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FREE JACOBI PROCESS

N. DEMNI¹

ABSTRACT. In this paper, we define and study free Jacobi processes of parameters $\lambda > 0$ and $0 < \theta \leq 1$, as the limit of the complex version of the matrix Jacobi process already defined by Y. Doumerc. In the first part, we focus on the stationary case for which we compute the law (that does not depend on time) and derive, for $\lambda \in]0, 1]$ and $1/\theta \geq \lambda + 1$ a free SDE analogous to the classical one. In the second part, we generalize this result under an additional condition. To proceed, we set a recurrence formula for the moments of the process using free stochastic calculus. This will also be used to compute the p. d. e. satisfied by the Cauchy transform of the free Jacobi's law.

1. INTRODUCTION

Free stochastic calculus with respect to the free additive Brownian motion was introduced by Kummerer and Speicher and then was developed by Biane and Speicher (cf [5]) who gave a "free Itô formula" for polynomials and a special class of functions. They first established results for simple (piecewise constant) "bi-processes" (the prefix "bi" is due to the non-commutativity, namely the integrand can be multiplied both on the right and on the left) belonging to some norm vector space. Next, they extend continuously the results by controlling the operator norm of the stochastic integral, more precisely, they established the L^∞ version of the free BDG inequality which extends the L^p free BDG inequality already set by Pisier and Xu (cf [20]) to the case $p = \infty$. Few years later, M. Capitaine and C. Donati defined and studied free Wishart processes. A free SDE similar to the classical one (i. e. of the complex Wishart processes) is provided. Here, we will define the free Jacobi process $(J_t)_{t \geq 0}$ with parameters (λ, θ) as the limit of the complex matrix Jacobi process with parameters (p, q) when the size of the matrix goes to infinity. Then, we will derive, for suitable parameters, the free SDE verified by this process. The rest of the paper consists of 5 sections which are devoted to the following topics: in section 2, we introduce the free Jacobi process with parameters $\theta \in]0, 1]$ and $\lambda > 0$, denoted $FJP(\lambda, \theta)$. In section 3, we use the free stochastic calculus and the polar decomposition to derive a free SDE provided that J and $P - J$ remain injective. Once deriving this SDE, we will give conditions ensuring the required injectivity. Section 4 considers the stationary case: we will compute the J_t 's law using Nica and Speicher's result on the compression by a free projection (cf [19] or [22] p. 87). Section 5 deals with a more general setting and is concerned with the free Jacobi process $(J_t)_{t \geq 0}$ to which we extend our result assuming in addition that J_0 is invertible : to do this, we set a

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Key Words: Unitary Brownian matrix, free Jacobi process, free additive Brownian motion, free multiplicative Brownian motion, polar decomposition.

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recurrence formula for the Jacobi moments which enables us to get a lower bound of $\tilde{\Phi}(\log(P - J_t))$ for all $t > 0$ where P and $\tilde{\Phi}$ are respectively the unit and the state of the so-called "compressed space". Note that this free SDE involves a complex free Brownian motion which is the (normalized) limit of the complex non self-adjoint Brownian motion involved in its classical counterpart. In the last section, we compute the p. d. e. satisfied by the Cauchy transform of the free Jacobi's law. Hopelessly, no results concerning the Jacobi's law has been deduced.

2. COMPLEX MATRIX JACOBI PROCESS AND FREE JACOBI PROCESS

We refer to [14] for facts about real matrix Jacobi processes. In the remainder of this paper, we are interested in the complex case. Let Θ be a $d \times d$ unitary Brownian matrix, i. e, a unitary matrix-valued process such that :

- $\Theta_0 = I_d$
- $(\Theta_{t_i} \Theta_{t_{i-1}}^{-1}, 1 \leq i \leq n)$ are independent for any collection $0 < t_1 < \dots < t_n$,
- $\Theta_t \Theta_s^{-1}, s < t$ have the same distribution as Θ_{t-s} (cf [4]),

and $X = \Pi_{m,p} \Theta$ is the left corner of Θ , then $J := XX^*$ is a $m \times m$ complex matrix Jacobi process starting at I_m and such that $0 \leq X_t X_t^* \leq I_m$. If X_t is the $m \times p$ left corner of $\tilde{Z} \Theta_t$ where \tilde{Z} is a $d \times d$ unitary random matrix independent of Θ , then XX^* is a $m \times m$ complex matrix Jacobi process starting from $X_0 X_0^*$ where X_0 is the left corner of \tilde{Z} . Let us take a $d(m) \times d(m)$ unitary Brownian matrix $Y_t(d(m))$ and let us consider the left corner $X_t = \Pi_{m,p(m)} Y_t(d(m))$ such that :

$$\lim_{m \rightarrow \infty} \frac{m}{d(m)} = \lambda \theta, \quad \lim_{m \rightarrow \infty} \frac{m}{p(m)} = \lambda > 0$$

Then, we may write:

$$XX_t^* \oplus 0_{d(m)-m} = P_m Y_t(d(m)) Q_m Y_t^*(d(m)) P_m$$

where P_m, Q_m are two projections defined by:

$$P_m = \begin{pmatrix} I_m & \\ & 0 \end{pmatrix} \quad Q_m = \begin{pmatrix} I_{p(m)} & \\ & 0 \end{pmatrix}$$

Recall that the unitary Brownian matrix, considered as a non-commutative variable in the non-commutative space $\mathcal{A}_d := \bigcap_{p>0} L^p(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}) \otimes \mathcal{M}_d(\mathbb{C})$ equipped with the normalized trace expectation $\mathbb{E} \otimes tr_d$, converges in distribution to the free multiplicative Brownian motion (cf [4]) Y in some non commutatif space (\mathcal{A}, Φ) , that is :

$$\lim_{d \rightarrow \infty} \mathbb{E}[tr_d(Y_{t_1}(d) \dots Y_{t_n}(d))] = \Phi(Y_{t_1} \dots Y_{t_n})$$

for any collection of times t_1, \dots, t_n , which implies that

$$\lim_{n \rightarrow \infty} \mathbb{E}[tr_d(Y_t(d)^k)] = \Phi(Y_t^k), \quad k \geq 0$$

Recall also that Y is unitary, $Y_0 = I$ (the unit of \mathcal{A}), has free left increments, that is, for a collection of times $0 < t_1 < t_2 < \dots < t_n$, $Y_{t_n} Y_{t_{n-1}}^{-1}, \dots, Y_{t_2} Y_{t_1}^{-1}$ are free and that

the law ν_t of Y_t is given by its Σ -transform:

$$\Sigma_{\nu_t}(z) = e^{\frac{t}{2} \frac{1+z}{1-z}}, \quad \nu_{t+s} = \nu_t \boxtimes \nu_s$$

where \boxtimes denotes the free multiplicative convolution (cf [3], [4]). Furthermore, we have :

$$\begin{aligned} \lim_{m \rightarrow \infty} \operatorname{tr}_{d(m)}(P_m) &= \lambda\theta \\ \lim_{m \rightarrow \infty} \operatorname{tr}_{d(m)}(Q_m) &= \theta, \quad \theta \in]0, 1[\\ \lim_{m \rightarrow \infty} \mathbb{E}(\operatorname{tr}_m(X_t X_t^*)) &= \lim_{m \rightarrow \infty} \left[\frac{d(m)}{m} \mathbb{E}(\operatorname{tr}_{d(m)}(P_m Y_t(d(m)) Q_m Y_t^*(d(m)) P_m)) \right]. \end{aligned}$$

Furthermore, if $(U_t(n))_{t \geq 0}$ is a family of independent unitary random matrices such that the distribution of $U_t(n)$ is equal to that of $V U_t(n) V^*$ for any unitary matrix V (unitary invariant) and such that $U_t(n)$ converges in distribution, and if $(D_t(n))_{t \geq 0}$ is a family of constant matrices converging in distribution and such that $\sup_n \|D_t(n)\| < \infty$, then the families

$$\{U_s(n), U_s^*(n)\}_s, \{D_t(n), D_t^*(n), t \geq 0\}$$

are asymptotically free as $n \rightarrow \infty$ (cf [17] Theorem. 4. 3. 1. p. 157).

Thus, applying these results to the family $\{Y_t(d(m)) Y_s^{-1}(d(m))\}_{0 \leq s < t}$ which is unitary invariant since (Y_t) is right-left invariant, the family $\{P_m, Q_m\}$ and using increments freeness of Y mentioned above, one can see that:

$$\{(Y_t(d(m)))_{t \geq 0}, (Y_t^*(d(m)))_{t \geq 0}\}, \{P_m, Q_m\}$$

are asymptotically free, i. e, its limiting distribution in $(\mathcal{A}_{d(m)}, \mathbb{E} \otimes \operatorname{tr}_{d(m)})$ as m goes to infinity is the distribution of $\{(Y_t)_{t \geq 0}, (Y_t^*)_{t \geq 0}, P, Q\}$ in (\mathcal{A}, Φ) such that

- (1) Y is a free multiplicative Brownian motion in (\mathcal{A}, Φ) .
- (2) P is a projection with $\Phi(P) = \lambda\theta \leq 1$, $\theta \in]0, 1[$.
- (3) Q is a projection with $\Phi(Q) = \theta$.
- (4) $QP = PQ = \begin{cases} P & \text{if } \lambda \leq 1 \\ Q & \text{if } \lambda > 1 \end{cases}$
- (5) $\{(Y_t)_{t \geq 0}, (Y_t^*)_{t \geq 0}\}$ and $\{P, Q\}$ are free.

Hence, we deduce that the limiting distribution of the complex matrix Jacobi process $(J_t)_{t \geq 0}$ in $(\mathcal{A}_m, \mathbb{E} \otimes \operatorname{tr}_m)$ is the distribution of $(P Y_t Q Y_t^* P)_{t \geq 0}$ in the compressed non-commutative space $P \mathcal{A} P$ equipped with the state

$$\tilde{\Phi} = \frac{1}{\Phi(P)} \Phi|_{P \mathcal{A} P} = \frac{1}{\lambda\theta} \Phi|_{P \mathcal{A} P}.$$

Hence, we shall define the free Jacobi process as follows:

Definition 1. Let (\mathcal{A}, Φ) be a non-commutative probability space. Let $\theta \in]0, 1[$ and $\lambda > 0$ such that $\lambda\theta \leq 1$. Let P and Q be two projections such that

$$\Phi(Q) = \theta, \quad \Phi(P) = \lambda\theta, \text{ and } PQ = QP = \begin{cases} P & \text{if } \lambda \leq 1 \\ Q & \text{if } \lambda > 1 \end{cases}$$

Let Y be a free multiplicative Brownian motion such that $\{(Y_t)_{t \geq 0}, (Y_t^*)_{t \geq 0}\}$ and $\{P, Q\}$ is a free family in (\mathcal{A}, Φ) . We will say that a process J in a non-commutative space (B, Ψ) is a free Jacobi process with parameters (λ, θ) if its distribution in (B, Ψ) is equal to

the distribution of the process $(PY_t QY_t^* P)_{t \geq 0}$ in $(P \mathcal{A} P, (1/\Phi(P))\Phi|_{P \mathcal{A} P})$. This process starts at $J_0 = P$ if $\lambda \leq 1$ and $J_0 = Q$ if $\lambda > 1$.

