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Abstract

We are concerned with singular elliptic problems of the form $-\Delta u \pm p(d(x))g(u) = \lambda f(x,u) + \mu|\nabla u|^a$ in Ω, where Ω is a smooth bounded domain in $\mathbb{R}^N$, $d(x) = \mathrm{dist}(x, \partial\Omega)$, $\lambda > 0$, $\mu \in \mathbb{R}$, $0 < a \leq 2$, and f, k are nonnegative and nondecreasing functions. We assume that $p(d(x))$ is a positive weight with possible singular behavior on the boundary of Ω and that the nonlinearity g is unbounded around the origin. Taking into account the competition between the anisotropic potential $p(d(x))$, the convection term $|\nabla u|^a$, and the singular nonlinearity g, we establish various existence and nonexistence results.
2000 Mathematics Subject Classification: 35B50, 35J65, 58J55.
Key words: singular elliptic equation, convection term, anisotropic media, singular potential, maximum principle.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain with smooth boundary. We are concerned in this paper with singular elliptic problems of the following type

$$\begin{cases} -\Delta u \pm p(d(x))g(u) = \lambda f(x,u) + \mu|\nabla u|^a & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P)^{\pm}$$

where $d(x) = \mathrm{dist}(x, \partial\Omega)$, $\lambda > 0$, $\mu \in \mathbb{R}$, and $0 < a \leq 2$.

We refer the reader to the works of Serrin [26], Choquet-Bruhat and Leray [8], and Kazdan and Warner [23], which motivate the requirement that the nonlinearity $|\nabla u|^a$ grows at most quadratically. We also assume that

- $g \in C^1(0,\infty)$ is a positive decreasing function and

  $(g1)$ $\quad \lim_{t\to 0^+} g(t) = +\infty.$

- $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function which is nondecreasing with respect to the second variable and such that f is positive on $\overline{\Omega} \times (0,\infty)$. Furthermore, f is either linear or f is sublinear with respect to the second variable. This last case means that f fulfills the hypotheses

  $(f1)$ $\quad$ the mapping $(0,\infty) \ni t \longmapsto \dfrac{f(x,t)}{t}$ is nonincreasing for all $x \in \overline{\Omega}$;

  $(f2)$ $\quad \lim_{t\to 0^+} \dfrac{f(x,t)}{t} = +\infty$ and $\lim_{t\to+\infty} \dfrac{f(x,t)}{t} = 0$, uniformly for $x \in \overline{\Omega}$.

- $p : (0,+\infty) \to (0,+\infty)$ is nonincreasing and Hölder continuous.

Such singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids or boundary layer phenomena for viscous fluids (we refer for more details to [5, 6, 7, 11, 13, 14] and the more recent papers [9, 15, 20, 21, 22, 24, 27, 28, 31]). We also point out that, due to the meaning of the unknowns (concentrations, populations, etc.), only the positive solutions are relevant in most cases.

To the best of our knowledge, there does not exist a qualitative theory for the study of singular boundary value problems with nonlinearities in the Kato class $K_N^{\rm loc}(\mathbb{R}^N)$. This theory was introduced by Aizenman and Simon in [2] to describe wide classes of functions arising in Potential Theory. We refer to the recent paper [25] for existence and bifurcation results on Dirichlet boundary value problems with indefinite nonlinearities.

The results in this paper complete the study developed in [16] and [17] since here we deal with singular weights. One of our purposes is to give a necessary and sufficient condition on the weight p in order to obtain a classical solution of problems $(P)^{\pm}$. By classical solution we understand a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ that fulfills $(P)^{\pm}$.

Dealing with problem $(P)^+$ we show that a necessary condition in order to have classical solution is

$$\int_0^1 p(t)g(t)dt < +\infty. \tag{1.1}$$

In the case where f is sublinear, that is, f fulfills the hypotheses $(f1)$ and $(f2)$, condition (1.1) is also sufficient for existence of a classical solutions of $(P)^+$ provided λ and μ belong to a certain range (see Theorem 2.2). Obviously, (1.1) implies the following Keller-Osserman type condition around the origin (see the proof of Theorem 2.2)

$$(KO) \qquad \int_0^1 \left(\int_0^t \Phi(s)ds \right)^{-1/2} dt < +\infty \,, \text{ where } \Phi(s) = p(s)g(s), \text{ for all } s > 0.$$

As proved by Bénilan, Brezis and Crandall [4], condition (KO) is equivalent to the *property of compact support*, that is, for every $h \in L^1(\mathbb{R}^N)$ with compact support, there exists a unique $u \in W^{1,1}(\mathbb{R}^N)$ with compact support such that $\Delta u \in L^1(\mathbb{R}^N)$ and

$$-\Delta u + \Phi(u) = h \qquad \text{a.e. in } \mathbb{R}^N.$$

The results are completely different for problem $(P)^-$. Our results in this case generalize those established in [32], in the sense that in the present paper we do not prescribe the behavior of the singular nonlinearity g around the origin. Also, we proved in [16] that if $p \equiv 1$, then the existence of a classical solution to $(P)^-$ does not depend on the asymptotic behavior of g near the origin, whereas the exponent a of the convection term $|\nabla u|^a$ plays a crucial role. In our case, the potential $p(d(x))$ also affects the existence of classical solutions to $(P)^-$.

Many papers have been devoted to the case $p \equiv 1$ and $\mu = 0$ (see [10, 12, 15, 27] and the references therein). One of the first works in the literature dealing with singular weights in connection with singular nonlinearities is due tu Taliaferro [29]. In [29] the following problem has been considered

$$\begin{cases} -y'' = \varphi(x) y^{-\beta} & \text{in } (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{1.2}$$

where $\beta > 0$ and $\varphi(x)$ is positive and continuous on $(0,1)$. It was proved that problem (1.2) has solutions if and only if $\int_0^1 t(1-t)\varphi(t)dt < +\infty$. Later, Agarwal and O'Regan [1, Section 2] studied the more general problem

$$\begin{cases} H''(t) = -p(t)g(H(t)) & \text{in } (0,1), \\ H > 0 & \text{in } (0,1), \\ H(0) = H(1) = 0, \end{cases} \tag{1.3}$$

where g satisfies $(g1)$ and p is positive and continuous on $(0,1)$. It is shown in [1] that if

$$\int_0^1 t(1-t)p(t)dt < +\infty, \tag{1.4}$$

then (1.3) has at least one classical solution. In our framework, p is continuous at $t = 1$ so condition (1.4) reduces to

$$\int_0^1 tp(t)dt < +\infty. \tag{1.5}$$

We prove that assumption (1.5) is necessary in order that problem $(P)^-$ has classical solutions. Furthermore, we argue in Section 3 that the existence of a classical solution of $(P)^-$ when f is sublinear depends on the asymptotic behavior of the gradient term $|\nabla u|^a$. In this sense, we prove

that if $0 < a < 1$, then $(P)^-$ has at least one classical solution for all $\mu \in \mathbb{R}$. In turn, if $1 < a \leq 2$, then $(P)^-$ has no solutions for large values of μ.

Special attention is payed to the case where $a = 1$. This case was left as an open question in [16]. We prove in Theorem 3.3 that if Ω is a ball centered at the origin, then $(P)^-$ has at least one solution for all $\mu \in \mathbb{R}$, provided $a = 1$.

The existence of a solution to $(P)^{\pm}$ is achieved by the sub and super-solution method. In particular, the super-solution of $(P)^-$ is expressed in terms of H. In the case of pure power nonlinearities, a careful analysis of (1.3) allows us to give boundary estimates of the solution.

The outline of the paper is as follows. In Section 2 we give existence and nonexistence results for problem $(P)^+$. Section 3 concerns the problem $(P)^-$ in which we discuss separately the case where f is linear or sublinear. At the end of this Section we present, as an application of the obtained results, the case where $p(t) = t^{-\alpha}$ and $g(t) = t^{-\beta}$, and we give some estimates for the solution at the boundary. To make the results clearer, we assume that $\lambda = 1$ and f is sublinear. Thus, problem $(P)^-$ becomes

$$\begin{cases} -\Delta u = d(x)^{-\alpha} u^{-\beta} + f(x,u) + \mu |\nabla u|^a & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

2 The problem $(P)^+$

We first establish the following general nonexistence result related to problem $(P)^+$.

Theorem 2.1. *Assume that* $\int_0^1 p(t)g(t)dt = +\infty$. *Let* $\Phi : \overline{\Omega} \times [0, +\infty) \to \mathbb{R}$ *be a Hölder continuous function. Then the inequality boundary value problem*

$$\begin{cases} -\Delta u + p(d(x))g(u) \leq \Phi(x,u) + C\,|\nabla u|^2 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

has no classical solutions.

