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# ON THE CLASSIFICATION OF GALOIS OBJECTS OVER THE QUANTUM GROUP OF A NONDEGENERATE BILINEAR FORM

THOMAS AUBRIOT

**ABSTRACT.** We study Galois and bi-Galois objects over the quantum group of a nondegenerate bilinear form, including the quantum group  $\mathcal{O}_q(SL(2))$ . We obtain the classification of these objects up to isomorphism and some partial results for their classification up to homotopy.

## INTRODUCTION

Hopf-Galois extensions and objects are quantum analogues of principal fibre bundles and torsors. It is in general a difficult problem to classify these objects. Several authors have already contributed to this problem, mainly in the finite dimensional case ([Ma1], [Ma2], [Sa2], [PO], [Bi2]...). In this paper we study a very different class of infinite dimensional Hopf algebras, including the quantum group  $\mathcal{O}_q(SL(2))$  of functions over  $SL_2$ . We obtain the classification up to isomorphism and present some partial results for the classification up to homotopy. Homotopy for Hopf-Galois extensions was introduced by Kassel [K1] and developped with Schneider [KS] in order to classify Galois extensions up to a coarser equivalence relation than isomorphism. This relation is very useful for pointed Hopf algebras but it appears that, when the Hopf algebra is the quantum group  $\mathcal{O}_q(SL(2))$ , the classification up to homotopy is harder to obtain than the one up to isomorphism.

We consider the Hopf algebras  $\mathcal{B}(E)$  introduced by Dubois-Violette and Launer [DL] as the quantum groups of nondegenerate bilinear forms given by invertible matrices  $E$  over a field  $k$ . One simple and interesting example of such a Hopf algebra is the quantum group  $\mathcal{O}_q(SL(2))$  of functions over  $SL_2$ . Bichon [Bi1] has proved that the representation category of each Hopf algebra  $\mathcal{B}(E)$  is monoidally equivalent to the one of  $\mathcal{O}_q(SL(2))$ , where  $q$  is a solution of the equation

$$q^2 + \text{Tr}(E^{-1}E^t)q + 1 = 0.$$

The main ingredient of his proof is the construction of a  $\mathcal{B}(E)$ - $\mathcal{O}_q(SL(2))$ -bi-Galois object  $\mathcal{B}(E, E_q)$  for a well-chosen invertible matrix  $E_q$ . In fact, such Galois objects  $\mathcal{B}(E, F)$  can be defined even when  $k$  is only assumed to be a commutative ring. They are generic in the following sense: if  $k$  is a PID (principal ideal domain), for any  $\mathcal{B}(E)$ -Galois object  $Z$  there exist an integer  $m \geq 2$

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and an invertible matrix  $F \in GL_m(k)$  such that  $Z$  is isomorphic to  $\mathcal{B}(E, F)$ . Moreover, two such  $\mathcal{B}(E)$ -Galois objects  $\mathcal{B}(E, F_1)$  and  $\mathcal{B}(E, F_2)$  are isomorphic if and only if there exists  $P \in GL(k)$  such that  $F_1 = PF_2P^t$  ( $P^t$  denotes the transpose of  $P$ ). In the case when  $k$  is a field, we obtain a full classification up to isomorphism of the Galois objects of  $B(E)$ . As a consequence, the group of  $\mathcal{B}(E)$ -bi-Galois objects is trivial as well as the lazy cohomology group. Note that Ostrik [O] has recently classified module categories over representations of  $\mathcal{O}_q(SL(2))$ , which together with Ulbrich's and Schauenburg's work (see [U1], [U2] and [Sa3]) also yields a classification of Galois objects, but the tools used in [O] are very different from ours.

Concerning the classification up to homotopy, we prove a partial result. Namely, we show that two Galois objects  $\mathcal{B}(E, F_1)$  and  $\mathcal{B}(E, F_2)$  are homotopically equivalent if the matrices  $F_1^{-1}F_1^t$  and  $F_2^{-1}F_2^t$  have the same characteristic polynomial. In particular, any cleft  $\mathcal{O}_q(SL(2))$ -Galois object is homotopically trivial.

The paper is organized as follows. In Section 1 we recall some basic facts on Galois and bi-Galois extensions. Section 2 and 3 are devoted to the isomorphism problem for  $\mathcal{B}(E)$ -Galois objects, while Section 4 deals with the classification up to homotopy.

## 1. HOPF-GALOIS EXTENSIONS AND BI-GALOIS OBJECTS

Let  $k$  be a commutative ring. All objects in this paper belong to the tensor category of  $k$ -modules and the tensor product over  $k$  is denoted by  $\otimes$ . Let  $H$  be a Hopf algebra and  $Z$  be a left  $H$ -comodule algebra with coaction  $\delta : Z \rightarrow H \otimes Z$ . We define the subalgebra  $R = Z^{coH}$  of  **$H$ -coinvariant elements** of  $Z$  by

$$R = \{z \in Z \mid \delta(z) = 1 \otimes z\}.$$

The linear application  $\text{can} : Z \otimes_R Z \rightarrow H \otimes Z$  given by

$$\text{can}(z \otimes z') = \delta(z)(1 \otimes z')$$

for all  $z, z' \in Z$ , is called the **canonical map** of  $Z$ .

If  $Z$  is a left  $H$ -comodule algebra and  $R$  is a subalgebra of  $Z$ , then we say that  $R \subset Z$  is a  **$H$ -Galois extension** if the subalgebra of  $H$ -coinvariant elements is  $R$  and if the canonical map  $\text{can} : Z \otimes_R Z \rightarrow H \otimes Z$  of  $Z$  is an isomorphism. In this case, we also say that  $Z$  is an  $H$ -Galois extension of  $R$ . A Galois extension  $Z$  of  $R$  is said to be **faithfully flat** if  $Z$  is faithfully flat as a right or left  $R$ -module. An  **$H$ -Galois object** is an  $H$ -Galois extension of  $k$  which is  $k$ -faithfully flat.

A **morphism of Galois extensions** between two  $H$ -Galois extensions  $Z$  and  $Z'$  of  $R$  is a morphism of  $H$ -comodule algebras which is the identity on  $R$ . If  $Z'$  is faithfully flat, it is always an isomorphism. We denote  $\text{Gal}_R(H/k)$  the set of isomorphism classes of faithfully flat  $H$ -Galois extensions of  $R$ . If  $Z$  is a faithfully flat  $H$ -Galois extension of  $R$ , its isomorphism class in  $\text{Gal}_R(H/k)$  is denoted by  $[Z]$ . If one of the objects  $R$  or  $k$  is clear, we will omit it from the notation. In the same way, one can define right  $H$ -Galois extensions of  $R$  and

we denote  $\text{Gal}_R^r(H/k)$  the set of isomorphism classes of faithfully flat right  $H$ -Galois extensions. If  $H$  has a bijective antipode, then there is a bijection between the sets  $\text{Gal}_R(H/k)$  and  $\text{Gal}_R^r(H/k)$ .

Recall that, if  $H$  is a Hopf algebra,  $U$  is a right  $H$ -comodule and  $V$  a left  $H$ -comodule, the **cotensor product**  $U \square_H V$  is the kernel of the map

$$\delta_U \otimes \text{id}_V - \text{id}_U \otimes \delta_V : U \otimes V \rightarrow U \otimes H \otimes V,$$

(or the equalizer of the coactions of  $U$  and  $V$ ).