Equivalently, the law of J is the limiting distribution of a complex matrix Jacobi process when $\frac{m}{p(m)} \xrightarrow{m \rightarrow \infty} \lambda$ and $\frac{m}{d(m)} \xrightarrow{m \rightarrow \infty} \lambda\theta$.

We also define the free Jacobi process starting at J_0 :

Definition 2. Let Y be a free multiplicative Brownian motion and Z a unitary operator free with Y . Then, the process defined by $\tilde{Y} = YZ$ is a free multiplicative Brownian motion starting at $\tilde{Y}_0 = Z$. Moreover, if Z is free with $\{P, Q\}$, then the process \tilde{J} defined by :

$$\tilde{J}_t := P\tilde{Y}_t Q\tilde{Y}_t^* P$$

is called a free Jacobi process with parameters (λ, θ) and starting at $J_0 = PZQZ^*P$.

For the sake of simplicity, we will write Y for a free multiplicative Brownian motion starting at Y_0 and J for a free Jacobi process starting at J_0 .

3. FREE JACOBI PROCESS AND FREE STOCHASTIC CALCULUS

We refer to [4] and [5] for free stochastic calculus and notations. Let $(\mathcal{A}, (\mathcal{A}_t), \Phi)$ be a filtered W^* non-commutative probability space, i. e, \mathcal{A} is a von Neumann Algebra, Φ is a faithful normal tracial state and (\mathcal{A}_t) is an increasing family of unital, weakly closed \star -subalgebras of \mathcal{A} . Recall that a bi-process $(U_t = \sum_i A_t^i \otimes B_t^i) \in \mathcal{A} \otimes \mathcal{A}^{op}$ is adapted if $U_t \in \mathcal{A}_t \otimes \mathcal{A}_t$ for all $t \geq 0$. Furthermore, adapted bi-processes form a complex vector space that we endow with the norm:

$$\|U\| = \left(\int_0^\infty \|U_s\|_{L^\infty(\mathcal{A} \otimes \mathcal{A}^{op})} ds \right)^{1/2}$$

where $L^\infty(\mathcal{A} \otimes \mathcal{A}^{op})$ is the norm defined by:

$$\|\cdot\|_{L^\infty(\mathcal{A} \otimes \mathcal{A}^{op})} = \lim_{p \rightarrow \infty} \|\cdot\|_{L^p(\mathcal{A} \otimes \mathcal{A}^{op})}$$

The completion of this space is denoted by \mathcal{B}_∞^a . Recall also that, for fixed $t > 0$ and $U \in \mathcal{B}_\infty^a$, we have:

$$\int_0^t U_s \sharp dX_s = \int_0^\infty U_s \mathbf{1}_{[0,t]}(s) \sharp dX_s$$

where X is a free additive Brownian motion and

$$U_s \sharp dX_s := \sum_i A_s^i dX_s B_s^i$$

Consider the process $(J_t := PY_t QY_t^* P)_{t \geq 0}$, where P, Q are two projections as in the definition above, Y is a free multiplicative Brownian motion in (\mathcal{A}, Φ) . Recall that Y satisfies the free SDE (cf [4]):

$$dY_t = i dX_t Y_t - \frac{1}{2} Y_t dt, \quad Y_0 \in \mathcal{A}$$

where $(X_t)_{t \geq 0}$ is a free additive Brownian motion in (\mathcal{A}, Φ) . By free Itô's formula ([4], [5], [18]), we get :

$$\begin{aligned} d(Y_t Q Y_t^*) &= (dY_t) Q Y_t^* + Y_t Q (dY_t^*) + \Phi(Y_t Q Y_t^*) dt \\ &= i dX_t Y_t Q Y_t^* - i Y_t Q Y_t^* dX_t - Y_t Q Y_t^* + \theta dt \end{aligned}$$

since X_t is self-adjoint. Thus, the free Jacobi process satisfies:

$$\begin{aligned} dJ_t &= P(dY_t Q Y_t^*) P \\ &= iP dX_t Y_t Q Y_t^* P - i P Y_t Q Y_t^* dX_t P - J_t dt + \theta P dt \\ &= iP Y_t (Y_t^* dX_t Y_t) Q Y_t^* P - iP Y_t Q (Y_t^* dX_t Y_t) Y_t^* P - J_t dt + \theta P dt, \end{aligned}$$

since Φ is tracial and Y_t is unitary (by definition). Next, let us recall the characterisation of the free Brownian motion (cf [6], [8]) which is the analogue of the classical Lévy characterisation: Let $(\mathcal{A}_s, s \in [0, 1])$ be an increasing family of von Neumann subalgebras in a non-commutative probability space (\mathcal{A}, Φ) , and let $(Z_s = (Z_s^1, \dots, Z_s^m); s \in [0, 1])$ be an m -tuple of self-adjoint (\mathcal{A}_s) -adapted processes such that :

- (1) Z is bounded and $Z_0 = 0$.
- (2) $\Phi(Z_t^i | \mathcal{A}_s) = Z_s^i$ for all $1 \leq i \leq m$.
- (3) $\Phi(|Z_t^i - Z_s^i|^4) \leq K(t-s)^2$ for some constant K and for all $1 \leq i \leq m$.
- (4) For any $l, p \in \{1, \dots, m\}$ and all $A, B \in \mathcal{A}_s$, one has:

$$\Phi(A(Z_t^p - Z_s^p)B(Z_t^l - Z_s^l)) = \mathbf{1}_{\{p=l\}} \Phi(A)\Phi(B)(t-s) + o(t-s),$$

then Z is a m -dimensional free Brownian motion. Hence, we can see that :

Lemma 1. *The process $(S_t) := (\int_0^t Y_s^* dX_s Y_s)_{t \geq 0}$ is an \mathcal{A}_t - free Brownian motion.*

Proof: Recall first that (cf [5]) for any two adapted biprocesses N and M belonging to \mathcal{B}_∞^a :

$$(1) \quad \Phi \left\{ \int_0^t N_s \# dX_s \int_0^t M_s \# dX_s \right\} = \int_0^t \langle N_s, M_s^* \rangle ds,$$

where X denotes a free Brownian motion in (\mathcal{A}, Φ) and $\langle \cdot \rangle$ is the inner product in $L^2(\mathcal{A}, \Phi) \otimes L^2(\mathcal{A}, \Phi)$ (namely, if $N = a \otimes a'$ and $M = b \otimes b'$, then $M^* = (b')^* \otimes b^*$ and $\langle N, M^* \rangle = \Phi(ab')\Phi(a'b)$). So, we have to verify the three conditions mentioned above.

Note that for all $T > 0$, $Y_t^* \otimes Y_t \mathbf{1}_{[0, T]} \in \mathcal{B}_\infty^a$ since $\|Y_t\| = \|Y_t^*\| = 1$ (Here, $\|\cdot\|$ denotes the algebra or $L^\infty(\mathcal{A}, \Phi)$ norm defined by $\|x\| = \lim_{n \rightarrow \infty} \Phi(|x|^{2n})^{1/2n}$). Take $A, B \in \mathcal{A}$, then (using (1) in the third line) :

$$\begin{aligned} \Phi(A(S_t - S_s)B(S_t - S_s)) &= \Phi \left\{ \int_s^t A Y_r^* dX_r Y_r \int_s^t B Y_r^* dX_r Y_r \right\} \\ &= \Phi \left\{ \int_s^t (A Y_r^* \otimes Y_r) \# dX_r \int_s^t (B Y_r^* \otimes Y_r) \# dX_r \right\} \\ &= \int_s^t \Phi(Y_r A Y_r^*) \Phi(Y_r B Y_r^*) dr \\ &= \Phi(A)\Phi(B)(t-s), \end{aligned}$$

since Φ is tracial and Y_t is unitary. Hence, the third condition is fulfilled. For the second one, we follow in the same way and use again the fact that Y_t is unitary to get :

$$\Phi(|S_t - S_s|^4) \leq (t - s)^2,$$

The first one follows since $\left(\int_0^t Y_s^* dX_s Y_s\right)_{t \geq 0}$ defines an \mathcal{A}_t - martingale. \square

Thus, our free SDE is written:

$$\begin{aligned} dJ_t &= iP Y_t dS_t Q Y_t^* P - iP Y_t Q dS_t Y_t^* P - J_t dt + \theta P dt \\ &= iP Y_t (1 - Q) dS_t Q Y_t^* P - iP Y_t Q dS_t (1 - Q) Y_t^* P - J_t dt + \theta P dt \end{aligned}$$

where 1 denotes the unit of \mathcal{A} . On the other hand, we can see that $P - J_t$ is positive. Indeed, since Y_t is unitary, $Q^2 = Q$ and $P^2 = P$, one has :

$$P - J_t = (P Y_t - P Y_t Q)(Y_t^* P - Q Y_t^* P) := C^* C$$

Hence, using the following polar decompositions :

$$\begin{aligned} Q Y_t^* P &= R_t \sqrt{J_t} \\ C &= (1 - Q) Y_t^* P = V_t \sqrt{P - J_t}, \end{aligned}$$

our free SDE becomes :

$$dJ_t = \sqrt{P - J_t} (i V_t^* dS_t R_t) \sqrt{J_t} + \sqrt{J_t} ((i R_t)^* dS_t V_t) \sqrt{P - J_t} + (\theta P - J_t) dt$$

Remark. From the polar decomposition, we can see that :

$$(2) \quad \sqrt{J_t} R_t^* V_t \sqrt{P - J_t} = P Y_t Q (1 - Q) Y_t^* P = 0$$

since $Q = Q^2$.

