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# Residual flux-based a posteriori error estimates for finite volume discretizations of inhomogeneous, anisotropic, and convection-dominated problems

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## Abstract

We derive in this paper a posteriori error estimates for finite volume discretizations of convection–diffusion–reaction equations, particularly focusing on the cases of inhomogeneous and anisotropic diffusion–dispersion tensors and of convection dominance. Our estimates are valid for any cell-centered finite volume scheme on a mesh consisting of simplices or rectangular parallelepipeds and extensions to general polygonal/polyhedral meshes including the nonmatching ones are provided as well. The estimates are established in the energy (semi-)norm for a locally postprocessed approximate solution preserving the finite volume fluxes and are of residual type. We prove that they are reliable (yield a global upper bound on the error), efficient (give a local lower bound on the error times an efficiency constant), and robust in the sense that the efficiency constant only depends on local variations in the coefficients and becomes optimal as the local Péclet number gets sufficiently small. The derived estimators are fully computable (all occurring constants are evaluated explicitly), so that they can serve both as indicators for adaptive refinement or for the actual control of the error. Numerical experiments confirm their accuracy.

**Key words:** convection–diffusion–reaction equation, finite volume method, a posteriori error estimates, convection dominance, inhomogeneous and anisotropic diffusion, nonmatching grids

**AMS subject classifications:** 65N15, 76M12, 76S05

## 1 Introduction

We consider in this paper the convection–diffusion–reaction problem

$$-\nabla \cdot (\mathbf{S}\nabla p) + \nabla \cdot (p\mathbf{w}) + rp = f \quad \text{in } \Omega, \quad (1.1a)$$

$$p = g \quad \text{on } \Gamma_D, \quad (1.1b)$$

$$-\mathbf{S}\nabla p \cdot \mathbf{n} = u \quad \text{on } \Gamma_N, \quad (1.1c)$$

where  $\mathbf{S}$  is a diffusion–dispersion tensor,  $\mathbf{w}$  is a given velocity field,  $r$  is a reaction function,  $f$  is a source term, and  $g$  and  $u$  prescribe the Dirichlet and Neumann boundary conditions, respectively. We suppose that  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , is a polygonal (meaning alternatively also polyhedral) domain (open, bounded, and connected set), that  $\Gamma_D \cap \Gamma_N = \emptyset$ ,  $\Gamma_D \cup \Gamma_N = \Gamma := \partial\Omega$ , that  $|\Gamma_D| \neq 0$ , where

$|\Gamma_D|$  is the measure of the set  $\Gamma_D$ , and that  $\Gamma_{\text{in}} := \{\mathbf{x} \in \partial\Omega; \mathbf{w} \cdot \mathbf{n} < 0\} \subset \Gamma_D$ . Finally,  $\mathbf{n}$  stands for the unit normal vector of  $\partial\Omega$ , outward to  $\Omega$ . Our purpose is to derive a posteriori error estimates for finite volume discretizations of the problem (1.1a)–(1.1c), emphasizing the cases where  $\mathbf{S}$  is inhomogeneous and anisotropic (nonconstant full-matrix) and where  $\mathbf{w}$  is dominating.

Residual a posteriori error estimates are nowadays well established for discretizations of the pure diffusion form of (1.1a)–(1.1c) (i.e.  $\mathbf{w} = r = 0$ ) by the finite element method, cf. for example the survey by Verfürth [41]. In the majority of the cases, however, the analysis is only given for  $\mathbf{S}$  being an identity matrix; an in-depth analysis for the general inhomogeneous and anisotropic diffusion tensor was presented by Bernardi and Verfürth [12]. In recent years a posteriori error estimates have been extended to convection–diffusion problems as well. We cite in particular Verfürth [42], who derived estimates in the energy norm for the conforming Galerkin method and its stabilized version (the SUPG method of Franca et al. [28]). His estimates are both reliable and efficient and moreover the efficiency constant becomes optimal as the local Péclet number gets sufficiently small. Similar results have been obtained in the frame of nonconforming finite element methods by El Alaoui and Ern in [19] for the inhomogeneous pure diffusion case and by El Alaoui et al. in [20] for the convection–diffusion case.

The theory is much less developed for finite volume methods. For vertex-centered schemes, the analogy with the finite element case is usually exploited in order to obtain a posteriori error estimates—this is e.g. the case of the works of Bergam et al. [11] or Lazarov and Tomov [31]. Still less work has been done for cell-centered schemes. Agouzal and Oudin [6] simply note that one can exploit the relations between the lowest-order Raviart–Thomas mixed finite element, the lowest-order nonconforming finite element, and the cell-centered finite volume methods on triangular meshes (cf. Arnold and Brezzi [7] and Baranger et al. [9]) in order to obtain an error indicator under a saturation-like hypothesis. Rigorous a posteriori error estimates are obtained by Achdou et al. [3], however only for two particular schemes. Equivalence of the discrete forms of the schemes with some finite element ones is used for this purpose. Nicaise [32] gives rigorous a posteriori estimates for a Morley interpolant (cf. Ciarlet [16]) of the original piecewise constant finite volume approximation. Finally, yet a different approach, yielding an estimate in the  $L^1$ -norm, independent of the size of the diffusion tensor, is given by Ohlberger [33, 34].

The purpose of this paper is to develop a sufficiently general and unique framework for a posteriori error estimation in the finite volume method. The derived estimates are first of all independent of particular schemes. Any cell-centered finite volume scheme, cf. Eymard et al. [22] and [21, 23], the schemes proposed in [17, 24, 27], “multi-point flux-approximation” schemes [1, 2, 18], and even combined finite volume–finite element schemes [25, 26] can be considered. The only requirement we have is that the scheme was written as a conservation equation over each computational cell, prescribing in particular the discrete diffusive and convective fluxes. Next, the estimates have the same form from a pure diffusion case to the full convection–diffusion–reaction one and a particular emphasis is put on their optimality in cases where  $\mathbf{S}$  is inhomogeneous and anisotropic and which are convection- or reaction-dominated. Arbitrary meshes are studied and, for the sake of completeness of the analysis, general inhomogeneous Dirichlet or Neumann boundary conditions are considered. The derived estimators are associated with the mesh cells, are easily computable, and they are also fully computable in the sense that all occurring constants are evaluated explicitly. This in particular means that they can serve not only as indicators for adaptive refinement, which is the usual practice, but also for the actual control of the error. Independent theory, though inspired by conforming [41] or nonconforming [19] residual estimates, is developed and in particular no equivalence with a different (finite element) case is necessary. Also, no additional regularity of the weak solution, neither any saturation assumption, is needed.

The basis of our approach is to exploit the particular feature of finite volume schemes, which is

the conservativity of the discrete fluxes over the sides (edges if  $d = 2$ , faces if  $d = 3$ ) of the mesh. Inspired by the results of Eymard et al. [23] and of the author [45], we first build a postprocessed approximate solution  $\tilde{p}_h$  which preserves exactly the finite volume discrete diffusive fluxes and whose mean or point value is in each cell fixed by the original constant approximation. By such a construction, the diffusion–dispersion tensor times the negative of the gradient of  $\tilde{p}_h$  lies in the  $\mathbf{H}(\text{div}, \Omega)$  space, as for the mixed finite element methods, cf. Roberts and Thomas [37] or Brezzi and Fortin [13]. Moreover, using the fixation of  $\tilde{p}_h$  by the original cell values, we are able to prove its convergence and to give a priori error estimates, under the condition that the original finite volume scheme satisfies some necessary properties. The construction of the postprocessed approximate solution is particularly easy for simplices (triangles if  $d = 2$  and tetrahedra if  $d = 3$ ) or, when  $\mathbf{S}$  is diagonal, for rectangular parallelepipeds (rectangles if  $d = 2$ , rectangular parallelepipeds if  $d = 3$ ); in this case  $\tilde{p}_h$  is a second-order polynomial, whose gradient (times minus  $\mathbf{S}$ ) is constructed as in the lowest-order Raviart–Thomas mixed finite element method, cf. [13, 37]. The crucial advantage that  $\mathbf{S}\nabla\tilde{p}_h$  is in  $\mathbf{H}(\text{div}, \Omega)$  is however compensated by the fact that  $\tilde{p}_h$  is nonconforming in the sense that  $\tilde{p}_h \notin H^1(\Omega)$ . The Oswald interpolation operator, recently analyzed in detail by Burman and Ern [14], is then employed to construct a conforming interpolate, which will be used in the a posteriori analysis. We describe all the above results in detail in Section 3, after collecting some preliminary remarks, notation, assumptions, and details on the continuous problem in Section 2.

Section 4 is then devoted to the a posteriori error estimates for the special case of matching meshes (containing no hanging nodes) consisting of simplices or rectangular parallelepipeds. Our estimates give a global upper bound on the error in  $p - \tilde{p}_h$  in the energy (semi-)norm and the only important tools for establishing them are the finite volume conservation equation over each computational cell and the Poincaré–Friedrichs and trace inequalities. Our estimates consist of several independent estimators, the principal of which penalize the nonconformity (jumps) in  $\tilde{p}_h$  and its residual. For pure diffusion problems, only these estimators (plus possibly still a Neumann boundary one) are present. When there is some convection, an additional upwinding estimator appears, and a reaction quadrature estimator may be present as well. We next prove that the principal (nonconformity and residual) estimators represent local lower bounds for the error as well, where in particular the efficiency constants are of the form  $c_1 + c_2 \min\{\text{Pe}, \varrho\}$ , where  $\text{Pe}$  (the local Péclet number) and  $\varrho$  are given below by (4.7) and where  $c_1, c_2$  only depend on local variations in  $\mathbf{S}$  (i.e. on local inhomogeneities and anisotropies), on local variations in  $\mathbf{w}$  and  $r$ , on the data, and on the shape-regularity parameter of the mesh. These estimators are thus in particular optimally efficient as the local Péclet number gets sufficiently small. We are not able to obtain similar results for the upwinding estimator, but numerical experiments suggest that this estimator represents a higher-order term as soon as the local Péclet number gets sufficiently small. We refer for a detailed discussion, as well as for several other remarks, to Section 4.3.

After having presented in detail the case of matching meshes consisting of simplices or rectangular parallelepipeds, we next in Section 5 discuss generalizations to meshes consisting of arbitrary polygons/polyhedrons, which may be nonmatching and consist of elements which are not necessarily convex. The first approach is to consider higher-order polynomials, but it reveals much more practical, at least for pure diffusion problems, to subdivide each mesh element into simplices and then directly use the theory established previously. Finally, we illustrate the accuracy of the derived estimates in Section 6 on several numerical experiments and conclude by some technical lemmas in Section 7.

## 2 Preliminaries

We introduce here the notation, define admissible partitions of the domain  $\Omega$ , recall some inequalities which will be important in the sequel, state the assumptions on the data, and finally give details on the continuous problem (1.1a)–(1.1c). In what follows, we conceptually denote by  $C_A$ ,  $c_A$  constants basically dependent on a quantity  $A$  but always independent of the discretization parameter  $h$  whose definition is given below.

### 2.1 Function spaces

For a domain  $S \subset \mathbb{R}^d$ , we denote by  $L^l(S)$  and  $\mathbf{L}^l(S) = [L^l(S)]^d$  the Lebesgue spaces, by  $(\cdot, \cdot)_S$  the  $L^2(S)$  or  $\mathbf{L}^2(S)$  inner product, and by  $\|\cdot\|_S$  the associated norm. We mean by  $|S|$  the  $d$ -dimensional Lebesgue measure of  $S$ , by  $|\sigma|$  the  $(d-1)$ -dimensional Lebesgue measure of  $\sigma \subset \mathbb{R}^{d-1}$ , and in particular by  $|\mathbf{s}|$  the length of a segment  $\mathbf{s}$ . At the same time,  $|A|$  is the cardinality of a set  $A$ . Next,  $H^l(S)$  is the Sobolev space of functions with square-integrable weak derivatives up to order  $l$ ;  $H_{\mathbb{D}}^1(S)$  is a subspace of  $H^1(S)$  of functions with traces vanishing on  $\Gamma_{\mathbb{D}}$  and  $H^{1/2}(\Gamma_{\mathbb{D}})$  is the trace space on  $\Gamma_{\mathbb{D}}$ . Next,  $\mathbf{H}(\text{div}, S)$  is the space of vector functions with square-integrable weak divergences,  $\mathbf{H}(\text{div}, S) = \{\mathbf{v} \in \mathbf{L}^2(S); \nabla \cdot \mathbf{v} \in L^2(S)\}$ , and  $\langle \cdot, \cdot \rangle_{\partial S}$  stands for  $(d-1)$ -dimensional inner product on  $\partial S$ . Finally, for a partition  $\mathcal{T}_h$  of  $\Omega$  as defined below, we will use the “broken Sobolev space”  $H^l(\mathcal{T}_h)$ ,  $H^l(\mathcal{T}_h) := \{\varphi \in L^2(\Omega); \varphi|_K \in H^l(K) \forall K \in \mathcal{T}_h\}$  (we use the notation  $H^l(K)$  instead of the more correct  $H^l(K^\circ)$ ). We refer to Adams [4] and Thomas [40] for details.

### 2.2 Partitions of the domain

We consider in this paper partitions  $\mathcal{T}_h$  of  $\Omega$  into closed polygons/polyhedrons such that  $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$  and such that the intersections of their interiors are pairwise empty. A particular attention will be paid to cases where the elements of the partition are either simplices or rectangular parallelepipeds and where  $\mathcal{T}_h$  is matching, contains no hanging nodes, i.e. where if  $K, L \in \mathcal{T}_h$ ,  $K \neq L$ , then  $K \cap L$  is either an empty set or a common face, edge, or vertex of  $K$  and  $L$ . In the general case, we admit nonmatching grids, i.e. the case where there exist two different polygons/polyhedrons  $K, L \in \mathcal{T}_h$  such that their intersection is not an empty set but it is not a common vertex, edge, or face of  $K$  and  $L$ , and we also allow for nonconvex elements  $K$ .

For  $K \in \mathcal{T}_h$ , we denote by  $\mathcal{E}_K$  the set of such subsets  $\sigma$  of  $\partial K$  that there exists  $L \in \mathcal{T}_h$  such that  $\sigma = \sigma_{K,L} := \partial K \cap \partial L$  has a positive  $(d-1)$ -dimensional Lebesgue measure. We call  $\sigma$  a side and  $K$  and  $L$  neighbors. We note that for matching meshes consisting of convex elements,  $\sigma$  are simply the sides in the geometrical sense, which is not necessarily the case on general polygonal meshes, see Figure 1. The set  $\mathcal{E}_K$  is completed by the (geometrical) sides of  $K$  lying at the boundary; we suppose in this case that each  $\sigma$  lies entirely either in  $\bar{\Gamma}_{\mathbb{D}}$  or in  $\bar{\Gamma}_{\mathbb{N}}$ . We next denote by  $\mathcal{E}_h$  the set of all sides of  $\mathcal{T}_h$ , by  $\mathcal{E}_h^{\text{int}}$  the set of interior, and by  $\mathcal{E}_h^{\text{ext}}$  the set of exterior sides. Also, let  $\mathcal{E}_h^{\mathbb{N}}$  stand for the sides contained in  $\bar{\Gamma}_{\mathbb{N}}$  and  $\mathcal{E}_h^{\mathbb{D}}$  for those contained in  $\bar{\Gamma}_{\mathbb{D}}$ . A family of meshes  $\{\mathcal{T}_h\}_h$  is parameterized by  $h := \max_{K \in \mathcal{T}_h} h_K$ , where  $h_K$  stands for the diameter of  $K$ ; we also use the notation  $h_\sigma$  for the diameter of  $\sigma \in \mathcal{E}_h$ . Finally, for general polygonal meshes, we denote by  $\widehat{\mathcal{T}}_K$  the partition of  $K \in \mathcal{T}_h$  by the elements of  $\widehat{\mathcal{T}}_h$  of the Assumption (B3) below.

For matching meshes consisting of simplices or rectangular parallelepipeds, we make the following shape regularity assumption:

*Assumption (A) (Shape regularity of matching meshes consisting of simplices or rectangular parallelepipeds)*

*There exists a constant  $\kappa_{\mathcal{T}} > 0$  such that  $\min_{K \in \mathcal{T}_h} \kappa_K \geq \kappa_{\mathcal{T}}$  for all  $h > 0$ , where  $\kappa_K := |K|/h_K^d$ .*

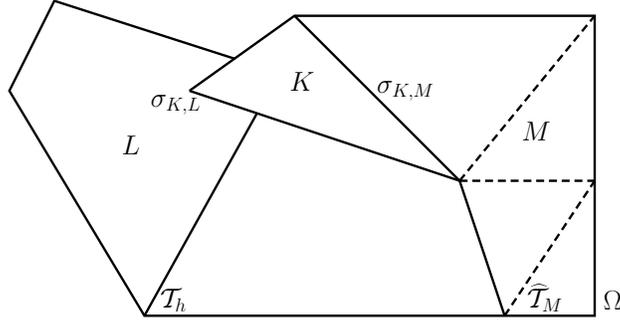


Figure 1: Considered meshes and notation

Let  $\rho_K$  denote the diameter of the largest ball inscribed in  $K$ . Then Assumption (A) is equivalent to the usual requirement of the existence of a constant  $\theta_{\mathcal{T}} > 0$  such that  $\max_{K \in \mathcal{T}_h} h_K / \rho_K \leq \theta_{\mathcal{T}}$  for all  $h > 0$ .

In the general case, we make the following shape regularity assumption:

*Assumption (B) (Shape regularity of arbitrary polygonal meshes)*

(B1) *There exists a constant  $\kappa_{\mathcal{T}} > 0$  such that  $\min_{K \in \mathcal{T}_h} \kappa_K \geq \kappa_{\mathcal{T}}$  for all  $h > 0$ , where  $\kappa_K := |K|/h_K^d$ ;*

(B2) *there exists a constant  $\xi_{\mathcal{T}} > 0$  such that  $\max_{K \in \mathcal{T}_h} |\mathcal{E}_K| \leq \xi_{\mathcal{T}}$  for all  $h > 0$ ;*

(B3) *there exist matching refinements  $\widehat{\mathcal{T}}_h$  of  $\mathcal{T}_h$  which consist of simplices or rectangular parallelepipeds and satisfy Assumption (A).*

### 2.3 Poincaré, Friedrichs, and trace inequalities

Let  $K$  be a convex polygon/polyhedron,  $\sigma$  its (geometrical) side, and  $\varphi \in H^1(K)$ . Three inequalities will play an essential role in the derivation of our a posteriori error estimates.

First, the Poincaré inequality states that

$$\|\varphi - \varphi_K\|_K^2 \leq C_{P,K} h_K^2 \|\nabla \varphi\|_K^2, \quad (2.1)$$

where  $\varphi_K$  is the mean of  $\varphi$  over  $K$  given by  $\varphi_K := (\varphi, 1)_K / |K|$  and where the constant  $C_{P,K}$  can for each convex  $K$  be evaluated as  $d/\pi$ , cf. [35, 10].

Second, the generalized Friedrichs inequality states that

$$\|\varphi - \varphi_\sigma\|_K^2 \leq C_{F,K,\sigma} h_K^2 \|\nabla \varphi\|_K^2, \quad (2.2)$$

where  $\varphi_\sigma$  is the mean of  $\varphi$  over  $\sigma \in \mathcal{E}_K$  given by  $\varphi_\sigma := \langle \varphi, 1 \rangle_\sigma / |\sigma|$ . It follows from [43, Lemma 4.1] that  $C_{F,K,\sigma} = 3d$  for a simplex  $K$  and its side  $\sigma$ . For a general convex  $K$ ,  $C_{F,K,\sigma}$  can be evaluated from the proof of [22, Lemma 9.4] or [21, Lemma 2] and [23, Lemma 3] in the form  $C h_K^{d-1} / |\sigma|$  with  $C$  only depending on  $d$  and  $\kappa_K$ .

