density $\rho_G(r, t)$ satisfy the following system, using polar coordinates

$$\begin{aligned} \frac{\partial \rho_G}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho_G v_G) &= 0 \\ \frac{\partial}{\partial t} (\rho_G v_G) + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho_G v_G^2) &= - \frac{\partial p}{\partial r} \end{aligned} \quad (2.1)$$

Setting $\rho(r, t) = r^2 \rho_G(r, t)$, $v(r, t) = v_G(r, t)$ we have

$$\begin{aligned} \frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial r} + \rho \frac{\partial v}{\partial r} &= 0 \\ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} &= - \frac{1}{\rho} \frac{\partial p_1}{\partial r} \end{aligned} \quad (2.2)$$

where $p_1(r, t)$ is an auxiliary function connected to the state equation of the gas $p(r, t)$ by $\frac{\partial p_1}{\partial r} = r^2 \frac{\partial p}{\partial r}$.

In (2.2) we choose the auxiliary function $p_1(r, t)$ such that $p_1 = \rho^\gamma$, $\gamma > 1$.

In this way, we have to consider the following system

$$\begin{aligned} \frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial r} + \rho \frac{\partial v}{\partial r} &= 0 \\ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \gamma \rho^{\gamma-2} \frac{\partial p}{\partial r} &= 0 \end{aligned} \quad (2.3)$$

(2.3) is equivalent to

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ v \end{pmatrix} + A \cdot \frac{\partial}{\partial r} \begin{pmatrix} \rho \\ v \end{pmatrix} = 0, \quad (2.4)$$

A being the matrix (2, 2)

$$A = \begin{pmatrix} v & \rho \\ \gamma \rho^{\gamma-2} & v \end{pmatrix}$$

The eigenvalues of A (characteristics speeds) are $\lambda = v - c$ and $\mu = v + c$, $c = \sqrt{\gamma \rho^{\gamma-1}}$. Since $\lambda < \mu$, system (2.4) is hyperbolic. Then there exists two functions $W(\rho, v)$ and $Z(\rho, v)$ (Riemann invariants) such that

$$W(\rho, v) = \text{constant on } \frac{dX^1}{dt} = \lambda \quad (2.5)$$

$$Z(\rho, v) = \text{constant on } \frac{dX^2}{dt} = \mu. \quad (2.6)$$

$W(\rho, v)$ is determined by the system $\frac{dv}{c/\rho} = \frac{d\rho}{1}$, the vector $R_1 = (1, c/\rho)$ being the eigenvector associated to the eigenvalue μ . Thus

$$W(\rho, v) = v - \frac{2c}{\gamma - 1} \quad (2.7)$$

In the same way, the Riemann invariant $Z(\rho, v)$ corresponding to λ is given by

$$Z(\rho, v) = v + \frac{2c}{\gamma - 1} \quad (2.8)$$

The functions $W(\rho, v) = W(t, r)$ and $Z(\rho, v) = Z(t, r)$ satisfy the following system equivalent to system (2.4)

$$\begin{aligned} \frac{\partial W}{\partial t} + \lambda(W, Z) \frac{\partial W}{\partial r} &= 0 \\ \frac{\partial Z}{\partial t} + \mu(W, Z) \frac{\partial Z}{\partial r} &= 0 \end{aligned} \quad (2.9)$$

where $\lambda(W, Z)$ and $\mu(W, Z)$ are given by

$$\begin{aligned} \lambda &= -\left(\frac{\gamma-3}{4}\right)Z + \left(\frac{\gamma+1}{4}\right)W \\ \mu &= \left(\frac{\gamma+1}{4}\right)Z - \left(\frac{\gamma-3}{4}\right)W \end{aligned} \quad (2.10)$$

as can be seen from (2.7) and (2.8).

It is well known that a sufficient condition in order that (2.9) is authentically nonlinear is that $\frac{\partial \lambda}{\partial W} > 0$ and $\frac{\partial \mu}{\partial Z} > 0$, which is the case here according to (2.10).

Integration along the characteristics defined by

$$\frac{dX^1}{dt} = \lambda(W, Z), \quad X^1(0) = \beta$$

gives

$$X^1_{(0,\beta)}(t) = \beta + \int_0^t \lambda(W(s, X^1(s)), Z(s, X^1(s))) ds \quad (2.11)$$

and therefore the solution of the value initial problem

$$W_t + \lambda(W, Z)W_r = 0, \quad W(0, r) = W_0(r) \quad (2.12)$$

can be written as

$$W(t, r) = W_0\left(X^1_{(0,\beta)}(0)\right) = W_0(\beta), \quad (2.13)$$

where $\beta = r - \int_0^t \lambda(W(s, X^1(s)), Z(s, X^1(s))) ds$.