Proposition 1. *Suppose that J_t and $P - J_t$ are injective operators in $P\mathcal{A}P$. Then, the following holds :*

- $R_t P = R_t$ and $V_t P = V_t$.
- $R_t^* R_t = P$ and $V_t^* V_t = P$.
- $P R_t^* V_t P = P V_t^* R_t P = 0$.

Proof: Recall first that if T is an operator in \mathcal{A} , then the support E of T is the orthogonal projection on $(\ker T)^\perp = \overline{\text{Im } T^*}$ and verifies $TE = T$ (cf Appendix III in [13]). Furthermore, if we consider the polar decomposition of T , namely :

$$T = A|T| = A(T^*T)^{1/2},$$

then, E is also the support of A and the latter is partially isometric, that is $A^*A = E$ and $AA^* = F$ where F is the support of T^* . Thus, the two first assertions follow if we prove that P is the support of both J_t and $P - J_t$. Indeed, the injectivity of J_t in $P\mathcal{A}P$ implies that $\ker J_t = \ker P$. Thus, we claim that P is the support of J_t ($(\ker P)^\perp = \text{Im } P$) and the same result holds for $P - J_t$. For the third, when J_t and $P - J_t$ are invertible, it is obvious. Else, (2) is written in $P\mathcal{A}P$:

$$0 = \sqrt{J_t} R_t^* V_t \sqrt{P - J_t} = \sqrt{J_t} (P R_t^* V_t P) \sqrt{P - J_t}$$

Since both J_t and $P - J_t$ are injective operators in $P\mathcal{A}P$, then :

$$(P R_t^* V_t P) \sqrt{P - J_t} = 0 \Rightarrow \sqrt{P - J_t} (P V_t^* R_t P) = 0 \Rightarrow (P V_t^* R_t P) = 0$$

Corollary 1. *Under the same assumption of Proposition 1, the process $(W_t)_{t \geq 0}$ defined by $W_t := (i/\sqrt{\Phi(P)}) \int_0^t (PV_s^* \otimes R_s P) \sharp dS_s$ is a $P\mathcal{A}_t P$ -complex free Brownian motion.*

Proof: Let us first recall that a process $Z : \mathbb{R}_+ \rightarrow \mathcal{A}$ is a complex (\mathcal{A}_t) -Brownian motion if it can be written $Z = \frac{X^1 + iX^2}{\sqrt{2}}$, where (X^1, X^2) is a 2-dimensional (\mathcal{A}_t) -free Brownian motion. Note also that $(iZ_t)_{t \geq 0}$ is still a complex (\mathcal{A}_t) -Brownian motion since $(-X^2)$ is an (\mathcal{A}_t) -free Brownian motion. So, we shall show that :

$$\left(\tilde{W}_t := (1/\sqrt{\Phi(P)}) \int_0^t (PV_s^* \otimes R_s P) \sharp dS_s \right)_{t \geq 0}$$

is a $P\mathcal{A}_t P$ -complex free Brownian motion. To do this, it suffices to show that :

$$\begin{aligned} X_t^1 &= \frac{\tilde{W}_t + \tilde{W}_t^*}{\sqrt{2}} = \frac{1}{\sqrt{2\Phi(P)}} \left(\int_0^t (PV_s^* \otimes R_s P) \sharp dS_s + \int_0^t (PR_s^* \otimes V_s P) \sharp dS_s \right) \\ X_t^2 &= \frac{\tilde{W}_t - \tilde{W}_t^*}{\sqrt{2}i} = \frac{1}{\sqrt{2\Phi(P)}i} \left(\int_0^t (PV_s^* \otimes R_s P) \sharp dS_s - \int_0^t (PR_s^* \otimes V_s P) \sharp dS_s \right) \end{aligned}$$

define two free (\mathcal{A}_t) -free Brownian motions using the characterisation of free Brownian motion. We will do this for X_1 . Note that, since R_t and V_t are partially isometric, then:

$$(PV_t^* \otimes R_t P \mathbf{1}_{[0,T]})_{t \geq 0} \quad \text{and} \quad (PR_t^* \otimes V_t P \mathbf{1}_{[0,T]})_{t \geq 0} \in B_\infty^a \quad \forall T > 0$$

Hence, the first condition follows since $\left(\int_0^t (PV_s^* \otimes R_s P) \sharp dS_s \right)_{t \geq 0}$ and $\left(\int_0^t (PR_s^* \otimes V_s P) \sharp dS_s \right)_{t \geq 0}$ are $P\mathcal{A}_t P$ -martingales. For $A, B \in \mathcal{A}_s$ and using (1), one has :

$$\begin{aligned} &\tilde{\Phi}(PAP(X_t^1 - X_s^1)PBP(X_t^1 - X_s^1)) = \\ &\frac{1}{2\Phi(P)} \tilde{\Phi} \left(\int_s^t (PAPV_u^* \otimes R_u P) \sharp dS_u + \int_s^t (PAPR_u^* \otimes V_u P) \sharp dS_u \right) \\ &\left(\int_s^t (PBPV_u^* \otimes R_u P) \sharp dS_u + \int_s^t (PBPR_u^* \otimes V_u P) \sharp dS_u \right) \\ &= \frac{1}{2\Phi^2(P)} \Phi \left(\int_s^t (PAPV_u^* \otimes R_u P) \sharp dS_u + \int_s^t (PAPR_u^* \otimes V_u P) \sharp dS_u \right) \\ &\left(\int_s^t (PBPV_u^* \otimes R_u P) \sharp dS_u + \int_s^t (PBPR_u^* \otimes V_u P) \sharp dS_u \right) \\ &= \frac{1}{2} \int_s^t [\tilde{\Phi}(PAPV_u^* V_u P) \tilde{\Phi}(R_u PBP R_u^*) + \tilde{\Phi}(PAPR_u^* R_u P) \tilde{\Phi}(V_u PBP V_u^*)] du \\ &+ \frac{1}{2} \int_s^t [\tilde{\Phi}(PAPV_u^* R_u P) \tilde{\Phi}(R_u PBP V_u^*) + \tilde{\Phi}(PAPR_u^* V_u P) \tilde{\Phi}(V_u PBP R_u^*)] du \\ &= \frac{1}{2} \int_s^t [\tilde{\Phi}(PAP) \tilde{\Phi}(PBP) + \tilde{\Phi}(PAP) \tilde{\Phi}(PBP)] du + \\ &\frac{1}{2} \int_s^t [\tilde{\Phi}(APV_u^* R_u P) \tilde{\Phi}(BPV_u^* R_u P) + \tilde{\Phi}(APR_u^* V_u P) \tilde{\Phi}(BPR_u^* V_u P)] du \\ &= \tilde{\Phi}(PAP) \tilde{\Phi}(PBP)(t - s), \end{aligned}$$

since $PV_u^*R_uP = PR_u^*V_uP = 0$ and since R_u and V_u are partially isometric (Proposition 1). Using the same arguments, we can see that the same result holds for X^2 . Furthermore, one has :

$$\begin{aligned}
& \tilde{\Phi}(PAP(X_t^1 - X_s^1)PBP(X_t^2 - X_s^2)) = \\
& \frac{1}{2\tilde{\Phi}(P)} \tilde{\Phi} \left(\int_s^t (PAPV_u^* \otimes R_uP) \sharp dS_u + \int_s^t (PAPR_u^* \otimes V_uP) \sharp dS_u \right) \\
& \left(\int_s^t (PBPV_u^* \otimes R_uP) \sharp dS_u - \int_s^t (PBPR_u^* \otimes V_uP) \sharp dS_u \right) \\
& = \frac{1}{2} \int_s^t [-\tilde{\Phi}(PAPV_u^*V_uP)\tilde{\Phi}(R_uPBP R_u^*) + \tilde{\Phi}(PAPR_u^*R_uP)\tilde{\Phi}(V_uPBP V_u^*)] du \\
& + \frac{1}{2} \int_s^t [\tilde{\Phi}(PAPV_u^*R_uP)\tilde{\Phi}(R_uPBP V_u^*) - \tilde{\Phi}(PAPR_u^*V_uP)\tilde{\Phi}(V_uPBP R_u^*)] du \\
& = \frac{1}{2} \int_s^t [-\tilde{\Phi}(PAP)\tilde{\Phi}(PBP) + \tilde{\Phi}(PAP)\tilde{\Phi}(PBP)] du + \\
& \frac{1}{2} \int_s^t [\tilde{\Phi}(APV_u^*R_uP)\tilde{\Phi}(BPV_u^*R_uP) - \tilde{\Phi}(APR_u^*V_uP)\tilde{\Phi}(BPR_u^*V_uP)] du = 0
\end{aligned}$$

which finishes the proof. Substituting R_t and V_t by R_tP and V_tP and using Corollary (1), we get :

Theorem 1. *Let J_0 such that J_0 and $P - J_0$ are injective operators in $P\mathcal{A}P$ and let $T := \inf\{s, \ker(J_s) \neq \ker P \text{ or } \ker(P - J_s) \neq \ker P\}$. Then, for all $t < T$, we have*

$$(3) \quad \begin{cases} dJ_t &= \sqrt{\lambda\theta}\sqrt{P - J_t}dW_t\sqrt{J_t} + \sqrt{\lambda\theta}\sqrt{J_t}dW_t^*\sqrt{P - J_t} + (\theta P - J_t) dt \\ J_0 &= PY_0QY_0^*P \end{cases}$$

where $(W_t)_{t \geq 0}$ is a $P\mathcal{A}P$ -complex free Brownian motion.

In the remainder of this paper, we will try to find the range of (λ, θ) ensuring the injectivity of both J_t and $P - J_t$. We first investigate the stationary case then deal with the general setting.

4. FREE JACOBI PROCESS : THE STATIONARY CASE

In this section, we will give some interest in the particular case when Y_0 is Haar distributed, that is $\Phi(Y_0^k) = \delta_{k0}$. Then Y_t remains Haar distributed for all $t > 0$. Thus, the law of J_t does not depend on time and such a process is called a stationary free Jacobi process. We will compute the law of J_t and try to find values of λ and θ such that there is no Dirac mass at 0 and 1, i. e, neither 0, nor 1 belongs to the discrete spectrum of J_t .