Finally, the trace inequality states that

$$\|\varphi - \varphi_\sigma\|_\sigma^2 \leq C_{t,K,\sigma} h_K \|\nabla \varphi\|_K^2, \quad (2.3a)$$

$$\|\varphi - \varphi_K\|_\sigma^2 \leq \widetilde{C}_{t,K,\sigma} h_K \|\nabla \varphi\|_K^2, \quad (2.3b)$$

cf. the proof of [22, Lemma 9.4] or [21, Lemma 2] and [23, Lemma 1]. It has in particular been shown in [32, Lemma 3.5] that  $C_{t,K,\sigma} = C_{t,d}h_K/h_\sigma$  for a simplex, where  $C_{t,d} \approx 1.55416$  if  $d = 2$  and  $C_{t,d} \approx 11.53557$  if  $d = 3$ . Similarly, it follows from the proof of [43, Lemma 4.1] that  $\tilde{C}_{t,K,\sigma} = 3dh_K|\sigma|/|K|$  for a simplex. Finally,  $\tilde{C}_{t,K,\sigma}$  only depends on  $d$  and  $\kappa_K$  for a general convex  $K$  and its side  $\sigma$ .

The above inequalities are valid for nonconvex polygons/polyhedrons  $K$  or when  $\sigma$  is only a general part of  $\partial K$  such that  $|\sigma| > 0$  as well, cf. [22, Lemma 10.4]. The dependencies on  $K$  and  $\sigma$  are in this case, however, more complicated in general.

## 2.4 Assumptions on the data

We suppose that there exists a basic partition  $\tilde{\mathcal{T}}_h$  of  $\Omega$  such that the data of the problem (1.1a)–(1.1c) are related to  $\tilde{\mathcal{T}}_h$  in the following way:

*Assumption (C) (Data)*

(C1)  $\mathbf{S}_K := \mathbf{S}|_K$  is a constant, symmetric, bounded, and uniformly positive definite tensor for all  $K \in \tilde{\mathcal{T}}_h$ , i.e.

$$\begin{aligned} \mathbf{S}_K \mathbf{v} \cdot \mathbf{v} &\geq c_{\mathbf{S},K} \mathbf{v} \cdot \mathbf{v}, \quad c_{\mathbf{S},K} > 0 \quad \forall \mathbf{v} \in \mathbb{R}^d, \quad \forall K \in \tilde{\mathcal{T}}_h, \\ \|\mathbf{S}_K\| &= C_{\mathbf{S},K}, \quad C_{\mathbf{S},K} > 0 \quad \forall K \in \tilde{\mathcal{T}}_h; \end{aligned}$$

(C2)  $\mathbf{w} \in \mathbf{H}(\text{div}, \Omega)$  and each component of  $\mathbf{w}$  is a polynomial of degree at most  $k$  on each  $K \in \tilde{\mathcal{T}}_h$ , such that  $|\mathbf{w}|_K| \leq C_{\mathbf{w},K}$ ,  $C_{\mathbf{w},K} \geq 0$ , for all  $K \in \tilde{\mathcal{T}}_h$ ;

(C3)  $r$  is a polynomial of degree at most  $k$  on each  $K \in \tilde{\mathcal{T}}_h$ ;

(C4)  $(\frac{1}{2}\nabla \cdot \mathbf{w} + r)|_K \geq c_{\mathbf{w},r,K}$  and  $|\nabla \cdot \mathbf{w}|_K| + |r|_K| \leq C_{\mathbf{w},r,K}$ ,  $c_{\mathbf{w},r,K} \geq 0$ ,  $C_{\mathbf{w},r,K} \geq 0$ , for all  $K \in \tilde{\mathcal{T}}_h$ ;

(C5)  $f$  is a polynomial of degree at most  $k$  on each  $K \in \tilde{\mathcal{T}}_h$ ;

(C6)  $g \in H^{1/2}(\Gamma_D)$ ;

(C7)  $u \in L^2(\Gamma_N)$ ;

(C8) if  $c_{\mathbf{w},r,K} = 0$ , then  $C_{\mathbf{w},r,K} = 0$ .

The assumptions that  $\mathbf{S}$  is piecewise constant and that  $\mathbf{w}$ ,  $r$ , and  $f$  are piecewise polynomial are made for the sake of simplicity and are usually satisfied in practice. If the functions at hand do not fulfill these requirements, interpolation can be used. Also, note that Assumption (C8) allows  $c_{\mathbf{w},r,K} = 0$  but  $\mathbf{w}|_K \neq 0$ .

## 2.5 Continuous problem

Let  $\mathcal{T}_h$  be, as throughout the whole paper, a refinement of  $\tilde{\mathcal{T}}_h$ . We define a bilinear form  $\mathcal{B}$  by

$$\mathcal{B}(p, \varphi) := \sum_{K \in \mathcal{T}_h} \{ (\mathbf{S}\nabla p, \nabla \varphi)_K + (\nabla \cdot (p\mathbf{w}), \varphi)_K + (rp, \varphi)_K \} \quad p, \varphi \in H^1(\mathcal{T}_h) \quad (2.4)$$

and the corresponding energy (semi-)norm by

$$\|\varphi\|_\Omega^2 := \sum_{K \in \mathcal{T}_h} \|\varphi\|_K^2, \quad \|\varphi\|_K^2 := c_{\mathbf{S},K} \|\nabla \varphi\|_K^2 + c_{\mathbf{w},r,K} \|\varphi\|_K^2 \quad \varphi \in H^1(\mathcal{T}_h). \quad (2.5)$$

In this way  $\mathcal{B}(\cdot, \cdot)$  and  $\|\cdot\|_\Omega$  are well-defined for  $p, \varphi \in H^1(\Omega)$  as well as for  $p, \varphi$  that are only piecewise regular. We remark that  $\|\cdot\|_\Omega$  is always a norm on  $H_D^1(\Omega)$ , whereas it is a norm on  $H^1(\mathcal{T}_h)$  only when  $c_{\mathbf{w},r,K} > 0$  for all  $K \in \mathcal{T}_h$ .

The weak formulation of the problem (1.1a)–(1.1c) is then to find  $p \in H^1(\Omega)$  with  $p|_{\Gamma_D} = g$  in the sense of traces such that

$$\mathcal{B}(p, \varphi) = (f, \varphi)_\Omega - \langle u, \varphi \rangle_{\Gamma_N} \quad \forall \varphi \in H_D^1(\Omega). \quad (2.6)$$

Assumptions (C1)–(C7), the Green theorem, and the Cauchy–Schwarz inequality imply that

$$\mathcal{B}(\varphi, \varphi) \geq \|\varphi\|_\Omega^2 \quad \forall \varphi \in H_D^1(\Omega) \quad (2.7)$$

and

$$\begin{aligned} \mathcal{B}(p, \varphi) \leq & \left( \max_{K \in \mathcal{T}_h} \left\{ \frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}} \right\} + \max_{K \in \mathcal{T}_h} \left\{ \frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}} \right\} \right) \|p\|_\Omega \|\varphi\|_\Omega \\ & + \max_{K \in \mathcal{T}_h} \left\{ \frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{S},K}}} \right\} \|p\|_\Omega \|\varphi\|_\Omega \quad \forall p, \varphi \in H^1(\mathcal{T}_h). \end{aligned} \quad (2.8)$$

Problem (2.6) under Assumption (C) in particular admits a unique solution.

**Remark 2.1** (Notation). *In estimate (2.8), if  $c_{\mathbf{w},r,K} = 0$ , the term  $C_{\mathbf{w},r,K}/c_{\mathbf{w},r,K}$  should be evaluated as zero; since Assumption (C8) for this case gives  $C_{\mathbf{w},r,K} = 0$ , the term with  $C_{\mathbf{w},r,K}$  in fact does not even enter the estimate. To simplify notation, we will systematically use the convention  $0/0 = 0$  throughout the text.*

**Remark 2.2** (Different partitions). *In order to avoid ambiguities, we clarify here the relations between the different partitions of  $\Omega$  considered in the paper. First of all, the partition  $\tilde{\mathcal{T}}_h$  is only considered so as describe the different parts of  $\Omega$  with different data. In porous media, typically, the elements of  $\tilde{\mathcal{T}}_h$  may describe the different geological layers. Next,  $\mathcal{T}_h$  is the “computational” partition and we naturally suppose that  $\mathcal{T}_h$  is a refinement of  $\tilde{\mathcal{T}}_h$ . Finally, for arbitrary (nonmatching) polygonal meshes, we may sometimes need a matching refinement  $\hat{\mathcal{T}}_h$  of  $\mathcal{T}_h$  which consist of simplices or rectangular parallelepipeds. Hence, in this last case,  $\hat{\mathcal{T}}_h$  is a refinement of  $\mathcal{T}_h$ , which is a refinement of  $\tilde{\mathcal{T}}_h$ .*

### 3 Finite volume schemes and postprocessing

We first introduce here the finite volume method for the problem (1.1a)–(1.1c). We next construct a locally postprocessed approximation  $\tilde{p}_h$  which will serve as the basis for our a posteriori error estimates. Since  $\tilde{p}_h$  will be nonconforming, the Oswald interpolation operator will be useful in the subsequent analysis. We finally present its extension to arbitrary grids and boundary conditions.

#### 3.1 The finite volume method

A general cell-centered finite volume scheme for the problem (1.1a)–(1.1c) can be written in the following form: find  $p_K$ ,  $K \in \mathcal{T}_h$ , the approximations to  $p$  such that

$$\sum_{\sigma \in \mathcal{E}_K} S_{K,\sigma} + \sum_{\sigma \in \mathcal{E}_K} W_{K,\sigma} + r_K p_K |K| = f_K |K| \quad \forall K \in \mathcal{T}_h, \quad (3.1)$$

where  $f_K := (f, 1)/|K|$ ,  $r_K := (r, 1)/|K|$ , and where  $S_{K,\sigma}$  and  $W_{K,\sigma}$  are, respectively, the diffusive and convective fluxes through the sides  $\sigma$  of an element  $K$ , functions of  $p_K$ ,  $K \in \mathcal{T}_h$ , of the mesh,

and of the data. For the a posteriori error estimates presented in this paper, we do not need the specific form of the diffusive and convective fluxes; our analysis however relies on the finite volume concept of their continuity, imposing that  $S_{K,\sigma_{K,L}} = -S_{L,\sigma_{K,L}}$  and  $W_{K,\sigma_{K,L}} = -W_{L,\sigma_{K,L}}$  for all  $\sigma_{K,L} \in \mathcal{E}_h^{\text{int}}$ . Note also that for nonmatching grids,  $\sigma$  is not necessarily a side of  $K$  in the geometrical sense, see Section 2.2. To fix ideas, we give an example.

When  $\mathcal{T}_h$  is admissible in the sense of [22, Definition 9.1], which is satisfied e.g. when  $\mathcal{T}_h$  is matching, consists of convex elements, and when there exist points  $\mathbf{x}_K$  in the interior of each  $K \in \mathcal{T}_h$  such that the straight lines connecting  $\mathbf{x}_K$  and  $\mathbf{x}_L$  for two neighboring elements  $K$  and  $L$  are orthogonal to  $\sigma_{K,L} = \partial K \cap \partial L$  and finally when an analogous orthogonality condition holds on the Dirichlet part of the boundary, and under the additional assumption that  $\mathbf{S}_K = s_K Id$  for all  $K \in \mathcal{T}_h$ , the following choices for  $S_{K,\sigma}$  and  $W_{K,\sigma}$  are possible:

$$S_{K,\sigma} = -s_{K,L} \frac{|\sigma_{K,L}|}{d_{K,L}} (p_L - p_K) \quad \sigma = \sigma_{K,L} \in \mathcal{E}_h^{\text{int}}, \quad (3.2)$$

$$S_{K,\sigma} = -s_K \frac{|\sigma|}{d_{K,\sigma}} (g_\sigma - p_K) \quad \sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\text{D}}, \quad (3.3)$$

$$S_{K,\sigma} = u_\sigma |\sigma| \quad \sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\text{N}}, \quad (3.4)$$

$$W_{K,\sigma} = p_\sigma w_{K,\sigma} \quad \sigma \in \mathcal{E}_K. \quad (3.5)$$

Here  $d_{K,L} = |\mathbf{x}_K - \mathbf{x}_L|$ , the Euclidean distance of  $\mathbf{x}_K$  and  $\mathbf{x}_L$ ,  $d_{K,\sigma}$  is the Euclidean distance of  $\mathbf{x}_K$  and  $\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\text{D}}$ , and

$$g_\sigma := \frac{\langle g, \mathbf{1} \rangle_\sigma}{|\sigma|}, \quad u_\sigma := \frac{\langle u, \mathbf{1} \rangle_\sigma}{|\sigma|}, \quad w_{K,\sigma} := \langle \mathbf{w} \cdot \mathbf{n}, \mathbf{1} \rangle_\sigma \quad (3.6)$$

are, respectively, the mean value of the Dirichlet boundary condition  $g$  on the side  $\sigma \in \mathcal{E}_h^{\text{D}}$ , the mean value of the Neumann boundary condition  $u$  on the side  $\sigma \in \mathcal{E}_h^{\text{N}}$ , and the flux of the velocity field  $\mathbf{w}$  through the side  $\sigma$  of  $K$ . For the value  $s_{K,L}$ , we have two choices, corresponding respectively to the arithmetic and harmonic averaging in the diffusion–dispersion tensor,

$$s_{K,L} := \frac{s_K + s_L}{2}, \quad (3.7a)$$

$$s_{K,L} := \frac{2s_K s_L}{s_K + s_L}. \quad (3.7b)$$

Finally,  $p_\sigma$  is the weighted upwind value, given for interior or Dirichlet boundary sides by

$$p_\sigma := \nu_\sigma \hat{p}_\sigma + (1 - \nu_\sigma) \bar{p}_\sigma, \quad (3.8)$$

where  $\hat{p}_\sigma$  is the upwind value and  $\bar{p}_\sigma$  the centered value, given respectively by

$$\hat{p}_\sigma := \begin{cases} p_K & \text{if } w_{K,\sigma} \geq 0 \\ p_L & \text{if } w_{K,\sigma} < 0 \end{cases}, \quad \bar{p}_\sigma := \frac{1}{2}(p_K + p_L) \quad (3.9)$$

if  $\sigma$  is an interior side between elements  $K$  and  $L$  and

$$\hat{p}_\sigma := \begin{cases} p_K & \text{if } w_{K,\sigma} \geq 0 \\ g_\sigma & \text{if } w_{K,\sigma} < 0 \end{cases} \quad \bar{p}_\sigma := \frac{1}{2}(p_K + g_\sigma) \quad (3.10)$$

if  $\sigma \in \mathcal{E}_K$  is a Dirichlet boundary side. Finally,

$$p_\sigma := p_K \quad (3.11)$$

if  $\sigma \in \mathcal{E}_K$  is a Neumann boundary side. In the above definitions,  $\nu_\sigma \in [0, 1]$  is a coefficient of the amount of upstream weighting. The full-upwind scheme (with  $\nu_\sigma = 1$  for all  $\sigma \in \mathcal{E}_h$ ) is classical, see [22, Section 9]. The introduction of the parameter  $\nu_\sigma$  has been motivated by its successful use in finite volume or combined finite volume–finite element schemes, cf. [15, 26]. In the present case, the choice for  $\nu_\sigma$  to still guarantee the stability of the scheme while reducing the excessive numerical diffusion added by the full upstream weighting is (for  $w_{K,\sigma} \neq 0$ , otherwise  $\nu_\sigma := 1$ )

$$\nu_\sigma := \begin{cases} 1 - \min \left\{ \frac{2s_{K,L}|\sigma_{K,L}|}{d_{K,L}|w_{K,\sigma}|}, 1 \right\} & \text{if } \sigma_{K,L} \in \mathcal{E}_h^{\text{int}}, \\ 1 - \min \left\{ \frac{2s_K|\sigma|}{d_{K,\sigma}|w_{K,\sigma}|}, 1 \right\} & \text{if } \sigma \in \mathcal{E}_h^{\text{D}} \text{ and } w_{K,\sigma} > 0, \\ 1 & \text{if } \sigma \in \mathcal{E}_h^{\text{D}} \text{ and } w_{K,\sigma} < 0. \end{cases} \quad (3.12)$$

**Remark 3.1** (Local Péclet upstream weighting). *The local Péclet upstream weighting (3.8)–(3.12) for the scheme (3.1)–(3.7b) on polygonal meshes  $\mathcal{T}_h$  admissible in the sense of [22, Definition 9.1] guarantees, when  $\mathbf{S}_K = s_K \text{Id}$  for all  $K \in \mathcal{T}_h$ ,  $\Gamma_{\text{N}} = \emptyset$ , and under appropriate assumptions on  $\mathbf{w}$ ,  $r$ , and  $f$ , the discrete maximum principle, while reducing the excessive numerical diffusion added by the full upstream weighting, cf. [26, Theorem 4.5] and [22, Proposition 9.2]. Moreover, the inequality*

$$\sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} p_\sigma w_{K,\sigma} p_K \geq \frac{1}{2} \sum_{K \in \mathcal{T}_h} p_K^2 (\nabla \cdot \mathbf{w}, 1)_K \quad (3.13)$$

holds, see [45, Lemma 6.5]. Another possibility for still more precise discretization of the convection term would be to define the centered value  $\bar{p}_\sigma$  by the value that takes a linear function given by  $p_K$  in  $\mathbf{x}_K$  and by  $p_L$  in  $\mathbf{x}_L$  on  $\sigma_{K,L} \in \mathcal{E}_h^{\text{int}}$  common to  $K, L \in \mathcal{T}_h$ , i.e. to take into account the distances of  $\mathbf{x}_K$  and  $\mathbf{x}_L$  from the side  $\sigma_{K,L}$  instead of putting  $\bar{p}_\sigma := (p_K + p_L)/2$ , and similarly at the boundary. While under appropriate modification of (3.12) and under the above-mentioned assumptions, the discrete maximum principle would still hold, this would not in general be the case for the inequality (3.13). This inequality is however very useful already while proving the existence and uniqueness of the discrete solution when there is no discrete maximum principle (recall that this principle can in general only be satisfied if  $\mathbf{S}$  is not anisotropic and under some conditions on the mesh), cf. [26, Theorems 4.3 and 4.4] and [22, Lemma 9.2].

**Remark 3.2** (Inhomogeneous diffusion–dispersion tensors). *Similarly as for the convection term, instead of directly using (3.7a) or (3.7b), we may employ the distances of  $\mathbf{x}_K$  and  $\mathbf{x}_L$  from the side  $\sigma_{K,L} \in \mathcal{E}_h^{\text{int}}$  common to  $K, L \in \mathcal{T}_h$  for the discretization of inhomogeneous diffusion–dispersion tensors.*

## 3.2 Postprocessing

The approximate finite volume solution is only piecewise constant, given by the value  $p_K$  in each  $K \in \mathcal{T}_h$ . In order to derive a posteriori error estimates, we construct in this paper a postprocessed approximation which has more regularity. We use for this purpose the additional knowledge that we have from a finite volume scheme: the fluxes.

Developing the ideas of Eymard et al. [23] or of the author [45], the most general postprocessed

finite volume approximation  $\tilde{p}_h$  that we consider in this paper is given by the requirements

$$\tilde{p}_h \in H^1(\mathcal{T}_h), \quad (3.14a)$$

$$-\mathbf{S}\nabla\tilde{p}_h \in \mathbf{H}(\text{div}, \Omega), \quad (3.14b)$$

$$(1 - \mu_K) \frac{(\tilde{p}_h, 1)_K}{|K|} + \mu_K \tilde{p}_h(\mathbf{x}_K) = p_K \quad \forall K \in \mathcal{T}_h, \quad (3.14c)$$

$$\langle -\mathbf{S}\nabla\tilde{p}_h|_K \cdot \mathbf{n}, 1 \rangle_{\partial K} = \sum_{\sigma \in \mathcal{E}_K} S_{K,\sigma} \quad \forall K \in \mathcal{T}_h. \quad (3.14d)$$

Here,  $\mu_K = 0$  or  $1$ , in the dependence on whether the particular scheme (3.1) represents by  $p_K$  the approximate mean value on  $K \in \mathcal{T}_h$  or the approximate point value in a point  $\mathbf{x}_K$  (which we for simplicity assume inside  $K$ ) and also on the expected regularity of  $\tilde{p}_h$ . For general elements, it is not clear how to practically find such  $\tilde{p}_h$ , especially when an additional desirable property is that  $\tilde{p}_h$  was a polynomial on each  $K \in \mathcal{T}_h$ . And, of course, we must guarantee that  $\tilde{p}_h$  was a good approximation of the weak solution  $p$  as soon as the finite volume scheme (3.1) is convergent. We treat in the following sections these questions.