As a direct consequence, we obtain :

Corollary 2.1. *Assume that* $\int_0^1 p(t)g(t)dt = +\infty$. *Then problem* $(P)^+$ *has no classical solutions.*

Proof of the theorem. We apply an idea found e.g. in [30]. It is readily seen that it suffices to prove the Theorem only for the case $C > 0$. Arguing by contradiction, we assume that the

boundary value inequality problem (2.1) has a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$. By the Gelfand transform $v = e^{Cu} - 1$ we find

$$\begin{aligned} \Delta v &= Ce^{Cu}\left(\Delta u + C|\nabla u|^2\right) \geq Ce^{Cu}\left[p(d(x))g(u) - \Phi(x,u)\right] \\ &= C(v+1)\left[p(d(x))g\left(\frac{\ln(v+1)}{C}\right) - \Phi\left(x, \frac{\ln(v+1)}{C}\right)\right]. \end{aligned} \tag{2.2}$$

Since v is continuous on $\overline{\Omega}$ and $v > 0$ in Ω, we deduce that

$$-\Delta v \leq \Psi(x,v) \leq C_0 \quad \text{ in } \Omega,$$

where $\Psi(x,v) = C(v+1)\Phi\left(x, \frac{\ln(v+1)}{C}\right)$. A straightforward argument based on the maximum principle combined with the observation that $v = 0$ on $\partial\Omega$ shows that $v \leq c_0 d(x)$ in Ω.

For $\varepsilon > 0$ small enough, consider an open set Ω_ε with smooth boundary such that $\Omega_\varepsilon \supset \{x \in \Omega;\ \text{dist}\,(x, \partial\Omega) > \varepsilon\}$. By integration in (2.2) we find

$$-\int_{\partial\Omega_\varepsilon} \frac{\partial v}{\partial \nu_\varepsilon} ds + C\int_{\Omega_\varepsilon} (v+1)p(d(x))g\left(\frac{\ln(v+1)}{C}\right) dx \leq \int_{\Omega_\varepsilon} \Psi(x,v)dx \leq C_0|\Omega|.$$

Therefore

$$\begin{aligned} \int_{\partial\Omega_\varepsilon} \frac{\partial v}{\partial \nu_\varepsilon} ds &\geq C\int_{\Omega_\varepsilon} (v+1)p(d(x))g\left(\frac{\ln(v+1)}{C}\right) dx - C_0|\Omega| \\ &\geq C\int_{\Omega_\varepsilon} p(d(x))g\left(\frac{v}{C}\right) dx - C_0|\Omega|. \end{aligned} \tag{2.3}$$

Since $v \leq c_0 d(x)$ in Ω, and $\int_0^1 p(t)g(t)dt = +\infty$, it follows that the integral in the right-hand side of (2.3) diverges as $\varepsilon \to 0^+$. Hence

$$\lim_{\varepsilon \to 0^+} \int_{\partial\Omega_\varepsilon} \frac{\partial v}{\partial \nu_\varepsilon} ds = +\infty.$$

But this contradicts the maximum principle (see [18, Lemma 3.4]) because $\limsup_{t\to 0^-} \frac{v(x_0+t\nu)}{t} < 0$, for all $x_0 \in \partial\Omega$. □

Before stating our existence results, we recall the following auxiliary tool (see [17, Lemma 2.1] for a complete proof).

Lemma 2.1. *Let $\Psi : \overline{\Omega} \times (0,+\infty) \to \mathbb{R}$ be a Hölder continuous function such that the mapping $(0,+\infty) \ni s \longmapsto \dfrac{\Psi(x,s)}{s}$ is strictly decreasing for each $x \in \Omega$. Assume that there exist v, $w \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

(a) $\Delta w + \Psi(x,w) \leq 0 \leq \Delta v + \Psi(x,v)$ *in* Ω;

(b) $v, w > 0$ *in* Ω *and* $v \leq w$ *on* $\partial\Omega$;

(c) $\Delta v \in L^1(\Omega)$ *or* $\Delta w \in L^1(\Omega)$.

Then $v \leq w$ in Ω.

Next, we prove that (1.1) is sufficient for the existence of a classical solution to $(P)^+$ provided $\mu \leq 0$ and $\lambda > 0$ is sufficiently large. We have

Theorem 2.2. *Assume that* $\int_0^1 p(t)g(t)dt < +\infty$.

(i) *If* $\mu = -1$, *then there exists* $\lambda^* > 0$ *such that* $(P)^+$ *has at least one classical solution if* $\lambda > \lambda^*$ *and no solution exists if* $0 < \lambda < \lambda^*$.

(ii) *If* $\mu = +1$ *and* $0 < a < 1$, *then there exists* $\lambda^* > 0$ *such that* $(P)^+$ *has at least one classical solution for all* $\lambda > \lambda^*$ *and no solution exists if* $0 < \lambda < \lambda^*$.

Proof. (i) We split the proof into several steps.

Step 1: Existence of a solution for λ *large.* By virtue of [27, Lemma 4] (see also ([28, Theorem 2]), the problem

$$\begin{cases} -\Delta U = \lambda f(x,U) & \text{in } \Omega, \\ U > 0 & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

has at least one classical solution U_λ, for all $\lambda > 0$. Using the regularity of f it follows that $U_\lambda \in C^2(\overline{\Omega})$ and there exist $c_1, c_2 > 0$ depending on λ such that

$$c_1 d(x) \le U_\lambda(x) \le c_2 d(x) \quad \text{in } \Omega. \tag{2.5}$$

Fix $\lambda > 0$ and observe that U_λ is a super-solution of $(P)^+$. The main point is to find a sub-solution $\underline{u}_\lambda$ of $(P)^+$ such that $\underline{u}_\lambda \le U_\lambda$ in Ω. For this purpose, let $\Phi(t) = p(t)g(t)$, $t > 0$, and define

$$\Psi : [0, +\infty) \to [0, +\infty), \quad \Psi(t) = \int_0^t \frac{1}{\sqrt{2\int_0^s \Phi(\tau)d\tau}} ds.$$

Remark first that Ψ is well defined, since $\Phi \in L^1(0,1)$. Indeed, there exists $m > 0$ such that $\Phi(s) \ge m$, for all $0 < s < 1$. This yields $(\int_0^s \Phi(\tau)d\tau)^{-1/2} \le (\sqrt{ms})^{-1}$, for all $0 < s < 1$ which implies the Keller-Osserman condition (KO) around the origin:

$$\int_0^1 \left(\int_0^t \Phi(s)ds \right)^{-1/2} dt < +\infty.$$

We claim that Ψ is a bijective map. Indeed, Ψ is increasing and if $M := \Phi(1)$, then

$$\int_0^s \Phi(\tau)d\tau \le \int_0^1 \Phi(\tau)d\tau + M(s-1), \quad \forall s \ge 1.$$

Thus, there exists $c > 0$ such that

$$\int_0^s \Phi(\tau)d\tau \le Ms + c, \quad \forall s \ge 1.$$

It follows that

$$\Psi(t) \ge \int_1^t \frac{1}{\sqrt{2(Ms+c)}} ds \ge \frac{1}{M}(\sqrt{2(Mt+c)} - c_1), \quad \forall\, t \ge 1.$$

This gives $\lim_{t\to+\infty}\Psi(t)=+\infty$ and the claim follows.

Let $h:[0,+\infty)\to[0,+\infty)$ be the inverse of Ψ. Then h satisfies

$$\begin{cases} h>0 & \text{in } (0,+\infty),\\ h'(t)=\sqrt{2\int_0^{h(t)}\Phi(s)ds} & \text{in } (0,+\infty),\\ h''(t)=\Phi(h(t)) & \text{in } (0,+\infty),\\ h(0)=h'(0)=0. \end{cases} \tag{2.6}$$

Hence $h\in C^2(0,+\infty)\cap C^1[0,+\infty)$. Let $\varphi_1>0$ be the first eigenfunction of $(-\Delta)$ in $H_0^1(\Omega)$. It is well known that there exists $C>0$ such that

$$Cd(x)\le\varphi_1\le\frac{1}{C}d(x)\quad \text{for all } x\in\Omega. \tag{2.7}$$

The key result for this part of the proof is the following.

