



HAL
open science

On the number of clusters for planar graphs

Jean-Michel Billiot, Franck Corset, Eric Fontenas

► **To cite this version:**

Jean-Michel Billiot, Franck Corset, Eric Fontenas. On the number of clusters for planar graphs. 2006.
hal-00078098

HAL Id: hal-00078098

<https://hal.science/hal-00078098>

Preprint submitted on 19 Jun 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

J.-M. Billiot · F. Corset · E. Fontenas

On the number of clusters for planar graphs

the date of receipt and acceptance should be inserted later

Abstract The Tutte polynomial is a powerful analytic tool to study the structure of planar graphs. In this paper, we establish some relations between the number of clusters per bond for planar graph and its dual : these relations bring into play the coordination number of the graphs. The factorial moment measure of the number of clusters per bond are given using the derivative of the Tutte polynomial. Examples are presented for simple planar graph. The cases of square, triangular, honeycomb, Archimedean and Laves lattices are discussed.

1 Introduction

It is well known that the Tutte polynomial is a powerful tool in order to explore phase transition via the Lee and Yang theorem. It allows the exact calculation of the partition function using Fortuin-Kasteleyn representation of the Potts model for a large family of graphs. This representation shows a strong link between phase transition and percolation purposes. We recall the famous paper [12] on exact critical values for site and bond percolation using matching and dual graph and the number of cluster per site or per bond. A result of Kesten is proving that bond percolation thresholds of a dual pair of periodic planar lattices sum 1 and then that the exact bond percolation threshold of the self dual square lattice is 0.5 [8]. Wierman [13] establishes exact bond percolation threshold for the triangular and honeycomb lattice and [14] makes recent progress on bond percolation on

LabSAD, EA 3698, BSHM, Université Pierre Mendès France, 1251 Avenue Centrale, BP 47, 38040 Grenoble Cedex 9, France
E-mail: Jean-Michel.billiot@upmf-grenoble.fr

Archimedean and Laves lattices. As the chromatic polynomial is a particular case of the Tutte polynomial, since the proof of the four colour theorem for planar graph [2,3], a large number of conjectures have been proposed on the real, integer and complex zeros of the chromatic polynomial for different class of graphs, see [7]. In fact, Sokal [11] prove that complex zeros of chromatic and Tutte polynomial are dense in the complex plane. A large class of graphs has been studied : for graphs of bounded degree we can see the work of Procacci et al. [10] and for graphs and matroids determined by their Tutte polynomials see [5] and references therein.

The paper is organized as follows. First, we recall the definition of Tutte polynomial of a graph and its dual. In a second part, some relations between the number of clusters per bond for planar graph and its dual are established using the coordination number. The factorial moment measure of the number of clusters per bond are given using the derivative of the Tutte polynomial. The cases of self-dual graphs, square, triangular, honeycomb, Archimedean and Laves lattices are discussed in the end of this paper.

2 Tutte Polynomial

2.1 Definition

Let be $G = (V, E)$ an undirected planar graph, where V is the set of vertices and E is the set of edges. The main theorem of this paper is based on the Tutte polynomial of a graph G . Given a connected graph G , the Tutte polynomial denoted by $T(G, x, y)$ can be obtained by computing the four following rules :

- (1) If G has no edges, then $T(G, x, y) = 1$.
- (2) If e is an edge of G that is neither a loop nor an isthmus, then

$$T(G, x, y) = T(G'_e, x, y) + T(G''_e, x, y)$$

where G'_e is the graph G with the edge e deleted and G''_e is the graph G with the edge e contracted.

- (3) If e is an isthmus, then $T(G, x, y) = xT(G'_e, x, y)$
- (4) If e is a loop, then $T(G, x, y) = yT(G''_e, x, y)$.

Example : The Tutte polynomial associated with a cycle of length n , C_n , is given by

$$T(C_n, x, y) = \sum_{k=1}^{n-1} x^k + y.$$

2.2 Dual graph

Given a connected planar graph G with n vertices and m edges, we assume that this graph has f faces. The Euler relation gives the following equality :

$$f = m - n + \alpha$$

with α is Euler-Poincaré characteristic. In this paper, we are interested in planar lattice, which corresponds to $\alpha = 1$. We can define the dual graph G^* of G : for this, it's necessary to add an exterior virtual face to the graph G . Thus, each vertex of G^* corresponds to a face of G and two vertices of G^* are connected by an edge if the corresponding faces in G have a boundary edge in common in G . We can write $n^* = f + 1$, $m^* = m$ where n^* is the number of vertices of G^* and m^* the number of edges of G^* .

Then, we have the following relation between the two Tutte polynomial $T(G, x, y) = T(G^*, y, x)$. For example, for self-dual graphs that means $G = G^*$, the Tutte polynomial is symmetric in x and y . But, we don't have reciprocal result. In fact, for example we can find different planar graphs having the same Tutte polynomial (see Figure 1 and [5] for details).