Our method differs from the ones used by both Capitaine and Casalis (cf [8]) and Collins ($P = Q$, cf [11]). It relies on Nica and Speicher's result concerning the compression by free projections. Namely, authors considered PaP for any operator $a \in \mathcal{A}$ free with P (cf [19], [22]). This condition is fulfilled for $a = Y_tQY_t^*$ since Y_t is Haar unitary. In fact, the following classical result holds (cf [17]):

Lemma 2. *If U is Haar unitary and \mathcal{B} is a sub-algebra which is free with U , then, $\forall A, B \in \mathcal{B}$, A and UBU^* are free.*

Thus, denoting the law of a in (\mathcal{A}, Φ) by μ_a , the law of J_t in $(P\mathcal{A}P, \tilde{\Phi})$ is given by (cf [22]):

$$\mu = \boxplus^r \mu_{\alpha a}$$

where $\Phi(P) = \alpha$, $r = 1/\alpha$ and \boxplus denotes the free additive convolution. In our case, $\alpha = \lambda\theta$ and $a = Y_t Q Y_t^*$ so that for all $k \geq 1$, $\Phi(a^k) = \Phi(Q) = \theta$ since Φ is tracial and Q is a projection. Thus,

$$\mu_a = (1 - \theta)\delta_0 + \theta\delta_1$$

Next, recall that the R-transform linearize the free additive convolution, so :

$$R(z) = rR_{\alpha a}(z) = R_a(\alpha z)$$

where the last equality follows from the definition of the R-transform and the multilinearity of free cumulants. Furthermore,

$$G_a(z) = \frac{1}{z} + \sum_{k \geq 1} \frac{\Phi(a^k)}{z^{k+1}} = \frac{z + \theta - 1}{z(z - 1)},$$

Hence,

$$K_a(z) = \frac{z + 1 + \sqrt{(z - 1)^2 + 4\theta z}}{2z},$$

and finally,

$$R_a(z) = K_a(z) - \frac{1}{z} = \frac{z - 1 + \sqrt{(z - 1)^2 + 4\theta z}}{2z}.$$

Consequently, we get :

$$\begin{aligned} R(z) &= \frac{\alpha z - 1 + \sqrt{(\alpha z - 1)^2 + 4\theta \alpha z}}{2\alpha z} \\ &= \frac{z - r + \sqrt{(z - r)^2 + 4z/\lambda}}{2z}, \end{aligned}$$

which implies that :

$$K(z) = \frac{z + (2 - r) + \sqrt{(z - r)^2 + 4z/\lambda}}{2z}.$$

Thus,

$$G(z) = \frac{(2 - r)z + (1/\lambda - 1) + \sqrt{Az^2 - Bz + C}}{2z(z - 1)},$$

where $A = r^2$, $B = 2(r + (r - 2)/\lambda)$ et $C = (1 - 1/\lambda)^2$. Hence, the J_t 's law is given by:

$$\mu(dx) = a_0\delta_0(dx) + a_1\delta_1(dx) + g(x)dx,$$

where

$$\begin{aligned} a_0 &= \lim_{y \rightarrow 0^+} -y\Im[G(iy)], & a_1 &= \lim_{y \rightarrow 0^+} -y\Im[G(1 + iy)] \\ g(x) &= \lim_{y \rightarrow 0^+} -\frac{1}{\pi}\Im[G(x + iy)] & \text{for some } x \in (0, 1), \end{aligned}$$

Remark. The third equality holds whenever $\lim_{z \in D \rightarrow x} \Im(G(z)) = \Im(G(x))$ where D is the upper half-plane and $x \in \mathbb{R}$ (cf [10]). In fact, if we denote by

$$G(z) = \int \frac{dF(\zeta)}{z - \zeta}$$

the Cauchy transform of a distribution function F , then the following inversion formula holds (cf [17], [22]):

$$F(b) - F(a) = \lim_{y \rightarrow 0^+} -\frac{1}{\pi} \int_a^b \Im(G(x + iy)) dx$$

for any two continuity points a, b of F (weak convergence). Silverstein and Choi showed that if the limit above exists, then F is differentiable and dF has the density function with respect to the Lebesgue measure given by $F'(x) = \lim_{y \rightarrow 0^+} - (1/\pi) \Im(G(x + iy))$. See [10] for more details.

From [9], we deduce that :

$$\mu_{l,k}(dx) = \max(0, 1 - l) \delta_0(dx) + \max(0, 1 - k) \delta_1(dx) + g(x) \mathbf{1}_{[x_-, x_+]} dx$$

where $l = 1/\lambda, k = (1 - \theta/(\lambda\theta))$ and

$$\begin{aligned} x_{\pm} &= \left(\sqrt{\theta(1 - \lambda\theta)} \pm \sqrt{\lambda\theta(1 - \theta)} \right)^2 \\ g(x) &= \frac{\sqrt{(x - x_-)(x_+ - x)}}{2\lambda\theta\pi x(1 - x)} \end{aligned}$$

Note that conditions $\lambda \in [0, 1], 1/\theta \geq \lambda + 1$ implies that $l \geq 1, k \geq 1$ so that $a_0 = b_0 = 0$. Hence, for $\lambda = 1, x_- = 0$ and if $\theta = 1/\lambda + 1$, then $x_+ = 1$. Consequently, we have :

Proposition 2. $\forall \lambda \in]0, 1], 1/\theta \geq \lambda + 1$ ($\theta \in]0, 1/2]$ for instance), J_t and $P - J_t$ are injective operators in the compressed space $P\mathcal{A}P$. For $\lambda \in]0, 1[$ and $1/\theta > \lambda + 1$, these operators are invertible in $P\mathcal{A}P$. Moreover, J_t is a solution of (3).

Remarks. 1/In [9], authors omit the normalizing constant $\sqrt{A}/2\pi = (2\pi\lambda\theta)^{-1}$, however one can compute it as follows : since

$$\frac{\sqrt{(x_+ - x)(x - x_-)}}{x(1 - x)} = \frac{\sqrt{(x_+ - x)(x - x_-)}}{x} + \frac{\sqrt{(x_+ - x)(x - x_-)}}{1 - x}$$

Then , the normalizing constant is given by :

$$\begin{aligned} K^{-1} &= \int_{x_-}^{x_+} \frac{\sqrt{(x_+ - x)(x - x_-)}}{x} dx + \int_{1-x_+}^{1-x_-} \frac{\sqrt{(1 - x_- - x)(x - 1 + x_+)}}{x} dx \\ &:= I(x_-, x_+) + I(1 - x_+, 1 - x_-) \end{aligned}$$

Moreover, using the variable change $u = (x - x_-)/(x_+ - x_-)$ and the integral representation of ${}_2F_1$ (cf [16]), one has :

$$\begin{aligned} I(x_-, x_+) &= \frac{(x_+ - x_-)^2}{x_+} B\left(\frac{3}{2}, \frac{3}{2}\right) {}_2F_1\left(1, \frac{3}{2}, 3, \frac{x_+ - x_-}{x_+}\right) \\ &\stackrel{(1)}{=} \frac{\pi}{4} \frac{(x_+ - x_-)^2}{x_+ + x_-} {}_2F_1\left(\frac{1}{2}, 1, 2, \left(\frac{x_+ - x_-}{x_+ + x_-}\right)^2\right) \\ &\stackrel{(2)}{=} \frac{\pi}{2} (x_+ + x_- - 2\sqrt{x_+ x_-}) \end{aligned}$$

where in (1) we used (cf [16], p. 1070) :

$${}_2F_1(a, b, 2b, z) = (1 - z/2)^{-a} {}_2F_1(a/2, (a+1)/2, b+1/2; (z/2 - z)^2), \quad 0 < |z| < 1,$$

and in (2), we used

$${}_2F_1\left(\frac{1}{2}, 1, 2; z\right) = 2 \frac{1 - \sqrt{1-z}}{z}, \quad 0 < |z| < 1.$$

As a result, $K^{-1} = \pi(1 - \sqrt{x_+ x_-} - \sqrt{(1-x_+)(1-x_-)}) = 2\pi\lambda\theta$. In the same way, setting $m_n(t) = \tilde{\Phi}(J_t^n), \forall n \geq 2$, one can see:

$$\begin{aligned} m_n - m_{n+1} &= K \int_{x_-}^{x_+} x^{n-1} \sqrt{(x_+ - x)(x - x_-)} dx \\ &= \frac{K\pi(x_+ - x_-)^2}{8} x^{n-1} {}_2F_1\left(1 - n, 3/2, 3; \frac{x_+ - x_-}{x_+}\right) \\ m_1 - m_2 &= \frac{K\pi}{8} (x_+ - x_-)^2, \quad 1 - m_1 = KI(x_-, x_+). \end{aligned}$$

Thus,

$$1 - m_n = \frac{K\pi}{8} (x_+ - x_-)^2 \left[1 + \frac{1}{x_+} \left(1 + \sum_{\substack{k=0 \\ k \neq 1}}^{n-1} x_+^k {}_2F_1\left(1 - k, 3/2, 3; \frac{x_+ - x_-}{x_+}\right) \right) \right]$$

One can also compute m_n by expanding $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k$ or use the integral representation of the Appell function F_1 (see ch. I in [15]).