Lemma 2.2. *There exist two positive constants $c>0$ and $M>0$ such that $\underline{u}_\lambda:=Mh(c\varphi_1)$ is a sub-solution of $(P)^+$ provided $\lambda>0$ is large enough.*

Proof. Since $h\in C^1[0,\infty)$ and $h(0)=0$, we can take $c>0$ small enough such that

$$h(c\varphi_1)\le d(x)\quad \text{in } \Omega. \tag{2.8}$$

By Hopf's maximum principle, there exist $\delta>0$ and $\omega\subset\subset\Omega$ such that $|\nabla\varphi_1|\ge\delta$ in $\Omega\setminus\omega$. Let

$$M=\max\{1,2(c\delta)^{-2}\}. \tag{2.9}$$

Since

$$\lim_{d(x)\to0^+}\Big\{-p(d(x))g(h(c\varphi_1))+Mc\lambda_1\varphi_1h'(c\varphi_1)+(Mch'(c\varphi_1)|\nabla\varphi_1|)^a\Big\}=-\infty,$$

we can assume that

$$-p(d(x))g(h(c\varphi_1))+Mc\lambda_1\varphi_1h'(c\varphi_1)+(Mch'(c\varphi_1)|\nabla\varphi_1|)^a<0\quad \text{in } \Omega\setminus\omega. \tag{2.10}$$

We are now able to show that $\underline{u}_\lambda:=Mh(c\varphi_1)$ is a sub-solution of $(P)^+$ provided $\lambda>0$ is sufficiently large. Indeed, using the monotonicity of g and (2.8) we have

$$\begin{aligned} &-\Delta\underline{u}_\lambda+p(d(x))g(\underline{u}_\lambda)+|\nabla\underline{u}_\lambda|^a=\\ &\quad=-Mc^2p(h(c\varphi_1))g(h(c\varphi_1))|\nabla\varphi_1|^2+Mc\lambda_1\varphi_1h'(c\varphi_1)\\ &\qquad+p(d(x))g(Mh(c\varphi_1))+(Mch'(c\varphi_1)|\nabla\varphi_1|)^a\\ &\quad\le p(d(x))g(h(c\varphi_1))(1-Mc^2|\nabla\varphi_1|^2)+Mc\lambda_1\varphi_1h'(c\varphi_1)+(Mch'(c\varphi_1)|\nabla\varphi_1|)^a. \end{aligned} \tag{2.11}$$

Taking into account the definition of M and (2.10), we find

$$\begin{aligned} &-\Delta \underline{u}_\lambda + p(d(x))g(\underline{u}_\lambda) + (|\nabla \underline{u}_\lambda|)^a \\ &\quad \leq -p(d(x))g(h(c\varphi_1)) + Mc\lambda_1\varphi_1 h'(c\varphi_1) + (Mch'(c\varphi_1)|\nabla\varphi_1|)^a < 0 \quad \text{in } \Omega\setminus\omega. \end{aligned} \tag{2.12}$$

On the other hand, from (2.11) and for all $x \in \omega$ we have

$$\begin{aligned} -\Delta \underline{u}_\lambda + p(d(x))g(\underline{u}_\lambda) + |\nabla \underline{u}_\lambda|^a \leq & p(d(x))g(h(c\varphi_1)) + Mc\lambda_1\varphi_1 h'(c\varphi_1) \\ & + (Mch'(c\varphi_1)|\nabla\varphi_1|)^a. \end{aligned} \tag{2.13}$$

Since $\varphi_1 > 0$ in $\overline{\omega}$ and f is positive on $\overline{\omega}\times(0,+\infty)$, we may choose $\lambda > 0$ such that

$$\lambda \min_{x\in\overline{\omega}} f(x, Mh(c\varphi_1)) \geq \max_{x\in\overline{\omega}} \Big\{ p(d(x))g(h(c\varphi_1)) + Mc\lambda_1\varphi_1 h'(c\varphi_1) + (Mch'(c\varphi_1)|\nabla\varphi_1|)^a \Big\}. \tag{2.14}$$

From (2.13) and (2.14) we deduce

$$-\Delta \underline{u}_\lambda + p(d(x))g(\underline{u}_\lambda) + |\nabla \underline{u}_\lambda|^a \leq \lambda f(x, \underline{u}_\lambda) \quad \text{in } \omega. \tag{2.15}$$

Now, relations (2.12) and (2.15) show that $\underline{u}_\lambda = Mh(c\varphi_1)$ is a sub-solution of $(P)^+$ provided $\lambda > 0$ satisfies (2.14). This finishes the proof of our Lemma. □

Using Lemma 2.1, it follows that $\underline{u}_\lambda \leq U_\lambda$ in Ω and by standard elliptic arguments (see [18]) we obtain a classical solution u_λ of $(P)^+$ such that $\underline{u}_\lambda \leq u_\lambda \leq U_\lambda$ in Ω.

Step 2: Nonexistence for $\lambda > 0$ small. We first remark that

$$\lim_{t\to 0^+} (f(x,t) - p(d(x))g(t)) = -\infty \quad \text{uniformly for } x \in \Omega.$$

Hence, there exists $t_0 > 0$ such that

$$f(x,t) - p(d(x))g(t) < 0, \quad \text{for all } (x,t) \in \Omega\times(0,t_0). \tag{2.16}$$

On the other hand, the assumption $(f1)$ yields

$$\frac{f(x,t) - p(d(x))g(t)}{t} \leq \frac{f(x,t)}{t} \leq \frac{f(x,t_0)}{t_0}, \tag{2.17}$$

for all $(x,t) \in \Omega\times[t_0,+\infty)$. Let $m = \max_{x\in\overline{\Omega}} \frac{f(x,t_0)}{t_0}$. Combining (2.16) with (2.17) we find

$$f(x,t) - p(d(x))g(t) < mt, \quad \text{for all } (x,t) \in \Omega\times(0,+\infty). \tag{2.18}$$

Set $\lambda_0 = \min\{1, \lambda_1/2m\}$. We claim that problem $(P)^+$ has no classical solution for $0 < \lambda \leq \lambda_0$. Indeed, assume by contradiction that u_0 is a classical solution of $(P)^+$ with $\lambda \in (0,\lambda_0]$. Then, according to (2.18), u_0 is a sub-solution of

$$\begin{cases} -\Delta u = \dfrac{\lambda_1}{2} u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.19}$$

By Lemma 2.1 we have $u_0 \leq U_\lambda$ in Ω. Furthermore, from (2.7) and (2.5) we get $cu_0 \leq \varphi_1$ in Ω for some positive constant $c > 0$. Note that cu_0 is still a sub-solution of (2.19) while φ_1 is a super-solution of (2.19). By standard elliptic arguments, problem (2.19) has a solution $u \in C^2(\overline{\Omega})$. Multiplying by φ_1 in (2.19) and integrating on Ω we have

$$-\int_\Omega \varphi_1 \Delta u dx = \frac{\lambda_1}{2} \int_\Omega u\varphi_1 dx,$$

that is,

$$\lambda_1 \int_\Omega u\varphi_1 dx = -\int_\Omega u\Delta\varphi_1 dx = \frac{\lambda_1}{2} \int_\Omega u\varphi_1 dx.$$

The above equality yields $\int_\Omega u\varphi_1 dx = 0$, but this is clearly a contradiction, since u and φ_1 are both positive on Ω. It follows that $(P)^+$ has no classical solutions for $0 < \lambda \leq \lambda_0$.

Step 3: Dependence on $\lambda > 0$. Set

$$A = \left\{\lambda > 0;\ \text{problem } (P)^+ \text{ has at least one classical solution}\right\}.$$

From the above arguments we deduce that A is nonempty and $\lambda^* := \inf A$ is positive. We show that if $\lambda \in A$, then $(\lambda, +\infty) \subseteq A$. To this aim, let $\lambda_1 \in A$ and $\lambda_2 > \lambda_1$. If u_{λ_1} is a solution of $(P)^+$ with $\lambda = \lambda_1$, then u_{λ_1} is a sub-solution of $(P)^+$ with $\lambda = \lambda_2$ while U_{λ_2} defined in (2.4) for $\lambda = \lambda_2$ is a super-solution. Moreover, we have

$$\Delta U_{\lambda_2} + \lambda_2 f(x, U_{\lambda_2}) \leq 0 \leq \Delta u_{\lambda_1} + \lambda_2 f(x, u_{\lambda_1}) \quad \text{in } \Omega,$$

$$U_{\lambda_2}, u_{\lambda_1} > 0 \quad \text{in } \Omega,$$

$$U_{\lambda_2} = u_{\lambda_1} = 0 \quad \text{on } \partial\Omega,$$

$$\Delta U_{\lambda_2} \in L^1(\Omega).$$

Again by Lemma 2.1 we get $u_{\lambda_1} \leq U_{\lambda_2}$ in Ω. Therefore, problem $(P)^+$ with $\lambda = \lambda_2$ has at least one classical solution. Since $\lambda \in A$ was arbitrary, we conclude that $(\lambda^*, +\infty) \subset A$. This completes the proof of (i).