Similarly we have

$$Z(t, r) = Z_0\left(X^2_{(0,\alpha)}(0)\right) = Z_0(\alpha) \quad (2.14)$$

where

$$X^2_{(0,\alpha)}(t) = \alpha + \int_0^t \mu(W(s, X^2(s)), Z(s, X^2(s))) ds.$$

Proposition 2.1. *Let us suppose that $W'_0(\beta) < 0$ or $Z'_0(\alpha) < 0$. Then the solution of system (2.3) is defined on a finite interval $[0, T[$.*

□

Proof

Differentiation of (2.5) and (2.11) with respect to β gives

$$\frac{dX^1_\beta}{dt} = \lambda_\beta(W, Z) \quad \text{with} \quad X^1_\beta(t=0) = 1. \quad (2.15)$$

In the same way

$$\frac{dX^2_\alpha}{dt} = \mu_\alpha(W, Z) \quad \text{with} \quad X^2_\alpha(t=0) = 1. \quad (2.16)$$

But $\lambda_\beta = \lambda_W W_\beta + \lambda_Z Z_\beta = \lambda_W W'_0(\beta)$ and $\mu_\alpha = \mu_Z Z'_0(\alpha)$. Using these results in (2.15) and (2.16) and integrating w.r.t. t along the characteristics gives

$$X_\beta^1(t) = 1 + \int_0^t \lambda_W W_0'(\beta) dt = 1 + \left(\frac{\gamma+1}{4} \right) W_0'(\beta) t \quad (2.17)$$

$$X_\alpha^2(t) = 1 + \int_0^t \mu_Z Z_0'(\alpha) dt = 1 + \left(\frac{\gamma+1}{4} \right) Z_0'(\alpha) t \quad (2.18)$$

From (2.17) we see that $X_\beta^1(t_1) = 0$ for $t_1 = \frac{-4}{(\gamma+1)W_0'(\beta)} > 0$. Similarly $X_\alpha^2(t_2) = 0$ for $t_2 = \frac{-4}{(\gamma+1)Z_0'(\alpha)} > 0$. Hence $\frac{\partial W}{\partial r}(t, r)$ becomes infinite for $T = \inf\{t_1, t_2\}$ since $\frac{\partial W}{\partial r} = W_\beta \cdot \frac{d\beta}{dX} = \frac{W_0'(\beta)}{X_\beta}$. \square

We can deduce from this result the following facts.

Consequence 1: Proposition 2.1 implies that system (2.9) admits an unique C^1 solution on $[0, T[$ for all $r \in \mathbb{R}^+$ and for initial data $\rho_G(0, r) = \rho_0(r)$ and $v_G(0, r) = v_0(r)$ belonging to $C^1(\mathbb{R}^+)$.

Consequence 2: Radius of the drop

From (1.8) we get the ODE giving the radius of the drop

$$\frac{dR(t)}{dt} = \frac{v_G(t, R(t))\rho_G(t, R(t))}{\rho_G(t, R(t)) - \rho_L}, \quad R(0) = R_0. \quad (2.19)$$

The Cauchy problem (2.19) has an unique solution $R(t)$ on a maximal time interval $[0, T^*[$ with $T^* \leq T$ for initial data $\rho_0(r)$ and $v_0(r)$ ensuring that $W_0'(r) < 0$ or $Z_0'(r) < 0$.

3. MASS FRACTION OF THE LIQUID

The mass fraction of the liquid Y_{L1} satisfies the conservation equation of specie (1.3) which can be rewritten as

$$\partial_t Y_{L1} - \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} Y_{L1} \right) + f(Y_{L1}) = 0 \quad (3.1)$$

using polar coordinates, the diffusion constant D_{12} being taken equal to 1. Then (3.1) is equivalent to

$$\partial_t Y_{L1} - \Delta Y_{L1} - \frac{2}{r} \frac{\partial}{\partial r} Y_{L1} + f(Y_{L1}) = 0, \quad \text{for } 0 < r < s(t) \quad (3.2)$$

where $s(t) = R(t)$ is the radius of the drop determined in section II.

The boundary condition at the surface $s(t)$ is given by the Rankine-Hugoniot condition connected to the thermodynamic equilibrium i.e. formula (1.10). With the change of variable $r = R(t)x$, the function $Y_{L1}(t, r)$ becomes the function $Y_{L1}(t, R(t)x) = u(t, x)$ which satisfies the following initial boundary value (i.b.v.) problem

$$\partial_t u - a(t) \left(\Delta u + \frac{2}{x} \partial_x u \right) - x \frac{R'(t)}{R(t)} \partial_x u + f(u) = 0, \quad 0 < x < 1, \quad t > 0 \quad (3.3)$$

$$|\lim_{x \rightarrow 0^+} x u_x(t, x)| < \infty, \quad u_x(t, 1) + k(t) u(t, 1) = 0 \quad (3.4)$$

$$u(0, x) = u_0(x) \quad (3.5)$$

where

$$a(t) = \frac{1}{R^2(t)}, \quad k(t) = \frac{R(t)R'(t)(K_1 - 1)}{K_2 \rho_G(t, R(t)) - K_3}. \quad (3.6)$$

Denote by $\Omega =]0, 1[$ and H for the Hilbert space defined by

$$H = \{v : \Omega \rightarrow \mathbb{R}, \text{ measurable and such that } \int_0^1 x^2 v^2(x) dx < +\infty\}.$$

Note that H is the closure of $C^0(\bar{\Omega})$ w.r.t. the norm $\|v\|_H = \left(\int_0^1 x^2 v^2(x) dx \right)^{1/2}$. We also introduce the real Hilbert space $V = \{v \in H \mid v' \in H\}$. In the following, we shall often use the fact that V is the closure of $C^1(\bar{\Omega})$ w.r.t. the norm $\|v\|_V = \left(\|v\|_H^2 + \|v'\|_H^2 \right)^{1/2}$. V is continuously embedded in H . Identifying H with his dual H' , one has $V \subset H \subset V'$ with continuous injections. Note also that the norms $\|\cdot\|_H$ and $\|\cdot\|_V$ can be defined, respectively, from the inner products $\langle u, v \rangle = \int_0^1 x^2 u(x)v(x) dx$ and $\langle u, v \rangle + \langle u', v' \rangle$.

We then have the following results, the proofs of which can be found in [6].