2/ One can compute $\tilde{\Phi}(\log(P - J_t))$ as follows : let $0 \leq z \leq 1$ and $\lambda \in]0, 1[$, $1/\theta > \lambda + 1$. Then, one can see that:

$$-\frac{d}{dz} \tilde{\Phi}(\log(P - zJ_t)) = -\frac{1}{z} \left(\frac{1}{z} G\left(\frac{1}{z}\right) - 1 \right) = \frac{(1 + 1/\lambda)z - r + \sqrt{Cz^2 - Bz + A}}{2z(1-z)}$$

Note that this derivative is well defined for $z = 0$ and $z = 1$. It follows that :

$$(4) \quad 2\tilde{\Phi}(\log(P - J_t)) = - \int_0^1 \frac{(1 + 1/\lambda)z - r + \sqrt{Cz^2 - Bz + A}}{z(1-z)} dz$$

Note first that $Cz^2 - Bz + A > 0 \forall \lambda \in]0, 1[$, $1/\theta > \lambda + 1$ since $x_+, x_- \in]0, 1[$ are the roots of $Az^2 - Bz + C$ (so that $z < 1/x_+$). In order to evaluate the integral in the right, we use the variable change $\sqrt{A}(1 - uz) = \sqrt{Cz^2 - Bz + A}$, which gives :

$$z = \frac{2Au - B}{Au^2 - C}, \quad 1 - z = \frac{Au^2 - 2Au + B - C}{Au^2 - C}, \quad dz = -2A \frac{Au^2 - Bu + C}{(Au^2 - C)^2} du.$$

Moreover, since $A - B + C = A(1 - \theta(\lambda + 1))^2 \geq 0$ and $\theta(1 + \lambda) \leq 1$, then the roots of $Au^2 - 2Au + B - C = 0$ are given by : $u_{\pm} = 1 \pm (1 - \theta(\lambda + 1))$. On the other hand, $B/2A = (1/2)(x_+ + x_-) = \theta(\lambda + 1 - 2\lambda\theta)$. Hence our expression factorizes to :

$$\begin{aligned} \tilde{\Phi}(\log(P - J_t)) &= -\frac{1}{\sqrt{A}} \int_{B/2A}^{\theta(\lambda+1)} \frac{(u - \theta(\lambda + 1))(Au^2 - Bu + C)}{(u^2 - C/A)(Au^2 - 2Au + B - C)} du \\ &= -\frac{1}{\sqrt{A}} \int_{B/2A}^{\theta(\lambda+1)} \frac{(Au^2 - Bu + C)}{(u^2 - C/A)(u - u_+)} du \\ &= \int_{B/2A}^{\theta(\lambda+1)} \frac{C_1}{u - \theta(1 - \lambda)} + \frac{C_2}{u + \theta(1 - \lambda)} + \frac{C_3}{u - u_+} du \end{aligned}$$

for some constants C_1, C_2, C_3 depending on both λ, θ , given by :

$$C_1 = 1, \quad C_2 = 1/\lambda, \quad C_3 = \frac{1 - \theta(\lambda + 1)}{\lambda\theta}$$

Thus, one gets :

$$\begin{aligned} \tilde{\Phi}(\log(P - J_t)) &= -[C_1 \log(u - \theta(1 - \lambda)) + C_2 \log(u + \theta(1 - \lambda)) + C_3 \log(u_+ - u)]_{B/2A}^{\theta(\lambda+1)} \\ &= \log(1 - \theta) + \frac{1}{\lambda} \log(1 - \lambda\theta) - C_3 \log \left[\frac{(1 - \theta(\lambda + 1))}{1 - \theta(\lambda + 1) + \lambda\theta^2} \right] \\ &= (1 + C_3) \log(1 - \theta) + \left(\frac{1}{\lambda} + C_3 \right) \log(1 - \lambda\theta) - C_3 \log(1 - \theta(\lambda + 1)) \\ &= \frac{(1 - \theta) \log(1 - \theta) + (1 - \lambda\theta) \log(1 - \lambda\theta) - (1 - \theta(\lambda + 1)) \log(1 - \theta(\lambda + 1))}{\lambda\theta} \end{aligned}$$

Note that the result extends for all $\lambda \in]0, 1[$, $1/\theta \geq \lambda + 1$. Since $P - J$ is still a $FJP(\lambda\theta/(1 - \theta), 1 - \theta)$, then :

$$\tilde{\Phi}(\log(J_t)) = \frac{\theta \log \theta + (1 - \lambda\theta) \log(1 - \lambda\theta) - \theta(1 - \lambda) \log(\theta(1 - \lambda))}{\lambda\theta}$$

which agrees with Rouault's result (cf [21]).

3/ In [14], Doumerc finds that for $p(m) \geq m + 1$ and $q(m) \geq m + 1$ where $q(m) = d(m) - p(m)$, the Jacobi process satisfies the following SDE:

$$dJ_t = \sqrt{I_m - J_t} dB_t \sqrt{J_t} + \sqrt{J_t} dB_t^T \sqrt{I_m - J_t} + (p(m)I_m - (p(m) + q(m))J_t) dt$$

where (B_t) is a real $m \times m$ Brownian matrix. The complex version of this process verifies:

$$dJ_t = \sqrt{I_m - J_t} dB_t \sqrt{J_t} + \sqrt{J_t} dB_t^* \sqrt{I_m - J_t} + (p(m)I_m - (p(m) + q(m))J_t) dt$$

where (B_t) is a complex $m \times m$ Brownian matrix. Heuristically, if we consider the ratio $dJ_t/(d(m))$ and let m go to infinity, then this SDE converges weakly (up to a constant) to its free counterpart, since normalized complex Brownian matrix converges in distribution

to the free complex Brownian motion. It is also worthnoting that conditions $p(m) \geq m+1$ and $q(m) \geq m+1$ are in agreement with $\lambda \in [0, 1[$ and $1/\theta \geq \lambda + 1$.

4/ From a combinatorial point of view, the result of Nica and Speicher reads :

$$\Phi(J_t^n) = \sum_{\pi \in NC(n)} k_\pi(P, \dots, P) \theta^{n+1-|\pi|}$$

where $NC(n)$ denotes the set of non-crossing partitions of $\{1, \dots, n\}$, $|\pi|$ is the cardinality of π and k_π is the corresponding mixed cumulant (see [22] for more details).

5. FREE JACOBI PROCESS : THE GENERAL SETTING.

In this section, we will suppose that $\lambda \leq 1$ and $1/\theta \geq \lambda + 1$. Let J_0 in $P\mathcal{A}P$ such that $0 < J_0 < P$, means J_0 and $P - J_0$ are invertible in $P\mathcal{A}P$. Then, by continuity of paths, the result of theorem 1 holds for $t < T$, that is :

$$\begin{cases} dJ_t = U_t \sharp dX_t + V_t \sharp dY_t + (\theta P - J_t) dt \\ J_0 = PY_0 QY_0^* P, \quad 0 < J_0 < P \end{cases}$$

where

$$\begin{aligned} W_t &= \frac{X_t + iY_t}{\sqrt{2}} \\ U_t &= \sqrt{\frac{\lambda\theta}{2}} (\sqrt{P - J_t} \otimes \sqrt{J_t} + \sqrt{J_t} \otimes \sqrt{P - J_t}) = \sum_{i=1}^2 A_t^i \otimes B_t^i \\ V_t &= i\sqrt{\frac{\lambda\theta}{2}} (\sqrt{P - J_t} \otimes \sqrt{J_t} - \sqrt{J_t} \otimes \sqrt{P - J_t}) = \sum_{i=1}^2 C_t^i \otimes D_t^i \end{aligned}$$

and X and Y are two free $P\mathcal{A}P$ -free-Brownian motions. Now, let us recall that for any operator Z , we set (cf [5]):

$$\begin{aligned} \partial Z^n &= \sum_{k=0}^{n-1} Z^k \otimes Z^{n-k-1} \\ \Delta_U(Z^n) &= 2 \sum_{i,j} Z^k A_t^i B_t^j Z^{n-k-l-2} \tilde{\Phi}(B_t^i Z^l A_t^j) \end{aligned}$$

where $U = \sum_i A^i \otimes B^i$ is an adapted bi-process.

Proposition 3. *Let X, Y be two free free-Brownian motions, U, V be two adapted integrable bi-processes and K an adapted process in $P\mathcal{A}P$. Let*

$$dM_t = U_t \sharp dX_t + V_t \sharp dY_t + K_t dt$$

then, for every polynomial R , we have :

$$\begin{aligned} dR(M_t) &= (\partial R(M_t) \sharp U_t) \sharp dX_t + (\partial R(M_t) \sharp V_t) \sharp dY_t + \partial R(M_t) \sharp K_t dt \\ &+ \frac{1}{2} (\Delta_U R(M_t) + \Delta_V R(M_t)) dt \end{aligned}$$

Proof: When $V = 0 \otimes 0$, this is the free Itô's formula stated in [5]. By linearity, it suffices to prove the formula for monomials. To do this, we shall proceed by induction. Hence, we assume that:

$$\begin{aligned} dM_t^n &= \partial M_t^n \sharp (U_t \sharp dX_t) + \partial M_t^n \sharp (V_t \sharp dY_t) + \partial M_t^n \sharp K_t dt + \\ &\quad \frac{1}{2} (\Delta_U M_t^n + \Delta_V M_t^n) dt \end{aligned}$$

By free integration by parts formula ([4]), we have:

$$\begin{aligned} dM_t^{n+1} &= d(M_t M_t^n) = dM_t M_t^n + M_t dM_t^n + (dM_t)(dM_t^n) \\ &= ((1 \otimes M_t^n + M_t \partial M_t^n) \sharp U_t) \sharp dX_t + ((1 \otimes M_t^n + M_t \partial M_t^n) \sharp V_t) \sharp dY_t \\ &\quad + (1 \otimes M_t + M_t \partial M_t^n) \sharp K_t dt + \frac{1}{2} M_t (\Delta_U M_t^n + \Delta_V M_t^n) dt + (dM_t)(dM_t^n) \end{aligned}$$

On the other hand, we can easily see that :

$$1 \otimes M_t^n + M_t \partial M_t^n = 1 \otimes M_t^n + \sum_{k=1}^n M_t^k \otimes M_t^{n-k} = \partial M_t^{n+1},$$

Then, using the fact that $(dX)(dY) = 0$ by the freeness of X and Y , we get :

$$(dM_t)(dM_t^n) = \sum_{i,j} \sum_{l=0}^{n-1} A_t^i B_t^j M_t^{n-l-1} \tilde{\Phi}(B_t^i M_t^l A_t^j) + \sum_{i,j} \sum_{l=0}^{n-1} C_t^i D_t^j M_t^{n-l-1} \tilde{\Phi}(D_t^i M_t^l C_t^j)$$

Moreover,

$$\begin{aligned} M_t \Delta_U(M_t^n) &= 2 \sum_{i,j} \sum_{\substack{k,l \geq 0 \\ k+l \leq n-2}} M_t^{k+1} A_t^i B_t^j M_t^{n-k-l-2} \tilde{\Phi}(B_t^i M_t^l A_t^j) \\ &= 2 \sum_{i,j} \sum_{l=0}^{n-2} \sum_{k=1}^{n-l-1} M_t^k A_t^i B_t^j M_t^{n-k-l-1} \tilde{\Phi}(B_t^i M_t^l A_t^j) \end{aligned}$$

and the same holds for

$$M_t \Delta_V(M_t^n) = 2 \sum_{i,j} \sum_{l=0}^{n-2} \sum_{k=1}^{n-l-1} M_t^k C_t^i D_t^j M_t^{n-k-l-1} \tilde{\Phi}(C_t^i M_t^l D_t^j)$$