(ii) *Step 1: Existence of a solution for λ large.*

According to Lemma 2.2, there exists $\lambda^* > 0$ such that $(P)^+$ has a sub-solution $\underline{u}_\lambda$ for $\lambda > \lambda^*$ and $\mu = -1$. Then $\underline{u}_\lambda$ is also a sub-solution in case $\mu = +1$, provided $\lambda > \lambda^*$. Let us construct now a super-solution. By [27, Lemma 4], for all $\lambda > \lambda^*$ there exists $v_\lambda \in C^2(\overline{\Omega})$ a solution of

$$\begin{cases} -\Delta v = \lambda f(x, v) + 1 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $0 < a < 1$, we can choose $M = M(\lambda) > 1$ large enough such that $M > M^a |\nabla v_\lambda|^a$ in Ω. Then, using $(f1)$ we obtain

$$-\Delta(Mv_\lambda) = \lambda M f(x, v_\lambda) + M \geq \lambda f(x, Mv_\lambda) + |\nabla(Mv_\lambda)|^a \quad \text{in } \Omega.$$

Hence $\overline{u}_\lambda := Mv_\lambda \in C^2(\overline{\Omega})$ is a super-solution of $(P)^+$ for all $\lambda > \lambda^*$. On the other hand, since $\Delta\overline{u}_\lambda + \lambda f(x, \overline{u}_\lambda) \leq 0 \leq \Delta\underline{u}_\lambda + \lambda f(x, \underline{u}_\lambda)$ in Ω, by Lemma 2.1 we get $\underline{u}_\lambda \leq \overline{u}_\lambda$ and finally, problem $(P)^+$ has at least one solution for all $\lambda > \lambda^*$.

Step 2: Nonexistence for $\lambda > 0$ small. We first extend Lemma 2.1 in the following way :

Lemma 2.3. *Let $0 < a < 1$ and $\Psi : \overline{\Omega} \times (0, +\infty) \to \mathbb{R}$ be a Hölder continuous function such that the mapping $(0, +\infty) \ni s \longmapsto \dfrac{\Psi(x,s)}{s}$ is strictly decreasing for each $x \in \Omega$. Assume that there exist v, $w \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

(a) $\quad \Delta w + \Psi(x, w) + |\nabla w|^a \leq 0 \leq \Delta v + \Psi(x, v) + |\nabla v|^a$ in Ω;

(b) $\quad v, w > 0$ in Ω and $v < w$ on $\partial\Omega$.

Then $v \leq w$ in Ω.

Proof. Assume by contradiction that the inequality $v \leq w$ does not hold throughout Ω and let $\varphi = \frac{v}{w}$. Clearly, $\varphi < 1$ on $\partial\Omega$ and

$$-\nabla \cdot \left[w^2 \nabla\varphi\right] = -w\Delta v + v\Delta w.$$

Let $x_0 \in \Omega$ denote a point of maximum of φ. In particular $\nabla\varphi(x_0) = 0$, $-\Delta\varphi(x_0) \geq 0$ and it follows that

$$0 \leq [-w\Delta v + v\Delta w](x_0).$$

Since $w(x_0) < v(x_0)$, it follows from assumption (a), the properties of Ψ and the above inequality that

$$0 < \left[|\nabla v|^a w - |\nabla w|^a v\right](x_0).$$

Since $\nabla\varphi(x_0) = 0$, we finally obtain

$$0 < \left[\left(\frac{v}{w}\right)^a w - v\right] |\nabla w|^a(x_0) = v^a \left(w^{1-a} - v^{1-a}\right) |\nabla w|^a(x_0),$$

contradicting $w(x_0) < v(x_0)$. □

Next, we assume by contradiction that there exists a sequence of solutions u_n of (P^+) associated to a parameter $\lambda_n \to 0^+$. A simple calculation shows that $w(x) = A(R^2 - |x|^2)$ is positive and satisfies the inequality $\Delta w + f(x, w) + |\nabla w|^a \leq 0$ in Ω, where $A, R > 0$ are large constants. In particular, it follows from Lemma 2.3 that $0 < u_n \leq w$ whenever $\lambda_n \leq 1$. Let $x_n \in \Omega$ denote a maximum point of u_n. Then $\nabla u_n(x_n) = 0$ and $-\Delta u_n(x_n) \geq 0$. Letting $d_n = d(x_n)$, $M_n = u_n(x_n)$, it follows from (P^+) that

$$p(d_n) g(M_n) \leq \lambda_n f(x_n, M_n) \leq C\lambda_n,$$

which yields a contradiction as $n \to \infty$.

The rest of the proof of (ii) follows as in the case $\mu = -1$ and Theorem 2.2 is now complete. □

3 The problem $(P)^-$

3.1 A nonexistence result

We first prove :

Theorem 3.1. *Assume that* $\int_0^1 tp(t)dt = +\infty$. *Then the inequality boundary value problem*

$$\begin{cases} -\Delta u + C|\nabla u|^2 \geq p(d(x))g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

has no classical solutions.

As a direct consequence, we obtain :

Corollary 3.1. *Assume that* $\int_0^1 tp(t)dt = +\infty$. *Then the problem* $(P)^-$ *has no classical solutions.*

Proof of the theorem. It suffices to prove the Theorem only for $C > 0$. We argue by contradiction and assume that there exists $u \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution of (3.1). Using $(g1)$, we can find $c_1 > 0$ such $\underline{u} := c_1\varphi_1$ verifies

$$-\Delta \underline{u} + C|\nabla \underline{u}|^2 \geq p(d(x))g(\underline{u}) \quad \text{in } \Omega.$$

Since g is decreasing, we easily obtain

$$u \geq \underline{u} \quad \text{in } \Omega. \tag{3.2}$$

We make in (3.1) the change of variable $v = 1 - e^{-Cu}$. Therefore

$$\begin{cases} -\Delta v = C(1-v)\left(C|\nabla u|^2 - \Delta u\right) \geq C(1-v)p(d(x))g\left(-\dfrac{\ln(1-v)}{C}\right) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

In order to avoid the singularities in (3.3) let us consider the approximated problem

$$\begin{cases} -\Delta v = C(1-v)p(d(x))g\left(\varepsilon - \dfrac{\ln(1-v)}{C}\right) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

with $0 < \varepsilon < 1$. Clearly v is a super-solution of (3.4). Furthermore, by (3.2) and the fact that $\lim_{t\to 0^+} \frac{1-e^{-Ct}}{t} = C > 0$, there exists $c_2 > 0$ such that $v \geq c_2\varphi_1$ in Ω. On the other hand, there exists $0 < c < c_2$ such that $c\varphi_1$ is a sub-solution of (3.4) and obviously $c\varphi_1 \leq v$ in Ω. Then, the problem (3.4) has a solution $v_\varepsilon \in C^2(\overline{\Omega})$ such that

$$c\varphi_1 \leq v_\varepsilon \leq v \quad \text{ in } \Omega. \tag{3.5}$$

Multiplying by φ_1 in (3.4) and integrating we find

$$\lambda_1 \int_\Omega \varphi_1 v_\varepsilon dx = C \int_\Omega (1 - v_\varepsilon)\varphi_1 p(d(x)) g\left(\varepsilon - \frac{\ln(1 - v_\varepsilon)}{C}\right) dx.$$

Using (3.5) we obtain

$$\begin{aligned} M =: \lambda_1 \int_\Omega \varphi_1 v dx &\geq C \int_\Omega (1 - v)\varphi_1 p(d(x)) g\left(-\frac{\ln(1 - v)}{C}\right) dx \\ &\geq C_1 \int_{\Omega_\delta} \varphi_1 p(d(x)) dx, \end{aligned} \tag{3.6}$$

where $\Omega_\delta \supset \{x \in \Omega;\ d(x) < \delta\}$, for some $\delta > 0$ sufficiently small. Since $\varphi_1(x)$ behaves like $d(x)$ in Ω_δ and $\int_0^1 tp(t)dt = +\infty$, by (3.6) we find a contradiction. Hence, problem (3.1) has no classical solutions and the proof is now complete. □

3.2 Existence results for $(P)^-$ in the sublinear case on f

Our aim here is to give existence results concerning $(P)^-$ in case where f is sublinear. Nevertheless, we prove that condition (1.5) suffices to guarantee the existence of a classical solution for μ belonging to a certain range.

In this case the existence of a solution is strongly dependent on the exponent a. To better understand this dependence, we assume $\lambda = 1$ but the same results hold for any $\lambda > 0$ (note only that the bifurcation point μ^* in the following theorem is dependent on λ).