A bilinear map  $\sigma : H \times H \rightarrow k$  is a **right invertible cocycle** for the Hopf algebra  $H$  if  $\sigma$  is convolution-invertible and satisfies the relations

$$\sigma(x_{(1)}y_{(1)}, z)\sigma(x_{(2)}, y_{(2)}) = \sigma(x, y_{(1)}z_{(1)})\sigma(y_{(2)}, z_{(2)})$$

and

$$\sigma(1, x) = \sigma(x, 1) = \varepsilon(x),$$

for all  $x, y, z \in H$ . Here  $\varepsilon$  denotes the counit of  $H$  and we have used Sweedler's notation  $x_{(1)} \otimes x_{(2)}$  for the comultiplication. Note that we use right cocycles whose definition is different from the one of left cocycles (see [Mo]). We denote  $\sigma^{-1}$  the inverse of  $\sigma$  for the convolution ;  $\sigma^{-1}$  is a left cocycle.

Recall ([Mo, Chapter 7]) that if  $H$  is a Hopf algebra,  $\sigma : H \times H \rightarrow k$  an invertible cocycle, one can define the Hopf algebra  $H^\sigma$  as the coalgebra  $H$  with the twisted product

$$x \cdot_\sigma y = \sigma^{-1}(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}\sigma(x_{(3)}, y_{(3)})$$

and the  $H$ -comodule algebra  $H_\sigma$  as the left  $H$ -comodule  $H$  with the twisted product

$$x \cdot_\sigma y = x_{(1)}y_{(1)}\sigma(x_{(2)}, y_{(2)}),$$

for any  $x, y \in H$ . The  $H$ -comodule algebra  $H_\sigma$  is an  $H$ -Galois extension of  $k$  and all such Galois extensions are called **cleft Galois extensions**. If  $H$  is  $k$ -faithfully flat, it is a **cleft Galois object**.

Kassel and Schneider [KS] (see also [K1]) have defined an equivalence relation denoted  $\sim$  and called **homotopy** on the class of faithfully flat Galois extensions of  $R$ . Two Hopf-Galois extensions are homotopy equivalent if there exists a polynomial path between these extensions. More precisely, let  $k[t]$  be the algebra of polynomials with coefficients in the ground ring  $k$ . For any  $k$ -module  $V$ , we denote  $V[t] = V \otimes k[t]$  and for  $i \in \{0, 1\}$  we denote  $[i] : V[t] \rightarrow V$  the  $k$ -linear map sending  $vt^n$  to  $vi^n$ . These two maps  $[i]$  induce two maps

$$[i]_* : \text{Gal}_{R[t]}(H[t], k[t]) \rightarrow \text{Gal}_R(H, k),$$

for  $i = 0, 1$ . We say that two  $H$ -Galois extensions  $Z_0$  and  $Z_1 \in \text{Gal}_R(H/k)$  are **homotopy equivalent** if there exists  $Z \in \text{Gal}_{R[t]}(H[t]/k[t])$  such that  $[i]_*(Z) = Z_i$  for  $i \in \{0, 1\}$ . We denote  $\mathcal{H}_R(H)$  the set of homotopy classes of faithfully flat left  $H$ -Galois extensions of  $R$ .

Kassel and Schneider [KS, Proposition 1.6, Corollary 1.11] have proved that twists of homotopy equivalent Galois objects are still homotopy equivalent. In fact, the twist is a particular case of the cotensor product by a bi-Galois object. We generalize this result now.

Let  $H$  and  $K$  be Hopf algebras. An  $H$ - $K$ -**bi-Galois object** is a  $H$ - $K$ -bi-comodule algebra  $Z$  which is a Galois object with respect to the right and the left coactions. By work of Schauenburg [Sa1] (see also [Sa3]), the set of bi-Galois objects is a groupoid with the multiplication given by the cotensor product. In particular, when  $H = K$ , the cotensor product over  $H = K$  puts a structure of group on the set of isomorphism classes of  $H$ - $H$ -bi-Galois objects. If  $Z$  is an  $H$ - $K$ -bi-Galois object, the cotensor product yields a bijective map  $\varphi_Z : \text{Gal}_k(K) \rightarrow \text{Gal}_k(H)$  defined by

$$\varphi_Z([A]) = [Z \square_K A]$$

for any left  $K$ -Galois object  $A$  (see [Sa1] and [Sa3] for details).

**Proposition 1.** *For any  $H$ - $K$ -bi-Galois object  $Z$ , the map  $\varphi_Z$  induces a bijective map  $\overline{\varphi_Z} : \mathcal{H}_k(K) \rightarrow \mathcal{H}_k(H)$  between the homotopy classes of left  $K$ -Galois objects and of left  $H$ -Galois objects.*

*Proof.* Let  $A_0, A_1$  be homotopically equivalent  $H$ -Galois objects via the  $H[t]$ -Galois object  $A$ . Then  $Z[t] = Z \otimes k[t]$  is an  $H[t]$ - $K[t]$ -bi-Galois object and  $Z[t] \square_{K[t]} A$  is an homotopy between  $Z \square_K A_0$  and  $Z \square_K A_1$ . There exists a  $K$ - $H$ -bi-Galois object  $Z^{-1}$  inverse of  $Z$  for the groupoid structure of bi-Galois objects and the map  $\overline{\varphi_{Z^{-1}}} : \mathcal{H}(H) \rightarrow \mathcal{H}(K)$  induced by  $Z^{-1}$  is the inverse of the map  $\overline{\varphi_Z} : \mathcal{H}(K) \rightarrow \mathcal{H}(H)$  induced by  $Z$ .  $\square$

## 2. THE HOPF ALGEBRA $\mathcal{B}(E)$ AND THE COMODULE ALGEBRA $\mathcal{B}(E, F)$

Let  $k$  be a commutative ring,  $n \geq 1$  an integer and  $E = (E_{ij})_{1 \leq i, j \leq n} \in GL_n(k)$ . Following [DL], we define  $\mathcal{B}_k(E)$  (or  $\mathcal{B}(E)$  when the base ring is clear) as the  $k$ -algebra generated by  $n^2$  variables  $a_{ij}$ ,  $1 \leq i, j \leq n$ , submitted to the matrix relations

$$E^{-1} a^t E a = I_n = a E^{-1} a^t E,$$

where  $E^{-1}$  is the inverse matrix of  $E$ ,  $a$  is the matrix  $(a_{ij})$ ,  $I_n$  the identity matrix of size  $n$  and  $a^t$  denotes the transpose of the matrix  $a$ .

The algebra  $\mathcal{B}(E)$  is a Hopf algebra with comultiplication  $\Delta$  defined by

$$\Delta(a_{ij}) = \sum_{k=1}^n a_{ik} \otimes a_{kj},$$

counit  $\varepsilon$  defined by  $\varepsilon(a_{ij}) = \delta_{ij}$ , for any  $i, j = 1, \dots, n$ , where  $\delta_{ij}$  is Kronecker's symbol, and antipode  $S$  defined by the matrix identity  $S(a) = E^{-1} a^t E$ .

Note that if  $n = 1$ , the Hopf algebra  $\mathcal{B}(E)$  is isomorphic to  $k[\mathbb{Z}/2\mathbb{Z}]$ , whose Galois objects are  $k[\mathbb{Z}/2\mathbb{Z}]_\sigma$  for  $\sigma \in H^2(\mathbb{Z}/2\mathbb{Z}, k^*)$ . In the following, we will only consider the cases where  $n \geq 2$ . In this case, this Hopf algebra is the quantum group of the bilinear (but non necessarily symmetric) form defined by the matrix  $E$ , in the sense that  $\mathcal{B}(E)$  is the universal Hopf algebra such that the bilinear form is a comodule map (for details see [DL]).