Consequently, we get :

$$\frac{1}{2} M_t (\Delta_V(M_t^n) + \Delta_U(M_t^n)) + (dM_t)(dM_t^n) = \frac{1}{2} (\Delta_V(M_t^{n+1}) + \Delta_U(M_t^{n+1}))$$

which completes the proof. ■

5.1. A recurrence formula for free Jacobi moments.

Corollary 2. *Let $m_n(t) := \tilde{\Phi}(J_t^n)$ for $n \geq 2$ and $t < T$. Then, we have the following recurrence relation:*

$$m_n(t) = m_n(0) - n \int_0^t m_n(s) ds + n\theta \int_0^t m_{n-1}(s) ds + \lambda\theta n \sum_{k=0}^{n-2} \int_0^t m_{n-k-1}(s) (m_k(s) - m_{k+1}(s)) ds$$

or equivalently,

$$\frac{dm_n(t)}{dt} = -nm_n(t) + n\theta m_{n-1}(t) + \lambda\theta n \sum_{k=0}^{n-2} m_{n-k-1}(t)(m_k(t) - m_{k+1}(t))$$

Proof: Using Proposition (3), we get:

$$dJ_t^n = \text{martingale} + \sum_{k=0}^{n-1} J_t^k (\theta P - J_t) J_t^{n-k-1} dt + \frac{1}{2} (\Delta_U(J_t^n) + \Delta_V(J_t^n)) dt$$

Next, we compute

$$\begin{aligned} \Delta_U(J_t^n) &= 2 \sum_{i,j=1}^2 \sum_{\substack{k,l \geq 0 \\ k+l \leq n-2}} J_t^k A_t^i B_t^j J_t^{n-k-l-2} \tilde{\Phi}(B_t^i J_t^l A_t^j) \\ &\stackrel{(1)}{=} 2 \sum_{i,j=1}^2 \sum_{\substack{k,l \geq 0 \\ k+l \leq n-2}} A_t^i B_t^j J_t^{n-l-2} \tilde{\Phi}(B_t^i J_t^l A_t^j) \\ &= 2 \sum_{i,j=1}^2 \sum_{l=0}^{n-2} (n-l-1) A_t^i B_t^j J_t^{n-l-2} \tilde{\Phi}(B_t^i J_t^l A_t^j) \\ &= 2 \sum_{i,j=1}^2 \sum_{l=0}^{n-2} (l+1) A_t^i B_t^j J_t^l \tilde{\Phi}(B_t^i J_t^{n-l-2} A_t^j) \\ &\stackrel{(2)}{=} 2 \sum_{l=0}^{n-2} (l+1) \left[\frac{\lambda\theta}{2} (P - J_t) J_t^l \tilde{\Phi}(J_t^{n-l-1}) + \frac{\lambda\theta}{2} J_t^{l+1} \tilde{\Phi}(J_t^{n-l-2} (P - J_t)) \right] \\ &\quad + \lambda\theta \sqrt{P - J_t} \sqrt{J_t} J_t^l \tilde{\Phi}(J_t^{n-l-2} \sqrt{J_t} \sqrt{P - J_t}) \end{aligned}$$

where in both (1) and (2), we used the fact that A_t^i, B_t^j and J_t commute $\forall i, j \in \{1, 2\}$. Similarly, we get :

$$\begin{aligned} \Delta_V(J_t^n) &= 2 \sum_{l=0}^{n-2} (l+1) \left[\frac{\lambda\theta}{2} (P - J_t) J_t^l \tilde{\Phi}(J_t^{n-l-1}) + \frac{\lambda\theta}{2} J_t^{l+1} \tilde{\Phi}(J_t^{n-l-2} (P - J_t)) \right] \\ &\quad - \lambda\theta \sqrt{P - J_t} \sqrt{J_t} J_t^l \tilde{\Phi}(J_t^{n-l-2} \sqrt{J_t} \sqrt{P - J_t}) \end{aligned}$$

Thus, we have:

$$\frac{1}{2} (\Delta_U(J_t^n) + \Delta_V(J_t^n)) = \lambda\theta \sum_{l=0}^{n-2} (l+1) \left[(P - J_t) J_t^l \tilde{\Phi}(J_t^{n-l-1}) + J_t^{l+1} \tilde{\Phi}(J_t^{n-l-2} (P - J_t)) \right]$$

Taking the expectation, it yields

$$\begin{aligned}
\frac{1}{2}\tilde{\Phi}(\Delta_U(J_t^n) + \Delta_V(J_t^n)) &= \lambda\theta \times \\
&\left(\sum_{l=0}^{n-2} (l+1)[\tilde{\Phi}((P - J_t)J_t^l)\tilde{\Phi}(J_t^{n-l-1})] + \sum_{l=0}^{n-2} (l+1)[\tilde{\Phi}(J_t^{l+1})\tilde{\Phi}(J_t^{n-l-2}(P - J_t))] \right) \\
&= \lambda\theta \times \left(\sum_{l=0}^{n-2} (l+1)[\tilde{\Phi}((P - J_t)J_t^l)\tilde{\Phi}(J_t^{n-l-1})] + \sum_{l=0}^{n-2} (n-l-1)[\tilde{\Phi}(J_t^{n-l-1})\tilde{\Phi}(J_t^l(P - J_t))] \right) \\
&= n\lambda\theta \sum_{l=0}^{n-2} [\tilde{\Phi}((P - J_t)J_t^l)\tilde{\Phi}(J_t^{n-l-1})] \\
&= n\lambda\theta \sum_{l=0}^{n-2} [m_{n-l-1}(t)(m_l(t) - m_{l+1}(t))]
\end{aligned}$$

Furthermore, since $P - J_t$ and J_t commute, we can easily see that :

$$\tilde{\Phi} \left(\sum_{k=0}^{n-1} J_t^k (\theta P - J_t) J_t^{n-k-1} \right) = n\tilde{\Phi}(J_t^{n-1}(\theta P - J_t))$$

and the proof is complete. \square

Remarks. 1/ In the last proof, we intertwined integrals and expectation since powers of J_t are still positive self-adjoint operators.

2/ For $n = 1$, we obviously have:

$$\tilde{\Phi}(J_t) = \tilde{\Phi}(J_0) + \theta t - \int_0^t \tilde{\Phi}(J_s) ds$$

which implies that : $m_1(t) = (\tilde{\Phi}(J_0) - \theta)e^{-t} + \theta$. We can also deduce this result from the definition of the free Jacobi process, using moments of Y_t (cf [4]) and the following formula :

$$\tau(a_1 b_1 a_2 b_2) = \tau(a_1 a_2) \tau(b_1) \tau(b_2) + \tau(b_1 b_2) \tau(a_1) \tau(a_2) - \tau(a_1) \tau(a_2) \tau(b_1) \tau(b_2),$$

where $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are free families in some non commutative probability space (\mathcal{B}, τ) .

3/ In the stationary case and for $\lambda = 1$, this formula reads :

$$(1 - \theta)m_n = \theta \sum_{k=0}^{n-1} m_{n-k-1} (m_k - m_{k+1})$$

For $\theta = 1/2$, one has :

$$m_n = \sum_{k=0}^{n-1} m_{n-k-1} (m_k - m_{k+1})$$

since $m_0 = 1$. Note also that J_t is a Beta random variable $B(1/2, 1/2)$, it follows that:

$$\frac{\Gamma(n + 1/2)}{n!} = m_n = \frac{1}{2\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{\Gamma(n - k - 1/2)\Gamma(k + 1/2)}{(n - k - 1)!(k + 1)!} = \sum_{k=1}^n \frac{\Gamma(n - k + 1/2)\Gamma(k - 1/2)}{2\sqrt{\pi}(n - k)!k!}$$

Using duplication formula, we get :

$$\begin{aligned} \frac{(2n)!}{(n!)^2} &= 2 \sum_{k=1}^n \frac{(2n-2k)!(2k-2)!}{((n-k)!)^2 k!(k-1)!} \\ &= \sum_{k=1}^n \frac{(2n-2k)!(2k-2)!}{((n-k)!)^2 k!(k-1)!} + \sum_{k=1}^n \frac{(2n-2k)!(2k-2)!}{(n-k)!(n-k+1)!((k-1)!)^2} \\ &= (n+1) \sum_{k=1}^n \frac{(2n-2k)}{(n-k+1)((n-k)!)^2} \frac{(2k-2)!}{k((k-1)!)^2} \end{aligned}$$

Recall that the Catalan numbers are given by : $D_p = 1/(p+1) \binom{2p}{p}$. Thus, our recurrence formula is equivalent to $D_n = \sum_{k=1}^n D_{n-k} D_{k-1}$.

Proposition 4. *If J_0 is invertible, then for all $\lambda \in]0, 1]$, $1/\theta \geq 1 + \lambda$ and $t \geq 0$, $P - J_t$ and J_t are injective operators .*

Proof: It is known that for a self - adjoint operator $a \in P \mathcal{A} P$ such that $0 < \|a\| < P$, we have:

$$\log(P - a) = - \sum_{n=1}^{\infty} \frac{a^n}{n}$$

Since $\tilde{\Phi}(a^n) = \int x^n \mu(dx)$ for a positive compact supported measure μ , we get :

$$\tilde{\Phi}(\log(P - a)) = - \sum_{n=1}^{\infty} \frac{\tilde{\Phi}(a^n)}{n} = -\tilde{\Phi}(a) - \sum_{n=2}^{\infty} \frac{\tilde{\Phi}(a^n)}{n}$$

Thus, substituting moments of J_t , one has (cf Corollary (2)):

$$\begin{aligned} \tilde{\Phi}(\log(P - J_t)) &= \tilde{\Phi}(\log(P - J_0)) - \theta t + \int_0^t \tilde{\Phi} \left(\sum_{n=1}^{\infty} J_s^n \right) ds - \theta \int_0^t \tilde{\Phi} \left(\sum_{n=1}^{\infty} J_s^n \right) ds \\ &\quad - \lambda \theta \int_0^t \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} \tilde{\Phi}(J_s^{n-1-k}) \tilde{\Phi}(J_s^k (P - J_s)) ds \\ &= \tilde{\Phi}(\log(P - J_0)) - \theta t + (1 - \theta) \int_0^t \tilde{\Phi}(J_s (P - J_s)^{-1}) ds - \lambda \theta \int_0^t \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \tilde{\Phi}(J_s^n) \tilde{\Phi}(J_s^k (P - J_s)) ds \\ &= \tilde{\Phi}(\log(P - J_0)) - \theta t + (1 - \theta) \int_0^t \tilde{\Phi}(J_s (P - J_s)^{-1}) ds - \lambda \theta \int_0^t \tilde{\Phi}(J_s (P - J_s)^{-1}) \tilde{\Phi}(P) ds \\ &= \tilde{\Phi}(\log(P - J_0)) - \theta t + (1 - \theta - \lambda \theta) \int_0^t \tilde{\Phi}(J_s (P - J_s)^{-1}) ds \\ &= \tilde{\Phi}(\log(P - J_0)) - (1 - \lambda \theta) t + (1 - \theta - \lambda \theta) \int_0^t \tilde{\Phi}((P - J_s)^{-1}) ds \end{aligned}$$