Lemma 3.1. *For every $v \in C^1([0, 1])$, $\epsilon > 0$ and $x \in [0, 1]$ we have*

$$\|v\|_0^2 = \frac{1}{2} \|v'\|_0^2 + v^2(1)$$

$$v^2(1) \leq \epsilon \|v'\|_0^2 + C_\epsilon \|v\|_0^2$$

$$|v(x)| \leq 2 \|v\|_1, \quad |xv(x)| \leq \sqrt{5} \|v\|_1$$

where $C_\epsilon = 3 + \frac{1}{\epsilon}$ and $\|\cdot\|_0 = \|\cdot\|_H$, $\|\cdot\|_1 = \|\cdot\|_V$. □

Lemma 3.2. *The embedding $V \subset H$ is compact.* □

Remark 3.1. Lemma 3.1 proves that $\left(\|v'\|_0^2 + v^2(1) \right)^{1/2}$ and $\|v\|_1$ are two equivalent norms on V since

$$\frac{2}{3}\|v\|_1^2 \leq v^2(1) + \|v'\|_0^2 \leq 5\|v\|_1^2, \quad \forall v \in V.$$

□

Remark 3.2. We have

$$xv(x) \in C^0([0, 1]), \quad \forall v \in V.$$

Indeed, on one hand $\lim_{x \rightarrow 0^+} xv(x) = 0, \forall v \in V$ (see [1], p.128), and on the other hand $v|_{[\epsilon, 1]} \in C^0([\epsilon, 1]), \forall \epsilon, 0 < \epsilon < 1$ since we have $H^1(\epsilon, 1) \subset C^0([\epsilon, 1])$ and $\epsilon\|v\|_{H^1(\epsilon, 1)} \leq \|v\|_1 \forall v \in V, 0 < \epsilon < 1$.

□

If X is any Banach space, we denote by $\|\cdot\|_X$ its norm, and by X' the dual space of X . We denote by $L^p(0, T; X), 1 \leq p \leq \infty$ the standard Banach space of real functions $u : (0, T) \rightarrow X$, measurable, such that

$$\|u\|_{L^p(0, T; X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p} < +\infty, \quad \text{for } 1 \leq p < \infty$$

and

$$\|u\|_{L^\infty(0, T; X)} = \text{ess sup}_{0 < t < T} \|u(t)\|_X, \quad \text{for } p = \infty.$$

Let $u(t), u'(t), u_x(t), u_{xx}(t)$ denote $u(t, x), \frac{\partial u}{\partial t}(t, x), \frac{\partial u}{\partial x}(t, x), \frac{\partial^2 u}{\partial x^2}(t, x)$ respectively.

We shall make the following assumptions:

- (H₁) $u_0 \in H$
- (H₂) $a, k \in W^{1, \infty}(0, T), a(t) \geq a_0 > 0$
- (F₁) $f \in C(\mathbb{R}, \mathbb{R})$
- (F₂) There exists positive constants C_1, C'_1, C_2 and $p, 1 < p < 3$ such that
 - (i) $uf(u) \geq C_1|u|^p - C'_1$
 - (ii) $|f(u)| \leq C_2(1 + |u|^{p-1})$

Let $u \in C^2([0, T] \times [0, 1])$ be a solution of problem (3.3)-(3.5). Then, after multiplying equation (3.3) by $x^2v, v \in V$ w.r.t. the scalar product of the space H , integrating by parts and taking into account boundary condition (3.4), we get

$$\frac{d}{dt} \langle u(t), v \rangle + a(t) \int_0^1 x^2 u_x v_x dx + a(t)k(t)u(1)v(1) - \frac{R'(t)}{R(t)} \int_0^1 x^3 u_x v dx + \langle f(u), v \rangle = 0$$

The weak formulation of the ibv problem (3.3)-(3.6) can be given in the following way:

Find $u(t)$ defined in the open set $(0, T)$ such that $u(t)$ satisfies the following variational problem

$$\frac{d}{dt} \langle u(t), v \rangle + \tilde{a}(t; u(t), v) + \langle f(u(t)), v \rangle = 0, \quad \forall v \in V \quad (3.7)$$

and the initial condition

$$u(0) = u_0 \quad (3.8)$$

where

$$\tilde{a}(t; u, v) = a(t) \int_0^1 x^2 u_x v_x dx + a(t)k(t)u(1)v(1) - \frac{R'(t)}{R(t)} \int_0^1 x^3 u_x v dx, \quad u, v \in V. \quad (3.9)$$

We then have the following lemma, the proof of which can be found in [2]

Lemma 3.3. *There exists constants K_T , α_T and β_T depending on T such that*

$$|\tilde{a}(t; u, v)| \leq K_T \|u\|_1 \|v\|_1, \quad \text{for all } u, v \in V, \quad (3.10)$$

$$\tilde{a}(t; u, u) \geq \alpha_T \|u\|_1^2 - \beta_T \|u\|_0^2, \quad u, v \in V. \quad (3.11)$$

□

We then have the following theorem

Theorem 3.1. *Let $T > 0$ and assumptions $(H_1), (H_2), (F_1), (F_2)$ hold true. Then, there exists a solution u of problem (3.7), (3.8) such that*

$$u \in L^2(0, T; V) \cap L^\infty(0, T; H), \quad x^{2/p}u \in L^p(Q_T)$$

$$tu \in L^\infty(0, T; V), \quad tu_t \in L^2(0, T; H).$$

Furthermore, if f satisfies the additional condition

$$(f(u) - f(v))(u - v) \geq -\delta |u - v|^2,$$

for all $u, v \in \mathbb{R}$, for some $\delta \in \mathbb{R}$, then the solution is unique.