### 3.2.1 A postprocessed approximate solution based on the solution of local Neumann problems

We may define  $\tilde{p}_h$  as the weak solution of the following local Neumann problems:

$$-\nabla \cdot \mathbf{S}\nabla\tilde{p}_h = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} S_{K,\sigma} \quad \forall K \in \mathcal{T}_h, \quad (3.15a)$$

$$(1 - \mu_K) \frac{(\tilde{p}_h, 1)_K}{|K|} + \mu_K \tilde{p}_h(\mathbf{x}_K) = p_K \quad \forall K \in \mathcal{T}_h, \quad (3.15b)$$

$$-\mathbf{S}\nabla\tilde{p}_h|_K \cdot \mathbf{n} = \frac{S_{K,\sigma}}{|\sigma|} \quad \forall \sigma \in \mathcal{E}_K, \quad \forall K \in \mathcal{T}_h. \quad (3.15c)$$

It is immediate to see that such  $\tilde{p}_h$  verifies (3.14a)–(3.14d) and it turns out that it is straightforward to find it in two particular cases.

### 3.2.2 Postprocessing on simplices for general diffusion–dispersion tensors

For matching triangular or tetrahedral meshes, it is immediate to show that if  $\mathbf{S} = Id$ ,  $\tilde{p}_h$  given by (3.15a)–(3.15c) is a piecewise second-order polynomial of the form  $\tilde{p}_h|_K = a_K(x^2 + y^2) + b_Kx + c_Ky + d_K$  if  $d = 2$  and  $\tilde{p}_h|_K = a_K(x^2 + y^2 + z^2) + b_Kx + c_Ky + d_Kz + e_K$  if  $d = 3$ . If  $\mathbf{S} \neq Id$ , then  $\tilde{p}_h$  verifying (3.15a)–(3.15c) is a full second-order polynomial on each  $K \in \mathcal{T}_h$  (the symmetry of  $\mathbf{S}$  is important in this case). Simply, (3.15c) imposes the normal derivative of  $\tilde{p}_h$  on the sides of  $K$ , (3.15b) imposes the mean of  $\tilde{p}_h$  over  $K$  or a point value in  $\mathbf{x}_K$ , and (3.15a) follows by the Green theorem and by the fact that  $-\nabla \cdot \mathbf{S}_K \nabla \tilde{p}_h$  is constant. Since  $-\mathbf{S}_M \nabla \tilde{p}_h|_M \cdot \mathbf{n}$  only differs by the sign on the two elements  $K, L = M$  sharing an interior side  $\sigma$ , which follows from the mass conservation or continuity of the finite volume diffusive fluxes  $S_{K,\sigma}$ , the important requirement (3.14b) is automatically verified. Note finally that the proposed postprocessing is local on each element and that one only has to build a second-order polynomial on each  $K \in \mathcal{T}_h$  from the prescribed degrees of freedom, so that its cost is negligible. The postprocessed approximation  $\tilde{p}_h$  is however in general not included in  $H^1(\Omega)$  but only in  $H^1(\mathcal{T}_h)$ .

### 3.2.3 Postprocessing on rectangular parallelepipeds for diagonal diffusion–dispersion tensors

For  $\mathbf{S}$  diagonal and matching meshes consisting of rectangles or rectangular parallelepipeds,  $\tilde{p}_h$  verifying (3.15a)–(3.15c) is a piecewise second-order polynomial of the form  $\tilde{p}_h|_K = a_K x^2 + b_K y^2 + c_K x + d_K y + e_K$  if  $d = 2$  and  $\tilde{p}_h|_K = a_K x^2 + b_K y^2 + c_K z^2 + d_K x + e_K y + f_K z + g_K$  if  $d = 3$ . Again, the postprocessing is local and the new variable  $\tilde{p}_h$  is nonconforming in the sense that it is in general not included in  $H^1(\Omega)$  but only in  $H^1(\mathcal{T}_h)$ .

**Remark 3.3** (Comparison with mixed finite elements). *The basis for the a posteriori error estimates given in [45] for the lowest-order Raviart–Thomas mixed finite element method on simplicial meshes was the construction of  $\tilde{p}_h$  such that  $-\mathbf{S}_K \nabla \tilde{p}_h|_K = \mathbf{u}_h|_K$  and  $(\tilde{p}_h, 1)_K / |K| = p_h|_K$  for all  $K \in \mathcal{T}_h$ , where  $\mathbf{u}_h$  and  $p_h$  are, respectively, the mixed vector and scalar approximations. Let  $\Gamma_D = \partial\Omega$  and  $g = 0$ . It is proved in the above reference that in this case the means of the “mixed finite element”  $\tilde{p}_h$  on interior sides are continuous and its means on exterior sides are equal to zero, so that we have at least some continuity of  $\tilde{p}_h$ . Since we do not dispose of a similar proof for  $\tilde{p}_h$  obtained from the finite volume method, we only suppose here that  $\tilde{p}_h \in H^1(\mathcal{T}_h)$ . Note however that for pure diffusion problems ( $\mathbf{w} = r = 0$ ), Baranger et al. [9] and Agouzal et al. [5] have shown that the cell-centered finite volume scheme (3.1)–(3.7b) on triangular and rectangular meshes  $\mathcal{T}_h$  admissible in the sense of [22, Definition 9.1] only differs from lowest-order Raviart–Thomas mixed finite elements by numerical integration (at least when  $\mathbf{S}$  is scalar and when Dirichlet boundary conditions are considered). In the general case and on simplicial meshes, lowest-order mixed finite elements are directly, without any numerical integration, equivalent to a particular finite volume scheme, see [44].*

### 3.2.4 A priori error estimate for the postprocessed approximation

We give in this section an a priori error estimate for the postprocessed approximation  $\tilde{p}_h$  given by (3.15a)–(3.15c).

**Theorem 3.4** (A priori error estimate for  $\tilde{p}_h$ ). *Let a basic partition  $\tilde{\mathcal{T}}_h$  be given and let its refinement  $\mathcal{T}_h$  be matching, consisting of convex elements, and satisfying Assumptions (B1)–(B2). Let  $p$ , the weak solution of the problem (1.1a)–(1.1c) given by (2.6), be such that  $p \in H^2(\tilde{\mathcal{T}}_h)$  and  $\mathbf{S} \nabla p \in \mathbf{H}(\text{div}, \Omega)$ . Next, let the finite volume scheme (3.1) be such that*

$$\sum_{K \in \mathcal{T}_h} h_K \left( \sum_{\sigma \in \mathcal{E}_K} \|(S_{K,\sigma} + \langle \mathbf{S} \nabla p \cdot \mathbf{n}, 1 \rangle_\sigma) |\sigma|^{-1} \|_\sigma \right)^2 \leq Ch^2, \\ \sum_{K \in \mathcal{T}_h} \|p - p_K\|_K^2 \leq Ch^2,$$

where the constant  $C > 0$  only depends on  $p$ ,  $d$ ,  $\kappa_{\mathcal{T}}$ ,  $\xi_{\mathcal{T}}$ , the data, and  $\Omega$ . Finally, let  $\tilde{p}_h$  be the postprocessed solution given by (3.15a)–(3.15c). Then

$$\sum_{K \in \mathcal{T}_h} \|\nabla(p - \tilde{p}_h)\|_K^2 \leq Ch^2, \\ \|p - \tilde{p}_h\|_\Omega^2 \leq Ch^2,$$

where the constant  $C > 0$  only depends on  $p$ ,  $d$ ,  $\kappa_{\mathcal{T}}$ ,  $\xi_{\mathcal{T}}$ , the data, and  $\Omega$ .

PROOF:

We first study the estimate on the gradient, following the proof of [23, Theorem 2]. We have, for each  $K \in \mathcal{T}_h$ ,

$$\begin{aligned}
c_{\mathbf{S},K} \|\nabla(p - \tilde{p}_h)\|_K^2 &\leq (\mathbf{S}\nabla(p - \tilde{p}_h), \nabla(p - \tilde{p}_h))_K \\
&= -(\nabla \cdot \mathbf{S}\nabla p, (p - \tilde{p}_h))_K + \langle \mathbf{S}\nabla(p - \tilde{p}_h) \cdot \mathbf{n}, (p - \tilde{p}_h) \rangle_{\partial K} \\
&\leq \|\nabla \cdot \mathbf{S}\nabla p\|_K C_{\mathbf{P},K}^{1/2} h_K \|\nabla(p - \tilde{p}_h)\|_K \\
&\quad + \sum_{\sigma \in \mathcal{E}_K} \tilde{C}_{t,K,\sigma}^{1/2} h_K^{1/2} \|\mathbf{S}\nabla(p - \tilde{p}_h) \cdot \mathbf{n}\|_\sigma \|\nabla(p - \tilde{p}_h)\|_K,
\end{aligned}$$

using Assumption (C1), the Green theorem, the fact that  $\|\nabla(p - \tilde{p}_h)\|_K$  does not depend on shifting  $\tilde{p}_h$  by a constant and fixing temporarily  $(\tilde{p}_h, 1)_K$  by  $(p, 1)_K$  instead of (3.15b), which makes the term  $(\nabla \cdot \mathbf{S}\nabla \tilde{p}_h, (p - \tilde{p}_h))_K$  disappear by virtue of (3.15a), and finally using the Schwarz inequality, the Poincaré inequality (2.1), and the trace inequality (2.3b). Thus

$$c_{\mathbf{S},K} \|\nabla(p - \tilde{p}_h)\|_K \leq C_{\mathbf{P},K}^{1/2} h_K \|\nabla \cdot \mathbf{S}\nabla p\|_K + h_K^{1/2} \sum_{\sigma \in \mathcal{E}_K} \tilde{C}_{t,K,\sigma}^{1/2} \|\mathbf{S}\nabla(p - \tilde{p}_h) \cdot \mathbf{n}\|_\sigma. \quad (3.16)$$

Next, the triangle inequality and (3.15c) imply that

$$\begin{aligned}
\|\mathbf{S}\nabla(p - \tilde{p}_h) \cdot \mathbf{n}\|_\sigma &\leq \|\mathbf{S}\nabla p \cdot \mathbf{n} - \langle \mathbf{S}\nabla p \cdot \mathbf{n}, 1 \rangle_\sigma |\sigma|^{-1}\|_\sigma \\
&\quad + \|(\langle \mathbf{S}\nabla p \cdot \mathbf{n}, 1 \rangle_\sigma + S_{K,\sigma}) |\sigma|^{-1}\|_\sigma
\end{aligned}$$

and we have

$$\begin{aligned}
\|\mathbf{S}\nabla p \cdot \mathbf{n} - \langle \mathbf{S}\nabla p \cdot \mathbf{n}, 1 \rangle_\sigma |\sigma|^{-1}\|_\sigma &\leq C_{\mathbf{S},K} \|\nabla p \cdot \mathbf{n} - \langle \nabla p \cdot \mathbf{n}, 1 \rangle_\sigma |\sigma|^{-1}\|_\sigma \\
&\leq C_{\mathbf{S},K} C_K^{1/2} h_K^{1/2} \|p\|_{H^2(K)}
\end{aligned}$$

using Assumption (C1) and [23, Lemma 2]; here the constant  $C_K$  only depends on  $d$  and  $\kappa_K$ . Thus, squaring (3.16), summing over all  $K \in \mathcal{T}_h$ , and using the triangle, Cauchy–Schwarz, and the above inequalities gives

$$\begin{aligned}
\sum_{K \in \mathcal{T}_h} c_{\mathbf{S},K} \|\nabla(p - \tilde{p}_h)\|_K^2 &\leq \sum_{K \in \mathcal{T}_h} \frac{3}{c_{\mathbf{S},K}} \left\{ C_{\mathbf{P},K} h_K^2 \|\nabla \cdot \mathbf{S}\nabla p\|_K^2 \right. \\
&\quad + C_{\mathbf{S},K}^2 C_K h_K^2 \|p\|_{H^2(K)}^2 \left( \sum_{\sigma \in \mathcal{E}_K} \tilde{C}_{t,K,\sigma}^{1/2} \right)^2 \\
&\quad \left. + h_K \left( \sum_{\sigma \in \mathcal{E}_K} \tilde{C}_{t,K,\sigma}^{1/2} \|(S_{K,\sigma} + \langle \mathbf{S}\nabla p \cdot \mathbf{n}, 1 \rangle_\sigma) |\sigma|^{-1}\|_\sigma \right)^2 \right\}.
\end{aligned}$$

Using the assumptions of the theorem, this concludes the proof of the bound for the error in the gradient.

We now turn to the  $L^2(\Omega)$ -estimate on  $p - \tilde{p}_h$ . We have

$$\begin{aligned}
\|p - \tilde{p}_h\|_\Omega^2 &= \sum_{K \in \mathcal{T}_h} \|p - \tilde{p}_h\|_K^2 \leq 2 \sum_{K \in \mathcal{T}_h} \{ \|p - p_K\|_K^2 + \|p_K - \tilde{p}_h\|_K^2 \} \\
&\leq 2 \sum_{K \in \mathcal{T}_h} \{ \|p - p_K\|_K^2 + C_{\mathbf{P},K} h_K^2 \|\nabla \tilde{p}_h\|_K^2 \} \\
&\leq 2 \sum_{K \in \mathcal{T}_h} \{ \|p - p_K\|_K^2 + 2C_{\mathbf{P},K} h_K^2 \|\nabla(\tilde{p}_h - p)\|_K^2 + 2C_{\mathbf{P},K} h_K^2 \|\nabla p\|_K^2 \},
\end{aligned} \quad (3.17)$$

using the triangle and Cauchy–Schwarz inequalities, (3.15b), and the Poincaré inequality (2.1) when  $\mu_K = 0$  or its continuous version when  $\mu_K = 1$ . The assumptions of the theorem and the previously obtained bound on the gradient of  $p - \tilde{p}_h$  conclude the proof of this second estimate and thus of the whole theorem.  $\square$

**Remark 3.5** (Meaning of Theorem 3.4 and an example). *Theorem 3.4 simply guarantees that if  $S_{K,\sigma}$  from the finite volume scheme (3.1) are  $O(h)$  approximations of the diffusive fluxes through the sides  $\sigma$  of all elements  $K \in \mathcal{T}_h$  and if the original piecewise constant approximation given by  $p_K$  in each  $K \in \mathcal{T}_h$  converges to  $p$  as  $O(h)$  in the  $L^2(\Omega)$ -norm, then  $\tilde{p}_h$  converges to  $p$  as  $O(h)$  in the broken  $H^1(\Omega)$ -norm. Let  $\mathbf{S} = Id$ ,  $\Gamma_N = \emptyset$ , and  $p \in H^2(\Omega)$ . Then this is for example the case, under some additional regularity assumptions on  $\mathbf{w}$ ,  $r$ , and  $g$  and under the assumption that  $d_{K,L} \approx h_K \approx h_L$  for all  $\sigma_{K,L} \in \mathcal{E}_h^{\text{int}}$  and  $d_{K,\sigma} \approx h_K$  for all  $\sigma \in \mathcal{E}_h^{\text{D}}$  and appropriate  $K$ , for the scheme (3.1)–(3.10) (with  $\nu_\sigma = 1$  for all  $\sigma \in \mathcal{E}_h$ ) on polygonal meshes  $\mathcal{T}_h$  admissible in the sense of [22, Definition 9.1], see [22, Theorem 9.4 and Remark 9.12].*

### 3.2.5 Convergence for the postprocessed approximation

For the sake of completeness, we state here the following result, discussing the convergence of  $\tilde{p}_h$  when  $p \in H^1(\Omega)$  only.

**Theorem 3.6** (Convergence of  $\tilde{p}_h$ ). *Let  $\mathbf{w} = r = 0$ ,  $\mathbf{S} = Id$ ,  $\Gamma_N = \emptyset$ , and  $g = 0$ . Let  $\mathcal{T}_h$  be admissible in the sense of [22, Definition 9.1] and satisfying Assumptions (B1)–(B2) and let the finite volume scheme be given by (3.1)–(3.7b). Let finally  $p$  be the weak solution of the problem (1.1a)–(1.1c) given by (2.6) and let  $\tilde{p}_h$  be the postprocessed solution given by (3.15a)–(3.15c). Then*

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \|\nabla(p - \tilde{p}_h)\|_K^2 &\rightarrow 0 \quad \text{as } h \rightarrow 0, \\ \|p - \tilde{p}_h\|_\Omega^2 &\rightarrow 0 \quad \text{as } h \rightarrow 0. \end{aligned}$$

The convergence of the gradient of  $\tilde{p}_h$  to the gradient of  $p$  was proved in [23]. The convergence of  $\tilde{p}_h$  to  $p$  then follows using the estimate (3.17).

### 3.3 Oswald interpolation operator

As the finite volume postprocessed approximation  $\tilde{p}_h$  is only contained in  $H^1(\mathcal{T}_h)$ , we will need below its conforming (continuous, contained in  $H^1(\Omega)$ ) interpolant. We will use for this purpose the Oswald interpolant, previously considered in [14, 29, 30], as well as in [3, 19]. We only consider in this section matching meshes  $\mathcal{T}_h$ , consisting of simplices or rectangular parallelepipeds and satisfying Assumption (A), and  $\tilde{p}_h$  piecewise polynomial—see the next section for the generalizations.

If  $\mathcal{T}_h$  consist of simplices, let  $\mathbb{P}_l(\mathcal{T}_h)$  denote the space of polynomials of total degree at most  $l$  on each simplex, without any continuity requirement at the sides. Similarly, if  $\mathcal{T}_h$  consist of rectangular parallelepipeds, let  $\mathbb{P}_l(\mathcal{T}_h)$  denote the space of piecewise polynomials of degree at most  $l$  in each variable. The Oswald interpolation operator  $\mathcal{I}_{\text{Os}} : \mathbb{P}_l(\mathcal{T}_h) \rightarrow \mathbb{P}_l(\mathcal{T}_h) \cap H^1(\Omega)$  is defined as follows: given a function  $\varphi_h \in \mathbb{P}_l(\mathcal{T}_h)$ , the value of  $\mathcal{I}_{\text{Os}}(\varphi_h)$  is prescribed at the Gauss–Lobatto nodes on rectangular parallelepipeds and suitable (e.g. Lagrangian on triangles when  $l = 2$ ) nodes on simplices, see [14, Sections 3 and 5] of  $\mathbb{P}_l(\mathcal{T}_h) \cap H^1(\Omega)$  by the average of the values of  $\varphi_h$  at this node,

$$\mathcal{I}_{\text{Os}}(\varphi_h)(V) = \frac{1}{|\mathcal{T}_V|} \sum_{K \in \mathcal{T}_V} \varphi_h|_K(V),$$

where  $\mathcal{T}_V$  is the set of such  $K \in \mathcal{T}_h$  that contain the node  $V$ . Let  $[\varphi_h]$  be the jump of a function  $\varphi_h$  across a side  $\sigma$ : if  $\sigma = \partial K \cap \partial L$ , then  $[\varphi_h]$  is the difference of the value of  $\varphi_h$  in  $K$  and  $L$  (the order of  $K$  and  $L$  has no influence on what follows) and  $[\varphi_h] := 0$  otherwise. The following result has been proved by Burman and Ern [14], see Lemmas 3.2 and 5.3 and Remark 3.2 in this reference:

**Lemma 3.7** (Oswald interpolation operator). *Let  $\mathcal{T}_h$  be matching, consisting of simplices or rectangular parallelepipeds, and satisfying Assumption (A). Let next  $\varphi_h \in \mathbb{P}_l(\mathcal{T}_h)$  and let  $\mathcal{I}_{\text{Os}}(\varphi_h)$  be constructed as described above. Then*

$$\begin{aligned} \|\varphi_h - \mathcal{I}_{\text{Os}}(\varphi_h)\|_K^2 &\leq C_1 \sum_{\sigma; \sigma \cap K \neq \emptyset} h_\sigma \|[\varphi_h]\|_\sigma^2, \\ \|\nabla(\varphi_h - \mathcal{I}_{\text{Os}}(\varphi_h))\|_K^2 &\leq C_2 \sum_{\sigma; \sigma \cap K \neq \emptyset} h_\sigma^{-1} \|[\varphi_h]\|_\sigma^2, \end{aligned}$$

where the constants  $C_1, C_2$  only depend on the space dimension  $d$ , on the maximal polynomial degree  $l$ , and on the shape regularity parameter  $\kappa_{\mathcal{T}}$ .