Hence, we deduce that, if $\lambda \in]0, 1]$ and $1/\theta \geq 1 + \lambda$ -which ensures that $1 - \theta - \lambda \theta \geq 0$ and $1 - \lambda \theta \geq 0$ - and if J_0 is an invertible operator, then:

$$\tilde{\Phi}(\log(P - J_t)) + (1 - \lambda \theta) t \geq \tilde{\Phi}(\log(P - J_0)) > -\infty \quad \forall t < T$$

which gives the injectivity of $P - J_t, \forall t \geq 0$. The second assertion follows since one can easily see from (3) that, if J is a free Jacobi process with parameters $FJP(\lambda, \theta)$ starting at $0 < J_0 < P$, then $P - J$ is still a free Jacobi process with parameters $(\lambda\theta/1 - \theta, 1 - \theta)$ starting at $0 < P - J_0 < P$, and since $\frac{1}{\theta} \geq \lambda + 1 \Rightarrow \frac{\lambda\theta}{1-\theta} \leq 1$ and $\lambda \leq 1 \Rightarrow (\lambda - 1)\theta \leq 0 \Rightarrow (\lambda\theta)/(1 - \theta) + 1 = (\theta(\lambda - 1) + 1)/(1 - \theta) \leq 1/(1 - \theta)$.

Corollary 3. *Under the same conditions of proposition (4), the $FJP(\lambda, \theta)$ satisfies for all $t \geq 0$ the following SDE:*

$$dJ_t = \sqrt{\lambda\theta}\sqrt{P - J_t}dW_t\sqrt{J_t} + \sqrt{\lambda\theta}\sqrt{J_t}dW_t^* \sqrt{P - J_t} + (\theta P - J_t) dt$$

where W is a complex free Brownian motion.

5.2. Free martingales polynomials. In this paragraph, we consider a stationary $FJP(1, 1/2)$ starting at J_0 , the law of which is the Beta law $B(1/2, 1/2)$. Recall that that a \mathcal{A}_t -adapted free process $(X_t)_{t \geq 0}$ is a \mathcal{A} -free martingale if and only if $\Phi(X_t | \mathcal{A}_s) = X_s$ (cf [1],[4], [7]).

Proposition 5. *Let \mathcal{J}_t denotes the von Neumann subalgebra generated by $(J_s, s \leq t)$ and let $0 < r < 1$. Then, the process $R_t := ((1 + re^t)P - 2re^t J_t)((1 + re^t)^2 P - 4re^t J_t)^{-1}$ is a \mathcal{J}_t -free martingale.*

Proof : R_t can be written as :

$$\begin{aligned} R_t &= \left[\frac{P}{1 + re^t} - 2 \frac{re^t}{(1 + re^t)^2} J_t \right] \left[P - \frac{4re^t}{(1 + re^t)^2} J_t \right]^{-1} \\ &= \left[\frac{1 - re^t}{2(1 + re^t)} P + \frac{1}{2} \left(P - \frac{4re^t}{(1 + re^t)^2} J_t \right) \right] \left[P - \frac{4re^t}{(1 + re^t)^2} J_t \right]^{-1} := \frac{1 - re^t}{2(1 + re^t)} H_t + \frac{P}{2} \end{aligned}$$

where

$$H_t = \left[P - \frac{4re^t}{(1 + re^t)^2} J_t \right]^{-1} = \sum_{n \geq 0} \frac{(4re^t)^n}{(1 + re^t)^{2n}} J_t^n$$

since $4re^t < (1 + re^t)^2$ and $0 \leq J_t \leq P$ for all $t > 0$. It follows that :

$$2dR_t = \frac{1 - re^t}{1 + re^t} dH_t - \frac{2re^t}{(1 + re^t)^2} H_t dt$$

On the other hand, one has for $1 \leq l \leq n - 1$ and $n \geq 2$:

$$\tilde{\Phi}(J_t^{n-l}) = \frac{\Gamma(n - l + 1/2)}{\sqrt{\pi}(n - l)!}, \quad \tilde{\Phi}(J_t^{n-l-1}(P - J_t)) = \frac{\Gamma(n - l - 1/2)}{2\sqrt{\pi}(n - l)!}$$

From the proof of Proposition (3), we deduce that, for all $t > 0$:

$$\begin{aligned}
dJ_t^n &= M_t + n\left(\frac{P}{2} - J_t\right)J_t^{n-1}dt + \frac{1}{2}(\Delta_U(J_t^n) + \Delta_V(J_t^n)) \\
&= M_t + \left[n\left(\frac{P}{2} - J_t\right)J_t^{n-1} + \sum_{l=1}^{n-1} l[(P - J_t)J_t^{l-1}\tilde{\Phi}(J_t^{n-l}) + J_t^l\tilde{\Phi}(J_t^{n-l-1}(P - J_t))] \right] dt \\
&= M_t + \left[n\left(\frac{P}{2} - J_t\right)J_t^{n-1} + \frac{1}{2\sqrt{\pi}} \sum_{l=1}^{n-1} l \left[(P - J_t)J_t^{l-1} \frac{\Gamma(n-l+1/2)}{(n-l)!} + \frac{J_t^l \Gamma(n-l-1/2)}{(n-l)!} \right] \right] dt \\
&= M_t + \left[n\left(\frac{P}{2} - J_t\right)J_t^{n-1} + \frac{1}{2\sqrt{\pi}} \left[\sum_{l=1}^{n-1} l \frac{\Gamma(n-l+1/2)}{(n-l)!} J_t^{l-1} - \sum_{l=1}^{n-1} l(n-l-1) \frac{\Gamma(n-l-1/2)}{(n-l)!} J_t^l \right] \right] dt \\
&= M_t + \left[n\left(\frac{P}{2} - J_t\right)J_t^{n-1} + \frac{1}{2\sqrt{\pi}} \left[\sum_{l=1}^{n-1} l \frac{\Gamma(n-l+1/2)}{(n-l)!} J_t^{l-1} - \sum_{l=2}^{n-1} (l-1)(n-l) \frac{\Gamma(n-l+1/2)}{(n-l+1)!} J_t^{l-1} \right] \right] dt \\
&= M_t + \left[n\left(\frac{P}{2} - J_t\right)J_t^{n-1} + \frac{1}{2\sqrt{\pi}} \left[\sum_{l=2}^{n-1} n \frac{\Gamma(n-l+1/2)}{(n-l+1)!} J_t^{l-1} + \frac{\Gamma(n-1/2)}{(n-1)!} P \right] \right] dt \\
&= M_t + \left[n\left(\frac{P}{2} - J_t\right)J_t^{n-1} + \frac{n}{2\sqrt{\pi}} \sum_{l=1}^{n-1} \frac{\Gamma(n-l+1/2)}{(n-l+1)!} J_t^{l-1} \right] dt \\
&= M_t - nJ_t^n dt + \frac{n}{2\sqrt{\pi}} \sum_{l=1}^n \frac{\Gamma(n-l+1/2)}{(n-l+1)!} J_t^{l-1} dt
\end{aligned}$$

where M_t stands for the martingale part. Note that this holds for $n = 1$. Thus :

$$\begin{aligned}
FV(dH_t) &= \sum_{n \geq 0} \frac{(4re^t)^n}{(1+re^t)^{2n}} FV(dJ_t^n) + \frac{1-re^t}{1+re^t} \sum_{n \geq 0} \frac{n(4re^t)^n}{(1+re^t)^{2n}} J_t^n dt \\
&= -\frac{2re^t}{1+re^t} \sum_{n \geq 0} \frac{n(4re^t)^n}{(1+re^t)^{2n}} J_t^n dt + \frac{1}{2\sqrt{\pi}} \sum_{n \geq 0} \frac{n(4re^t)^n}{(1+re^t)^{2n}} \sum_{l=1}^n \frac{\Gamma(n-l+1/2)}{(n-l+1)!} J_t^{l-1} dt \\
&= -\frac{2re^t}{1+re^t} \sum_{n \geq 0} \frac{n(4re^t)^n}{(1+re^t)^{2n}} J_t^n dt + \frac{1}{2} \sum_{l \geq 1} \frac{(4re^t)^l}{(1+re^t)^{2l}} \sum_{n \geq 0} \frac{(n+l)(1/2)_n}{(n+1)!} \frac{(4re^t)^n}{(1+re^t)^{2n}} J_t^{l-1} dt \\
&= -\frac{2re^t}{1+re^t} \sum_{n \geq 0} \frac{n(4re^t)^n}{(1+re^t)^{2n}} J_t^n dt + \frac{2re^t}{(1+re^t)^2} \sum_{l \geq 0} \frac{(4re^t)^l}{(1+re^t)^{2l}} \sum_{n \geq 0} \frac{(n+l+1)(1/2)_n}{(n+1)!} \frac{(4re^t)^n}{(1+re^t)^{2n}} J_t^l dt \\
&= -\frac{2re^t}{1+re^t} \sum_{n \geq 0} \frac{n(4re^t)^n}{(1+re^t)^{2n}} J_t^n dt + \frac{2re^t}{(1+re^t)^2} \sum_{l \geq 0} \frac{(4re^t)^l}{(1+re^t)^{2l}} J_t^l \sum_{n \geq 0} \frac{(1/2)_n}{n!} \frac{(4re^t)^n}{(1+re^t)^{2n}} dt \\
&+ \frac{1}{2} \sum_{l \geq 0} \frac{l(4re^t)^l}{(1+re^t)^{2l}} J_t^l \sum_{n \geq 0} \frac{(1/2)_n}{(n+1)!} \frac{(4re^t)^{n+1}}{(1+re^t)^{2n+2}} dt
\end{aligned}$$

Then, using that (cf [2]) :

$${}_1\mathcal{F}_0(a, z) := \sum_{p=0}^{\infty} (a)_p \frac{z^p}{p!} = (1-z)^{-a}, \quad |z| < 1,$$

one has for all t such that $re^t < 1$:

$$\sum_{n \geq 0} \frac{(1/2)_n}{n!} \frac{(4re^t)^n}{(1+re^t)^{2n}} dt = \left(1 - \frac{4re^t}{(1+re^t)^2}\right)^{-1/2} = \frac{1+re^t}{1-re^t}$$

and similarly, from $\int_0^u dz/\sqrt{1-z} = 2 - 2\sqrt{1-u}$,

$$\frac{1}{2} \sum_{n \geq 0} \frac{(1/2)_n}{(n+1)!} \frac{(4re^t)^{n+1}}{(1+re^t)^{2n+2}} = 1 - \left(1 - \frac{4re^t}{(1+re^t)^2}\right)^{1/2} = \frac{2re^t}{1+re^t}$$

The result follows from an easy computation. ■

Corollary 4. *Let T_k denotes the Tchebycheff polynomial of the first kind (cf [2]) defined by :*

$$T_k(x) = \cos(k \arccos(x)), \quad k \geq 0, \quad x \in]-1, 1[$$

Thus the process $S(k)$ defined by $S_t(k) := e^{kt}T_k(2J_t - P)$ is a \mathcal{J}_t -free martingale.