□

Proof: We divide it in several steps.

Step1. The Galerkin method: denote by $\{w_j\}$, $j = 1, 2, \dots$ an orthonormal basis in the separable Hilbert space V . We find $u_m(t)$ of the form

$$u_m(t) = \sum_{j=1}^m c_{mj}(t)w_j \quad (3.12)$$

where $c_{mj}(t)$ satisfy the following system of nonlinear differential equations

$$\langle u'_m(t), w_j \rangle + \tilde{a}(t; u_m(t), w_j) + \langle f(u_m(t)), w_j \rangle = 0, \quad 1 \leq j \leq m \quad (3.13)$$

and the initial condition

$$u_m(0) = u_{0m} \quad (3.14)$$

where

$$u_{0m} \rightarrow u_0 \text{ strongly in } H. \quad (3.15)$$

It is clear that for each m , there exists a solution $u_m(t)$ of the form (3.12), which satisfies (3.13) and (3.14) almost everywhere on $0 \leq t \leq T_m$, for some T_m , $0 < T_m \leq T$. The following estimates allow us to take $T_m = T$ for all m .

Step2. A priori estimates.

(a) the first estimate:

Multiplying the j^{th} equation of the system (3.13) by $c_{mj}(t)$ and summing up w.r.t. j , we have

$$\frac{1}{2} \frac{d}{dt} \|u_m(t)\|_0^2 + \tilde{a}(t; u_m(t), u_m(t)) + \langle f(u_m(t)), u_m(t) \rangle = 0 \quad (3.16)$$

Using assumption (H₂), (F₂,i), Lemma 3.1 and Remark 3.1, it follows from (3.16) that

$$\frac{d}{dt} \|u_m(t)\|_0^2 + 2\alpha_T \|u_m(t)\|_1^2 + 2C_1 \int_0^1 x^2 |u_m(t, x)|^p dx \leq \frac{2C'_1}{3} + 2\beta_T \|u_m(t)\|_0^2 \quad (3.17)$$

Integrating (3.17), with (3.15), we have

$$S_m(t) \leq C_0 + \frac{2}{3} TC'_1 + 2\beta_T \int_0^t S_m(s) ds, \quad (3.18)$$

where

$$S_m(t) = \|u_m(t)\|_0^2 + 2\alpha_T \int_0^t \|u_m(s)\|_1^2 ds + 2C_1 \int_0^t ds \int_0^1 x^2 |u_m(s, x)|^p dx, \quad (3.19)$$

and C_0 is a constant depending only on u_0 with $\|u_{0m}\|_0^2 \leq C_0 \forall m$.

By Gronwall's lemma, we obtain from (3.18)

$$S_m(t) \leq \left(C_0 + \frac{2}{3} TC'_1 \right) \exp(2\beta_T t) \leq M_T, \quad \forall m, \forall t, 0 \leq t \leq T_m \leq T, \quad (3.20)$$

i.e. $T_m = T$.

In the following M_T will denote a generic constant depending only on T .

(b) the second estimate: replacing w_j by $t^2 u_m$ in (3.8) gives

$$\begin{aligned}
& \|tu'_m\|_0^2 + \frac{1}{2} \frac{d}{dt} \left[a(t) \|tu_m\|_0^2 + a(t)k(t)t^2u_m^2(1) \right] + \frac{1}{2} \frac{d}{dt} \left[t^2 \int_0^1 x^2 \hat{f}(u_m) dx \right] \\
&= \|u_{mx}\|_0^2 \frac{d}{dt} [t^2 a(t)] + \frac{1}{2} u_m^2(1) \frac{d}{dt} (t^2 a(t)k(t)) + 2t \int_0^1 x^2 \hat{f}(u_m) dx \\
&\quad + \frac{R'(t)t^2}{R(t)} \int_0^1 x^3 u_{mx} u'_m dx
\end{aligned} \tag{3.21}$$

where

$$\hat{f}(z) = \int_0^z f(y) dy. \tag{3.22}$$

Integrating (3.21) w.r.t. time variable from 0 to t , we shall have, after some rearrangements

$$\begin{aligned}
& 2 \int_0^t \|su'_m(s)\|_0^2 ds + a(t) \|tu_{mx}(t)\|_0^2 + a(t)t^2u_m^2(t,1) \\
&= a(t)(1-k(t))t^2u_m^2(t,1) + \int_0^t [s^2a(s)]' \|u_{mx}(s)\|_0^2 ds \\
&+ \int_0^t [s^2a(s)k(s)]' u_m^2(s,1) ds + 2 \int_0^t \frac{R'(s)}{R(s)} s^2 \langle xu_{mx}(s), u'_m(s) \rangle ds \\
&+ 4 \int_0^t s ds \int_0^1 x^2 \hat{f}(u_m(s,x)) dx - 2t^2 \int_0^1 x^2 \hat{f}(u_m) dx.
\end{aligned} \tag{3.23}$$

By means of assumption (H₂) and Remark 3.1, we get

$$a(t) \|tu_m(t)\|_0^2 + a(t)t^2u_m^2(t,1) \geq \frac{2}{3} a_0 \|tu_m(t)\|_1^2 \quad \forall t \in [0, T], \forall m. \tag{3.24}$$

Let us choose $\epsilon > 0$ such that

$$\|a\|_\infty \|1-k\|_\infty \epsilon < \frac{a_0}{3} \tag{3.25}$$

where $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(0,T)}$.