### 3.4 A conforming interpolate for arbitrary grids and boundary conditions

On the basis of the Oswald interpolating operator of the previous section, we define here a conforming interpolate  $\mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)$  for arbitrary grids and boundary conditions.

Let thus  $\tilde{p}_h$  be the postprocessed finite volume approximation given by (3.14a)–(3.14d) and let  $\tilde{p}_h$  be in addition a polynomial. If  $\mathcal{T}_h$  is matching and consists of simplices or rectangular parallelepipeds, let  $\mathcal{I}_{\text{Os}}(\tilde{p}_h)$  be the Oswald interpolate of  $\tilde{p}_h$ , see the previous section. If  $\mathcal{T}_h$  is nonmatching or consists of arbitrary polygons/polyhedrons, let  $\mathcal{I}_{\text{Os}}(\tilde{p}_h)$  be defined on  $\widehat{\mathcal{T}}_h$  (cf. Assumption (B3)), taking  $\tilde{p}_h$  as a piecewise polynomial on  $\widehat{\mathcal{T}}_h$ . We define  $\mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h) \in H^1(\Omega)$ , differing from  $\mathcal{I}_{\text{Os}}(\tilde{p}_h)$  only on such  $K \in \mathcal{T}_h$  that contain a boundary side, by

$$\mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)|_{\Gamma_D} = g \quad \text{in the sense of traces,} \quad (3.18a)$$

$$\langle \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h) \mathbf{w} \cdot \mathbf{n}, 1 \rangle_\sigma = W_{K,\sigma} \quad \forall \sigma \in \mathcal{E}_h^N. \quad (3.18b)$$

Note that if  $\mathbf{w} \cdot \mathbf{n}$  is constant but nonzero on a given  $\sigma \in \mathcal{E}_h^N$ , then (3.18b) prescribes the mean value of  $\mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)$  on this side by  $W_{K,\sigma} / \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_\sigma$ , i.e. by  $p_\sigma$  for the scheme (3.1)–(3.12).

## 4 An a posteriori error estimate for simplices or rectangular parallelepipeds

We present in this section our a posteriori estimate on the error between the weak solution  $p$  and the postprocessed approximate finite volume solution  $\tilde{p}_h$  defined in Section 3.2. We only consider here matching meshes consisting of simplices or rectangular parallelepipeds. Also, for meshes consisting of rectangular parallelepipeds,  $\mathbf{S}$  is supposed to be diagonal. Finally, we suppose that  $\tilde{p}_h$  is given by (3.15a)–(3.15c) and that the diffusive fluxes through Neumann boundary sides are given by (3.4), (3.6). The other cases are postponed to Section 5 below. We first state the a posteriori error estimate and its efficiency, then give several remarks, and finally present the proofs of the estimate and of its efficiency.

## 4.1 A posteriori error estimate

We state here our a posteriori error estimate. Let us first put

$$m_K^2 := \min \left\{ C_{P,K} \frac{h_K^2}{c_{S,K}}, \frac{4}{c_{\mathbf{w},r,K}} \right\}$$

for all  $K \in \mathcal{T}_h$ , where  $C_{P,K}$  is the constant from the Poincaré inequality (2.1). We define the *residual estimator*  $\eta_K$  associated with an element  $K$  by

$$\eta_K := m_K \|f + \nabla \cdot \mathbf{S} \nabla \tilde{p}_h - \nabla \cdot (\tilde{p}_h \mathbf{w}) - r \tilde{p}_h\|_K. \quad (4.1)$$

Let  $\mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)$  be the modification of the Oswald interpolate  $\mathcal{I}_{\text{Os}}(\tilde{p}_h)$  at the boundary given by (3.18a)–(3.18b), see Sections 3.3 and 3.4. Denote  $v := \tilde{p}_h - \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)$ . Then the *nonconformity estimators*  $\zeta_{1,K}, \zeta_{2,K}$  associated with an element  $K$  are given by

$$\zeta_{1,K} := \|v\|_K, \quad (4.2a)$$

$$\zeta_{2,K} := \min \left\{ \frac{\|\mathbf{S} \nabla v\|_K}{\sqrt{c_{S,K}}} + \frac{2\|\nabla \cdot (v \mathbf{w})\|_K + \|rv\|_K}{\sqrt{c_{\mathbf{w},r,K}}}, \right. \\ \left. \frac{\|\mathbf{S} \nabla v\|_K + \sqrt{C_{P,K}} h_K \|\nabla v \cdot \mathbf{w}\|_K}{\sqrt{c_{S,K}}} + \frac{2\|v \nabla \cdot \mathbf{w}\|_K + \|rv\|_K}{\sqrt{c_{\mathbf{w},r,K}}} \right\}. \quad (4.2b)$$

Next, let

$$m_\sigma^2 := \min \left\{ \max_{K; \sigma \in \mathcal{E}_K} \left\{ C_{F,K,\sigma} \frac{|\sigma| h_K^2}{|K| c_{S,K}} \right\}, \max_{K; \sigma \in \mathcal{E}_K} \left\{ \frac{|\sigma|}{|K| c_{\mathbf{w},r,K}} \right\} \right\} \quad (4.3)$$

for interior or Dirichlet boundary sides. Here  $C_{F,K,\sigma}$  is the constant from the generalized Friedrichs inequality (2.2). The *upwinding estimator*  $\iota_K$  associated with an element  $K$  is given by

$$\iota_K := \sum_{\sigma \in \mathcal{E}_K \setminus \mathcal{E}_h^N} m_\sigma \|(W_{K,\sigma} - \langle \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h) \mathbf{w} \cdot \mathbf{n}, 1 \rangle_\sigma) |\sigma|^{-1}\|_\sigma. \quad (4.4)$$

Finally, the *reaction quadrature estimator*  $v_K$  associated with an element  $K$  is given by

$$v_K := \frac{1}{\sqrt{c_{\mathbf{w},r,K}}} \|r_K p_K - (r \tilde{p}_h, 1)_K |K|^{-1}\|_K \quad (4.5)$$

and the *Neumann boundary estimator*  $\lambda_K$  associated with an element  $K$  by

$$\lambda_K := 0 + \frac{1}{\sqrt{c_{S,K}}} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^N} \sqrt{C_{t,K,\sigma}} \sqrt{h_K} \|u_\sigma - u\|_\sigma, \quad (4.6)$$

where  $C_{t,K,\sigma}$  is the constant from the trace inequality (2.3a). We then have the following a posteriori error estimate:

**Theorem 4.1** (A posteriori error estimate on simplices or rectangular parallelepipeds). *Let  $p$  be the weak solution of the problem (1.1a)–(1.1c) given by (2.6). Let  $\mathcal{T}_h$  be matching, consisting of simplices or rectangular parallelepipeds, and satisfying Assumption (A). When  $\mathcal{T}_h$  consists of rectangular parallelepipeds, let in addition  $\mathbf{S}$  be diagonal. Let finally a finite volume scheme of the form (3.1) be given, let  $\tilde{p}_h$  be its postprocessed solution given by (3.15a)–(3.15c), and let the diffusive fluxes through Neumann boundary sides be given by (3.4), (3.6). Then*

$$\|p - \tilde{p}_h\|_\Omega \leq \left\{ \sum_{K \in \mathcal{T}_h} \zeta_{1,K}^2 \right\}^{1/2} + \left\{ \sum_{K \in \mathcal{T}_h} (\eta_K + \zeta_{2,K} + \iota_K + v_K + \lambda_K)^2 \right\}^{1/2}.$$

## 4.2 Efficiency of the estimate

Let the local Péclet number  $\text{Pe}_K$  and  $\varrho_K$  be given by

$$\text{Pe}_K := h_K \frac{C_{\mathbf{w},K}}{c_{\mathbf{S},K}}, \quad \varrho_K := \frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{w},r,K}} \sqrt{c_{\mathbf{S},K}}}. \quad (4.7)$$

Next, let

$$\begin{aligned} \alpha_{*,K} &:= c_{\mathbf{S},K} + 2c_{\mathbf{S},K} \left( \frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}} + 2\varrho_K \right)^2, & \beta_{*,K} &:= c_{\mathbf{w},r,K} + 4 \frac{C_{\mathbf{w},r,K}^2}{c_{\mathbf{w},r,K}}, \\ \alpha_{\#,K} &:= c_{\mathbf{S},K} + 2c_{\mathbf{S},K} \left( \frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}} + \text{Pe}_K \sqrt{C_{\text{P},K}} \right)^2, & \beta_{\#,K} &:= c_{\mathbf{w},r,K} + 4 \frac{C_{\mathbf{w},r,K}^2}{c_{\mathbf{w},r,K}}, \end{aligned}$$

and, for  $\varphi \in H^1(K)$ ,

$$\|\varphi\|_{*,K}^2 := \alpha_{*,K} \|\nabla \varphi\|_K^2 + \beta_{*,K} \|\varphi\|_K^2, \quad \|\varphi\|_{\#,K}^2 := \alpha_{\#,K} \|\nabla \varphi\|_K^2 + \beta_{\#,K} \|\varphi\|_K^2.$$

Finally, let

$$c_{\mathbf{S},\omega_K} := \min_{L; L \cap K \neq \emptyset} c_{\mathbf{S},L}, \quad c_{\mathbf{w},r,\omega_K} := \min_{L; L \cap K \neq \emptyset} c_{\mathbf{w},r,L}.$$

The theorem below discusses the efficiency of the a posteriori error estimators of Section 4.1.

**Theorem 4.2** (Efficiency of the a posteriori error estimators on simplices or rectangular parallelepipeds). *Suppose that the assumptions of Theorem 4.1 are verified. Then, for the residual estimator  $\eta_K$  on each  $K \in \mathcal{T}_h$ , there holds*

$$\eta_K \leq C_3 \|\|p - \tilde{p}_h\|_K \left\{ \left( \frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}} + \frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}} \right) + \min\{\text{Pe}_K, \varrho_K\} \right\}, \quad (4.8)$$

where the constant  $C_3$  only depends on the space dimension  $d$ , on the shape regularity parameter  $\kappa_{\mathcal{T}}$ , and on the polynomial degree  $k$  of  $f$ ,  $\mathbf{w}$ , and  $r$  (see Lemma 4.9 below). For the nonconformity estimators  $\zeta_{1,K}, \zeta_{2,K}$  on each  $K \in \mathcal{T}_h$ , we have

$$\begin{aligned} \zeta_{1,K}^2 + \zeta_{2,K}^2 &\leq C_4 \min \left\{ \frac{\alpha_{*,K}}{c_{\mathbf{S},\omega_K}} + \min \left\{ \frac{\beta_{*,K}}{c_{\mathbf{w},r,\omega_K}}, \frac{\beta_{*,K} h_K^2}{c_{\mathbf{S},\omega_K}} \right\}, \right. \\ &\quad \left. \frac{\alpha_{\#,K}}{c_{\mathbf{S},\omega_K}} + \min \left\{ \frac{\beta_{\#,K}}{c_{\mathbf{w},r,\omega_K}}, \frac{\beta_{\#,K} h_K^2}{c_{\mathbf{S},\omega_K}} \right\} \right\} \sum_{L; L \cap K \neq \emptyset} \|\|p - \tilde{p}_h\|_L^2 \\ &\quad + C_4 \beta_{*,K} \inf_{s_h \in \mathbb{P}_2(\mathcal{T}_h) \cap H^1(\Omega)} \sum_{L; L \cap K \neq \emptyset} \|p - s_h\|_L^2 \\ &\quad + 2 \max \left\{ \|\|\mathcal{I}_{\text{Os}}(\tilde{p}_h) - \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)\|_{*,K}^2, \|\|\mathcal{I}_{\text{Os}}(\tilde{p}_h) - \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)\|_{\#,K}^2 \right\}, \end{aligned} \quad (4.9)$$

where the constant  $C_4$  only depends on  $d$  and  $\kappa_{\mathcal{T}}$  (see Lemma 4.10 below).

## 4.3 Various remarks

We give several remarks in this section.

### 4.3.1 Nature of the estimate and comparisons with different estimators

The basis for our a posteriori error estimates is the construction of the postprocessed approximation  $\tilde{p}_h$ , described in Section 3.2. This approximation is more regular than the original piecewise constant finite volume approximation and we recall that it converges as  $O(h)$  to  $p$  in the broken  $H^1(\Omega)$ -norm under appropriate assumptions, see Theorem 3.4 and Remark 3.5. Then, the a posteriori error estimate in the (semi-)norm (2.5) of Theorem 4.1 shall be read as an  $L^2(\Omega)$  a posteriori error estimate for the approximate gradient given by  $\nabla\tilde{p}_h$  plus an  $L^2(\Omega)$  a posteriori error estimate for  $\tilde{p}_h$  itself as soon as  $c_{\mathbf{w},r,K} > 0$  for all  $K \in \mathcal{T}_h$ .

Let us now compare the given a posteriori error estimate for finite volumes with a posteriori error estimates used in the Galerkin finite element method. The postprocessed approximation  $\tilde{p}_h$  has the crucial advantage over Galerkin finite element approximations that the normal traces of  $-\mathbf{S}\nabla\tilde{p}_h$  are by (3.15c), and by the finite volume continuity of the discrete fluxes, continuous across interior sides. Hence the side error estimators penalizing the mass balance common in Galerkin finite element methods (cf. [42]) do not appear here at all. This advantage is however compensated by the fact that  $\tilde{p}_h$  is nonconforming in the sense that it is not included in  $H^1(\Omega)$ , so that the estimators (4.2a) and (4.2b), penalizing in fact the jumps in  $\tilde{p}_h$  (cf. (4.12) in the proof of Lemma 4.10 below) come in place. Next, whereas in the lowest-order Galerkin finite element method,  $\nabla \cdot \mathbf{S}_K \nabla p_h|_K$  is always equal to zero on all  $K \in \mathcal{T}_h$ , the element residuals (4.1) give a good sense even when the original finite volume solution is elementwise constant.

Another interesting comparison is with the estimators proposed by Nicaise [32] for the finite volume method. His basic idea is quite similar to the one presented in this work: first postprocess the original piecewise constant finite volume approximation. He uses for this purpose the Morley interpolant, cf. Ciarlet [16]. However, only the means of the fluxes of this interpolant through the mesh sides are continuous, so that, in the general case, one has to penalize both the improper mass balance of  $\mathbf{S}\nabla\tilde{p}_h$  and the nonconformity of  $\tilde{p}_h$ . We note however that in certain cases, the Morley interpolant is conforming (contained in  $H^1(\Omega)$ ), so that the nonconformity penalization disappears. Another remark in this comparison may be that the postprocessed approximation presented in [32] has to be constructed differently in dependence on whether convection and reaction are present. This at the one hand permits to prove the efficiency of the estimates (see the next section for the discussion of the efficiency of our estimates), but it at the other one complicates the implementation. Finally, the question of the a priori error estimates (convergence) of the postprocessed approximation is not investigated in [32].

### 4.3.2 The estimate and its efficiency

We discuss briefly in this remark the a posteriori error estimate of Theorem 4.1 and its efficiency treated in Theorem 4.2, especially with respect to inhomogeneities, anisotropies, and convection or reaction dominance. We refer to a more detailed discussion in [45, Section 5.2].

Let us first discuss the residual and nonconformity estimators. First of all, Theorem 4.2 proves that our estimate is optimal with respect to the inhomogeneities in the sense that its efficiency does not depend on the maximal ratio of the inhomogeneities over the whole domain, but for a given element  $K$ , only on the maximal ratio over  $K$  and elements sharing a vertex with  $K$ ; this is the meaning of the term  $\alpha_{*,K}/c_{\mathbf{S},\omega_K}$  from Theorem 4.2. The efficiency with respect to anisotropy is expressed by  $C_{\mathbf{S},K}/c_{\mathbf{S},K}$  and is always local in a given  $K \in \mathcal{T}_h$ . Next, one can easily see that the efficiency gets to optimal values as soon as only the local Péclet number (4.7), not the global one, gets sufficiently small. Also, notice that the estimate is optimal in the reaction-dominated case as well, since the quantities  $C_{\mathbf{w},r,K}/c_{\mathbf{w},r,K}$  and  $\beta_{*,K}/c_{\mathbf{w},r,\omega_K}$  remain well bounded in the limit. We finally remark that for the nonconformity estimators, the above statements are

only valid up to higher-order terms (the part  $\inf_{s_h \in \mathbb{P}_2(\mathcal{T}_h) \cap H^1(\Omega)}$ ), which is however only present when  $\frac{1}{2} \nabla \cdot \mathbf{w} + r > 0$ . Also, the error in the approximation of the boundary conditions is left aside from these considerations by the last term of the estimate (4.9) on  $\zeta_{1,K}^2 + \zeta_{2,K}^2$ .

Next, note that the reaction quadrature estimator is zero whenever  $r$  is piecewise constant and  $\mu_K$  from (3.15b) is set to zero. If this is not the case, its influence is as that of a quadrature formula—see Section 6.2 below for an example of its influence in a numerical experiment. One could eliminate this estimator by employing directly  $\tilde{p}_h$  in the discretization of the reaction term in the finite volume scheme (3.1). In what concerns the Neumann boundary estimator, it only penalizes the fact that the Neumann boundary condition is not exactly satisfied by the finite volume approximation when it is not given by a piecewise constant function. It has a form similar to Neumann boundary estimators from the finite element method, see [42].

Finally, the fact that the upwinding estimator cannot in general give a lower bound for the error is quite obvious. Let us for clearness consider the scheme (3.1)–(3.12) on a mesh  $\mathcal{T}_h$  admissible in the sense of [22, Definition 9.1] and let  $\mathbf{w} \cdot \mathbf{n}$  be constant on each mesh side. Then it is not difficult to imagine a situation where  $p = \tilde{p}_h$ , whereas  $p_{\sigma_{K,L}} - \langle \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h), 1 \rangle_{\sigma_{K,L}} / |\sigma_{K,L}|$ , the difference of the mean value of  $\mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)$  on a side  $\sigma_{K,L} \in \mathcal{E}_h^{\text{int}}$  and of the combination of  $p_K$  and  $p_L$ , is generally nonzero. The numerical experiments presented below in Section 6.2 however show that this estimator represents a higher-order term as soon as the local Péclet number gets small and when the upstream weighting (3.8)–(3.12), modified in the sense of Remark 3.1, is used. Similarly to the reaction quadrature estimator, this estimator would completely disappear while employing  $\mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)$  in the discretization of the convective term. We however notice that such convective flux does not seem too much appealing for practical computations: whereas the scheme (3.1)–(3.12) on admissible triangular meshes is very simple and leads to a 4-point stencil, employing  $\mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)$  in the discretization of the convection term would complicate the definition of the scheme and imply instead of 4 at around 20 nonzero values on each matrix row.