If  $q \in k^*$  is an invertible element of the ring  $k$ , let  $E_q \in GL_2(k)$  be the matrix defined by

$$E_q = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix}.$$

The Hopf algebra  $\mathcal{B}(E_q)$  is isomorphic to the Hopf algebra  $\mathcal{O}_q(SL(2))$  (see the definition of  $\mathcal{O}_q(SL(2))$  in [K2]).

Let  $n, m \geq 1$  be integers and let  $E \in GL_n(k)$ ,  $F \in GL_m(k)$  be invertible scalar matrices. Following Bichon [Bi1], we define the algebra  $\mathcal{B}(E, F)$  as the  $k$ -algebra generated by  $n \times m$  variables  $z_{ij}$ ,  $i = 1, \dots, n$ ;  $j = 1, \dots, m$ , submitted to the matrix relations

$$F^{-1}z^t E z = I_m, \quad z F^{-1} z^t E = I_n,$$

where  $z$  is the matrix of generators  $z_{ij}$  and  $I_m, I_n$  are the identity matrices of size  $m, n$  respectively. We consider the  $k$ -algebra morphism  $\delta : \mathcal{B}(E, F) \rightarrow \mathcal{B}(E) \otimes \mathcal{B}(F)$ , defined by

$$(1) \quad \delta(z_{ij}) = \sum_{k=1}^n a_{ik} \otimes z_{kj},$$

for any  $i = 1, \dots, n$  and  $j = 1, \dots, m$ , that endows  $\mathcal{B}(E, F)$  with a left  $\mathcal{B}(E)$ -comodule algebra structure.

In the same way, we have a  $k$ -algebra map  $\rho : \mathcal{B}(E, F) \rightarrow \mathcal{B}(E, F) \otimes \mathcal{B}(F)$  defined by

$$\rho(z_{ij}) = \sum_{k=1}^n z_{ik} \otimes b_{kj},$$

where the  $b_{ij}$ 's stands for the canonical generators of  $\mathcal{B}(F)$ . The algebra morphism  $\rho$  endows  $\mathcal{B}(E, F)$  with a right comodule algebra structure and  $\mathcal{B}(E, F)$  is a  $\mathcal{B}(E)$ - $\mathcal{B}(F)$ -bicomodule algebra.

Bichon has proved [Bi1, Propositions 3.3, 3.4] that if  $k$  is a field and if  $\text{Tr}(E^{-1}E^t) = \text{Tr}(F^{-1}F^t)$ , then  $\mathcal{B}(E, F)$  is a  $\mathcal{B}(E)$ - $\mathcal{B}(F)$ -bi-Galois object. Note that the matrices of form  $F^{-1}F^t$  appear in Riehm's work [R] on the classification of bilinear form. Precisely, for any nondegenerate bilinear map  $\beta : V \times V \rightarrow k$  given by an invertible matrix  $F$ , the matrix  $\sigma = F^{-1}F^t$  is called the **asymmetry** of  $\beta$ . Over a commutative ring, Bichon's result extends to the following proposition.

**Proposition 2.** *The canonical map of  $\mathcal{B}(E, F)$  considered as a left (resp right)  $\mathcal{B}(E)$ -comodule algebra (resp  $\mathcal{B}(F)$ -comodule algebra) is bijective.*

*Moreover, if  $\mathcal{B}(E, F)$  is  $k$ -faithfully flat, it is a  $\mathcal{B}(E)$ - $\mathcal{B}(F)$ -bi-Galois object.*

*Proof.* The proof is the same as for [Bi1, Propositions 3.3, 3.4].  $\square$

Together with Proposition 1, this yields the following corollary.

**Corollary 3.** *Assume that  $k$  is a field. Let  $E$  be an invertible matrix and  $q \in k^*$  such that  $\text{Tr}(E^{-1}E^t) = -q - q^{-1}$ , then there is a bijection between  $\mathcal{H}_k(\mathcal{B}(E))$  and  $\mathcal{H}_k(\mathcal{O}_q(SL(2)))$ .*

### 3. CLASSIFICATION UP TO ISOMORPHISM

The  $\mathcal{B}(E)$ -comodule algebras  $\mathcal{B}(E, F)$  are generic in the following sense.

**Theorem 4.** *Let  $k$  be a PID,  $n \geq 2$  an integer,  $E \in GL_n(k)$  and  $Z$  be a  $\mathcal{B}(E)$ -Galois object. Then there exist an integer  $m \geq 2$  and an invertible matrix  $F \in GL_m(k)$  such that  $\text{Tr}(F^{-1}F^t) = \text{Tr}(E^{-1}E^t)$  and such that  $Z$  is isomorphic to  $\mathcal{B}(E, F)$  as a  $\mathcal{B}(E)$ -Galois object.*

*Proof.* Let  $\text{Comod-}\mathcal{B}(E)$  be the monoidal category of right  $\mathcal{B}(E)$ -comodules, with the tensor product  $\otimes$  over  $k$ , and  $\text{Mod}(k)$  be the monoidal category of  $k$ -modules. Following Ulbrich [U1], [U2] and Schauenburg [Sa3], to any  $\mathcal{B}(E)$ -Galois object  $Z$ , we associate the fibre functor  $\omega_Z : \text{Comod-}\mathcal{B}(E) \rightarrow \text{Mod}(k)$  defined by

$$\omega_Z(V) = V \square_{\mathcal{B}(E)} Z$$

for any  $V \in \text{Comod-}\mathcal{B}(E)$ . The map  $Z \rightarrow \omega_Z$  defines a bijective correspondence between the left  $\mathcal{B}(E)$ -Galois objects (they are by definition faithfully flat) and the exact monoidal functors (= fibre functors)  $\text{Comod-}\mathcal{B}(E) \rightarrow \text{Mod}(k)$ . Moreover, the fibre functor  $\omega_Z$  sends comodules that are finitely generated projective  $k$ -modules to finitely generated projective  $k$ -modules (the functor  $\omega_Z$  preserves the duals). We denote by  $\psi_2 : \omega_Z(V) \otimes \omega_Z(V) \rightarrow \omega_Z(V \otimes V)$  and  $\psi_0 : \omega_Z(k) \rightarrow k$  the monoidal isomorphisms. Note that  $\psi_2 : (V \square_{\mathcal{B}(E)} Z) \otimes (V \square_{\mathcal{B}(E)} Z) \rightarrow (V \otimes V) \square_{\mathcal{B}(E)} Z$  is induced by the multiplication of  $Z$ .

The **fundamental comodule of  $\mathcal{B}(E)$** , denoted  $V_E$ , is the finite free  $k$ -module of rank  $n$  with basis  $(v_1, \dots, v_n)$  and endowed with the  $\mathcal{B}(E)$ -comodule structure defined by  $\delta(v_i) = \sum_{k=1}^n v_k \otimes a_{ki}$  for  $1 \leq i \leq n$ . The linear map  $\beta_E : V_E \otimes V_E \rightarrow k$  defined by  $\beta_E(v_i, v_j) = E_{ij}$  for  $1 \leq i, j \leq n$  is a  $\mathcal{B}(E)$ -comodule morphism and induces a map

$$\overline{\beta_E} : W \otimes W \xrightarrow{\psi_2} \omega_Z(V_E \otimes V_E) \xrightarrow{\omega_Z(\beta_E)} \omega_Z(k) \xrightarrow{\psi_0} k,$$

where  $W = \omega_Z(V_E)$ . Since  $V_E$  is free of finite rank,  $W$  is a finitely generated projective  $k$ -module. The base ring  $k$  being principal, it implies that  $W$  is a free  $k$ -module of finite rank, say  $m$ .