Using again Lemma 3.1, Remark 3.1 with $\epsilon > 0$ as in (3.25) and first estimate (3.20), we estimate without difficulty the terms in the r.h.s. of (3.23) as follows

$$\begin{aligned}
a(t)(1-k(t))t^2u_m^2(t,1) &\leq \|a\|_\infty \|1-k\|_\infty \left(\epsilon \|tu_m(t)\|_1^2 + C_\epsilon \|tu_m(t)\|_0^2 \right) \\
&\leq \frac{a_0}{3} \|tu_m(t)\|_1^2 + M_T
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
& \int_0^t (s^2a(s))' \|u_{mx}(s)\|_0^2 ds + \int_0^t (s^2a(s)k(s))' u_m^2(s,1) ds \\
&\leq [\|(t^2a)'\|_\infty + \|(t^2ak)'\|_\infty] \int_0^t [\|u_{mx}(s)\|_0^2 + u_m^2(s,1)] ds \\
&\leq 5 [\|(t^2a)'\|_\infty + \|(t^2ak)'\|_\infty] \int_0^t \|u_m(s)\|_1^2 ds \leq M_T
\end{aligned} \tag{3.27}$$

$$2 \left| \int_0^t s^2 \frac{R'(t)}{R(t)} \langle xu_{mx}(s), u'_m(s) \rangle ds \right| \leq \int_0^t \|su'_m(s)\|_0^2 ds + \left\| \frac{R'}{R} \right\|_\infty^2 \int_0^t \|su_m(s)\|_1^2 ds. \quad (3.28)$$

On the other hand, from assumptions (F₁) and (F₂), we have

$$-\hat{m}_0 = - \int_{-z_0}^{z_0} |f(y)| dy \leq \hat{f}(z) = \int_0^z f(y) dy \leq C_2 \left(|z| + \frac{|z|^p}{p} \right), \quad \forall z \in \mathbb{R} \quad (3.29)$$

where $z_0 = (C'_1/C_1)^{1/p}$.

Using the first estimate (3.20), (3.29) and Lemma 3.1, we obtain

$$\begin{aligned} & \left| 4 \int_0^t s ds \int_0^1 x^2 \hat{f}(u_m(s, x)) dx - 2t^2 \int_0^1 x^2 \hat{f}(u_m(t, x)) dx \right| \\ & \leq 4C_2 \int_0^t s ds \int_0^1 x^2 \left(|u_m(s, x)| + \frac{1}{p} |u_m(s, x)|^p \right) dx + 2t^2 \int_0^1 x^2 \hat{m}_0 dx \\ & \leq 4C_2 \int_0^t s \|u_m(s)\|_0 ds + \frac{4}{p} C_2 t \int_0^t ds \int_0^1 x^2 |u_m(s, x)|^p dx + \frac{2}{3} T^2 \hat{m}_0 \\ & \leq 2C_2 T \sqrt{M_T} + \frac{2C_2}{pC_1} T M_T + \frac{2}{3} T^2 \hat{m}_0 \leq M_T. \end{aligned} \quad (3.30)$$

Hence, we deduce from (3.23), (3.24), (3.26)-(3.28) and (3.30) that

$$\int_0^t \|su'_m(s)\|_0^2 ds + \frac{a_0}{3} \|tu_m(t)\|_1^2 \leq M_T + \left\| \frac{R'}{R} \right\|_\infty^2 \int_0^t \|su_m(s)\|_1^2 ds. \quad (3.31)$$

By Gronwall's lemma, it follows that

$$\int_0^t \|su'_m(s)\|_0^2 ds + \frac{a_0}{3} \|tu_m(t)\|_1^2 \leq M_T \exp \left(\left\| \frac{R'}{R} \right\|_\infty^2 \cdot \frac{3T}{a_0} \right) \leq M_T, \quad \forall t \in [0, T]. \quad (3.32)$$

On the other hand, by using (3.20) and assumption (F₂,ii) we have

$$\int_0^t ds \int_0^1 \left| x^{2/p'} f(u_m(s, x)) \right|^{p'} dx \leq (2C_2)^{p'} \int_0^t ds \int_0^1 x^2 |u_m(s, x)|^p dx \leq M_T \quad (3.33)$$

with $p' = \frac{p}{p-1}$.

step3: The limiting process.

By (3.20), (3.32) and (3.33) we deduce that there exists a subsequence of $\{u_m\}$, still denoted $\{u_m\}$ such that

$$\begin{aligned} u_m & \rightharpoonup u \text{ weakly } * \text{ in } L^\infty(0, T; H) \\ u_m & \rightharpoonup u \text{ weakly in } L^2(0, T; V) \\ x^{2/p} u_m & \rightharpoonup x^{2/p} u \text{ weakly in } L^p(Q_T) \\ tu_m & \rightharpoonup tu \text{ weakly } * \text{ in } L^\infty(0, T; V) \\ (tu_m)' & \rightharpoonup (tu)' \text{ weakly in } L^2(0, T; H). \end{aligned} \quad (3.34)$$

A compactness lemma ([5] p.57) and formula (3.34) enables us to extract from the sequence $\{u_m\}$ a subsequence still denoted by $\{u_m\}$ such that

$$tu_m \rightarrow tu \text{ strongly in } L^2(0, T; H). \quad (3.35)$$

The continuity of f implies

$$f(u_m(t, x)) \rightarrow f(u(t, x)) \text{ a.e. } (t, x) \in Q_T = (0, T) \times (0, 1) \quad (3.36)$$

since we have the convergence of $u_m(t, x)$ to $u(t, x)$ a.e. $(t, x) \in Q_T$.