### 4.3.3 The estimate for pure diffusion problems

For the sake of clarity, we give here the exact form of our a posteriori error estimate for the case where  $\mathbf{w} = r = 0$  in (1.1a)–(1.1c). Using that in this case  $-\nabla \cdot \mathbf{S}_K \nabla \tilde{p}_h = f_K$  for all  $K \in \mathcal{T}_h$ , which follows by imposing (3.15c) on a simplex or on a rectangular parallelepiped when  $\mathbf{S}$  is diagonal and by (3.1), the analysis for the convection–diffusion–reaction case simplifies to the a posteriori error estimate

$$\| \| p - \tilde{p}_h \| \|_\Omega \leq \left\{ \sum_{K \in \mathcal{T}_h} \zeta_{1,K}^2 \right\}^{1/2} + \left\{ \sum_{K \in \mathcal{T}_h} (\eta_K + \zeta_{2,K} + \lambda_K)^2 \right\}^{1/2},$$

where

$$\begin{aligned} \eta_K &:= h_K \sqrt{\frac{C_{\text{P},K}}{c_{\mathbf{S},K}}} \| f - f_K \|_K, \\ \zeta_{1,K} &:= \sqrt{c_{\mathbf{S},K}} \| \nabla(\tilde{p}_h - \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)) \|_K, \\ \zeta_{2,K} &:= \frac{\| \mathbf{S} \nabla(\tilde{p}_h - \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)) \|_K}{\sqrt{c_{\mathbf{S},K}}}, \\ \lambda_K &:= 0 + \frac{1}{\sqrt{c_{\mathbf{S},K}}} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\text{N}}} \sqrt{C_{\text{t},K,\sigma}} \sqrt{h_K} \| u_\sigma - u \|_\sigma. \end{aligned}$$

#### 4.4 Proof of the a posteriori error estimate

We give in this section a proof of Theorem 4.1. To begin with, the following bound for the error  $\|p - \tilde{p}_h\|_\Omega$  holds:

**Lemma 4.3** (Abstract framework). *Let the assumptions of Theorem 4.1 be verified and let  $s \in H^1(\Omega)$  such that  $s|_{\Gamma_D} = g$  in the sense of traces be arbitrary. Then*

$$\begin{aligned} \|p - \tilde{p}_h\|_\Omega \leq & \|s - \tilde{p}_h\|_\Omega + \sup_{\varphi \in H_D^1(\Omega), \|\varphi\|_\Omega=1} \{T_R(\varphi) + T_{NC}(\varphi) \\ & + T_U(\varphi) + T_{RQ}(\varphi) + T_{\Gamma_N}(\varphi)\}, \end{aligned}$$

where

$$\begin{aligned} T_R(\varphi) &:= \sum_{K \in \mathcal{T}_h} (f + \nabla \cdot \mathbf{S} \nabla \tilde{p}_h - \nabla \cdot (\tilde{p}_h \mathbf{w}) - r \tilde{p}_h, \varphi - \varphi_K)_K, \\ T_{NC}(\varphi) &:= \mathcal{B}(\tilde{p}_h - s, \varphi) + \sum_{K \in \mathcal{T}_h} \langle (s - \tilde{p}_h) \mathbf{w} \cdot \mathbf{n}, \varphi_K \rangle_{\partial K}, \\ T_U(\varphi) &:= \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \langle (W_{K,\sigma} - \langle s \mathbf{w} \cdot \mathbf{n}, 1 \rangle_\sigma) |\sigma|^{-1}, \varphi_K \rangle_\sigma, \\ T_{RQ}(\varphi) &:= \sum_{K \in \mathcal{T}_h} (r_K p_K - (r \tilde{p}_h, 1)_K |K|^{-1}, \varphi_K)_K, \\ T_{\Gamma_N}(\varphi) &:= \sum_{\sigma \in \mathcal{E}_h^N} \langle u_\sigma - u, \varphi - \varphi_\sigma \rangle_\sigma, \end{aligned}$$

and where  $\varphi_K$  is the mean of  $\varphi$  over  $K$ ,  $\varphi_K := (\varphi, 1)_K / |K|$ , and  $\varphi_\sigma$  is the mean of  $\varphi$  over  $\sigma$ ,  $\varphi_\sigma := \langle \varphi, 1 \rangle_\sigma / |\sigma|$ .

PROOF:

The triangle inequality implies

$$\|p - \tilde{p}_h\|_\Omega \leq \|p - s\|_\Omega + \|s - \tilde{p}_h\|_\Omega.$$

Now since  $(p - s) \in H_D^1(\Omega)$ , we can use the coercivity of the form  $\mathcal{B}(\cdot, \cdot)$  given by (2.7), so that

$$\begin{aligned} \|p - s\|_\Omega &\leq \frac{\mathcal{B}(p - s, p - s)}{\|p - s\|_\Omega} \leq \sup_{\varphi \in H_D^1(\Omega), \|\varphi\|_\Omega=1} \mathcal{B}(p - s, \varphi) \\ &= \sup_{\varphi \in H_D^1(\Omega), \|\varphi\|_\Omega=1} \{\mathcal{B}(p - \tilde{p}_h, \varphi) + \mathcal{B}(\tilde{p}_h - s, \varphi)\}. \end{aligned} \quad (4.10)$$

Let us consider an arbitrary  $\varphi \in H_D^1(\Omega)$ . We have, using the bilinearity of  $\mathcal{B}(\cdot, \cdot)$ , the definition (2.6) of the weak solution  $p$ , and the Green theorem in each  $K \in \mathcal{T}_h$ ,

$$\begin{aligned} &\mathcal{B}(p - \tilde{p}_h, \varphi) \tag{4.11} \\ &= (f, \varphi)_\Omega - \langle u, \varphi \rangle_{\Gamma_N} - \sum_{K \in \mathcal{T}_h} \{(\mathbf{S} \nabla \tilde{p}_h, \nabla \varphi)_K + (\nabla \cdot (\tilde{p}_h \mathbf{w}), \varphi)_K + (r \tilde{p}_h, \varphi)_K\} \\ &= \sum_{K \in \mathcal{T}_h} (f + \nabla \cdot \mathbf{S} \nabla \tilde{p}_h - \nabla \cdot (\tilde{p}_h \mathbf{w}) - r \tilde{p}_h, \varphi)_K - \sum_{K \in \mathcal{T}_h} \langle \mathbf{S} \nabla \tilde{p}_h \cdot \mathbf{n}, \varphi \rangle_{\partial K} - \langle u, \varphi \rangle_{\Gamma_N} \\ &= \sum_{K \in \mathcal{T}_h} \left\{ (f + \nabla \cdot \mathbf{S} \nabla \tilde{p}_h - \nabla \cdot (\tilde{p}_h \mathbf{w}) - r \tilde{p}_h, \varphi)_K + \sum_{\sigma \in \mathcal{E}_K} \langle u_\sigma - u, \varphi \rangle_{\sigma \cap \Gamma_N} \right\}. \end{aligned}$$

Note that we have in particular used the continuity of the normal trace of  $\mathbf{S}\nabla\tilde{p}_h$  on interior sides following from (3.15c) and from the finite volume continuity of the diffusive fluxes, i.e.

$$\langle (\mathbf{S}\nabla\tilde{p}_h \cdot \mathbf{n})|_K + (\mathbf{S}\nabla\tilde{p}_h \cdot \mathbf{n})|_L, \varphi \rangle_{\sigma_{K,L}} = \langle 0, \varphi \rangle_{\sigma_{K,L}} = 0 \quad \forall \sigma_{K,L} \in \mathcal{E}_h^{\text{int}},$$

the fact that  $\langle \mathbf{S}\nabla\tilde{p}_h \cdot \mathbf{n}, \varphi \rangle_\sigma = 0$  for  $\sigma \in \mathcal{E}_h^{\text{D}}$  following by  $\varphi \in H_D^1(\Omega)$ , and finally (3.4) and (3.15c) for Neumann boundary sides. If  $\sigma \in \mathcal{E}_h^{\text{N}}$ , notice moreover that  $\langle u_\sigma - u, \varphi \rangle_\sigma = \langle u_\sigma - u, \varphi - \varphi_\sigma \rangle_\sigma$ , since  $\langle u_\sigma - u, \varphi_\sigma \rangle_\sigma = 0$  by (3.6) (recall that  $\varphi_\sigma$  is a constant).

Now by the definition (3.1) of the finite volume scheme, it follows that

$$\begin{aligned} & (f + \nabla \cdot \mathbf{S}\nabla\tilde{p}_h - \nabla \cdot (\tilde{p}_h \mathbf{w}) - r\tilde{p}_h, \varphi_K)_K + \langle \tilde{p}_h \mathbf{w} \cdot \mathbf{n}, \varphi_K \rangle_{\partial K} - \varphi_K \sum_{\sigma \in \mathcal{E}_K} W_{K,\sigma} \\ & + (r\tilde{p}_h, \varphi_K)_K - \varphi_K r_K p_K |K| \pm \langle s \mathbf{w} \cdot \mathbf{n}, \varphi_K \rangle_{\partial K} = 0 \quad \forall K \in \mathcal{T}_h. \end{aligned}$$

To see this, recall that  $\varphi_K$  is the constant mean of  $\varphi$  over  $K$ , that  $(\nabla \cdot \mathbf{S}\nabla\tilde{p}_h, 1)_K = \langle \mathbf{S}\nabla\tilde{p}_h \cdot \mathbf{n}, 1 \rangle_{\partial K} = -\sum_{\sigma \in \mathcal{E}_K} S_{K,\sigma}$  by the Green theorem and by (3.15c), and that  $(\nabla \cdot (\tilde{p}_h \mathbf{w}), 1)_K = \langle \tilde{p}_h \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\partial K}$  again by the Green theorem. Hence we can subtract this term from each summand in (4.11). Noticing that  $\mathcal{B}(\tilde{p}_h - s, \varphi)$  appears directly as the second term of (4.10) concludes the proof.  $\square$

We now estimate the terms  $T_R$ ,  $T_{\text{NC}}$ ,  $T_U$ ,  $T_{\text{RQ}}$ , and  $T_{\Gamma_N}$  separately, putting  $s = \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)$  in Lemma 4.3.

**Lemma 4.4** (Residual estimate). *Let  $\varphi \in H_D^1(\Omega)$  be arbitrary. Then*

$$T_R(\varphi) \leq \sum_{K \in \mathcal{T}_h} \eta_K \|\varphi\|_K,$$

where  $\eta_K$  is given by (4.1).

PROOF:

The Schwarz inequality and Lemma 7.1 from Section 7 below imply

$$T_R(\varphi) \leq \sum_{K \in \mathcal{T}_h} \|f + \nabla \cdot \mathbf{S}\nabla\tilde{p}_h - \nabla \cdot (\tilde{p}_h \mathbf{w}) - r\tilde{p}_h\|_K \|\varphi - \varphi_K\|_K \leq \sum_{K \in \mathcal{T}_h} \eta_K \|\varphi\|_K. \quad \square$$

**Lemma 4.5** (Nonconformity estimate). *Let  $\varphi \in H_D^1(\Omega)$  be arbitrary. Then*

$$T_{\text{NC}}(\varphi) \leq \sum_{K \in \mathcal{T}_h} \zeta_{2,K} \|\varphi\|_K,$$

where  $\zeta_{2,K}$  is given by (4.2b).

PROOF:

Denote  $v := \tilde{p}_h - \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)$ . Then, for each  $K \in \mathcal{T}_h$ ,

$$\begin{aligned} & (\mathbf{S}\nabla v, \nabla \varphi)_K + (\nabla \cdot (v \mathbf{w}), \varphi)_K + (rv, \varphi)_K - \langle v \mathbf{w} \cdot \mathbf{n}, \varphi_K \rangle_{\partial K} \\ & = (\mathbf{S}\nabla v, \nabla \varphi)_K + (\nabla \cdot (v \mathbf{w}), \varphi - \varphi_K)_K + (rv, \varphi)_K \\ & \leq \|\mathbf{S}\nabla v\|_K \|\nabla \varphi\|_K + (2\|\nabla \cdot (v \mathbf{w})\|_K + \|rv\|_K) \|\varphi\|_K \\ & \leq \left( \frac{\|\mathbf{S}\nabla v\|_K}{\sqrt{c_{\mathbf{S},K}}} + \frac{2\|\nabla \cdot (v \mathbf{w})\|_K + \|rv\|_K}{\sqrt{c_{\mathbf{w},r,K}}} \right) \|\varphi\|_K, \end{aligned}$$

using the Green theorem in the first equality. Alternatively,

$$\begin{aligned}
& (\mathbf{S}\nabla v, \nabla\varphi)_K + (\nabla \cdot (v\mathbf{w}), \varphi - \varphi_K)_K + (rv, \varphi)_K \\
& \leq (\|\mathbf{S}\nabla v\|_K + \sqrt{C_{P,K}}h_K\|\nabla v \cdot \mathbf{w}\|_K)\|\nabla\varphi\|_K + (2\|v\nabla \cdot \mathbf{w}\|_K + \|rv\|_K)\|\varphi\|_K \\
& \leq \left( \frac{\|\mathbf{S}\nabla v\|_K + \sqrt{C_{P,K}}h_K\|\nabla v \cdot \mathbf{w}\|_K}{\sqrt{c_{S,K}}} + \frac{2\|v\nabla \cdot \mathbf{w}\|_K + \|rv\|_K}{\sqrt{c_{w,r,K}}} \right) \|\varphi\|_K,
\end{aligned}$$

employing the Poincaré inequality (2.1). Noticing the definition of  $\zeta_{2,K}$  by (4.2b) concludes the proof.  $\square$

**Lemma 4.6** (Upwinding estimate). *Let  $\varphi \in H_D^1(\Omega)$  be arbitrary. Then*

$$T_U(\varphi) \leq \sum_{K \in \mathcal{T}_h} \iota_K \|\varphi\|_K,$$

where  $\iota_K$  is given by (4.4).

PROOF:

We have

$$\begin{aligned}
T_U(\varphi) &= \sum_{\sigma_{K,L} \in \mathcal{E}_h^{\text{int}}} \langle (W_{K,\sigma_{K,L}} - \langle \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)\mathbf{w} \cdot \mathbf{n}_K, 1 \rangle_{\sigma_{K,L}}) |\sigma_{K,L}|^{-1}, \varphi_K - \varphi_L \rangle_{\sigma_{K,L}} \\
&\quad + \sum_{\sigma_K \in \mathcal{E}_h^{\text{D}}} \langle (W_{K,\sigma_K} - \langle \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)\mathbf{w} \cdot \mathbf{n}_K, 1 \rangle_{\sigma_K}) |\sigma_K|^{-1}, \varphi_K \rangle_{\sigma_K},
\end{aligned}$$

using that  $\mathbf{w} \cdot \mathbf{n}$  as well as  $\mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)$  are continuous across interelement sides, a similar continuity of the finite volume convective fluxes  $W_{K,\sigma}$ , and (3.18b). Let  $\sigma_{K,L} \in \mathcal{E}_h^{\text{int}}$  and let us put  $\varphi_{\sigma_{K,L}} := \langle \varphi, 1 \rangle_{\sigma_{K,L}} / |\sigma_{K,L}|$ . Then

$$\begin{aligned}
\|\varphi_K - \varphi_L\|_{\sigma_{K,L}} &\leq \|\varphi_K - \varphi_{\sigma_{K,L}}\|_{\sigma_{K,L}} + \|\varphi_L - \varphi_{\sigma_{K,L}}\|_{\sigma_{K,L}} \\
&\leq \max_{M=\{K,L\}} \left\{ C_{F,M,\sigma_{K,L}} \frac{|\sigma_{K,L}|h_M^2}{|M|c_{S,M}} \right\}^{1/2} (\|\varphi\|_K + \|\varphi\|_L)
\end{aligned}$$

by the triangle inequality and the first estimate of Lemma 7.2 from Section 7 below. At the same time,

$$\begin{aligned}
\|\varphi_K - \varphi_L\|_{\sigma_{K,L}} &\leq \|\varphi_K\|_{\sigma_{K,L}} + \|\varphi_L\|_{\sigma_{K,L}} \\
&\leq \max_{M=\{K,L\}} \left\{ \frac{|\sigma_{K,L}|}{|M|c_{w,r,M}} \right\}^{1/2} (\|\varphi\|_K + \|\varphi\|_L),
\end{aligned}$$

using the triangle inequality and the second estimate of Lemma 7.2. Similar estimates on  $\|\varphi_K\|_{\sigma_K}$  for  $\sigma_K \in \mathcal{E}_h^{\text{D}}$  follow directly from Lemma 7.2 using that  $\varphi_{\sigma_K} = 0$  on Dirichlet boundary sides by  $\varphi \in H_D^1(\Omega)$ . Hence

$$\begin{aligned}
T_U(\varphi) &\leq \sum_{\sigma \in \mathcal{E}_h \setminus \mathcal{E}_h^{\text{N}}} \left\{ m_\sigma \|(W_{K,\sigma} - \langle \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)\mathbf{w} \cdot \mathbf{n}, 1 \rangle_\sigma) |\sigma|^{-1}\|_\sigma \sum_{K;\sigma \in \mathcal{E}_K} \|\varphi\|_K \right\} \\
&= \sum_{K \in \mathcal{T}_h} \left\{ \sum_{\sigma \in \mathcal{E}_K \setminus \mathcal{E}_h^{\text{N}}} m_\sigma \|(W_{K,\sigma} - \langle \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)\mathbf{w} \cdot \mathbf{n}, 1 \rangle_\sigma) |\sigma|^{-1}\|_\sigma \right\} \|\varphi\|_K
\end{aligned}$$

with  $m_\sigma$  and  $\eta_\sigma$  given respectively by (4.3) and (4.4). Noticing the definition of  $\iota_K$  by (4.4) concludes the proof.  $\square$

**Lemma 4.7** (Reaction quadrature estimate). *Let  $\varphi \in H_D^1(\Omega)$  be arbitrary. Then*

$$T_{\text{RQ}}(\varphi) \leq \sum_{K \in \mathcal{T}_h} v_K \|\varphi\|_K,$$

where  $v_K$  is given by (4.5).

PROOF:

The Schwarz inequality and the definition of  $\|\cdot\|_K$  by (2.5) imply

$$T_{\text{R}}(\varphi) \leq \sum_{K \in \mathcal{T}_h} \|r_K p_K - (r \tilde{p}_h, 1)_K |K|^{-1} \|_K \|\varphi\|_K \leq \sum_{K \in \mathcal{T}_h} v_K \|\varphi\|_K. \quad \square$$

**Lemma 4.8** (Neumann boundary estimate). *Let  $\varphi \in H_D^1(\Omega)$  be arbitrary. Then*

$$T_{\Gamma_N}(\varphi) \leq \sum_{K \in \mathcal{T}_h} \lambda_K \|\varphi\|_K,$$

where  $\lambda_K$  is given by (4.6).

PROOF:

We have, using the trace inequality (2.3a),

$$\begin{aligned} T_{\Gamma_N}(\varphi) &= \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^N} \langle u_\sigma - u, \varphi - \varphi_\sigma \rangle_\sigma \\ &\leq \sum_{K \in \mathcal{T}_h} \|\nabla \varphi\|_K \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^N} C_{t,K,\sigma}^{1/2} h_K^{1/2} \|u_\sigma - u\|_\sigma \leq \sum_{K \in \mathcal{T}_h} \lambda_K \|\varphi\|_K. \quad \square \end{aligned}$$

Lemmas 4.3–4.8 and the Cauchy–Schwarz inequality prove Theorem 4.1.

## 4.5 Proof of the efficiency of the estimate

We give in this section a proof of Theorem 4.2.

**Lemma 4.9** (Efficiency of the residual estimator). *Let the assumptions of Theorem 4.1 be verified, let  $K \in \mathcal{T}_h$ , and let  $\eta_K$  be the residual estimator given by (4.1). There holds*

$$\eta_K \leq C_3 \|\|p - \tilde{p}_h\|_K \left\{ \left( \frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}} + \frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}} \right) + \min\{\text{Pe}_K, \varrho_K\} \right\},$$

where  $\text{Pe}_K$  and  $\varrho_K$  are given by (4.7) and where the constant  $C_3$  only depends on the space dimension  $d$ , on the shape regularity parameter  $\kappa_{\mathcal{T}}$ , and on the polynomial degree  $k$  of  $f$ ,  $\mathbf{w}$ , and  $r$ .

PROOF:

The proof follows the one given in [42] and adapted in [45] by the author. We recall it here for the sake of completeness.

Let  $\psi_K$  be the bubble function on  $K$ , given as the product of the linear functions that take the value 1 at one vertex of  $K$  and vanish at the other vertices, and let us denote  $v := (f + \nabla \cdot \mathbf{S} \nabla \tilde{p}_h - \nabla \cdot (\tilde{p}_h \mathbf{w}) - r \tilde{p}_h)$  on a given  $K \in \mathcal{T}_h$ . Note that  $v$  is a polynomial in  $K$  by Assumption (C).