Theorem 3.2. *Assume $\lambda = 1$, $\int_0^1 tp(t)dt < +\infty$ and conditions $(f1)$, $(f2)$, $(g1)$ and $0 < a \leq 2$ are fulfilled.*

(i) *If $0 < a < 1$, then problem $(P)^-$ has at least one solution, for all $\mu \in \mathbb{R}$;*

(ii) *If $1 < a \leq 2$, then there exists $\mu^* > 0$ such that $(P)^-$ has at least one classical solution for all $\mu < \mu^*$ and no solution exists if $\mu > \mu^*$.*

As a direct consequence, we obtain the following corollary, which can be compared to Theorem 2.2 :

Corollary 3.2. *Assume $\mu = \pm 1$, $\int_0^1 tp(t)dt < +\infty$ and conditions $(f1)$, $(f2)$, $(g1)$ and $0 < a \leq 2$ are fulfilled.*

(i) *If* $0 < a < 1$, *then problem* $(P)^-$ *has at least one solution, for all* $\lambda > 0$;

(ii) *If* $< 1 < a \leq 2$ *and* $\mu = -1$, *then problem* $(P)^-$ *has at least one solution, for all* $\lambda > 0$;

(iii) *If* $1 < a \leq 2$ *and* $\mu = +1$, *then there exists* $\lambda^* > 0$ *such that* $(P)^-$ *has at least one classical solution for all* $\lambda > \lambda^*$ *and no solution exists if* $\lambda < \lambda^*$.

Proof of the theorem. (i) Case $\mu > 0$. By [27, Lemma 4] there exists a classical solution ζ of the problem

$$\begin{cases} -\Delta\zeta = f(x,\zeta) & \text{in } \Omega, \\ \zeta > 0 & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Using the regularity of f we have $\zeta \in C^2(\overline{\Omega})$. Then, ζ is a sub-solution of $(P)^-$ provided $\mu > 0$. We focus now on finding a super-solution $\overline{u}_\mu$ of $(P)^-$ such that $\zeta \leq \overline{u}_\mu$ in Ω.

Let H be the solution of (1.3). Since H is concave, there exists $H'(0+) \in (0, +\infty]$. Taking $0 < b < 1$ small enough, we can assume that $H' > 0$ in $(0, b]$, so H is increasing on $[0, b]$. Multiplying by H' in (1.3) and integrating on $[t, b]$, we find

$$(H')^2(t) - (H')^2(b) = 2\int_t^b p(s)g(H(s))H'(s)ds \leq 2p(t)\int_{H(t)}^{H(b)} g(\tau)d\tau. \tag{3.8}$$

Using the monotonicity of g it follows that

$$(H')^2(t) \leq 2H(b)p(t)g(H(t)) + (H')^2(b), \quad \text{for all } \ 0 < t \leq b. \tag{3.9}$$

Hence, there exist $C_1, C_2 > 0$ such that

$$(H')(t) \leq C_1 p(t)g(H(t)), \quad \text{for all } 0 < t \leq b \tag{3.10}$$

and

$$(H')^2(t) \leq C_2 p(t)g(H(t)), \quad \text{for all } 0 < t \leq b. \tag{3.11}$$

Now we can proceed to construct a super-solution for $(P)^-$. First, we fix $c > 0$ such that

$$c\varphi_1 \leq \min\{b, d(x)\} \quad \text{ in } \ \Omega. \tag{3.12}$$

By Hopf's maximum principle, there exist $\omega \subset\subset \Omega$ and $\delta > 0$ such that

$$|\nabla\varphi_1| > \delta \quad \text{ in } \ \Omega \setminus \omega. \tag{3.13}$$

Moreover, since

$$\lim_{d(x)\to 0^+} \left\{c^2 p(c\varphi_1)g(H(c\varphi_1))|\nabla\varphi_1|^2 - 3f(x, H(c\varphi_1))\right\} = +\infty,$$

we can assume that

$$c^2 p(c\varphi_1) g(H(c\varphi_1))|\nabla\varphi_1|^2 \geq 3f(x, H(c\varphi_1)) \quad \text{in } \Omega \setminus \omega. \tag{3.14}$$

Let $M > 1$ be such that

$$Mc^2\delta^2 > 3. \tag{3.15}$$

Since $H'(0+) > 0$ and $0 < a < 1$, we can choose $M > 1$ such that

$$M\frac{(c\delta)^2}{C_1} H'(c\varphi_1) \geq 3\mu(McH'(c\varphi_1)|\nabla\varphi_1|)^a \quad \text{in } \Omega \setminus \omega,$$

where C_1 is the constant appearing in (3.10). By (3.10), (3.13) and (3.15) we derive

$$Mc^2 p(c\varphi_1) g(H(c\varphi_1))|\nabla\varphi_1|^2 \geq 3\mu(McH'(c\varphi_1)|\nabla\varphi_1|)^a \quad \text{in } \Omega \setminus \omega. \tag{3.16}$$

Since g is decreasing and $H'(c\varphi_1) > 0$ in $\overline{\omega}$, there exists $M > 0$ such that

$$Mc\lambda_1\varphi_1 H'(c\varphi_1) \geq 3p(d(x))g(H(c\varphi_1)) \quad \text{in } \omega. \tag{3.17}$$

In the same manner, using $(f2)$ and the fact that $\varphi_1 > 0$ in $\overline{\omega}$, we can choose $M > 1$ large enough such that

$$Mc\lambda_1\varphi_1 H'(c\varphi_1) \geq 3\mu(MH'(c\varphi_1)|\nabla\varphi_1|)^a \quad \text{in } \omega, \tag{3.18}$$

and

$$Mc\lambda_1\varphi_1 H'(c\varphi_1) \geq 3f(x, MH(c\varphi_1)) \quad \text{in } \omega. \tag{3.19}$$

For M satisfying (3.15)-(3.19), we prove that

$$\overline{u}_\mu(x) := MH(c\varphi_1(x)), \quad \text{for all } x \in \Omega, \tag{3.20}$$

is a super-solution of $(P)^-$. We have

$$-\Delta\overline{u}_\mu = Mc^2 p(c\varphi_1) g(H(c\varphi_1))|\nabla\varphi_1|^2 + Mc\lambda_1\varphi_1 H'(c\varphi_1) \quad \text{in } \Omega. \tag{3.21}$$

We first show that

$$Mc^2 p(c\varphi_1) g(H(c\varphi_1))|\nabla\varphi_1|^2 \geq p(d(x))g(\overline{u}_\mu) + f(x, \overline{u}_\mu) + \mu|\nabla\overline{u}_\mu|^a \quad \text{in } \Omega \setminus \omega. \tag{3.22}$$

Indeed, by (3.12), (3.13) and (3.15) we get

$$\begin{aligned} \frac{M}{3}c^2 p(c\varphi_1) g(H(c\varphi_1))|\nabla\varphi_1|^2 &\geq p(d(x))g(H(c\varphi_1)) \\ &\geq p(d(x))g(MH(c\varphi_1)) \\ &= p(d(x))g(\overline{u}_\mu) \quad \text{in } \Omega \setminus \omega. \end{aligned} \tag{3.23}$$

The assumption $(f1)$ and (3.14) produce

$$\begin{aligned}\frac{M}{3}c^2p(c\varphi_1)g(H(c\varphi_1))|\nabla\varphi_1|^2 &\geq Mf(x,H(c\varphi_1))\\ &\geq f(x,MH(c\varphi_1))\\ &= f(x,\overline{u}_\mu) \quad \text{ in } \Omega\setminus\omega.\end{aligned} \tag{3.24}$$

From (3.16) we obtain

$$\begin{aligned}\frac{M}{3}c^2p(c\varphi_1)g(H(c\varphi_1))|\nabla\varphi_1|^2 &\geq \mu(McH'(c\varphi_1)|\nabla\varphi_1|)^a\\ &= \mu|\nabla\overline{u}_\mu|^a \quad \text{ in } \Omega\setminus\omega.\end{aligned} \tag{3.25}$$

Now, relation (3.22) follows by (3.23), (3.24) and (3.25).