Applying a weak convergence lemma (see [5]) we obtain

$$x^{2/p'} \tilde{f}(u_m) \rightarrow x^{2/p'} \tilde{f}(u) \text{ weakly in } L^{p'}(Q_T).$$

Passing to the limit in (3.13), (3.14), then by (3.15), (3.34) and (3.36) one can show that the function $u(t)$ satisfies the i.b.v. problem (3.7), (3.8).

step4: Uniqueness of the solutions.

First of all, we shall need the following lemma, which is a slight improvement of a lemma used in [2] (see also [5])

Lemma 3.4. *Let w be the weak solution of the i.b.v. problem*

$$\begin{aligned} w_t - a(t)(w_{xx} + \frac{2}{x}w_x) &= \tilde{f}(t, x), \quad 0 < t < T, \quad 0 < x < 1, \\ \left| \lim_{x \rightarrow 0^+} xw_x(t, x) \right| &< +\infty, \quad w_x(t, 1) + k(t)w(t, 1) = 0, \quad w(0, x) = 0, \\ w &\in L^2(0, T; V) \cap L^\infty(0, T; H), \quad x^{2/p}w \in L^p(Q_T) \\ tw &\in L^\infty(0, T; V), \quad tw_t \in L^2(0, T; H). \end{aligned}$$

Then

$$\frac{1}{2} \|w(t)\|_0^2 + \int_0^t a(s) \left[\|w_x(s)\|_0^2 + k(s)w^2(s, 1) \right] ds - \int_0^t \langle \tilde{f}(s), w(s) \rangle ds = 0, \quad \text{a.e. } t \in (0, T)$$

□

Uniqueness of the solution will be deduced as follows. Let u and v be two weak solutions of (3.3)–(3.5). Then $w = u - v$ is a weak solution of the associated problem mentioned in Lemma 3.4, with the r.h.s. function $\tilde{f}(t, x) = \frac{xR'(t)}{R(t)}w_x - f(u) + f(v)$. Lemma 3.4 gives

$$\frac{1}{2} \|w(t)\|_0^2 + \int_0^t \tilde{a}(s; w(s), w(s)) ds + 2 \int_0^t \langle f(u(s)) - f(v(s)), w(s) \rangle ds = 0.$$

Using Lemma 3.3 and (F₃) we obtain

$$\|w(t)\|_0^2 + 2\alpha_T \int_0^t \|w(s)\|_1^2 ds \leq 2(\delta + \beta_T) \int_0^t \|w(s)\|_0^2 ds. \quad (3.37)$$

If $\delta + \beta_T \geq 0$ we have $\|w(t)\|_0 = 0$ by applying Gronwall's lemma. In the case where $\delta + \beta_T < 0$ the result is clearly still true.

This ends the proof of Theorem 3.1. □

For the boundedness of solutions, we shall make use of the following assumptions

- (H₁') $u_0 \in L^\infty(0, 1)$, $|u_0(x)| \leq M$, a.e. $x \in (0, 1)$
- (H₂') $a, k \in W^{1,\infty}(0, T)$, $a(t) \geq a_0 > 0$, $k(t) \geq k_0 > 0$
- (F₁') $uf(u) \geq 0 \quad \forall u \in \mathbb{R}$ such that $|u| \geq \|u_0\|_\infty$, for a.e. $x \in (0, 1)$.

We then have the following result

Theorem 3.2. *Let (H₁'), (H₂'), (F₁)-(F₃) and (F₁') hold. Then the unique weak solution of the ibv problem (3.7)-(3.9) as given by theorem 1, belongs to $L^\infty(Q_T)$.* □

Proof: Firstly, assuming that $u_0(x) \leq M$, $Z = u - M$ satisfies the i.b.v. problem

$$\partial_t Z - a(t) \left(\Delta Z + \frac{2}{x} \partial_x Z \right) - x \frac{R'(t)}{R(t)} \partial_x Z + f(Z + M) = 0, \quad 0 < x < 1, \quad t \in (0, T) \quad (3.38)$$

$$|\lim_{x \rightarrow 0^+} x Z_x(t, x)| < \infty, \quad Z_x(t, 1) + k(t)[Z(t, 1) + M] = 0 \quad (3.39)$$

$$Z(0, x) = u_0(x) - M \quad (3.40)$$

Multiplying equation (3.38) by $x^2 v$, for $v \in V$, integrating by parts w.r.t. variable x and taking into account boundary condition (3.39), one has after some rearrangements

$$\begin{aligned} \int_0^1 x^2 Z_t v dx + a(t) \int_0^1 x^2 Z_x v_x dx + a(t) k(t) Z(t, 1) v(1) - \frac{R'(t)}{R(t)} \int_0^1 x^3 Z_x v dx \\ + \int_0^1 x^2 f(Z + M) v dx = -M a(t) k(t) v(1), \quad \forall v \in V \end{aligned} \quad (3.41)$$

$\forall v \in V$ hence for $v = Z^+ = \frac{1}{2}(Z + |Z|)$ since $u_0 \in L^\infty(0, 1)$. Thus it follows that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 x^2 |Z^+|^2 dx + a(t) \int_0^1 x^2 |(Z^+)_x|^2 dx + a(t) k(t) |Z^+(t, 1)|^2 \\ - \frac{R'(t)}{R(t)} \int_0^1 x^3 Z_x^+ Z^+ dx + \int_0^1 x^2 f(Z^+ + M) Z^+ dx = -M a(t) k(t) Z^+(t, 1) \leq 0 \end{aligned}$$

since

$$\int_0^1 x^2 Z_t Z^+ dx = \int_{0, Z > 0} x^2 (Z^+)_t Z^+ dx = \frac{1}{2} \frac{d}{dt} \int_0^1 x^2 |Z^+|^2 dx.$$

On the other hand, by assumption (H₂') and Remark 3.1, we obtain

$$a(t) \int_0^1 x^2 |Z_x^+|^2 dx + a(t) k(t) |Z^+(t, 1)|^2 \geq \tilde{C}_0 \|Z^+(t)\|_1^2, \quad (3.42)$$

where $\tilde{C}_0 = \frac{2}{3} a_0 \min \{1, k_0\}$.