Set  $F_{ij} = \overline{\beta_E}(w_i \otimes w_j)$  for all  $1 \leq i, j \leq m$ . Writing the elements  $(w_j)_{1 \leq j \leq m}$  as elements of  $V_E \otimes Z$  and expanding them in the basis  $(v_1, \dots, v_n)$  of  $V_E$ , we see that there exist  $(t_{ij})_{i=1, \dots, n; j=1, \dots, m} \in Z$  such that  $w_j = \sum_{i=1}^n v_i \otimes t_{ij}$  for any  $j = 1, \dots, m$ . Since  $(w_j)_{1 \leq j \leq m}$  belong to the cotensor product  $V_E \square_{\mathcal{B}(E_q)} Z$ , the elements  $(t_{ij})_{i=1, \dots, n; j=1, \dots, m}$  satisfy the relations

$$(2) \quad \delta(t_{ij}) = \sum_{k=1}^n a_{ik} \otimes t_{kj}$$

for all  $1 \leq i \leq n$  and  $1 \leq j \leq m$ .

Since the monoidal isomorphism  $\psi_2$  is given by the multiplication of  $Z$ , the image of the base  $(w_j)_{1 \leq j \leq m}$  by the map  $\overline{\beta_E}$  is equal to

$$\begin{aligned} F_{ij} &= \overline{\beta_E}(w_i \otimes w_j) \\ &= \psi_0 \circ (\beta_E \otimes \text{id}) \circ \psi_2((\sum_{k=1}^n v_k \otimes t_{ki}) \otimes (\sum_{l=1}^n v_l \otimes t_{lj})) \\ &= \psi_0 \circ (\beta_E \otimes \text{id})(\sum_{k,l=1}^n (v_k \otimes v_l) \otimes t_{ki} t_{lj}) \\ &= \psi_0(\sum_{k,l=1}^n E_{kl} \otimes t_{ki} t_{lj}) \\ &= \sum_{k,l=1}^n E_{kl} t_{ki} t_{lj} \end{aligned}$$

for any  $1 \leq i, j \leq m$ . Putting  $T = (t_{ij})_{i=1, \dots, n; j=1, \dots, m}$  and  $F = (F_{ij})_{1 \leq i, j \leq m}$ , we obtain

$$(3) \quad F = T^t E T.$$

Let us now consider the  $k$ -linear map  $\nu : k \rightarrow V_E \otimes V_E$  defined by

$$\nu(1) = \sum_{i,j=1, \dots, n} E_{ij}^{-1} v_i \otimes v_j,$$

where  $E_{ij}^{-1}$  denotes the  $(i, j)$ -entry of the inverse matrix  $E^{-1}$ . Since this map is a  $\mathcal{B}(E)$ -comodule morphism, it induces a linear map

$$\bar{\nu} : k \xrightarrow{\psi_0^{-1}} \omega_Z(k) \xrightarrow{\omega_Z(\nu)} \omega_Z(V_E \otimes V_E) \xrightarrow{\psi_2^{-1}} \omega_Z(V_E) \otimes \omega_Z(V_E).$$

Let us compute  $\bar{\nu}(1)$ . We have

$$\begin{aligned} \bar{\nu}(1) &= \psi_2^{-1} \circ (\nu \otimes \text{id}) \circ \psi_0^{-1}(1) \\ &= \psi_2^{-1} \circ (\nu \otimes \text{id})(1 \otimes 1) \\ &= \psi_2^{-1}(\sum_{i,j=1}^n E_{ij}^{-1}(v_i \otimes v_j) \otimes 1) \\ &= \sum_{k,l=1}^n E_{kl}^{-1}(v_k \otimes 1) \otimes (v_l \otimes 1). \end{aligned}$$

Expanding  $\bar{\nu}(1)$  in the basis  $(w_j)_{1 \leq j \leq m}$  of  $\omega_Z(V_E)$ , we obtain a matrix  $(G_{ij})_{1 \leq i, j \leq m} \in M_m(k)$  such that

$$\bar{\nu}(1) = \sum_{i,j=1}^m G_{ij} w_i \otimes w_j = \sum_{i,j=1}^m \sum_{k,l=1}^n G_{ij} (v_k \otimes t_{ki}) \otimes (v_l \otimes t_{lj})$$

for all  $1 \leq i, j \leq m$ . Then we have

$$\sum_{k,l=1}^n E_{kl}^{-1}(v_k \otimes 1) \otimes (v_l \otimes 1) = \sum_{i,j=1}^m \sum_{k,l=1}^n G_{ij} (v_k \otimes t_{ki}) \otimes (v_l \otimes t_{lj}),$$

and then

$$E_{kl}^{-1} = \sum_{i,j=1}^m G_{ij} t_{ki} t_{lj},$$

which we can rewrite as

$$(4) \quad E^{-1} = T G T^t.$$

We now prove  $G = F^{-1}$ . We have

$$(\beta_E \otimes \text{id}_{V_E}) \circ (\text{id}_{V_E} \otimes \nu) = \text{id}_{V_E}$$

and

$$(\text{id}_{V_E} \otimes \beta_E) \circ (\nu \otimes \text{id}_{V_E}) = \text{id}_{V_E}$$

for  $\beta_E$  and  $\nu$ . Since  $\omega_Z$  is monoidal, we obtain

$$(\overline{\beta_E} \otimes \text{id}_W) \circ (\text{id}_W \otimes \bar{\nu}) = \text{id}_W$$

and

$$(\text{id}_W \otimes \overline{\beta_E}) \circ (\bar{\nu} \otimes \text{id}_W) = \text{id}_W$$



for  $\overline{\beta_E}$  and  $\bar{\nu}$ . That is for any basis vector  $w_i$  we have

$$\sum_{jk}^m F_{ij} G_{jk} w_k = w_i \quad \text{and} \quad \sum_{jk}^m G_{jk} F_{ki} w_j = w_i.$$

This implies that the matrix  $G$  is the inverse of  $F$ . Then Relations (3) and (4) yield the relations

$$(5) \quad F^{-1} T^t E T = I_m \quad \text{and} \quad T F^{-1} T^t E = I_n.$$

In the same way, we obtain

$$\beta_E \circ \nu(1) = \text{Tr}(E^{-1} E^t).$$

Since  $\omega_Z$  is monoidal,

$$\overline{\beta_E} \circ \bar{\nu}(1) = \text{Tr}(F^{-1} F^t)$$

has to be equal to  $\text{Tr}(E^{-1} E^t)$ . When  $\bar{k}$  is a field, Bichon has proved in [Bi1, Section 4] that, under this condition, the algebra  $\mathcal{B}_{\bar{k}}(E, F)$  is nonzero. Since our base ring  $k$  is a PID, it embeds into a field  $\bar{k}$ . It is clear that for any invertible matrices  $E, F$ , the algebras  $\mathcal{B}_k(E, F) \otimes_k \bar{k}$  and  $\mathcal{B}_{\bar{k}}(E, F)$  are isomorphic. Therefore,  $\mathcal{B}_k(E, F)$  is nonzero provided  $\text{Tr}(E^{-1} E^t) = \text{Tr}(F^{-1} F^t)$ .

In view of (5) the map

$$\varphi(z_{ij}) = t_{ij},$$

defines an algebra morphism  $\varphi : \mathcal{B}(E, F) \rightarrow Z$ . We claim that  $\varphi$  is an isomorphism of  $\mathcal{B}(E)$ -Galois objects. First to see that  $\varphi$  is a  $\mathcal{B}(E)$ -comodule morphism, it is enough to check it on the generators  $(z_{ij})$ . The definition of the coaction (1) and relation (2) give

$$(\text{Id} \otimes \varphi) \circ \delta_{\mathcal{B}(E, F)}(z_{ij}) = \sum_{k=1}^n a_{ik} \otimes t_{kj} = \delta_Z \circ \varphi(z_{ij})$$

for any  $1 \leq i \leq n$  and  $1 \leq j \leq m$ .