Then the equivalence of norms on finite-dimensional spaces, the inverse inequality (cf. e.g. [16, Theorem 3.2.6]), and the definition of  $\|\cdot\|_K$  by (2.5) give

$$\begin{aligned} c\|v\|_K^2 &\leq (v, \psi_K v)_K, \\ \|\psi_K v\|_K &\leq \|v\|_K, \\ \|\|\psi_K v\|\|_K &\leq C \min \left\{ \frac{h_K}{\sqrt{c_{\mathbf{S},K}}}, \frac{1}{\sqrt{c_{\mathbf{w},r,K}}} \right\}^{-1} \|v\|_K, \end{aligned}$$

with the constants  $c$  and  $C$  only depending on  $d$ ,  $\kappa_K$ , and the polynomial degree  $k$  of  $f$ ,  $\mathbf{w}$ , and  $r$ . Next, we immediately have (cf. the proof of Lemma 4.3)

$$\mathcal{B}(p - \tilde{p}_h, \psi_K v) = (v, \psi_K v)_K,$$

and, using (2.8),

$$\begin{aligned} \mathcal{B}(p - \tilde{p}_h, \psi_K v) &\leq \left( \frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}} + \frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}} \right) \|\|p - \tilde{p}_h\|\|_K \|\|\psi_K v\|\|_K \\ &\quad + \frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{S},K}}} \|\|p - \tilde{p}_h\|\|_K \|\psi_K v\|_K. \end{aligned}$$

Combining the above estimates, one comes to

$$\begin{aligned} c\|v\|_K^2 &\leq \|\|p - \tilde{p}_h\|\|_K \|v\|_K \\ &\quad \left\{ \left( \frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}} + \frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}} \right) C \min \left\{ \frac{h_K}{\sqrt{c_{\mathbf{S},K}}}, \frac{1}{\sqrt{c_{\mathbf{w},r,K}}} \right\}^{-1} + \frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{S},K}}} \right\}. \end{aligned}$$

Considering the definition of  $\eta_K$  by (4.1) and of  $\text{Pe}_K$  and  $\varrho_K$  by (4.7) concludes the proof.  $\square$

**Lemma 4.10** (Efficiency of the nonconformity estimators). *Let the assumptions of Theorem 4.1 be verified, let  $K \in \mathcal{T}_h$ , and let  $\zeta_{1,K}, \zeta_{2,K}$  be the nonconformity estimators given by (4.2a) and (4.2b), respectively. There holds*

$$\begin{aligned} \zeta_{1,K}^2 + \zeta_{2,K}^2 &\leq C_4 \min \left\{ \frac{\alpha_{*,K}}{c_{\mathbf{S},\omega_K}} + \min \left\{ \frac{\beta_{*,K}}{c_{\mathbf{w},r,\omega_K}}, \frac{\beta_{*,K} h_K^2}{c_{\mathbf{S},\omega_K}} \right\}, \right. \\ &\quad \left. \frac{\alpha_{\#,K}}{c_{\mathbf{S},\omega_K}} + \min \left\{ \frac{\beta_{\#,K}}{c_{\mathbf{w},r,\omega_K}}, \frac{\beta_{\#,K} h_K^2}{c_{\mathbf{S},\omega_K}} \right\} \right\} \sum_{L; L \cap K \neq \emptyset} \|\|p - \tilde{p}_h\|\|_L^2 \\ &\quad + C_4 \beta_{*,K} \inf_{s_h \in \mathbb{P}_2(\mathcal{T}_h) \cap H^1(\Omega)} \sum_{L; L \cap K \neq \emptyset} \|p - s_h\|_L^2 \\ &\quad + 2 \max \left\{ \|\|\mathcal{I}_{\text{Os}}(\tilde{p}_h) - \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)\|\|_{*,K}^2, \|\|\mathcal{I}_{\text{Os}}(\tilde{p}_h) - \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)\|\|_{\#,K}^2 \right\}, \end{aligned}$$

where the constants  $\alpha_{*,K}$ ,  $\beta_{*,K}$ ,  $\alpha_{\#,K}$ ,  $\beta_{\#,K}$ ,  $c_{\mathbf{S},\omega_K}$ , and  $c_{\mathbf{w},r,\omega_K}$  and  $\|\|\cdot\|\|_{*,K}$ ,  $\|\|\cdot\|\|_{\#,K}$  are defined in Section 4.2 and where the constant  $C_4$  only depends on the space dimension  $d$  and on the shape regularity parameter  $\kappa_{\mathcal{T}}$ .

PROOF:

One shows easily that

$$\begin{aligned} \zeta_{2,K} &\leq \min \left\{ \left( \frac{C_{\mathbf{S},K}}{\sqrt{c_{\mathbf{S},K}}} + 2 \frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{w},r,K}}} \right) \|\nabla v\|_K + 2 \frac{C_{\mathbf{w},r,K}}{\sqrt{c_{\mathbf{w},r,K}}} \|v\|_K, \right. \\ &\quad \left. \frac{C_{\mathbf{S},K} + \sqrt{C_{\mathbf{P},K}} h_K C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{S},K}}} \|\nabla v\|_K + 2 \frac{C_{\mathbf{w},r,K}}{\sqrt{c_{\mathbf{w},r,K}}} \|v\|_K \right\}, \end{aligned}$$

where  $v = \tilde{p}_h - \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)$ . Hence

$$\zeta_{1,K}^2 + \zeta_{2,K}^2 \leq \min \{ \|\|v\|\|_{*,K}^2, \|\|v\|\|_{\#,K}^2 \}$$

with  $\|\| \cdot \|\|_{*,K}$  and  $\|\| \cdot \|\|_{\#,K}$  defined in Section 4.2. Thus

$$\begin{aligned} \zeta_{1,K}^2 + \zeta_{2,K}^2 &\leq 2 \min \{ \|\|\tilde{p}_h - \mathcal{I}_{\text{Os}}(\tilde{p}_h)\|\|_{*,K}^2, \|\|\tilde{p}_h - \mathcal{I}_{\text{Os}}(\tilde{p}_h)\|\|_{\#,K}^2 \} \\ &\quad + 2 \max \{ \|\|\mathcal{I}_{\text{Os}}(\tilde{p}_h) - \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)\|\|_{*,K}^2, \|\|\mathcal{I}_{\text{Os}}(\tilde{p}_h) - \mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)\|\|_{\#,K}^2 \}. \end{aligned}$$

The rest of the proof, devoted to showing a bound on  $\|\|\tilde{p}_h - \mathcal{I}_{\text{Os}}(\tilde{p}_h)\|\|_{*,K}$ , is similar to that given in [45]. We recall it here for the sake of completeness. The proof for  $\|\|\tilde{p}_h - \mathcal{I}_{\text{Os}}(\tilde{p}_h)\|\|_{\#,K}$  can be established likewise.

Let henceforth  $C$  denote a constant only depending on  $d$  and on  $\kappa_{\mathcal{T}}$ , not necessarily the same at each occurrence. It follows directly from Lemma 3.7 and the definition of  $\|\| \cdot \|\|_{*,K}$  that

$$\|\|\tilde{p}_h - \mathcal{I}_{\text{Os}}(\tilde{p}_h)\|\|_{*,K}^2 \leq C \left( \alpha_{*,K} \sum_{\sigma; \sigma \cap K \neq \emptyset} h_\sigma^{-1} \|\|\tilde{p}_h\|\|_\sigma^2 + \beta_{*,K} \sum_{\sigma; \sigma \cap K \neq \emptyset} h_\sigma \|\|\tilde{p}_h\|\|_\sigma^2 \right). \quad (4.12)$$

We will next use the inequality

$$h_\sigma^{-1/2} \|\|\tilde{p}_h\|\|_\sigma \leq C \sum_{L; \sigma \in \mathcal{E}_L} \|\nabla(\tilde{p}_h - \varphi)\|_L.$$

This inequality was established in [3, Theorem 10] for simplicial meshes,  $\sigma \in \mathcal{E}_h^{\text{int}}$ , and an arbitrary  $\varphi \in H^1(\Omega)$ . It generalizes easily to rectangular parallelepipeds and taking into account that we have put  $[\varphi] = 0$  for  $\sigma \in \mathcal{E}_h^{\text{ext}}$ , it is immediately valid on exterior sides as well. This inequality implies that

$$h_\sigma^\gamma \|\|\tilde{p}_h\|\|_\sigma^2 \leq C \frac{h_\sigma^{\gamma+1}}{\min_{L; \sigma \in \mathcal{E}_L} c_{\mathbf{S},L}} \sum_{L; \sigma \in \mathcal{E}_L} c_{\mathbf{S},L} \|\nabla(\tilde{p}_h - p)\|_L^2, \quad (4.13)$$

where we put  $\gamma = -1, 1$ . Next, for an arbitrary  $s_h \in \mathbb{P}_2(\mathcal{T}_h) \cap H^1(\Omega)$ ,

$$\begin{aligned} h_\sigma^{1/2} \|\|\tilde{p}_h\|\|_\sigma &\leq h_\sigma C \sum_{L; \sigma \in \mathcal{E}_L} \|\nabla(\tilde{p}_h - s_h)\|_L \leq C \sum_{L; \sigma \in \mathcal{E}_L} h_L \|\nabla(\tilde{p}_h - s_h)\|_L \\ &\leq C \sum_{L; \sigma \in \mathcal{E}_L} \|\tilde{p}_h - s_h\|_L \leq C \sum_{L; \sigma \in \mathcal{E}_L} \|\tilde{p}_h - p\|_L + C \sum_{L; \sigma \in \mathcal{E}_L} \|p - s_h\|_L, \end{aligned}$$

using the inverse inequality given by [16, Theorem 3.2.6] and the triangle inequality. Hence

$$h_\sigma \|\|\tilde{p}_h\|\|_\sigma^2 \leq C \frac{1}{\min_{L; \sigma \in \mathcal{E}_L} c_{\mathbf{w},r,L}} \sum_{L; \sigma \in \mathcal{E}_L} c_{\mathbf{w},r,L} \|\tilde{p}_h - p\|_L^2 + C \sum_{L; \sigma \in \mathcal{E}_L} \|p - s_h\|_L^2 \quad (4.14)$$

holds as well, which gives a sense when all  $c_{\mathbf{w},r,L}$  for  $L$  such that  $\sigma \in \mathcal{E}_L$  are nonzero. Combining estimates (4.12)–(4.14) while estimating  $\min_{L; \sigma \in \mathcal{E}_L} c_L$  for a side  $\sigma$  such that  $\sigma \cap K \neq \emptyset$  from below by  $\min_{L; L \cap K \neq \emptyset} c_L$  concludes the proof.  $\square$

Lemmas 4.9–4.10 together prove Theorem 4.2.

## 5 Generalizations

We discuss in this section the generalizations of the a posteriori error estimate of Section 4 to post-processed approximations given by (3.14a)–(3.14d) instead of (3.15a)–(3.15c) and to grids which may be nonmatching (containing hanging nodes) or composed of arbitrary polygonal elements.

## 5.1 Higher-order polynomials on matching grids of simplices or rectangular parallelepipeds

In Section 4, we have considered  $\tilde{p}_h$  as piecewise second-order polynomials given by (3.15a)–(3.15c). Let now  $\Gamma_N = \emptyset$  for simplicity and let  $\tilde{p}_h$  be a (higher-order) polynomial on each  $K \in \mathcal{T}_h$  satisfying (3.14a)–(3.14d) instead of (3.15a)–(3.15c). One then easily verifies that the results of Section 4 carry over to this case; in particular Theorems 4.1 and 4.2 hold (the constants  $C_3$  and  $C_4$  from Theorem 4.2 now in addition depend on the polynomial degree of  $\tilde{p}_h$ ). In fact, the prescription of constant diffusive fluxes of  $\tilde{p}_h$  over the sides of  $K \in \mathcal{T}_h$  by (3.15c) was only used on Neumann boundary sides. So for triangular elements, we may in particular consider polynomials of 3-rd degree with the diffusive fluxes  $-\mathbf{S}_K \nabla \tilde{p}_h|_K \cdot \mathbf{n}|_\sigma$  over the edges  $\sigma$  of  $K \in \mathcal{T}_h$  given as 2-nd order polynomials on these edges. A similar construction with rectangular parallelepipeds may enable to consider general  $\mathbf{S}$ . However,  $\tilde{p}_h$  should be chosen carefully in order to guarantee the convergence to  $p$  together with its (broken) gradient, cf. Sections 3.2.4 and 3.2.5.

## 5.2 Higher-order polynomials on arbitrary polygonal grids

We show here that the previous idea generalizes to arbitrary grids as well.

Let us thus consider a (nonmatching) grid  $\mathcal{T}_h$  consisting of arbitrary polygons/polyhedrons and a cell-centered finite volume scheme of the form (3.1) on this grid. Let next  $\tilde{p}_h$  be a piecewise polynomial on  $\mathcal{T}_h$  given by (3.14a)–(3.14d). Under these assumptions, a closer inspection of the arguments of Section 4 shows that Theorem 4.1 still holds true (on the original grid  $\mathcal{T}_h$ ). In particular, the assumption that  $\mathcal{T}_h$  is matching and consists of simplices or rectangular parallelepipeds is not used in the proofs of Section 4.4. However, in the case where convection is present, an additional assumption that  $|\sigma| \approx h_K^{d-1}$  for all  $\sigma \in \mathcal{E}_K$  (now not necessarily the sides of  $K$  in the geometrical sense, see Section 2.2) is important in order to use the generalized Friedrichs inequality (2.2) in Lemma 7.2 below, used itself in the proof of Lemma 4.6. Finally, the first paragraph of the proof of Lemma 4.10 stays unchanged. Next, notice that

$$\|\|\tilde{p}_h - \mathcal{I}_{\text{Os}}(\tilde{p}_h)\|\|_{*,K}^2 = \sum_{L \in \widehat{\mathcal{T}}_K} \|\|\tilde{p}_h - \mathcal{I}_{\text{Os}}(\tilde{p}_h)\|\|_{*,L}^2,$$

which means that the second part of the proof of Lemma 4.10 is satisfied on the grid  $\widehat{\mathcal{T}}_h$ , whence the a posteriori error estimate of Theorem 4.1 on the grid  $\mathcal{T}_h$  is efficient as well, in the sense of Theorem 4.2.

We summarize the above results in the following theorem:

**Theorem 5.1** (A posteriori error estimate on arbitrary polygonal grids and its efficiency). *Let  $p$  be the weak solution of the problem (1.1a)–(1.1c) given by (2.6), with  $\Gamma_N = \emptyset$ . Let  $\mathcal{T}_h$  satisfy Assumption (B) and let  $\widehat{\mathcal{T}}_h$  be its matching refinement, consisting of simplices or rectangular parallelepipeds. Next, let a finite volume scheme of the form (3.1) on  $\mathcal{T}_h$  be given, let  $\tilde{p}_h$  be its postprocessed solution given by (3.14a)–(3.14d), let  $\tilde{p}_h$  be a polynomial of degree at most  $l$  on each  $K \in \mathcal{T}_h$ , and let  $\mathcal{I}_{\text{Os}}^\Gamma(\tilde{p}_h)$  be constructed as described in Sections 3.3 and 3.4. Also, if  $\mathbf{w} \neq 0$ , let  $|\sigma| \approx h_K^{d-1}$  for all  $\sigma \in \mathcal{E}_K$  and all  $K \in \mathcal{T}_h$ . Let finally the estimators  $\eta_K$ ,  $\zeta_{1,K}$ ,  $\zeta_{2,K}$ ,  $\iota_K$ , and  $\nu_K$  be given for all  $K \in \mathcal{T}_h$  respectively by (4.1), (4.2a), (4.2b), (4.4), and (4.5). Then*

$$\|\|p - \tilde{p}_h\|\|_\Omega \leq \left\{ \sum_{K \in \mathcal{T}_h} \zeta_{1,K}^2 \right\}^{1/2} + \left\{ \sum_{K \in \mathcal{T}_h} (\eta_K + \zeta_{2,K} + \iota_K + \nu_K)^2 \right\}^{1/2}. \quad (5.1)$$

Next, the estimates (4.8) and (4.9) from Theorem 4.2 are valid on each  $K \in \mathcal{T}_h$ . The constant  $C_3$  only depends on the space dimension  $d$ , on the shape of the elements  $K \in \mathcal{T}_h$  and in particular on the regularity parameter  $\kappa_{\mathcal{T}}$ , on the polynomial degree  $l$  of  $\tilde{p}_h$ , and on the polynomial degree  $k$  of  $f$ ,  $\mathbf{w}$ , and  $r$ . Similarly, the constant  $C_4$  only depends on  $d$ ,  $\kappa_{\hat{\mathcal{T}}}$ , and the polynomial degree  $l$  of  $\tilde{p}_h$ .

**Remark 5.2** (Further generalizations of Theorem 5.1). *We remark that for the a posteriori estimate (5.1) itself,  $\tilde{p}_h$  in fact does not necessarily have to be a polynomial on each  $K \in \mathcal{T}_h$ . On the contrary, the assumption that  $\tilde{p}_h$  is a polynomial of degree at most  $l$  on each  $K \in \mathcal{T}_h$  is necessary in the given proof of the efficiency. Also, we have only supposed  $\Gamma_N = \emptyset$  for the sake of simplicity.*

### 5.3 Pure diffusion problems and arbitrary polygonal grids: approximate solution of local Neumann problems

Since the higher-order polynomial postprocessing of the previous section may be difficult to carry out in practice, we present in this section another approach, motivated by the easy postprocessing (3.15a)–(3.15c) on simplices. We only consider here pure diffusion problems ( $\mathbf{w} = r = 0$ ).

Let thus  $\mathcal{T}_h$  be a (nonmatching) grid consisting of arbitrary polygons/polyhedrons and let  $\hat{\mathcal{T}}_h$  be its matching refinement consisting of simplices and satisfying Assumption (A) (cf. Assumption (B3)). Let a cell-centered finite volume scheme of the form (3.1) be given on  $\mathcal{T}_h$ . Instead of directly searching  $\tilde{p}_h$  verifying (3.14a)–(3.14d) or (3.15a)–(3.15c), let us consider finite volume or lowest-order Raviart–Thomas mixed finite element approximations (cf. Eymard et al. [22] and Roberts and Thomas [37] or Brezzi and Fortin [13]) of (3.15a)–(3.15c) on the simplicial grids  $\hat{\mathcal{T}}_K$  of  $K$  for all  $K \in \mathcal{T}_h$ . Now note that a finite volume scheme of the form (3.1)–(3.7b) on a simplicial mesh  $\hat{\mathcal{T}}_K$  of  $K$ , admissible in the sense of [22, Definition 9.1], under the additional condition that  $\mathbf{S}_K = s_K Id$ , preserves exactly the given Neumann fluxes  $S_{K,\sigma}$ ,  $\sigma \in \mathcal{E}_K$ . The same holds true for the lowest-order Raviart–Thomas mixed finite element scheme for (3.15a)–(3.15c) on  $\hat{\mathcal{T}}_K$ , without any restrictions on  $\hat{\mathcal{T}}_K$  or  $\mathbf{S}_K$ . Moreover, by (3.15b), the approximate values  $p_L$ ,  $L \in \hat{\mathcal{T}}_K$ , are closely related to the original  $p_K$ .