Using the monotonicity of $f(u) + \delta u$ and (F_1) we have

$$\begin{aligned} \int_0^1 x^2 f(Z^+ + M) Z^+ dx &= \int_0^1 x^2 [f(Z^+ + M) - f(M)] Z^+ dx + \int_0^1 f(M) x^2 Z^+ dx \\ &\geq -\delta \int_0^1 x^2 |Z^+|^2 dx + \int_0^1 f(M) x^2 Z^+ dx \geq -\delta \|Z^+\|_0^2. \end{aligned} \quad (3.43)$$

Hence it follows from (3.41)-(3.43) with Cauchy's inequality applied to the term $-\frac{R'(t)}{R(t)} \int_0^1 x^3 Z_x^+ Z^+ dx$

$$\frac{d}{dt} \|Z^+(t)\|_0^2 + \tilde{C}_0 \|Z^+(t)\|_1^2 \leq \left(\frac{1}{\tilde{C}_0} \left\| \frac{R'}{R} \right\|_\infty^2 + 2|\delta| \right) \|Z^+(t)\|_0^2. \quad (3.44)$$

Integrating (3.44), we get

$$\|Z^+(t)\|_0^2 \leq \|Z^+(0)\|_0^2 + \left(\frac{1}{\tilde{C}_0} \left\| \frac{R'}{R} \right\|_\infty^2 + 2|\delta| \right) \int_0^t \|Z^+(s)\|_0^2 ds. \quad (3.45)$$

Since $Z^+(0) = (u(0, x) - M)^+ = (u_0(x) - M)^+ = 0$, hence using Gronwall's lemma, we obtain $\|Z^+(t)\|_0 = 0$. Thus $u(t, x) \leq M$ a.e. $(t, x) \in Q_T$.

The case $u_0(x) \geq -M$ can be dealt with, in the same manner as above by considering $Z = u + M$ and $Z^- = \frac{1}{2}(|Z| - Z)$. Thus we get $Z^- = 0$ and hence $u(t, x) \geq -M$ a.e. $(t, x) \in Q_T$.

All in all, one obtains $|u(t, x)| \leq M$ a.e. $(t, x) \in Q_T$ and this ends the proof of Theorem 3.2.

4. NUMERICAL APPLICATIONS

For the numerical applications, we have taken in (2.3) $\gamma = 3$, so that equations (2.9) reduce to Burger's equations

$$\begin{cases} W_t + WW_r = 0, & W(0, r) = W_0(r) \\ Z_t + ZZ_r = 0, & Z(0, r) = Z_0(r). \end{cases} \quad (4.1)$$

We know that the Burger's equation

$$u_t + uu_r = 0, \quad u(0, r) = u_0(r)$$

admits the solution $u(t, r) = u_0(\xi(t, r))$ where $\xi(t, r)$ is defined by the parametrization $r = u_0(\xi)t + \xi$. In connection with (4.1), we have considered two examples. In the first example, we have chosen the initial conditions $W_0(r) = 1, r > 0; W_0(0) = 0$ and $Z_0(r) = 2, r > 0; Z_0(0) = 0$. The continuous solutions of (4.1) are given by

$$W(t, r) = \begin{cases} 1 & \text{if } 0 \leq t \leq r \\ \frac{r}{t} & \text{if } 0 \leq r \leq t \end{cases}$$

$$Z(t, r) = \begin{cases} 2 & \text{if } 0 \leq 2t \leq r \\ \frac{r}{t} & \text{if } 0 \leq r \leq 2t. \end{cases}$$

According to Section 2, the radius of the drop is given by formula (2.19) which we rewrite with initial conditial taken equal to 1 as

$$\frac{dR(t)}{dt} = \frac{v_G(t, R(t))\rho_G(t, R(t))}{\rho_G(t, R(t)) - \rho_L}, \quad R(0) = 1 \quad (4.2)$$

with

$$\begin{cases} v_G(t, r) = \frac{1}{2}(W(t, r) + Z(t, r)) \\ \rho_G(t, r) = \frac{1}{2\sqrt{3}r^2}(Z(t, r) - W(t, r)). \end{cases} \quad (4.3)$$

In figure 1 below, we have drawn the curve $t \rightarrow R(t)$ on the time interval $[0, 1]$ with a step $h = 0.05$ and $\rho_L = 0.9$.