The morphism  $\varphi$  is a morphism of  $\mathcal{B}(E)$ -comodule algebras, is the identity on the coinvariants elements  $k$  of  $Z$ , and Proposition 2 ensures that the comodule algebra  $\mathcal{B}(E, F)$  has a bijective canonical map. Moreover,  $Z$  is a faithfully flat Galois extension of  $k$ . Then by [Sn, Remark 3.11] the morphism  $\varphi$  is an isomorphism, and  $Z$  and  $\mathcal{B}(E, F)$  are isomorphic  $\mathcal{B}(E)$ -Galois objects.

It remains to prove that the size  $m$  of  $F \geq 2$ . First assume that  $m = 1$ . Then  $W = \omega_Z(V_E) \cong k$ . By [Sa1], [Sa3], there is an Hopf algebra  $K$  such that  $Z$  is a  $\mathcal{B}(E)$ - $K$ -bi-Galois object. Since there exists an inverse  $Z^{-1}$  of  $Z$  for the groupoid structure of bi-Galois objects, we have

$$V_E \cong V_E \square_{\mathcal{B}(E)} Z \square_K Z^{-1} \cong k \square_K Z^{-1}.$$

Since  $Z^{-1}$  is a Galois object, the image  $k \square_K Z^{-1}$  of the trivial comodule of dimension one is the algebra  $k \cong (Z^{-1})^{coH}$  of coinvariants. Then the size  $m$  of  $F$  is equal to one only if the size  $n$  of  $E$  is one. The same argument proves that  $m$  cannot be zero.

□

We now turn to the classification of the Galois objects  $\mathcal{B}(E, F)$ . The following lemma, implicit in [Bi1], will be useful.

**Lemma 5.** *Let  $k$  be a PID, let  $n, m \geq 2$  be integers, and  $E \in GL_n(k)$  and  $F \in GL_m(k)$  be invertible matrices. Assume that  $\mathcal{B}(E, F)$  is a  $\mathcal{B}(E)$ - $\mathcal{B}(F)$ -bi-Galois object and let  $\varphi : \text{Comod-}\mathcal{B}(E) \rightarrow \text{Comod-}\mathcal{B}(F)$  be the associated monoidal equivalence. Let  $V_E$  and  $V_F$  be the respective fundamental comodules of  $\mathcal{B}(E)$  and  $\mathcal{B}(F)$ . Then*

$$\varphi(V_E) \cong V_F$$

*Proof.* Let  $w_1, \dots, w_m$  be the canonical basis of  $V_F$ . Then we have a  $\mathcal{B}(F)$ -colinear map  $\theta_F : V_F \rightarrow \varphi(V_E)$  defined by

$$\theta_F(w_j) = \sum_{i=1}^n v_i \otimes z_{ij}.$$

Similarly, we have a  $\mathcal{B}(E)$ -colinear morphism  $\theta_E : V_E \rightarrow \varphi^{-1}(V_F)$  defined by

$$\theta_E(v_i) = \sum_{j=1}^m w_j \otimes t_{ji},$$

where the  $t_{ji}$ 's are the generators of  $\mathcal{B}(F, E)$ . It is easy to see that  $\varphi(\theta_E) \circ \theta_F$  is the canonical isomorphism  $V_F \rightarrow \varphi(\varphi^{-1}(V_F))$ . We deduce that  $\theta_F$  and  $\theta_E$  are monomorphisms and then that  $\theta_F$  is an isomorphism.  $\square$

As an immediate consequence of Lemma 5, we have the following necessary condition for  $\mathcal{B}(E)$ -Galois objects to be cleft.

**Corollary 6.** *Let  $k$  be a PID and  $n, m \geq 2$  be integers,  $E \in GL_n(k)$ ,  $F \in GL_m(k)$  and  $\mathcal{B}(E, F)$  be a cleft  $\mathcal{B}(E)$ -Galois object. Then  $m = n$ .*

*Proof.* If  $\mathcal{B}(E, F)$  is a cleft Galois object, the associated fibre functor is isomorphic as a functor to the forgetful functor and in particular preserves the rank of finite free modules.  $\square$

Let us now state our classification result for the extensions  $\mathcal{B}(E, F)$ .

**Theorem 7.** *Let  $k$  be a PID,  $n, m_1, m_2$  be integers  $\geq 2$  and  $E \in GL_n(k)$ ,  $F_1 \in GL_{m_1}(k)$ ,  $F_2 \in GL_{m_2}(k)$  be invertible matrices such that the algebras  $\mathcal{B}(E, F_1)$  and  $\mathcal{B}(E, F_2)$  are  $k$ -faithfully flat. Then the  $\mathcal{B}(E)$ -Galois objects  $\mathcal{B}(E, F_1)$  and  $\mathcal{B}(E, F_2)$  are isomorphic if and only if  $m_1 = m_2$  and there exists an invertible matrix  $P \in GL_{m_1}(k)$  such that  $F_1 = PF_2P^t$ .*

Note that, by [R] the bilinear forms associated to  $F_1$  and  $F_2$  are equivalent if and only if the asymmetries of  $F_1$  and  $F_2$  are similar.

*Proof.* As in the proof of [Bi1, Proposition 2.3], one shows that if  $P \in GL_m(k)$ , the  $\mathcal{B}(E)$ -comodule algebras  $\mathcal{B}(E, F)$  and  $\mathcal{B}(E, PFP^t)$  are isomorphic.

Conversely assume that  $\mathcal{B}(E, F_1)$  and  $\mathcal{B}(E, F_2)$  are  $k$ -faithfully flat: then Proposition 2 ensures that  $\mathcal{B}(E, F_1)$  and  $\mathcal{B}(E, F_2)$  are Galois objects. Let  $V_E$  be the fundamental  $\mathcal{B}(E)$ -comodule and let  $\beta_E : V_E \otimes V_E \rightarrow k$  be the linear map defined by  $E$ . Let  $\omega_1 = -\square_{\mathcal{B}(E)}\mathcal{B}(E, F_1)$  and  $\omega_2 = -\square_{\mathcal{B}(E)}\mathcal{B}(E, F_2)$  be the fibre functors associated to  $\mathcal{B}(E, F_1)$  and  $\mathcal{B}(E, F_2)$ .

By Lemma 5, the vector space  $\omega_1(V_E)$  has a basis  $(w_1^1, \dots, w_{m_1}^1)$  and  $\omega_2(V_E)$  has a basis  $(w_1^2, \dots, w_{m_2}^2)$ . The comodule algebra isomorphism  $\varphi : \mathcal{B}(E, F_1) \rightarrow$

$\mathcal{B}(E, F_2)$  induces an isomorphism  $\text{id} \otimes \varphi : \omega_1(V_E) \rightarrow \omega_2(V_E)$ . Then in particular the rank of these two free  $k$ -modules is the same, that is  $m_1 = m_2 = m$ . Let  $P \in GL_m(k)$  be the matrix of  $\text{id} \otimes \varphi$  in the bases  $(w_1^1, \dots, w_{m_1}^1)$  and  $(w_1^2, \dots, w_{m_2}^2)$ .