Proof : The generating function of these polynomials is given by (cf [2]) :

$$L(x, z) := \sum_{k=0}^{\infty} T_k(x) z^k = \frac{1-zx}{1-2zx+z^2}, \quad |z| < 1$$

Letting $z = re^t$ with $0 < r < e^{-t} < e^{-s}$ for $s < t$, then $L(2J_t - P, re^t)$ and $L(2J_s - P, re^s)$ converge, and using that R is a \mathcal{J}_t -martingale, one gets :

$$\sum_{k=0}^{\infty} \tilde{\Phi}(e^{kt}T_k(2J_t - P) | \mathcal{J}_s) r^k = \tilde{\Phi}(R_t | \mathcal{J}_s) = R_s = \sum_{k=0}^{\infty} T_k(2J_s - P) e^{ks} r^k$$

Taking the derivative of both sides at $r = 0$, we are done.

6. ON THE CAUCHY TRANSFORM OF THE FREE JACOBI PROCESS

Under the assumptions of the previous section, one has :

$$\tilde{\Phi}(J_t) = (\tilde{\Phi}(J_0) - \theta)e^{-t} + \theta$$

Then, a similar computation as in Proposition (4) gives for all u in the unit disk:

$$\begin{aligned} \tilde{\Phi}(\log(P - uJ_t)) &= -\sum_{n \geq 1} \frac{u^n}{n} \tilde{\Phi}(J_t^n) = -u\tilde{\Phi}(J_t) - \sum_{n \geq 2} \frac{u^n}{n} \tilde{\Phi}(J_t^n) \\ &= u(\tilde{\Phi}(J_0) - \tilde{\Phi}(J_t)) + \tilde{\Phi}(\log(P - uJ_0)) + \int_0^t \tilde{\Phi} \left(\sum_{n \geq 2} (uJ_s)^n \right) ds - \theta u \int_0^t \tilde{\Phi} \left(\sum_{n \geq 2} (uJ_s)^{n-1} \right) ds \\ &\quad - \lambda \theta \int_0^t \sum_{n \geq 2} \sum_{k=0}^{n-2} u^n \tilde{\Phi}(J_s^{n-k-1}) \tilde{\Phi}(J_s^k(P - J_s)) ds \end{aligned}$$

$$\begin{aligned} &= u(\tilde{\Phi}(J_0) - \theta)(1 - e^{-t}) + \tilde{\Phi}(\log(P - uJ_0)) + \int_0^t \tilde{\Phi}(u^2 J_s^2 (P - uJ_s)^{-1}) ds \\ &\quad - \theta u \int_0^t \tilde{\Phi}(uJ_s(P - uJ_s)^{-1}) ds - \lambda \theta u \int_0^t \tilde{\Phi} \left(\sum_{n \geq 0} (uJ_s)^{n+1} \right) \tilde{\Phi} \left(\sum_{k \geq 0} (uJ_s)^k (P - J_s) \right) ds \end{aligned}$$

Using the fact that $u^2 J_s^2 = (uJ_s - P)(uJ_s + P) + P$ and $uJ_s = (uJ_s - P) + P$, we get :

$$\begin{aligned} \tilde{\Phi}(\log(P - uJ_t)) &= u(\tilde{\Phi}(J_0) - \theta)(1 - e^{-t}) + \tilde{\Phi}(\log(P - uJ_0)) + (1 - \theta u) \int_0^t \tilde{\Phi}((P - uJ_s)^{-1}) ds \\ &\quad + \theta u t - \int_0^t \tilde{\Phi}((P + uJ_s)) ds - \lambda \theta u \int_0^t \tilde{\Phi}(uJ_s(P - uJ_s)^{-1}) \tilde{\Phi}((P - J_s)(P - uJ_s)^{-1}) ds \\ &= \tilde{\Phi}(\log(P - uJ_0)) + (1 - \theta u) \int_0^t \tilde{\Phi}((P - uJ_s)^{-1}) ds - t \\ &\quad - \lambda \theta u \int_0^t [\tilde{\Phi}((P - uJ_s)^{-1}) - 1][1 + (u - 1)\tilde{\Phi}(J_s(P - uJ_s)^{-1})] ds \\ &= \tilde{\Phi}(\log(P - uJ_0)) + (1 - \theta u - \lambda \theta u) \int_0^t \tilde{\Phi}((P - uJ_s)^{-1}) ds - (1 - \lambda \theta u)t \\ &\quad - \lambda \theta (u - 1) \int_0^t \tilde{\Phi}((P - uJ_s)^{-1}) \tilde{\Phi}(uJ_s(P - uJ_s)^{-1}) ds + \lambda \theta (u - 1) \int_0^t \tilde{\Phi}(uJ_s(P - uJ_s)^{-1}) ds \\ &= \tilde{\Phi}(\log(P - uJ_0)) + (1 - \theta u - \lambda \theta u + 2\lambda \theta (u - 1)) \int_0^t \tilde{\Phi}((P - uJ_s)^{-1}) ds - (1 - \lambda \theta u)t \\ &\quad - \lambda \theta (u - 1)t - \lambda \theta (u - 1) \int_0^t \tilde{\Phi}^2((P - uJ_s)^{-1}) ds \\ &= \tilde{\Phi}(\log(P - uJ_0)) + (1 - \theta u + \lambda \theta (u - 2)) \int_0^t \tilde{\Phi}((P - uJ_s)^{-1}) ds - (1 - \lambda \theta)t \\ &\quad + \lambda \theta (u - 1) \int_0^t \tilde{\Phi}^2((P - uJ_s)^{-1}) ds \end{aligned}$$

Setting $h_t(u) = \tilde{\Phi}((P - uJ_t)^{-1})$, then :

$$h_t(u) = \frac{1}{u} G_t \left(\frac{1}{u} \right)$$

where G_t denotes the Cauchy (or the G-) transform of the FJP 's law defined on \mathbb{C}^+ by :

$$G_t(z) = \int_0^1 \frac{\mu_t(dx)}{z-x}$$

Furthermore, we have:

$$-\frac{d}{du} \tilde{\Phi}(\log(P - uJ_t)) = \tilde{\Phi}(J_t(P - uJ_t)^{-1}) = \frac{1}{u}(h_t(u) - 1)$$

Thus, we get:

$$\begin{aligned} h_t(u) &= h_0(u) + \theta(1 - \lambda) \int_0^t u h_s(u) ds - (1 - \theta u + \lambda\theta(u - 2)) \int_0^t u h'_s(u) ds \\ &\quad + \lambda\theta \int_0^t u h_s^2(u) ds + 2\lambda\theta(u - 1) \int_0^t u h_s(u) h'_s(u) ds \end{aligned}$$

or equivalently :

$$\begin{aligned} G_t(1/u) &= G_0(1/u) + \theta(1 - \lambda)u \int_0^t G_s(1/u) ds + \lambda\theta \int_0^t G_s^2(1/u) ds + (1 - \theta u + \lambda\theta(u - 2)) \times \\ &\quad \int_0^t \left[G_s(1/u) + \frac{1}{u} G'_s(1/u) \right] ds + 2\lambda\theta \left(\frac{1-u}{u^2} \right) \int_0^t [u G_s^2(1/u) + G'_s(1/u) G_s(1/u)] \\ &= G_0(1/u) + (1 - 2\lambda\theta) \int_0^t G_s(1/u) ds + \lambda\theta \left(\frac{2}{u} - 1 \right) \int_0^t G_s^2(1/u) ds \\ &\quad + \frac{1 - \theta u + \lambda\theta(u - 2)}{u} \int_0^t G'_s(1/u) ds + \frac{2\lambda\theta(1-u)}{u^2} \int_0^t G_s(1/u) G'_s(1/u) ds \end{aligned}$$

As a consequence, G satisfies the p. d. e. :

Proposition 6.

$$\begin{aligned} G_t(z) &= G_0(z) + (1 - 2\lambda\theta) \int_0^t G_s(z) ds + \lambda\theta(2z - 1) \int_0^t G_s^2(z) ds \\ &\quad + ((1 - 2\lambda\theta)z - \theta(1 - \lambda)) \int_0^t G'_s(z) ds + 2\lambda\theta z(z - 1) \int_0^t G_s(z) G'_s(z) ds \end{aligned}$$

7. CONCLUSION AND OPEN QUESTIONS

Throughout this paper, one can see that the study of the free Jacobi process is more easier than the study of the complex matrix Jacobi process in the sense that, though both cases belong to a non commutative context, we get more powerful results on the law of the process in the infinite dimensional case, namely, the recurrence formula for the moments and the Cauchy transform. Yet, one would like to deepen our results and get much more precision on the law, that is, compute the discrete and continuous parts as done in the stationary case. For instance, let us interest in the Dirac mass at 1, then $-y\mathfrak{S}(G(1 + iy))$ involves $\mathfrak{S}(yG'(1 + iy))$ which can be unbounded. Hence, one can not intertwine limit and integral signs.

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