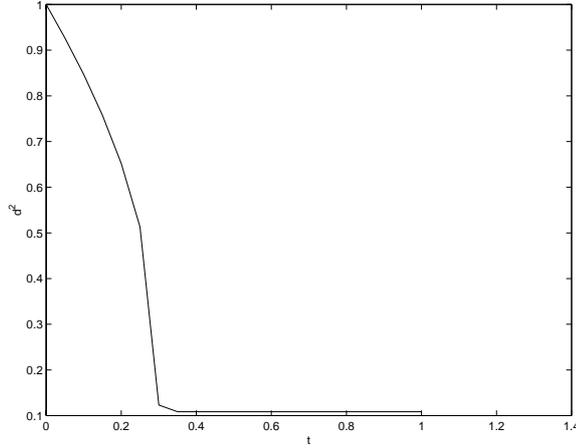


Fig. 1

For the second example we have chosen the experimental conditions made by the LCSR in the study of single drop evaporation, the drop being suspended from a silicate tube. Drops are made up of n-heptane fuel ($\rho_L = 683 \text{ kg/mm}^3$) in air at normalized atmospheric pressure and with an initial speed $v_G(0, r) = C_1 = 35 \text{ mm/s}$. The initial density $\rho_G(0, r)$ of the gas is taken as $C_2 = \frac{348}{T_0}$, $T_0 = 373 \text{ K}$. So The solution of (4.1) can be written as

$$W(t, r) = C_1 - \sqrt{3}C_2\xi^2, \quad \xi = \frac{1 + \sqrt{1 - 4(r - C_1t)\sqrt{3}C_2t}}{2\sqrt{3}C_2t},$$

$$Z(t, r) = C_1 + \sqrt{3}C_2\eta^2, \quad \eta = \frac{-1 + \sqrt{1 + 4(r - C_1t)\sqrt{3}C_2t}}{2\sqrt{3}C_2t}.$$

Figures 2 and 3 represent the velocity $v_G(t, r)$ and the pressure $\rho_G(t, r)$ given by (4.3) for $(t, r) \in (0, 1) \times (0, 1)$.

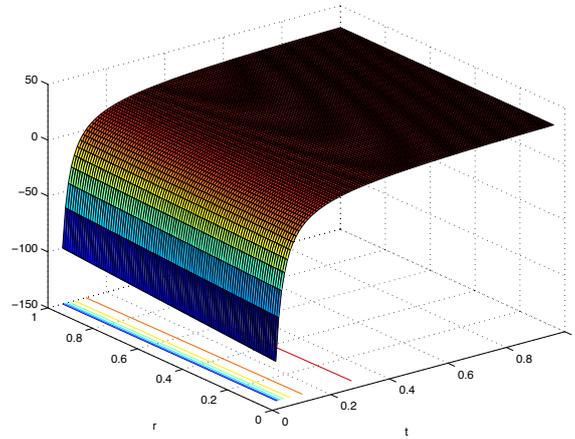


Fig. 2

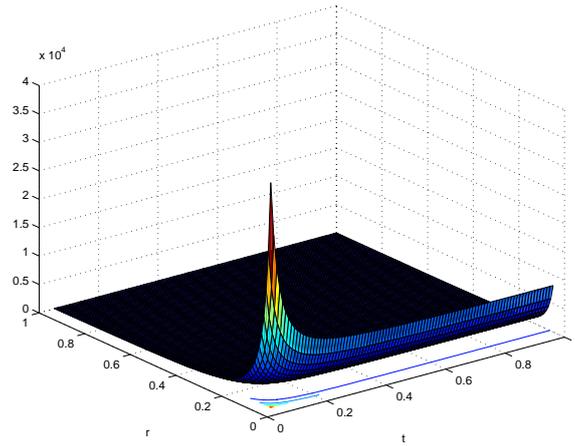


Fig. 3

The curve of the radius $t \rightarrow R(t)$ for this case is drawn in figure 4.

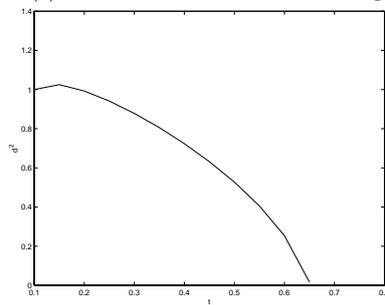


Fig. 4

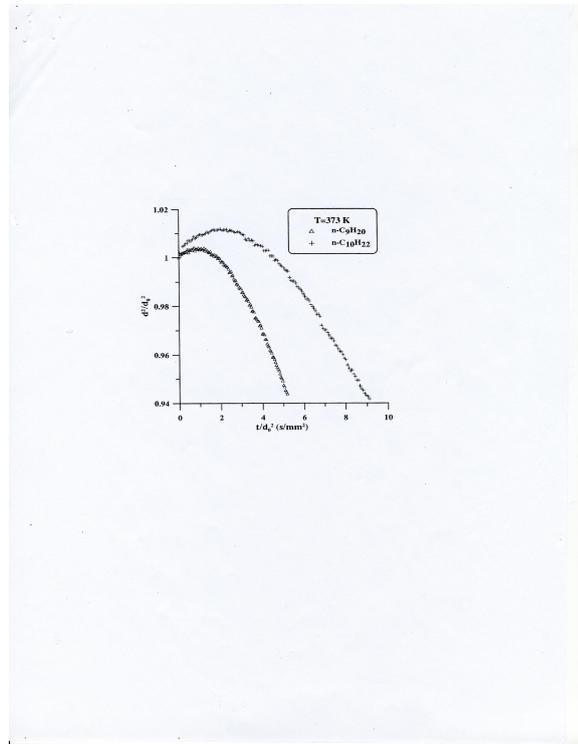


Fig. 5

Since $W'_0(\xi) < 0$ the maximal existence interval is finite (Proposition 2.1) as can be seen in our graphic. Let us remark that looking on the experimental curves made by the LCSR (figure 5) at the beginning, the function $t \rightarrow R(t)$ is increasing around the vicinity of $t = 0$. This fact is confirmed by our model which represents a good improvement of our previous model(see [1]) in which the velocity $v_G(t)$ was a given function of t .

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