The matrices of the bilinear maps  $\omega_1(\beta_E)$  and  $\omega_2(\beta_E)$ , in the bases  $(w_1^1, \dots, w_m^1)$  and  $(w_1^2, \dots, w_m^2)$ , are  $F_1$  and  $F_2$  respectively. Moreover, the isomorphism  $\varphi$  gives the relation

$$\omega_1(\beta_E) = \omega_2(\beta_E) \circ ((\text{id} \otimes \varphi) \otimes (\text{id} \otimes \varphi)).$$

That is for any  $i, j = 1, \dots, m$

$$\begin{aligned} \omega_1(\beta_E)(w_i^1 \otimes w_j^1) &= \omega_2(\beta_E) \left( \left( \sum_{k=1}^m P_{ik} w_k^2 \right) \otimes \left( \sum_{l=1}^m P_{jl} w_l^2 \right) \right) \\ (F_1)_{ij} &= \sum_{k,l=1}^m P_{ik} P_{jl} (F_2)_{kl}, \end{aligned}$$

or in matrix form  $F_1 = P F_2 P^t$ .  $\square$

**Remark 8.** As an application of Theorem 7, let us consider the case where the matrix  $F$  is symmetric. Let  $k$  be a PID, let  $n, m, p \geq 2$  be integers, and  $E \in GL_n(k)$ ,  $F \in GL_m(k)$  and  $G \in GL_p(k)$  be invertible matrices. Assume that  $F$  is symmetric and  $\mathcal{B}(E, F)$  is a Galois object. Then the Galois objects  $\mathcal{B}(E, F)$  and  $\mathcal{B}(E, G)$  are isomorphic if and only if  $G$  is symmetric of size  $p = m$ .

We now consider the case when  $k$  is a field. For any integer  $n \geq 2$ , and any invertible matrix  $E \in GL_n(k)$  we define

$$X_0(E) = \{F \in GL_m(k), m \geq 2, \text{Tr}(F^{-1}F^t) = \text{Tr}(E^{-1}E^t)\}.$$

Consider the equivalence relation  $\sim$  defined by  $F_1 \sim F_2$  if and only if there exists  $P \in GL(k)$  such that  $F_1 = P F_2 P^t$  and put  $X(E) = X_0(E) / \sim$ .

**Corollary 9.** *Assume that  $k$  is a field. Then for any  $n \geq 2$  and  $E \in GL_n(k)$ , there is a bijection  $\psi : X(E) \rightarrow \text{Gal}(\mathcal{B}(E))$  sending  $F$  onto  $[\mathcal{B}(E, F)]$ .*

*Proof.* Propositions 3.2, 3.3 and 3.4 in [Bi1] ensure that we have indeed this map  $\psi$ . Moreover,  $\psi$  is surjective by Theorem 4 and injective by Theorem 7.  $\square$

We also have the following result.

**Corollary 10.** *Assume that  $k$  is an algebraically closed field of characteristic zero. For any  $n \geq 2$  and  $E \in GL_n(k)$ , the group of  $\mathcal{B}(E)$ - $\mathcal{B}(E)$ -bi-Galois objects is trivial.*

*Proof.* Let  $Z$  be a  $\mathcal{B}(E)$ - $\mathcal{B}(E)$ -bi-Galois object. By Theorem 4, there exist  $m \geq 2$  and  $F \in GL_m(k)$  such that  $Z$  is isomorphic to  $\mathcal{B}(E, F)$  as a  $\mathcal{B}(E)$ -Galois object. Bichon [Bi1, Propositions 3.3, 3.4] has proved that  $\mathcal{B}(E, F)$  is a  $\mathcal{B}(E)$ - $\mathcal{B}(F)$ -bi-Galois object. Then by [Sa1, Theorem 3.5] the Hopf algebras  $\mathcal{B}(E)$  and  $\mathcal{B}(F)$  are isomorphic that is, by [Bi1, Theorem 5.3], there exists  $P \in GL(k)$  such that  $F = P^t E P$ . The matrix  $P$  enables us to construct an isomorphism of left  $\mathcal{B}(E)$ -Galois objects  $Z \cong \mathcal{B}(E, F) \cong \mathcal{B}(E)$ . Now since  $Z$  is a  $\mathcal{B}(E)$ -bi-Galois object, we know from [Sa1, Theorem 3.5] that there exists  $f \in \text{Aut}(\mathcal{B}(E))$  such that  $Z \cong \mathcal{B}(E)^f$  as  $\mathcal{B}(E)$ -bi-Galois objects. Such a bi-Galois object is trivial if and only if  $f$  is coinver. Since by [Bi1, Theorem 5.3] any Hopf automorphism of  $\mathcal{B}(E)$  is coinver, we are done.  $\square$

The lazy cohomology group of a Hopf algebra was introduced in [BC], where it was realized as a subgroup of the group of bi-Galois objects. Therefore, we have the following.

**Corollary 11.** *Assume that  $k$  is an algebraically closed field of characteristic zero. The lazy cohomology group of  $\mathcal{B}(E)$  is trivial for any  $E \in GL_n(k)$ .*

#### 4. GALOIS OBJECTS UP TO HOMOTOPY

In this section we study the homotopy theory of  $\mathcal{B}(E)$ -Galois objects. We assume that  $k$  is an algebraically closed field. For technical reasons we only consider  $\mathcal{O}_q(SL(2))$ -Galois objects (recall that  $\mathcal{O}_q(SL(2)) = \mathcal{B}(E_q)$ ). Since for any  $E \in GL_n(k)$  there exists a  $\mathcal{B}(E)$ - $\mathcal{B}(E_q)$ -bi-Galois object, Proposition 1 ensures that  $\mathcal{H}(\mathcal{B}(E)) \cong \mathcal{H}(\mathcal{B}(E_q))$  and then there is no loss of generality.

We begin, using Lemma 5, by giving a necessary condition for two  $\mathcal{B}(E_q)$ -Galois extensions to be homotopically equivalent.

**Proposition 12.** *Let  $m_0, m_1 \geq 2$  be integers, let  $F_0 \in GL_{m_0}(k)$ ,  $F_1 \in GL_{m_1}(k)$  and assume that  $\mathcal{B}(E_q, F_0)$  and  $\mathcal{B}(E_q, F_1)$  are  $\mathcal{B}(E_q)$ -Galois objects. If  $\mathcal{B}(E_q, F_0)$  and  $\mathcal{B}(E_q, F_1)$  are homotopically equivalent, then the matrices  $F_0$  and  $F_1$  have the same size  $m_0 = m_1$ .*

*Proof.* Let us consider two  $\mathcal{B}(E_q)$ -Galois objects  $\mathcal{B}(E_q, F_0)$  and  $\mathcal{B}(E_q, F_1)$  with homotopy  $\mathcal{B}_{k[t]}(E_q, F_t)$  (by Theorem 4, any  $\mathcal{B}_{k[t]}(E_q)$ -Galois object is of this form for some  $F_t \in GL_m(k[t])$ ). Then  $V_E \square_{\mathcal{B}_{k[t]}(E_q)} \mathcal{B}_{k[t]}(E_q, F_t)$  is a finite free  $k[t]$ -module of rank equal to the size of the matrix  $F_t$ , which does not depend on  $t$ . The evaluation at  $t = 0, 1$  gives  $m_0 = m_1$ .  $\square$

Let us state a sufficient condition for two  $\mathcal{B}(E)$ -Galois objects to be homotopically equivalent.