Next we prove that

$$Mc\lambda_1\varphi_1H'(c\varphi_1) \geq p(d(x))g(\overline{u}_\mu) + f(x,\overline{u}_\mu) + \mu|\nabla\overline{u}_\mu|^a \quad \text{ in } \omega. \tag{3.26}$$

From (3.17) and (3.18) we get

$$\begin{aligned}\frac{M}{3}c\lambda_1\varphi_1H'(c\varphi_1) &\geq p(d(x))g(H(c\varphi_1))\\ &\geq p(d(x))g(MH(c\varphi_1))\\ &= p(d(x))g(\overline{u}_\mu) \quad \text{ in } \omega\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}\frac{M}{3}c\lambda_1\varphi_1H'(c\varphi_1) &\geq \mu(McH'(c\varphi_1)|\nabla\varphi_1|)^a\\ &= \mu|\nabla\overline{u}_\mu|^a \quad \text{ in } \omega.\end{aligned} \tag{3.28}$$

Finally, from (3.19) we derive

$$\frac{M}{3}c\lambda_1\varphi_1H'(c\varphi_1) \geq f(x,MH(c\varphi_1)) = f(x,\overline{u}_\mu) \quad \text{ in } \omega. \tag{3.29}$$

Clearly, relation (3.26) follows from (3.27), (3.28) and (3.29).

Combining (3.21) with (3.22) and (3.26) we conclude that $\overline{u}_\mu$ is a super-solution of $(P)^-$. Thus, by Lemma 2.1 we obtain $\zeta \leq \overline{u}_\mu$ in Ω and by sub and super-solution method it follows that $(P)^-$ has at least one classical solution for all $\mu > 0$.

Case $\mu \leq 0$. We fix $\nu > 0$ and let $u_\nu \in C^2(\Omega)\cap C(\overline{\Omega})$ be a solution of $(P)^-$ for $\mu = \nu$. Then u_ν is a super-solution of $(P)^-$ for all $\mu \leq 0$. Set

$$m := \inf_{(x,t)\in\overline{\Omega}\times(0,+\infty)} \Big(p(d(x))g(t) + f(x,t)\Big).$$

Since $\lim_{t\to 0^+} g(t) = +\infty$ and the mapping $(0,+\infty) \ni t \longmapsto \min_{x\in\overline{\Omega}} f(x,t)$ is positive and nondecreasing, we deduce that m is a positive real number. Consider the problem

$$\begin{cases} -\Delta v = m + \mu|\nabla v|^a & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.30}$$

Clearly zero is a sub-solution of (3.30). Since $\mu \leq 0$, the solution w of the problem

$$\begin{cases} -\Delta w = m & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

is a super-solution of (3.30). Hence, (3.30) has at least one solution $v \in C^2(\Omega) \cap C(\overline{\Omega})$. We claim that $v > 0$ in Ω. Indeed, if not, we deduce that $\min_{x\in\overline{\Omega}} v$ is achieved at some point $x_0 \in \Omega$. Then $\nabla v(x_0) = 0$ and

$$-\Delta v(x_0) = m + \mu|\nabla v(x_0)|^a = m > 0, \quad \text{contradiction.}$$

Therefore, $v > 0$ in Ω. It is easy to see that v is sub-solution of $(P)^-$ and $-\Delta v \leq m \leq -\Delta u_\nu$ in Ω, which gives $v \leq u_\nu$ in Ω. Again by the sub and super-solution method we conclude that $(P)^-$ has at least one classical solution $u_\mu \in C^2(\Omega) \cap C(\overline{\Omega})$.

(ii) The proof follows the same steps as above. The only difference is that (3.16) and (3.18) are no more valid for any $\mu > 0$. The main difficulty when dealing with estimates like (3.16) is that $H'(c\varphi_1)$ may blow-up at the boundary. However, combining the assumption $1 < a \leq 2$ with (3.11), we can choose $\mu > 0$ small enough such that (3.16) and (3.18) hold. This implies that the problem $(P)^-$ has a classical solution provided $\mu > 0$ is sufficiently small.

Set

$$A = \left\{\mu > 0;\ \text{problem } (P)^- \text{ has at least one classical solution}\right\}.$$

From the above arguments, A is nonempty. Let $\mu^* = \sup A$. We first claim that if $\mu \in A$, then $(0, \mu) \subseteq A$. To this aim, let $\mu_1 \in A$ and $0 < \mu_2 < \mu_1$. If u_{μ_1} is a solution of $(P)^-$ with $\mu = \mu_1$, then u_{μ_1} is a super-solution of $(P)^-$ with $\mu = \mu_2$, while ζ defined in (3.7) is a sub-solution. Using Lemma 2.1 once more, we get $\zeta \leq u_{\mu_1}$ in Ω so $(P)^-$ has at least one classical solution for $\mu = \mu_2$. This proves the claim. Since $\mu_1 \in A$ was arbitrary, we conclude that $(0, \mu^*) \subset A$.

Next, we prove that $\mu^* < +\infty$. To this aim, we use the following result which is a consequence of Theorem 2.1 in [3].

Lemma 3.1. *Assume that $a > 1$. Then there exists a positive number $\bar{\sigma}$ such that the problem*

$$\begin{cases} -\Delta v \geq |\nabla v|^a + \sigma & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.31}$$

has no solutions for $\sigma > \bar{\sigma}$.

Consider $\mu \in A$ and let u_μ be a classical solution of $(P)^-$. Set $v = \mu^{1/(a-1)} u_\mu$. Using our assumption $1 < a \leq 2$, we deduce that v fulfills

$$\begin{cases} -\Delta v \geq |\nabla v|^a + m\mu^{1/(a-1)} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.32}$$

According to Lemma 3.1, we obtain $m\mu^{1/(a-1)} \leq \bar{\sigma}$, that is, $\mu \leq \left(\frac{\bar{\sigma}}{m}\right)^{a-1}$. This means that $\mu^* \leq \left(\frac{\bar{\sigma}}{m}\right)^{a-1}$, hence μ^* is finite. The existence of a solution in the case $\mu \leq 0$ can be achieved in the same manner as above.

This finishes the proof of Theorem 3.2. □

In what follows we discuss the case $a = 1$. Note that the method used in Theorem 3.2 does not apply here for large values of μ.

Assume that $\Omega = B_R(0)$ for some $R > 0$, where $B_R(0) = \{x \in \mathbb{R}^N;\ |x| < R\}$. In this case and with $\lambda = 1$, problem $(P)^-$ becomes

$$\begin{cases} -\Delta u = p(R - |x|)g(u) + f(x,u) + \mu|\nabla u| & |x| < R, \\ u > 0 & |x| < R, \\ u = 0 & |x| = R. \end{cases} \tag{3.33}$$

Theorem 3.3. *Assume that $\int_0^1 tp(t)dt < +\infty$. Then the problem (3.33) has at least one solution for all $\mu \in \mathbb{R}$.*

Proof. The case $\mu \leq 0$ is the same as in the proof of Theorem 3.2 (i). In what follows, we assume that $\mu > 0$. Using Theorem 3.2 (i) it is easy to see that there exists $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta \underline{u} = p(R - |x|)g(\underline{u}) & |x| < R, \\ \underline{u} > 0 & |x| < R, \\ \underline{u} = 0 & |x| = R. \end{cases}$$

It is obvious that $\underline{u}$ is a sub-solution of (3.33) for all $\mu > 0$. In order to provide a super-solution of (3.33) we consider the problem

$$\begin{cases} -\Delta u = p(R - |x|)g(u) + 1 + \mu|\nabla u| & |x| < R, \\ u > 0 & |x| < R, \\ u = 0 & |x| = R. \end{cases} \tag{3.34}$$

We need the following auxiliary result.

Lemma 3.2. *Problem (3.34) has at least one solution.*

Proof. We are looking for radially symmetric solution u of (3.34), that is, $u = u(r)$, $0 \leq r =$

$|x| \leq R$. In this case, problem (3.34) becomes

$$\begin{cases} -u'' - \dfrac{N-1}{r}u'(r) = p(R-r)g(u(r)) + 1 + \mu|u'(r)| & 0 \leq r < R, \\ u > 0 & 0 \leq r < R, \\ u(R) = 0. \end{cases} \tag{3.35}$$

This implies

$$-(r^{N-1}u'(r))' \geq 0 \quad \text{for all } \ 0 \leq r < R,$$

which yields $u'(r) \leq 0$ for all $0 \leq r < R$. Then (3.35) gives

$$-\left(u'' + \frac{N-1}{r}u'(r) + \mu u'(r)\right) = p(R-r)g(u(r)) + 1, \quad 0 \leq r < R.$$

We obtain

$$-(e^{\mu r}r^{N-1}u'(r))' = e^{\mu r}r^{N-1}\psi(r, u(r)), \quad 0 \leq r < R, \tag{3.36}$$

where

$$\psi(r,t) = p(R-r)g(t) + 1, \quad (r,t) \in [0,R) \times (0, +\infty).$$

From (3.36) we get

$$u(r) = u(0) - \int_0^r e^{-\mu t}t^{-N+1}\int_0^t e^{\mu s}s^{N-1}\psi(s, u(s))dsdt, \quad 0 \leq r < R. \tag{3.37}$$