Hence we can now write a “virtual finite volume scheme” of the form (3.1) on the mesh  $\hat{\mathcal{T}}_h$ , only inserting the discrete diffusive fluxes  $S_{L,\sigma}$  for all  $L \in \hat{\mathcal{T}}_h$  and  $\sigma \in \mathcal{E}_L$ , obtained from the local problems, into (3.1) written on  $\hat{\mathcal{T}}_h$ . Such a virtual scheme preserves the original diffusive fluxes  $S_{K,\sigma}$  for all  $K \in \mathcal{T}_h$  and  $\sigma \in \mathcal{E}_K$  and on each  $K \in \mathcal{T}_h$  approximately preserves the original value  $p_K$ . Now since the mesh  $\hat{\mathcal{T}}_h$  is simplicial, we can immediately use the postprocessing (3.15a)–(3.15c) on  $\hat{\mathcal{T}}_h$  in order to construct a piecewise second-order polynomial  $\tilde{p}_h$  and have all the results of Section 4 for  $\tilde{p}_h$ . Note in particular that if  $\hat{\mathcal{T}}_K$  for some  $K \in \mathcal{T}_h$  only contains  $K$ , i.e. when  $K$  is itself a simplex, then the present approach coincides with constructing  $\tilde{p}_h|_K$  directly by (3.15a)–(3.15c). One could also show a priori error estimates or convergence for  $\tilde{p}_h$  as in Sections 3.2.4 and 3.2.5.

We summarize the above results in the following theorem:

**Theorem 5.3** (A posteriori error estimate for pure diffusion problems on arbitrary polygonal grids and its efficiency). *Let  $p$  be the weak solution of the problem (1.1a)–(1.1c) given by (2.6), with  $\mathbf{w} = r = 0$ . Let  $\mathcal{T}_h$  satisfy Assumption (B) and let  $\hat{\mathcal{T}}_h$  be its matching refinement, consisting of simplices. Next, let a finite volume scheme of the form (3.1) on  $\mathcal{T}_h$  be given and let  $\tilde{p}_h$  be given by finite volume or lowest-order Raviart–Thomas mixed finite element approximations of (3.15a)–(3.15c) on the simplicial grids  $\hat{\mathcal{T}}_K$  of  $K$  for all  $K \in \mathcal{T}_h$  and the subsequent postprocessing by (3.15a)–(3.15c) on  $\hat{\mathcal{T}}_h$  in the sense described above. Let finally the diffusive fluxes through Neumann boundary sides of  $\hat{\mathcal{T}}_h$  be given by (3.4), (3.6) and let the estimators  $\eta_K$ ,  $\zeta_{1,K}$ ,  $\zeta_{2,K}$ , and  $\lambda_K$  be given for all  $K \in \hat{\mathcal{T}}_h$*

respectively by (4.1), (4.2a), (4.2b), and (4.6). Then

$$\| \|p - \tilde{p}_h\| \|_{\Omega} \leq \left\{ \sum_{K \in \hat{\mathcal{T}}_h} \zeta_{1,K}^2 \right\}^{1/2} + \left\{ \sum_{K \in \hat{\mathcal{T}}_h} (\eta_K + \zeta_{2,K} + \lambda_K)^2 \right\}^{1/2}.$$

Next, the estimates (4.8) and (4.9) from Theorem 4.2 are valid on each  $K \in \hat{\mathcal{T}}_h$ . The constant  $C_3$  only depends on the space dimension  $d$ , on the shape regularity parameter  $\kappa_{\hat{\mathcal{T}}}$ , and on the polynomial degree  $k$  of  $f$  and similarly, the constant  $C_4$  only depends on  $d$  and  $\kappa_{\hat{\mathcal{T}}}$ .

**Remark 5.4** (Relation to a posteriori error estimates based on the solution of local Neumann problems in the finite element method). *Bank and Weiser [8] derive a posteriori error estimates in the finite element method on the basis of approximate solution of local Neumann problems. In this case, the solutions of the local problems serve to define an a posteriori error estimator for the original finite element approximation, whereas we have in this section used the solutions of the local problems in order to define a new approximate solution, for which we can easily give an a posteriori estimate.*

## 6 Numerical experiments

We test our a posteriori error estimates on two model problems in this section. The first problem contains a strongly inhomogeneous diffusion–dispersion tensor and the second one is convection-dominated. In both cases, the analytical solution is known. We use the finite volume scheme (3.1)–(3.12), which we extend from triangular grids admissible in the sense of [22, Definition 9.1] to strictly Delaunay triangular meshes (the circumcircle of each triangle does not contain any vertex in its interior or on its boundary and no circumcenters of boundary triangles lie outside the domain or on its boundary), cf. Eymard et al. [22, Example 9.1]. The harmonic averaging (3.7b) for the diffusion–dispersion tensor is employed while modifying it in the sense of Remark 3.2, i.e. by taking into account the distances of the circumcenters  $\mathbf{x}_K$ ,  $K \in \mathcal{T}_h$ , from the sides of  $K$ . When  $\sigma_{K,L} \in \mathcal{E}_h^{\text{int}}$  and  $\mathbf{x}_K$  lies in  $L$  (which may happen for strictly Delaunay triangular meshes),  $s_{K,L} = s_L$ . When convection is present, we use the local Péclet upstream weighting (3.8)–(3.12), modified in the sense of Remark 3.1. The postprocessed approximate solution  $\tilde{p}_h$  is given by (3.15a)–(3.15c) with  $\mu_K = 1$  for all  $K \in \mathcal{T}_h$  and  $\mathbf{x}_K$  being the circumcenter of  $K$ , see Section 3.2.2.

### 6.1 Model problem with strongly inhomogeneous diffusion–dispersion tensor

This model problem is taken from [36, 18] and is motivated by the fact that in real applications, the diffusion–dispersion tensor  $\mathbf{S}$  may be discontinuous and strongly inhomogeneous. We consider in particular  $\Omega = (-1, 1) \times (-1, 1)$  and the equation (1.1a) with  $\mathbf{w} = 0$ ,  $r = 0$ , and  $f = 0$ . We suppose that  $\Omega$  is divided into four subdomains  $\Omega_i$  corresponding to the axis quadrants (in the counterclockwise direction) and that  $\mathbf{S}$  is constant and equal to  $s_i Id$  in  $\Omega_i$ . Under such conditions, analytical solution writing

$$p(r, \theta) = r^\alpha (a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

in each  $\Omega_i$  can be found. Here  $(r, \theta)$  are the polar coordinates in  $\Omega$ ,  $a_i$  and  $b_i$  are constants depending on  $\Omega_i$ , and  $\alpha$  is a parameter. This solution is continuous across the interfaces but only the normal component of its flux  $\mathbf{u} = -\mathbf{S}\nabla p$  is continuous; it finally exhibits a singularity in the origin. We assume Dirichlet boundary conditions given by this solution and consider two different cases:

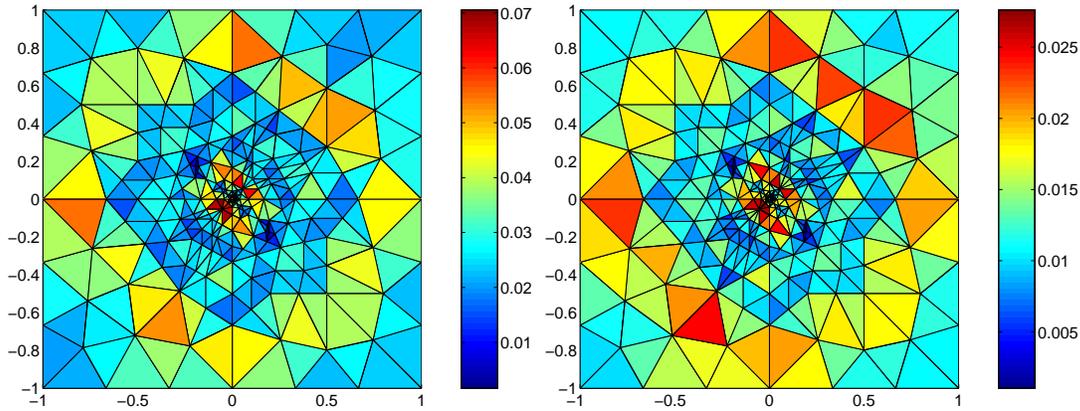


Figure 2: Estimated (left) and real (right) error distribution,  $\alpha = 0.53544095$

$$\alpha = 0.53544095$$

$$\begin{aligned} a_1 &= 0.44721360 & b_1 &= 1 \\ a_2 &= -0.74535599 & b_2 &= 2.33333333 \\ a_3 &= -0.94411759 & b_3 &= 0.55555556 \\ a_4 &= -2.40170264 & b_4 &= -0.48148148, \end{aligned}$$

$$\alpha = 0.12690207$$

$$\begin{aligned} a_1 &= 0.1 & b_1 &= 1 \\ a_2 &= -9.60396040 & b_2 &= 2.96039604 \\ a_3 &= -0.48035487 & b_3 &= -0.88275659 \\ a_4 &= 7.70156488 & b_4 &= -6.45646175 . \end{aligned}$$

In the first case,  $s_1 = s_3 = 5$ ,  $s_2 = s_4 = 1$ , whereas in the second one,  $s_1 = s_3 = 100$ ,  $s_2 = s_4 = 1$ . The original grid consisted of 112 triangles and we have refined it either uniformly (up to 4 refinements) or adaptively on the basis of our estimator. The latter case is a little complicated, since we need the refined mesh to still be strictly Delaunay, and hence the usual “longest edge” refinement or its variants cannot be used. We thus use the following concept: to the given set of vertices, we first add those which correspond to edges midpoints of such triangles where the estimated  $\|\cdot\|_{\Omega}$ -error is greater than the half of the maximum of the estimators. Then the *Triangle* mesh generator, see Shewchuk [38, 39], is used to produce a mesh which comprises the given set of vertices, respects the four subdomains, and guarantees a minimal angle (20 degrees in the given case). This mesh is then checked for the (uniform) strict Delaunay condition: the sum of the two opposite angles for each interior edge has to be less than or equal to  $\pi - \alpha$  where  $\alpha$  is a positive constant and similarly at the boundary. If it does not satisfies this condition, further vertices are added and the process is repeated until the mesh is (uniformly) strictly Delaunay. Since  $f = 0$ , the residual estimators  $\eta_K$  (4.1) are zero for each  $K \in \mathcal{T}_h$  (recall that this would be the case for general piecewise constant  $f$ , cf. Section 4.3.3), and hence the a posteriori error estimate is entirely given by the nonconformity estimators  $\zeta_{1,K}$ ,  $\zeta_{2,K}$  (4.2a)–(4.2b).

We give in Figure 2 an example of our a posteriori estimate on the error and its distribution and the actual error and its distribution on an adaptively refined mesh for the first test case. We can see that the predicted distribution is excellent and that in particular even in this case where the solution is smoother, the singularity is well recognized. Next, Figure 3 gives an example of

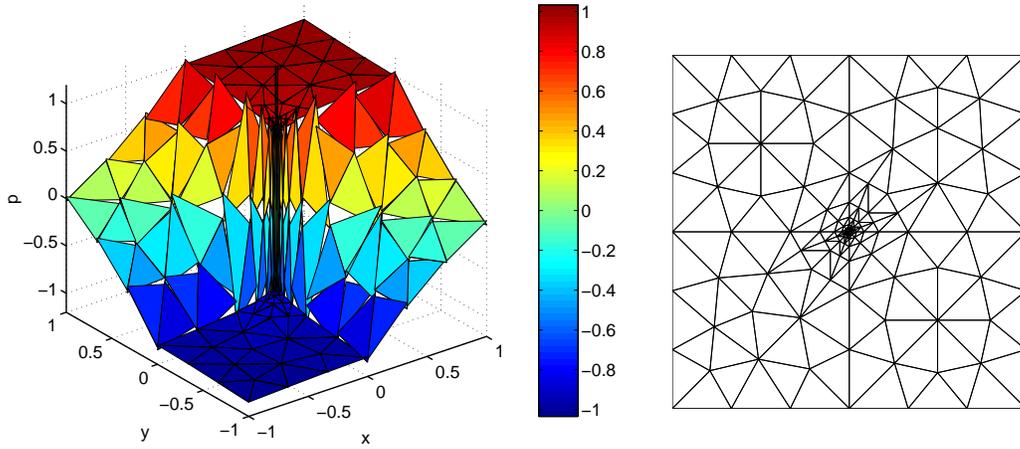


Figure 3: Approximate solution and the corresponding adaptively refined mesh,  $\alpha = 0.12690207$

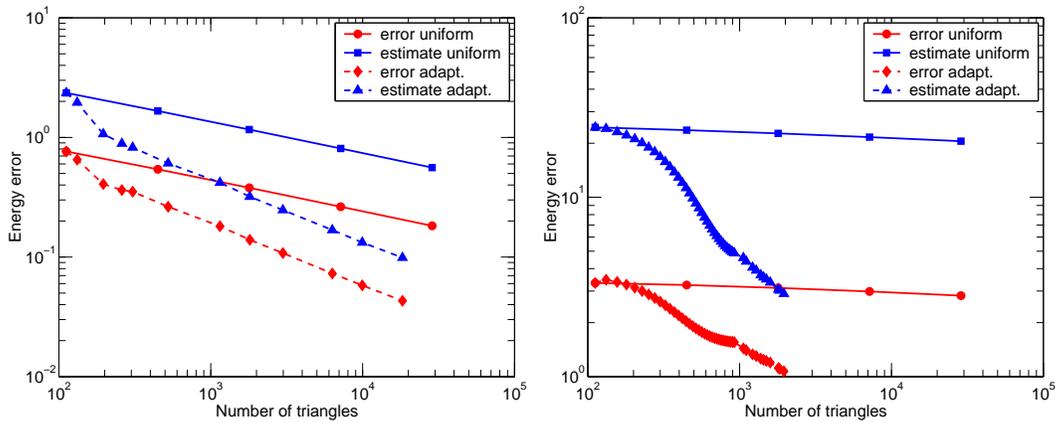


Figure 4: Estimated and real error against the number of elements in uniformly/adaptively refined meshes for  $\alpha = 0.53544095$  (left) and  $\alpha = 0.12690207$  (right)

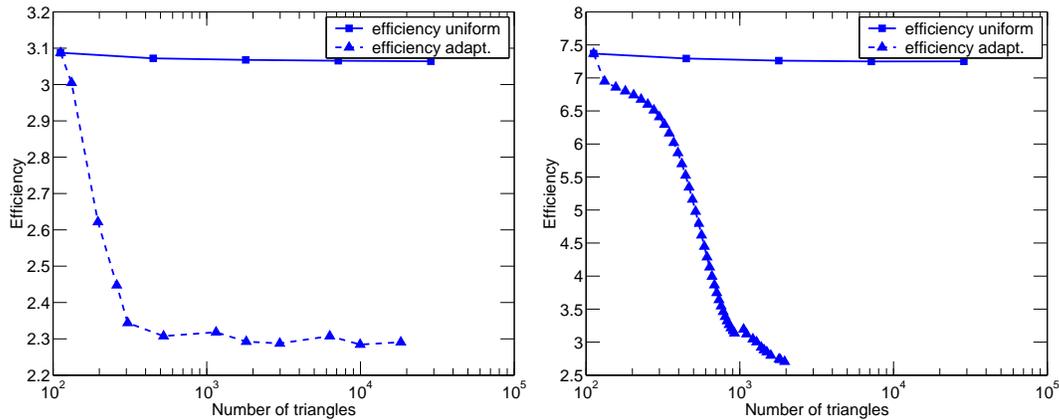


Figure 5: Overall efficiency of the a posteriori error estimates against the number of elements in uniformly/adaptively refined meshes for  $\alpha = 0.53544095$  (left) and  $\alpha = 0.12690207$  (right)

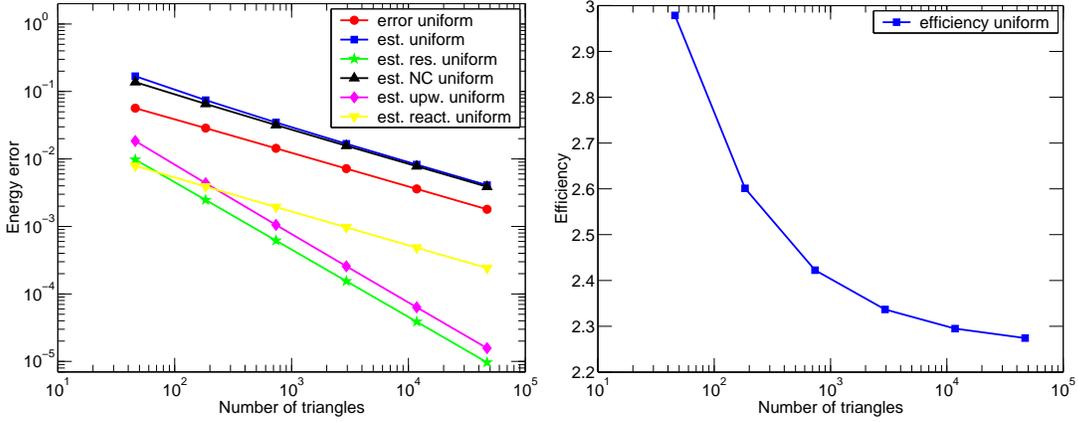


Figure 6: Estimated and real error and the different estimators (left) and overall efficiency (right) against the number of elements,  $\varepsilon = 1$ ,  $a = 0.5$

the approximate solution on an adaptively refined mesh and this mesh in the second test case. Here, the singularity is much more important and consequently the grid is highly refined around the origin (for an adaptively refined grid of 2000 triangles, the diameter of the smallest triangles near the origin is  $10^{-15}$  and 80% of the triangles are contained in the circle of radius 0.1). Figure 4 then reports the estimated and actual errors of the numerical solutions on uniformly/adaptively refined grids in the two test cases. The energy seminorm (2.5) was approximated with a 7-point quadrature formula in each triangle. It can be seen from these plots that one can substantially reduce the number of unknowns necessary to attain the prescribed precision using the derived a posteriori error estimates and adaptively refined grids. Finally, we can see in Figure 5 the efficiency plots for the two cases, giving the ratio of the estimated  $\|\cdot\|_{\Omega}$ -error to the real  $\|\cdot\|_{\Omega}$ -error. This quantity simply expresses how many times we have overestimated the actual error—recall that there are no undetermined multiplicative constants in our estimates. These plots show that our estimator is almost asymptotically exact, and this even for the cases with strong inhomogeneities. We refer for a further discussion on this point to [45, Section 5.3].

## 6.2 Convection-dominated model problem

This problem is a modification of a problem considered in [20]. We put  $\Omega = (0, 1) \times (0, 1)$ ,  $\mathbf{w} = (0, 1)$ , and  $r = 1$  in (1.1a) and consider three cases with  $\mathbf{S} = \varepsilon Id$  and  $\varepsilon$  equal to, respectively, 1,  $10^{-2}$ , and  $10^{-4}$ . The right-hand side term  $f$ , Neumann boundary conditions on the upper side, and Dirichlet boundary conditions elsewhere are chosen such that the solution was

$$p(x, y) = 0.5 \left( 1 - \tanh\left(\frac{0.5 - x}{a}\right) \right).$$

This solution is in fact one-dimensional and possesses an internal layer of width  $a$  which we set, respectively, equal to 0.5, 0.05, and 0.02. We start the computations from an unstructured grid of  $\Omega$  consisting of 46 triangles and refine it either uniformly (up to 5 refinements) or adaptively.

For  $\varepsilon = 1$  and  $a = 0.5$  (diffusion-dominated regime), estimated and actual errors in the energy norm (2.5), the different estimators, and the efficiency are reported in Figure 6. Note in particular that in this regime, the residual as well as upwinding estimators represent higher-order terms and that the influence of the reaction quadrature estimator is limited. Finally, our estimator reproduced very precisely the distribution of the error in this case.