**Theorem 13.** *Let  $k$  be an algebraically closed field,  $m_0, m_1 \geq 2$  be integers and  $F_0, F_1$  be invertible matrices of size  $m_0, m_1$  such that  $\text{Tr}(F_i^{-1}F_i^t) = -q - q^{-1}$  for  $i = 0, 1$ .*

*If  $m_0 = m_1$  and if  $F_0^{-1}F_0^t$  and  $F_1^{-1}F_1^t$  have the same characteristic polynomial, then the two  $\mathcal{O}_q(SL(2))$ -Galois objects  $\mathcal{B}(E_q, F_0)$  and  $\mathcal{B}(E_q, F_1)$  are homotopically equivalent.*

The rest of the section is devoted to the proof of the theorem. To this end, we construct a homotopy between the Galois objects, that is an  $\mathcal{O}_q(SL(2))[t]$ -Galois object over the polynomial ring  $k[t]$ . First, let us begin with some terminology. We will say that a matrix  $F \in GL_m(k)$  (here  $k$  is an arbitrary commutative ring) is **manageable** if  $F_{mm}^{-1} = 0$  and if the rightmost nonzero coefficient  $F_{mv}^{-1}$  in the bottom row is an invertible element of  $k$ . In the case of a manageable matrix, the proof of [Bi1, Proposition 3.4] still works and we obtain:

**Proposition 14.** *Assume that  $k$  is a commutative ring and let  $F \in GL_m(k)$  be a manageable matrix such that  $\text{Tr}(F^{-1}F^t) = -q - q^{-1}$ . Then  $\mathcal{B}(E_q, F)$  is a free  $k$ -module.*

The problem for constructing an homotopy is the following one.

(P) Let  $F_0, F_1 \in GL_m(k)$  be manageable matrices such that  $\text{Tr}(F_0^{-1}F_0^t) = \text{Tr}(F_1^{-1}F_1^t)$ . Find a matrix  $F(t) \in GL_m(k[t])$  such that

- (1)  $F(0) = F_0, F(1) = F_1$ .
- (2)  $\text{Tr}(F(t)^{-1}F(t)^t) = \text{Tr}(F_0^{-1}F_0^t) = \text{Tr}(F_1^{-1}F_1^t)$ .
- (3)  $F(t)$  is manageable.

Now assume that  $F_0$  and  $F_1$  have diagonal block decompositions with the same size:

$$F_0 = \left( \begin{array}{c|c} (F_0)_{11} & 0 \\ \hline 0 & (F_0)_{22} \end{array} \right), \quad F_1 = \left( \begin{array}{c|c} (F_1)_{11} & 0 \\ \hline 0 & (F_1)_{22} \end{array} \right),$$

that

$$\text{Tr}((F_0)_{11}^{-1}(F_0)_{11}^t) = \text{Tr}((F_1)_{11}^{-1}(F_1)_{11}^t)$$

and

$$\text{Tr}((F_0)_{22}^{-1}(F_0)_{22}^t) = \text{Tr}((F_1)_{22}^{-1}(F_1)_{22}^t)$$

and finally that each block is manageable. Then clearly Problem (P) reduces to the same problem for each block. This simple remark, combined with Riehm's work [R] on the structure of bilinear forms, will reduce our problem to the case of some "elementary" matrices.

We will use freely the following results of [R]. For any nondegenerate bilinear map  $\beta : V \times V \rightarrow k$  given by an invertible matrix  $F$ , and for any eigenvalue  $p \neq \pm 1$  of its asymmetry  $\sigma$ ,  $p^{-1}$  is also an eigenvalue of  $\sigma$  and the two characteristic spaces  $C_p$  and  $C_{p^{-1}}$  associated to  $p$  and  $p^{-1}$  are isotropic (for the bilinear form  $\beta$ ). The vector space  $V$  is the orthogonal sum of the subspaces  $C_1, C_{-1}$  and  $C_p \oplus C_{p^{-1}}$ , where  $p$  runs over all eigenvalues of  $\sigma$  different from  $\pm 1$ . Then there exists a basis of  $V$  such that the matrix of  $\sigma$  is a block matrix made of Jordan blocks of odd dimension with eigenvalue 1, Jordan blocks of even dimension with eigenvalue  $-1$  and pairs of blocks of eigenvalues  $p, p^{-1}$  and of the same dimension.

Assume that the asymmetries  $\sigma_0$  and  $\sigma_1$  associated to  $F_0$  and  $F_1$  have the same characteristic polynomial and that  $\sigma_1$  is diagonal. Then by [R], Problem (P) reduces to three cases.

- A.  $\sigma_0$  is a Jordan block of even dimension  $d$  with eigenvalue  $-1$  (and  $\sigma_1 = -I_d$ ),
- B.  $\sigma_0$  is a Jordan block of odd dimension  $d$  with eigenvalue 1 (and  $\sigma_1 = I_d$ ),
- C.  $\sigma_0$  is a diagonal block matrix made of two Jordan blocks of eigenvalues  $p, p^{-1}$  and of the same size  $d$  (and  $\sigma_1$  is diagonal with  $d$  diagonal coefficients equal to  $p$  and  $d$  equal to  $p^{-1}$ ).

Let us now look at the possible forms of a matrix  $F$  such that  $\sigma_0 = F^{-1}F^t$  for each of these three cases.

**Lemma 15.** *A) If  $\sigma_0$  is a Jordan block of even dimension with eigenvalue equal to  $-1$  and if there exists an invertible matrix  $F$  such that  $F^{-1}F^t = \sigma_0$ , then  $F$*

has the lower anti-triangular form

$$(6) \quad F = \begin{pmatrix} 0 & \cdots & \cdots & 0 & F_{1n} \\ \vdots & & \ddots & -F_{1n} & * \\ \vdots & \ddots & \ddots & * & * \\ 0 & F_{1n} & * & * & * \\ -F_{1n} & * & * & * & * \end{pmatrix}.$$

B) If  $\sigma_0$  is a Jordan block of odd dimension with eigenvalue equal to 1 and there exists an invertible matrix  $F$  such that  $F^{-1}F^t = \sigma_0$ , then  $F$  has the lower anti-triangular form

$$(7) \quad F = \begin{pmatrix} 0 & \cdots & \cdots & 0 & F_{1n} \\ \vdots & & \ddots & -F_{1n} & * \\ \vdots & \ddots & \ddots & * & * \\ 0 & -F_{1n} & * & * & * \\ F_{1n} & * & * & * & * \end{pmatrix}.$$

C) If  $\sigma_0$  is made of two Jordan blocks of eigenvalues  $p$  and  $p^{-1}$  and of size  $n$ , then the invertible matrix  $F$  defined by

$$(8) \quad \begin{pmatrix} 0 & I_n \\ J_p & 0 \end{pmatrix},$$

where  $I_n$  is the identity of size  $n$ ,  $J_p$  is a Jordan block of size  $n$  and eigenvalue  $p$  and  $0$  is the zero matrix, has an asymmetry similar to  $\sigma_0$ .

*Proof.* We say that the elements  $a_{i,n+1-i}$ , for  $1 \leq i \leq n$  of a matrix  $A \in M_n(k)$  lies on the anti-diagonal and we use obvious notion of lower and upper anti-triangular matrices.

A) Assume that  $F$  is a matrix such that  $F^{-1}F^t = \sigma_0$ , that is  $F^t = F\sigma_0$  or

$$(9) \quad \begin{cases} F_{i1} = -F_{1i} & \forall i = 1, \dots, n \\ F_{ji} = F_{i,j-1} - F_{ij} & \forall i = 1, \dots, n; j = 2, \dots, n \end{cases}$$

Let us consider the first row. The equation  $F_{11} = -F_{11}$  implies  $F_{11} = 0$ . Then, for any  $k = 2, \dots, n$ , we have from (9) the equations  $F_{1k} = -F_{k1}$  and  $F_{k1} = F_{1,k-1} - F_{1k}$  and then  $F_{1,k-1} = 0$ . Then the first row and the first column are equal to zero except the last terms  $F_{1n}$  and  $F_{n1}$ .