On the other hand, in view of Theorem 3.2 and using the fact that g is decreasing, there exists a unique solution $w \in C^2(B_R(0)) \cap C(\overline{B}_R(0))$ of the problem

$$\begin{cases} -\Delta w = p(R - |x|)g(w) + 1 & |x| < R, \\ w > 0 & |x| < R, \\ w = 0 & |x| = R. \end{cases} \tag{3.38}$$

Clearly, w is a sub-solution of (3.34). Due to the uniqueness and to the symmetry of the domain, w is radially symmetric, so, $w = w(r)$, $0 \leq r = |x| \leq R$. As above we get

$$w(r) = w(0) - \int_0^r t^{-N+1}\int_0^t s^{N-1}\psi(s, w(s))dsdt, \quad 0 \leq r < R. \tag{3.39}$$

We claim that there exists a solution $v \in C^2[0,R) \cap C[0,R]$ of (3.37) such that $v > 0$ in $[0,R)$. Let $A = w(0)$ and define the sequence $(v_k)_{k\geq 1}$ by

$$\begin{cases} v_k(r) = A - \displaystyle\int_0^r e^{-\mu t}t^{-N+1}\int_0^t e^{\mu s}s^{N-1}\psi(s, v_{k-1}(s))dsdt, & 0 \leq r < R,\ k \geq 1, \\ v_0 = w. \end{cases} \tag{3.40}$$

Note that v_k is decreasing in $[0,R)$ for all $k \geq 0$. From (3.39) and (3.40) we easily check that $v_1 \geq v_0$ and by induction we deduce $v_k \geq v_{k-1}$ for all $k \geq 1$. Hence

$$w = v_0 \leq v_1 \leq ... \leq v_k \leq ... \leq A \quad \text{in } B_R(0).$$

Thus, there exists $v(r) := \lim_{k\to\infty} v_k(r)$, for all $0 \leq r < R$ and $v > 0$ in $[0,R)$. We can now pass to the limit in (3.40) in order to get that v is a solution of (3.37). By classical regularity results we also obtain $v \in C^2[0,R) \cap C[0,R]$. This proves the claim.

We have obtained a super-solution v of (3.34) such that $v \geq w$ in $B_R(0)$. So, the problem (3.34) has at least one solution and the proof of our Lemma is now complete. □

Let u be a solution of the problem (3.34). For $M > 1$ we have

$$\begin{aligned} -\Delta(Mu) &= Mp(R-|x|)g(u) + M + \mu|\nabla(Mu)| \\ &\geq p(R-|x|)g(Mu) + M + \mu|\nabla(Mu)|. \end{aligned} \tag{3.41}$$

Since f is sublinear, we can choose $M > 1$ such that

$$M \geq f(x, M|u|_\infty) \quad \text{in } B_R(0).$$

Then $\overline{u}_\mu := Mu$ satisfies

$$-\Delta\overline{u}_\mu \geq p(R-|x|)g(\overline{u}_\mu) + f(x,\overline{u}_\mu) + \mu|\nabla\overline{u}_\mu| \quad \text{in } B_R(0).$$

It follows that $\overline{u}_\mu$ is a super-solution of (3.33). Since g is decreasing we easily deduce $\underline{u} \leq \overline{u}_\mu$ in $B_R(0)$ so, problem $(P)^-$ has at least one solution.

The proof of Theorem 3.3 is now complete. □

3.3 Existence results for $(P)^-$ in the linear case on f

In this section we turn to the study of problem $(P)^-$ when f is linear. More precisely, we consider the problem

$$\begin{cases} -\Delta u = p(d(x))g(u) + \lambda u + \mu|\nabla u|^a & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.42}$$

where $\lambda > 0$ and p, g are as in the previous sections. We assume in what follows that $0 < a < 1$.

Note that the existence results established in [27, Lemma 4] or [28] do not apply here since the mapping

$$\Psi(x,t) = p(d(x))g(t) + \lambda t, \qquad (x,t) \in \Omega \times (0,+\infty),$$

is not defined on $\partial\Omega \times (0,+\infty)$.

Theorem 3.4. *Assume that $\int_0^1 tp(t)dt < +\infty$ and conditions $(g1)$, $0 < a < 1$ are fulfilled. Then for $\mu \geq 0$ the problem (3.42) has solutions if and only if $\lambda < \lambda_1$.*

Proof. Fix $\lambda \in (0, \lambda_1)$ and $\mu \geq 0$. By Theorem 3.2 (i) there exists $u \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution of the problem

$$\begin{cases} -\Delta u = p(d(x))g(u) + \mu|\nabla u|^a & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Obviously, $\underline{u}_{\lambda\mu} := u$ is a sub-solution of (3.42). Since $\lambda < \lambda_1$, there exists $v \in C^2(\overline{\Omega})$ such that

$$\begin{cases} -\Delta v = \lambda v + 2 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $0 < a < 1$, we can choose $M > 0$ large enough such that

$$M > \lambda |u|_\infty \text{ and } \ M > \mu(M|\nabla v|)^a \text{ in } \Omega.$$

Then $w := Mv$ satisfies

$$-\Delta w \geq \lambda(u + w) + \mu|\nabla w|^a \quad \text{in } \Omega.$$

We claim that $\overline{u}_{\lambda\mu} := u + w$ is a super-solution of (3.42). Indeed, we have

$$-\Delta \overline{u}_{\lambda\mu} \geq p(d(x))g(u) + \lambda \overline{u}_{\lambda\mu} + \mu|\nabla u|^a + \mu|\nabla w|^a \quad \text{in } \Omega. \tag{3.43}$$

Using the assumption $0 < a < 1$ one can easily deduce

$$t_1^a + t_2^a \geq (t_1 + t_2)^a, \quad \text{for all } \ t_1, t_2 \geq 0.$$

Hence

$$|\nabla u|^a + |\nabla w|^a \geq (|\nabla u| + |\nabla w|)^a \geq |\nabla(u + w)|^a \quad \text{in } \Omega. \tag{3.44}$$

Combining (3.43) with (3.44) we obtain

$$-\Delta \overline{u}_{\lambda\mu} \geq p(d(x))g(\overline{u}_{\lambda\mu}) + \lambda \overline{u}_{\lambda\mu} + \mu|\nabla \overline{u}_{\lambda\mu}|^a \quad \text{in } \Omega.$$

Hence, $(\underline{u}_{\lambda\mu}, \overline{u}_{\lambda\mu})$ is an ordered pair of sub and super-solution of (3.42), so there exists a classical solution $u_{\lambda\mu}$ of (3.42), provided $\mu \geq 0$ and $0 < \lambda < \lambda_1$. Assume by contradiction that there exist $\lambda \geq \lambda_1$ and $\mu \geq 0$ such that the problem (3.42) has a classical solution $u_{\lambda\mu}$. If $m = \min_{x \in \overline{\Omega}} p(d(x))g(u_{\lambda\mu}) > 0$ it follows that $u_{\lambda\mu}$ is a super-solution of

$$\begin{cases} -\Delta u = \lambda u + m & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.45}$$

Clearly, zero is a sub-solution of (3.45), so there exists a classical solution u of (3.45) such that $u \leq u_{\lambda\mu}$ in Ω. By maximum principle and elliptic regularity we get $u > 0$ in Ω and $u \in C^2(\overline{\Omega})$. To raise a contradiction, we proceed as in the proof of Theorem 2.2 (ii).