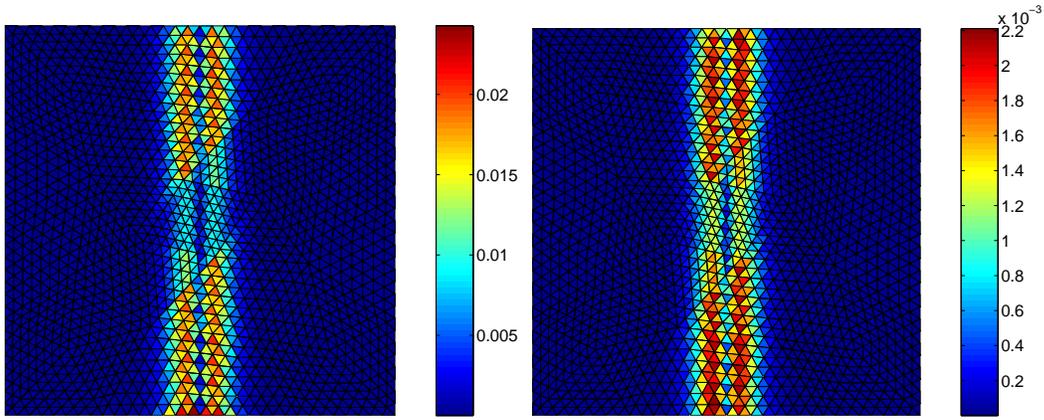


Figure 7: Estimated (left) and real (right) error distribution,  $\varepsilon = 10^{-2}$ ,  $a = 0.05$

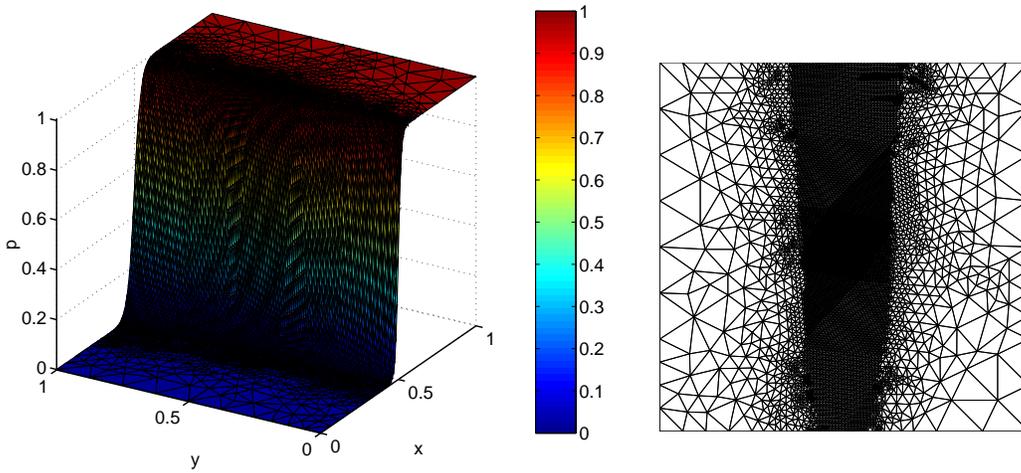


Figure 8: Approximate solution and the corresponding adaptively refined mesh,  $\varepsilon = 10^{-4}$ ,  $a = 0.02$

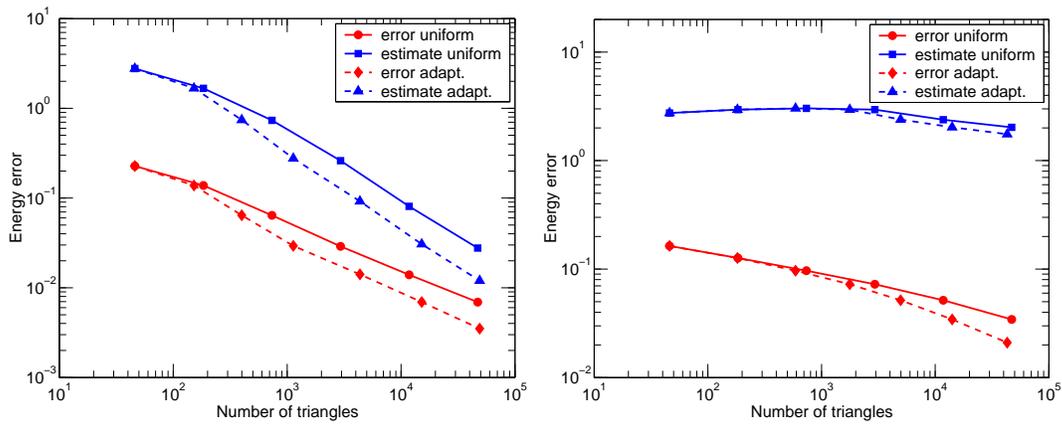


Figure 9: Estimated and real error against the number of elements in uniformly/adaptively refined meshes for  $\varepsilon = 10^{-2}$ ,  $a = 0.05$  (left) and  $\varepsilon = 10^{-4}$ ,  $a = 0.02$  (right)

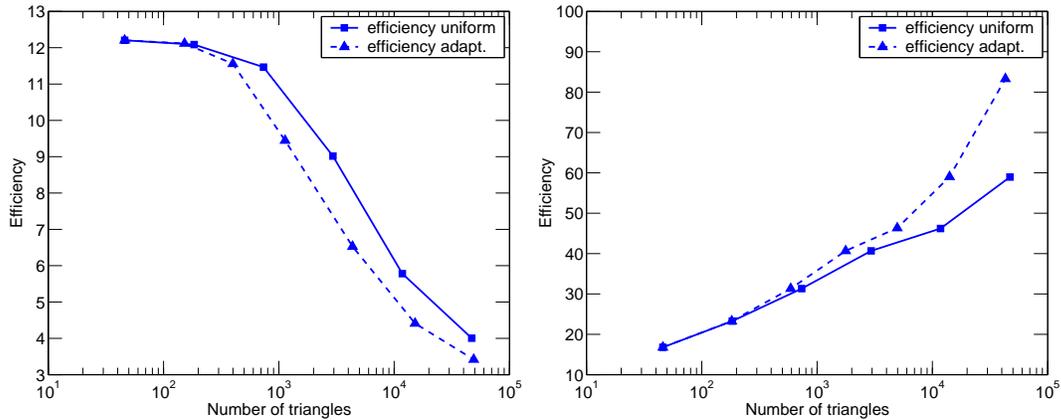


Figure 10: Overall efficiency of the a posteriori error estimates against the number of elements in uniformly/adaptively refined meshes for  $\varepsilon = 10^{-2}$ ,  $a = 0.05$  (left) and  $\varepsilon = 10^{-4}$ ,  $a = 0.02$  (right)

For  $\varepsilon = 10^{-2}$  and  $a = 0.05$  (convection-dominated regime on coarse meshes and diffusion-dominated regime with progressive refinement), still the distribution of the error is predicted very well, cf. Figure 7. Note in particular the correct localization of the error away from the center of the shock, as well as the sensitivity of our estimator to the shape of the elements. Next, an example of an adaptively refined mesh for  $\varepsilon = 10^{-4}$  and  $a = 0.02$  is given in Figure 8. The problem with keeping the refined mesh uniformly strictly Delaunay reveals as very severe in this case. For this reason, much more vertices are added at a time: the refinement criterion is set to 0.1 and 0.01 times the maximum of the estimators for  $\varepsilon = 10^{-2}$ ,  $a = 0.05$ , and  $\varepsilon = 10^{-4}$ ,  $a = 0.02$ , respectively. The estimated and real errors for these cases are plotted against the number of elements in uniformly/adaptively refined meshes in Figure 9. Again, one can see that we can substantially reduce the number of unknowns necessary to attain the prescribed precision using the derived a posteriori error estimates and adaptively refined grids. Finally the efficiency plots are given in Figure 10. In the first case, the efficiency is almost optimal for finest grids, whereas in the second one, not even the elements in the refined shock region start to leave the convection-dominated regime, a point where the efficiency would start to decrease.

## 7 Auxiliary results

We give in this section two auxiliary results that were needed in the paper.

**Lemma 7.1.** *Let  $K \in \mathcal{T}_h$ , let  $\varphi \in H^1(K)$ , and let  $\varphi_K$  be the mean of  $\varphi$  over  $K$  given by  $\varphi_K := (\varphi, 1)_K / |K|$ . Then*

$$\|\varphi - \varphi_K\|_K^2 \leq \min \left\{ C_{P,K} \frac{h_K^2}{c_{S,K}}, \frac{4}{c_{w,r,K}} \right\} \|\varphi\|_K^2.$$

PROOF:

The Poincaré inequality (2.1) and the definition of  $\|\cdot\|_K$  by (2.5) imply

$$\|\varphi - \varphi_K\|_K^2 \leq C_{P,K} h_K^2 \|\nabla \varphi\|_K^2 \leq C_{P,K} \frac{h_K^2}{c_{S,K}} \|\varphi\|_K^2.$$

Next, the estimate

$$\|\varphi - \varphi_K\|_K^2 \leq 4 \|\varphi\|_K^2 \leq \frac{4}{c_{w,r,K}} \|\varphi\|_K^2$$

follows from the triangle and Cauchy–Schwarz inequalities and the definition of  $\|\cdot\|_K$  by (2.5).  $\square$

**Lemma 7.2.** *Let  $K \in \mathcal{T}_h$ , let  $\varphi \in H^1(K)$ , and let  $\varphi_K$  be the mean of  $\varphi$  over  $K$  given by  $\varphi_K := (\varphi, 1)_K/|K|$  and  $\varphi_\sigma$  the mean of  $\varphi$  over  $\sigma \in \mathcal{E}_K$  given by  $\varphi_\sigma := \langle \varphi, 1 \rangle_\sigma/|\sigma|$ , respectively. Then*

$$\|\varphi_K - \varphi_\sigma\|_\sigma^2 \leq C_{F,K,\sigma} \frac{|\sigma|h_K^2}{|K|c_{S,K}} \|\varphi\|_K^2$$

and

$$\|\varphi_K\|_\sigma^2 \leq \frac{|\sigma|}{|K|c_{w,r,K}} \|\varphi\|_K^2.$$

PROOF:

Let us put  $\tilde{\varphi} := \varphi - \varphi_\sigma$  and  $\tilde{\varphi}_K := (\tilde{\varphi}, 1)_K/|K|$ . We now note that  $\tilde{\varphi}_\sigma := \langle \tilde{\varphi}, 1 \rangle_\sigma/|\sigma| = 0$  and that  $\nabla \tilde{\varphi} = \nabla \varphi$ , which allows us to estimate

$$\|\varphi_K - \varphi_\sigma\|_\sigma^2 = \tilde{\varphi}_K^2 |\sigma| \leq \frac{|\sigma|}{|K|} \|\tilde{\varphi}\|_K^2 \leq C_{F,K,\sigma} \frac{|\sigma|h_K^2}{|K|} \|\nabla \varphi\|_K^2 \leq C_{F,K,\sigma} \frac{|\sigma|h_K^2}{|K|c_{S,K}} \|\varphi\|_K^2,$$

using the generalized Friedrichs inequality (2.2) and the definition (2.5) of  $\|\cdot\|_K$ .

For the second estimate, we have

$$\|\varphi_K\|_\sigma^2 = |\sigma|\varphi_K^2 \leq \frac{|\sigma|}{|K|} \|\varphi\|_K^2 \leq \frac{|\sigma|}{|K|c_{w,r,K}} \|\varphi\|_K^2,$$

using the definition of  $\|\cdot\|_K$  by (2.5).  $\square$

## Acknowledgements

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## References

- [1] AAVATSMARK, I., BARKVE, T., BØE, Ø., AND MANNSETH, T. Discretization on unstructured grids for inhomogeneous, anisotropic media. I. Derivation of the methods. *SIAM J. Sci. Comput.* 19, 5 (1998), 1700–1716.
- [2] AAVATSMARK, I., BARKVE, T., BØE, Ø., AND MANNSETH, T. Discretization on unstructured grids for inhomogeneous, anisotropic media. II. Discussion and numerical results. *SIAM J. Sci. Comput.* 19, 5 (1998), 1717–1736.
- [3] ACHDOU, Y., BERNARDI, C., AND COQUEL, F. A priori and a posteriori analysis of finite volume discretizations of Darcy’s equations. *Numer. Math.* 96, 1 (2003), 17–42.
- [4] ADAMS, R. A. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.
- [5] AGOUZAL, A., BARANGER, J., MAITRE, J.-F., AND OUDIN, F. Connection between finite volume and mixed finite element methods for a diffusion problem with nonconstant coefficients. Application to a convection diffusion problem. *East-West J. Numer. Math.* 3, 4 (1995), 237–254.

- [6] AGOUZAL, A., AND OUDIN, F. A posteriori error estimator for finite volume methods. *Appl. Math. Comput.* 110, 2-3 (2000), 239–250.
- [7] ARNOLD, D. N., AND BREZZI, F. Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates. *RAIRO Modél. Math. Anal. Numér.* 19, 1 (1985), 7–32.
- [8] BANK, R. E., AND WEISER, A. Some a posteriori error estimators for elliptic partial differential equations. *Math. Comp.* 44, 170 (1985), 283–301.
- [9] BARANGER, J., MAÎTRE, J.-F., AND OUDIN, F. Connection between finite volume and mixed finite element methods. *RAIRO Modél. Math. Anal. Numér.* 30, 4 (1996), 445–465.
- [10] BEBENDORF, M. A note on the Poincaré inequality for convex domains. *Z. Anal. Anwendungen* 22, 4 (2003), 751–756.
- [11] BERGAM, A., MGHAZLI, Z., AND VERFÜRTH, R. Estimations a posteriori d’un schéma de volumes finis pour un problème non linéaire. *Numer. Math.* 95, 4 (2003), 599–624.
- [12] BERNARDI, C., AND VERFÜRTH, R. Adaptive finite element methods for elliptic equations with non-smooth coefficients. *Numer. Math.* 85, 4 (2000), 579–608.
- [13] BREZZI, F., AND FORTIN, M. *Mixed and hybrid finite element methods*, vol. 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.
- [14] BURMAN, E., AND ERN, A. Continuous interior penalty *hp*-finite element methods for advection and advection–diffusion equations. *Submitted to Math. Comp.* (2006).
- [15] CHÉNIER, E., EYMARD, R., AND NICOLAS, X. A finite volume scheme for the transport of radionuclides in porous media. *Comput. Geosci.* 8, 2 (2004), 163–172. Simulation of Transport Around a Nuclear Waste Disposal Site: The COUPLEX Test Cases.
- [16] CIARLET, P. G. *The Finite Element Method for Elliptic Problems*, vol. 4 of *Studies in Mathematics and its Applications*. North-Holland, Amsterdam, 1978.
- [17] COUDIÈRE, Y., VILA, J.-P., AND VILLEDIEU, P. Convergence rate of a finite volume scheme for a two-dimensional convection–diffusion problem. *M2AN Math. Model. Numer. Anal.* 33, 3 (1999), 493–516.
- [18] EIGESTAD, G. T., AND KLAUSEN, R. A. On the convergence of the multi-point flux approximation O-method: numerical experiments for discontinuous permeability. *Numer. Methods Partial Differential Equations* 21, 6 (2005), 1079–1098.
- [19] EL ALAOUI, L., AND ERN, A. Residual and hierarchical a posteriori error estimates for nonconforming mixed finite element methods. *M2AN Math. Model. Numer. Anal.* 38, 6 (2004), 903–929.
- [20] EL ALAOUI, L., ERN, A., AND BURMAN, E. A priori and a posteriori analysis of nonconforming finite elements with face penalty for advection–diffusion equations. *Submitted to IMA J. Numer. Anal.* (2005).
- [21] EYMARD, R., GALLOUËT, T., AND HERBIN, R. Convergence of finite volume schemes for semilinear convection diffusion equations. *Numer. Math.* 82, 1 (1999), 91–116.

- [22] EYMARD, R., GALLOUËT, T., AND HERBIN, R. Finite volume methods. In *Handbook of Numerical Analysis, Vol. VII*. North-Holland, Amsterdam, 2000, pp. 713–1020.
- [23] EYMARD, R., GALLOUËT, T., AND HERBIN, R. Finite volume approximation of elliptic problems and convergence of an approximate gradient. *Appl. Numer. Math.* 37, 1-2 (2001), 31–53.
- [24] EYMARD, R., GALLOUËT, T., AND HERBIN, R. A cell-centred finite-volume approximation for anisotropic diffusion operators on unstructured meshes in any space dimension. *IMA J. Numer. Anal.* 26, 2 (2006), 326–353.
- [25] EYMARD, R., HILHORST, D., AND VOHRALÍK, M. A combined finite volume–finite element scheme for the discretization of strongly nonlinear convection–diffusion–reaction problems on nonmatching grids. *Submitted* (2006).
- [26] EYMARD, R., HILHORST, D., AND VOHRALÍK, M. A combined finite volume–nonconforming/mixed-hybrid finite element scheme for degenerate parabolic problems. *To appear in Numer. Math.* (2006).
- [27] FAILLE, I. A control volume method to solve an elliptic equation on a two-dimensional irregular mesh. *Comput. Methods Appl. Mech. Engrg.* 100, 2 (1992), 275–290.
- [28] FRANCA, L. P., FREY, S. L., AND HUGHES, T. J. R. Stabilized finite element methods. I. Application to the advective-diffusive model. *Comput. Methods Appl. Mech. Engrg.* 95, 2 (1992), 253–276.
- [29] HOPPE, R. H. W., AND WOHLMUTH, B. Element-oriented and edge-oriented local error estimators for nonconforming finite element methods. *RAIRO Modél. Math. Anal. Numér.* 30, 2 (1996), 237–263.
- [30] KARAKASHIAN, O. A., AND PASCAL, F. A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems. *SIAM J. Numer. Anal.* 41, 6 (2003), 2374–2399.
- [31] LAZAROV, R., AND TOMOV, S. A posteriori error estimates for finite volume element approximations of convection–diffusion–reaction equations. *Comput. Geosci.* 6, 3-4 (2002), 483–503. Locally conservative numerical methods for flow in porous media.
- [32] NICAISE, S. A posteriori error estimations of some cell-centered finite volume methods. *SIAM J. Numer. Anal.* 43, 4 (2005), 1481–1503.
- [33] OHLBERGER, M. A posteriori error estimate for finite volume approximations to singularly perturbed nonlinear convection–diffusion equations. *Numer. Math.* 87, 4 (2001), 737–761.
- [34] OHLBERGER, M. A posteriori error estimates for vertex centered finite volume approximations of convection–diffusion–reaction equations. *M2AN Math. Model. Numer. Anal.* 35, 2 (2001), 355–387.
- [35] PAYNE, L. E., AND WEINBERGER, H. F. An optimal Poincaré inequality for convex domains. *Arch. Rational Mech. Anal.* 5 (1960), 286–292 (1960).
- [36] RIVIÈRE, B., WHEELER, M. F., AND BANAS, K. Part II. Discontinuous Galerkin method applied to single phase flow in porous media. *Comput. Geosci.* 4, 4 (2000), 337–341.

- [37] ROBERTS, J. E., AND THOMAS, J.-M. Mixed and hybrid methods. In *Handbook of Numerical Analysis, Vol. II*. North-Holland, Amsterdam, 1991, pp. 523–639.
- [38] SHEWCHUK, J. R. Triangle: engineering a 2d quality mesh generator and delaunay triangulator. In *Applied Computational Geometry: Towards Geometric Engineering*. Springer-Verlag, Berlin, 1996, pp. 203–222. Lecture Notes in Computer Science, Vol. 1148.
- [39] SHEWCHUK, J. R. Delaunay refinement algorithms for triangular mesh generation. *Comput. Geom.* 22, 1-3 (2002), 21–74. 16th ACM Symposium on Computational Geometry (Hong Kong, 2000).
- [40] THOMAS, J.-M. *Sur l'analyse numérique des méthodes d'éléments finis hybrides et mixtes*. Ph.D. dissertation, Université Pierre et Marie Curie (Paris 6), May 1977.
- [41] VERFÜRTH, R. *A review of a posteriori error estimation and adaptive mesh-refinement techniques*. Teubner-Wiley, Stuttgart, 1996.
- [42] VERFÜRTH, R. A posteriori error estimators for convection–diffusion equations. *Numer. Math.* 80, 4 (1998), 641–663.
- [43] VOHRALÍK, M. On the discrete Poincaré–Friedrichs inequalities for nonconforming approximations of the Sobolev space  $H^1$ . *Numer. Funct. Anal. Optim.* 26, 7–8 (2005), 925–952.
- [44] VOHRALÍK, M. Equivalence between lowest-order mixed finite element and multi-point finite volume methods on simplicial meshes. *To appear in M2AN Math. Model. Numer. Anal.* (2006).
- [45] VOHRALÍK, M. A posteriori error estimates for lowest-order mixed finite element discretizations of convection–diffusion–reaction equations. *Submitted* (2006).