For the second row and column, we have from the previous computation  $F_{12} = F_{21} = 0$  then  $F_{22} = F_{21} - F_{22} = 0$ . For any  $k = 3, \dots, n-1$  we have

$$\begin{cases} F_{2k} = F_{k1} - F_{k2} \\ F_{k2} = F_{2,k-1} - F_{2k}. \end{cases}$$

Then  $F_{2,k-1} = 0$  and, since  $F_{2,k-1} = F_{k-1,1} - F_{k-1,2}$ , we also have  $F_{k-1,2} = 0$ . Then for all  $k \leq n-2$  the entries  $F_{2,k}$  and  $F_{k,2}$  are equal to zero. In the same way, any coefficient lying above the anti-diagonal is equal to zero.

The coefficient  $F_{i,n+1-i}$  on the anti-diagonal satisfies the relation  $F_{i,n+1-i} = F_{n+1-i,i-1} - F_{n+1-i,i}$ . We also have

$$(10) \quad \begin{cases} F_{i,n+2-i} = F_{n+2-i,i-1} - F_{n+2-i,i} \\ F_{n+2-i,i} = F_{i,n+1-i} - F_{i,n+2-i}, \end{cases}$$

then

$$(11) \quad F_{n+2-i,i} = F_{i,n+1-i} - F_{n+2-i,i-1} + F_{n+2-i,i}$$

that is

$$(12) \quad F_{i,n+1-i} = F_{n+1-(i-1),(i-1)}.$$

Then the determinant of  $F$  is  $(F_{1n})^n$  and the matrix has the wanted form.

B) Let us now consider the case where  $\sigma_0$  is a Jordan block of odd dimension and eigenvalue 1 and  $F$  is a matrix such that  $F^{-1}F^t = \sigma_0$ , that is  $F^t = F\sigma_0$  or

$$(13) \quad \begin{cases} F_{1i} = F_{i1} & \forall i = 1, \dots, n \\ F_{ji} = F_{i,j-1} + F_{ij} & \forall i = 1, \dots, n; j = 2, \dots, n \end{cases}$$

Let us consider the first line. For any  $k = 2, \dots, n$  we have from (13) the equations  $F_{1k} = F_{k1}$  and  $F_{k1} = F_{1,k-1} + F_{1k}$  and then  $F_{1,k-1} = 0$ , since  $F_{1,k-1} = F_{k-1,1}$ , the first line and the first column are equal to zero except the last terms  $F_{1n} = F_{n1}$ . In the same way as for the previous case, we see that all the coefficients lying above the anti-diagonal must be zero. Moreover, in the same way as for (10) - (12), the anti-diagonal coefficient in position  $(i, n - i + 1)$  is  $(-1)^{i+1}F_{1n}$ ; the determinant is  $(F_{1n})^n$  and  $F$  has the wanted form.

C) Assume that  $\sigma_0$  is made of two Jordan blocks of eigenvalues  $p$  and  $p^{-1}$  and of size  $n$ . We define  $F$  by the relation (8). Its asymmetry is the matrix

$$\begin{pmatrix} J_p^{-1} & 0 \\ 0 & J_p^t \end{pmatrix}$$

which is similar to  $\sigma_0$ . □

*Proof of Theorem 13.* Let us construct the matrix  $F(t)$  solution of the problem (P).

*Cases A, B:* Let us consider a Jordan block  $\sigma_0$  with eigenvalue  $\pm 1$  and size more than two. By the previous lemma 15, the matrix  $F$  such that  $F^{-1}F^t = \sigma_0$  is an anti-triangular matrix of form (6) or (7). Consider the matrix  $F(t) \in GL_n(k[t])$  defined by

$$F(t)_{i,n+1-i} = F_{i,n+1-i}, \quad F(t)_{ij} = tF_{ij},$$

for any  $1 \leq i, j \leq n$  such that  $j \neq n + 1 - i$  (that is  $F(t)$  is equal to  $F$  on the anti-diagonal and to  $tF$  on the other coefficients).

To compute  $\text{Tr}(F(t)^{-1}F(t)^t)$  we have to know the diagonal coefficients of the asymmetry of  $F(t)$ , which are equal to products of the anti-diagonal coefficients of  $F(t)^{-1}$  and  $F(t)^t$ . Remark that if a matrix  $F(t)$  is lower anti-triangular, its inverse  $F(t)^{-1}$  is upper anti-triangular, and their anti-diagonal coefficients are related by

$$1 = (F(t)^{-1})_{i,n+1-i}(F(t))_{n+1-i,i},$$

for any  $i = 1, \dots, n$ . Since the anti-diagonal coefficients of  $F(t)$  do not depend on  $t$ , the ones of  $F(t)^{-1}$  do not depend on  $t$  either and we have

$$\text{Tr}(F(t)^{-1}F(t)^t) = \text{Tr}(F_0^{-1}F_0^t) = \text{Tr}(F_1^{-1}F_1^t).$$

From the definition of  $F(t)$ , we have  $F(0) = F_0$  and  $F(1)$  is a block matrix with anti-diagonal blocks. Then the asymmetry of  $F(1)$  is diagonal and then equal to  $\sigma_1$ .

Since  $F(t)^{-1}$  is upper anti-triangular and invertible,  $F(t)$  is manageable. Finally,  $F(t)$  is a solution of (P) in the cases A,B.

*Case C:* Let us now consider the case of two Jordan blocks of size  $n$  and eigenvalues  $p$  and  $p^{-1}$  and suppose that  $F$  has the form (8).

Consider the matrix  $F(t) \in GL_{2n}(k[t])$  defined by

$$\begin{pmatrix} 0 & I_n \\ J_p(t) & 0 \end{pmatrix},$$

where  $J_p(t)$  is the matrix with diagonal coefficients equal to  $p$  and upper diagonal coefficients  $(i, i+1)$  equal to  $t$ . The inverse  $F(t)^{-1}$  is

$$\begin{pmatrix} 0 & (J_p(t))^{-1} \\ I_n & 0 \end{pmatrix}$$

and then  $F(t)$  is manageable and the trace of its asymmetry is constant. Finally  $F(t)$  is a solution of (P) in the case C.  $\square$

**Corollary 16.** *All cleft  $\mathcal{O}_q(SL(2))$ -Galois objects are homotopically trivial.*

*Proof.* Let  $Z$  be a cleft Galois object of  $\mathcal{O}_q(SL(2))$ . Then by Theorem 4, the Galois object  $Z$  is isomorphic to  $\mathcal{B}(E_q, F)$  and by Corollary 6 the matrix  $F$  is a  $2 \times 2$  matrix with trace equal to  $-q - q^{-1}$ .

If  $q \neq 1$ , there exists  $P \in GL(k)$  such that  $F = PE_qP^t$ . Then the Galois object  $\mathcal{B}(E_q, F)$  is isomorphic to the trivial object  $\mathcal{B}(E_q)$ .

If  $q = 1$ , the two possible asymmetries are, up to similarity, a diagonal matrix  $\sigma_1$  with eigenvalue  $-1$  and multiplicity 2 associated to a matrix  $F_1$  or a Jordan block matrix  $\sigma_2$  of size 2 and eigenvalue  $-1$  associated to a matrix  $F_2$ . The two associated Galois objects  $\mathcal{B}(E_q, F_1)$  and  $\mathcal{B}(E_q, F_2)$  are nonisomorphic as the asymmetries are nonsimilar, but they are homotopically equivalent by Theorem 13 as the asymmetries have the same characteristic polynomial.  $\square$

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INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, C.N.R.S. - UNIVERSITÉ LOUIS PASTEUR, 7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE, FAX : +33 (0)3 90 24 03 28

*E-mail address:* aubriot@math.u-strasbg.fr