Multiplying by φ_1 in (3.45) and then integrating over Ω we find

$$-\int_\Omega \varphi_1 \Delta u = \lambda \int_\Omega u\varphi_1 + m\int_\Omega \varphi_1.$$

This implies $\lambda_1 \int_\Omega u\varphi_1 = \lambda \int_\Omega u\varphi_1 + m\int_\Omega \varphi_1$, which is a contradiction, since $\lambda \geq \lambda_1$ and $m > 0$. The proof of Theorem 3.4 is now complete. □

3.4 An application

We show here how the results in this section applies to the problem (1.6). Recall that if $\int_0^1 tp(t)dt < +\infty$ and μ belongs to a certain range, then Theorem 3.2 asserts that (1.6) has at least one classical solution u_μ satisfying $u_\mu \leq MH(c\varphi_1)$ in Ω, for some $M, c > 0$. Here H is the solution of

$$\begin{cases} H''(t) = -t^{-\alpha}H^{-\beta}(t), \quad \text{for all } 0 < t \leq b < 1, \\ H, H' > 0 \quad \text{in } (0, b], \\ H(0) = 0. \end{cases} \tag{3.46}$$

With the same idea as in the proof of Theorem 3.2, we can show that there exists $m > 0$ small enough such that $v := mH(c\varphi_1)$ satisfies

$$-\Delta v \leq d(x)^{-\alpha} v^{-\beta} \quad \text{in } \Omega. \tag{3.47}$$

Indeed, we have

$$-\Delta v = m[c^{2-\alpha}|\nabla\varphi_1|^2\varphi_1^{-\alpha}H^{-\beta}(c\varphi_1) + \lambda_1 c\varphi_1 H'(c\varphi_1)] \quad \text{in } \Omega.$$

Using (2.7) and (3.10), there exist two positive constants $c_1, c_2 > 0$ such that

$$-\Delta v \leq m[c_1|\nabla\varphi_1|^2 + c_2\varphi_1]d(x)^{-\alpha}H^{-\beta}(c\varphi_1) \quad \text{in } \Omega.$$

Clearly (3.47) holds if we choose $m > 0$ small enough such that $m[c_1|\nabla\varphi_1|^2 + c_2\varphi_1] < 1$ in Ω. Moreover, v is a sub-solution of (1.6) for all $\mu > 0$ and one can easily see that $v \leq u_\mu$ in Ω. Hence

$$mH(c\varphi_1) \leq u_\mu \leq MH(c\varphi_1) \quad \text{in } \Omega. \tag{3.48}$$

Now, a careful analysis of (3.46) together with (3.48) is used in order to obtain boundary estimates for the solution of (1.6). Our estimates complete the results in [19, Theorem 2.1] since here the potential $p(d(x))$ blows-up at the boundary.

Theorem 3.5. *The following properties hold true.*

(i) *If $\alpha \geq 2$, then the problem (1.6) has no classical solutions.*

(ii) *If $\alpha < 2$, then there exists $\mu^* \in (0, +\infty]$ (with $\mu^* = +\infty$ if $0 < a < 1$) such that problem (1.6) has at least one classical solution u_μ, for all $-\infty < \mu < \mu^*$. Moreover, for all $0 < \mu < \mu^*$, there exist $0 < \delta < 1$ and $C_1, C_2 > 0$ such that u_μ satisfies*

(ii1) *If $\alpha + \beta > 1$, then*

$$C_1 d(x)^{\frac{2-\alpha}{1+\beta}} \leq u_\mu(x) \leq C_2 d(x)^{\frac{2-\alpha}{1+\beta}}, \quad \text{for all } x \in \Omega; \tag{3.49}$$

(ii2) *If $\alpha + \beta = 1$, then*

$$C_1 d(x)(-\ln d(x))^{\frac{1}{2-\alpha}} \leq u_\mu(x) \leq C_2 d(x)(-\ln d(x))^{\frac{1}{2-\alpha}}, \tag{3.50}$$

for all $x \in \Omega$ with $d(x) < \delta$;

(ii3) *If $\alpha + \beta < 1$, then*

$$C_1 d(x) \leq u_\mu(x) \leq C_2 d(x), \quad \text{for all } x \in \Omega. \tag{3.51}$$

Proof. The existence and nonexistence of a solution to (1.6) follows directly from Theorems 3.1 and 3.2. We next prove the boundary estimates (3.49)-(3.51).

(ii1) Remark that

$$H(t) = \left(\frac{(1+\beta)^2}{(2-\alpha)(\alpha+\beta-1)}\right)^{1/(1+\beta)} t^{\frac{2-\alpha}{1+\beta}}, \qquad t > 0,$$

is a solution of (3.46) provided $\alpha + \beta > 1$. The conclusion in this case follows now from (3.48).

(ii2) Note that in this case problem (3.46) becomes

$$\begin{cases} H''(t) = -t^{-\alpha} H^{\alpha-1}(t), & \text{for all } 0 < t \leq b < 1, \\ H(0) = 0, \\ H > 0 \quad \text{in } (0, b]. \end{cases} \tag{3.52}$$

Since H is concave, it follows that

$$H(t) > tH'(t), \quad \text{for all } 0 < t \leq b. \tag{3.53}$$

Relations (3.52) and (3.53) yield

$$-H''(t) < t^{-1}(H'(t))^{\alpha-1}, \quad \text{for all } 0 < t \leq b.$$

Hence

$$-H''(t)(H'(t))^{1-\alpha} \leq \frac{1}{t}, \quad \text{for all } 0 < t \leq b. \tag{3.54}$$

Integrating in (3.54) over $[t,b]$ we get

$$(H')^{2-\alpha}(t) - (H')^{2-\alpha}(b) \leq (2-\alpha)(\ln b - \ln t), \quad \text{for all } \ 0 < t \leq b.$$

Hence, there exist $c_1 > 0$ and $\delta_1 \in (0,b)$ such that

$$H'(t) \leq c_1(-\ln t)^{\frac{1}{2-\alpha}}, \quad \text{for all } \ 0 < t \leq \delta_1. \tag{3.55}$$

Fix $t \in (0,\delta_1]$. Integrating over $[\varepsilon,t]$, $0 < \varepsilon < t$, in (3.55) we have

$$H(t) - H(\varepsilon) \leq c_1 t(-\ln t)^{\frac{1}{2-\alpha}} + \frac{c_1}{2-\alpha}\int_\varepsilon^t (-\ln s)^{\frac{\alpha-1}{2-\alpha}} ds. \tag{3.56}$$

Note that

$$\int_0^t (-\ln s)^{\frac{\alpha-1}{2-\alpha}} ds < +\infty \qquad \text{and} \quad \lim_{t\to 0^+} \frac{\int_0^t (-\ln s)^{\frac{\alpha-1}{2-\alpha}} ds}{t(-\ln t)^{\frac{1}{2-\alpha}}} = 0. \tag{3.57}$$

Thus, taking $\varepsilon \to 0^+$ in (3.56) we deduce that there exist $c_2 > 0$ and $\delta_2 \in (0,\delta_1)$ such that

$$H(t) \leq c_2 t(-\ln t)^{\frac{1}{2-\alpha}}, \quad \text{for all } \ 0 < t \leq \delta_2. \tag{3.58}$$

From (3.52) and (3.58) we obtain

$$-H''(t) \geq c_2^{\alpha-1} t^{-1}(-\ln t)^{\frac{\alpha-1}{2-\alpha}}, \quad \text{for all } \ 0 < t \leq \delta_2.$$

Integrating over $[t,\delta_2]$ in the above inequality we get

$$H'(t) \geq (2-\alpha)c_2^{\alpha-1}\left[(-\ln t)^{\frac{1}{2-\alpha}} - (-\ln \delta_2)^{\frac{1}{2-\alpha}}\right], \quad \text{for all } \ 0 < t \leq \delta_2.$$

Therefore, there exist $c_3 > 0$ and $\delta_3 \in (0,\delta_2)$ such that

$$H'(t) \geq c_3(-\ln t)^{\frac{1}{2-\alpha}}, \quad \text{for all } \ 0 < t \leq \delta_3.$$

With the same arguments as in (3.55)-(3.58) we obtain $c_4 > 0$ and $\delta_4 \in (0,\delta_3)$ such that

$$H(t) \geq c_4 t(-\ln t)^{\frac{1}{2-\alpha}}, \quad \text{for all } \ 0 < t \leq \delta_4. \tag{3.59}$$

The conclusion of (ii) in Theorem 3.5 follows now from (3.58) and (3.59).

(ii3) Using the fact that $H'(0+) \in (0,+\infty]$ and the inequality (3.53), we get the existence of $c > 0$ such that

$$H(t) > ct, \quad \text{for all } \ 0 < t \leq b.$$

This yields

$$-H''(t) \leq c^{-\beta} t^{-(\alpha+\beta)}, \quad \text{for all } \ 0 < t \leq b.$$

Since $\alpha + \beta < 1$, it follows that $H'(0+) < +\infty$, that is, $H \in C^1[0,b]$. Thus, there exists $c_1, c_2 > 0$ such that

$$c_1 t \leq H(t) \leq c_2 t, \quad \text{for all } \ 0 < t \leq b. \tag{3.60}$$

The conclusion in Theorem 3.5 (iii) follows directly from (3.60) and (3.48).

This completes the proof of Theorem 3.5. □

Acknowledgments. The authors are partially supported by Programme EGIDE-Brancusi PAI 08915PG between Université de Picardie Jules Verne and University of Craiova. This work has been completed while V.R. was visiting the Université de Picardie Jules Verne in February 2006. He thanks Prof. O. Goubet for invitation and for many constructive discussions.

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