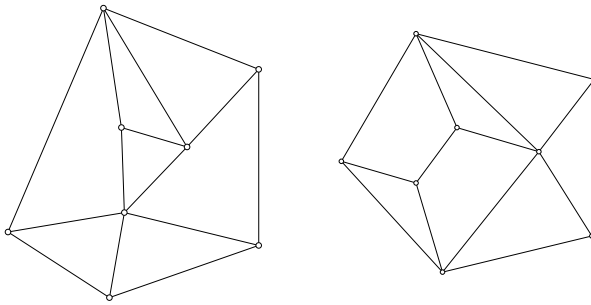


Fig. 1 two graphs with the same Tutte polynomial.

An other representation of the Tutte polynomial

We can introduce in the Tutte polynomial the notion of rank function r : the rank function of a graph G is a function on the subsets of E , which associates to each $A \subset E$ the value $r(A) = |V| - k(G, A)$ where V denotes the set of vertices and $k(G, A)$ is the number of connected components of the graph with vertex set V and edge set $A \subset E$. This definition provides an explicit formula for the Tutte polynomial :

$$T(G, x, y) = \sum_{A \subset E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)}.$$

In this paper, we just consider connected graph : in this case, if we set $n = |V|$, $m = |E|$, the Tutte polynomial may be written as

$$\begin{aligned} T(G, x, y) &= \sum_{A \subseteq E} (x-1)^{k(G,A)-1} (y-1)^{k(G,A)+|A|-n} \\ &= \frac{1}{(x-1)(y-1)^n} \sum_{A \subseteq E} (y-1)^{|A|} [(x-1)(y-1)]^{k(G,A)}. \end{aligned}$$

Two important values of the Tutte polynomial

(1) If we calculate this Tutte polynomial on the curve $(x-1)(y-1) = 1$ or more specially for $x = 1/p$ and $y = 1/(1-p)$, we obtain

$$T(G, 1/p, 1/(1-p)) = \frac{1}{(x-1)(y-1)^n} \sum_{k=0}^m C_m^k (y-1)^k = \frac{y^m}{(y-1)^n (x-1)}$$

which gives

$$T(G, 1/p, 1/(1-p)) = \frac{1}{(1-p)^{m-n+1} p^{n-1}}. \quad (1)$$

(2) Now, we look at the curve $(x-1)(y-1) = q$, where q is a real. For $x = 1 + q(1-p)/p$ and $y = 1/(1-p)$, the Tutte polynomial can be written as :

$$\begin{aligned} T(G, 1 + q \frac{(1-p)}{p}, \frac{1}{1-p}) &= \frac{(1-p)^{n-m-1} p^{1-n}}{q} \sum_{A \subseteq E} p^{|A|} (1-p)^{m-|A|} q^{k(G,A)} \\ &= \frac{T(G, 1/p, 1/(1-p))}{q} E(q^{k(G,p)}) \end{aligned} \quad (2)$$

where $E(q^k)$ denote the expectation of the random variable q^k following the binomial distribution. This relation is well known because it gives a connexion between the Tutte polynomial and the partition function of the Potts model using the Fortuin Kasteleyn representation.

By using this calcul, we give now a new result between the number of connected component for a graph G and its dual.

3 Number of cluster per bond

3.1 Definition and notations

We associate with each edge of the graph G a distribution : an edge will be white with probability p and black with probability $1-p$. We denote by $k(G, A, p)$ the number of connected components with white edges in A , subset of E .

For the dual graph, we can build a relation between A and a set A^* , subset of E^* , the set of the edges of G^* : let be e an edge of A . These edge is common on two faces of the graph G which correspond two vertices of G^* and these two vertices define an edge e^* in G^* . So, given a subset A of edges e of E , the set A^* of E^* is the complementary set of edges e^* thus defined. In opposite, the edge in G^* shall be white with probability $1 - p$ and black with probability p . Also, if the edge e in G is white, the edge e^* will be black in G^* . We denote by $k(G^*, A^*, 1 - p)$ the number of connected components with white edges in A^* , subset of E^* (see Figure 2).

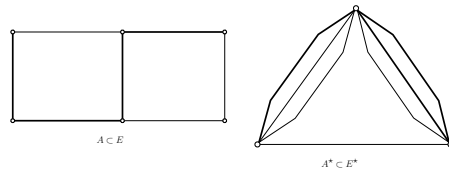


Fig. 2 Example of set A in G and corresponding A^* in G^* .

3.2 Connected components

We use the following theorem and introduce a new proof based on the Tutte polynomial. It seems that this relation already appears in [9] in order to obtain some correlation duality relations for Potts model on planar graphs.

Theorem 31 Let be G a connected planar graph and denote by G^* the dual graph of G .

For all subset A of E and corresponding subset A^* of E^* ,

$$k(G^*, A^*, 1 - p) - k(G, A, p) = |A| - n + 1 \quad (3)$$

where $|A|$ represents number of elements of A .

In this result, we can notice that the random variable $k(G^*, 1 - p) - k(G, p)$ follows a binomial distribution with modality set $\{1 - n, \dots, n^* - 1\}$, which

is symmetric for $n = n^*$.

Proof : the proof is based on the symmetry of the Tutte polynomial. For the graph G , on the curve $(x-1)(y-1) = q$, the equation (2) gives

$$T(G, 1 + q \frac{1-p}{p}, \frac{1}{1-p}) = (1-p)^{n-m-1} p^{1-n} \sum_{A \subseteq e(G)} p^{|A|} (1-p)^{n_A - |A|} q^{k(G, A, p) - 1}$$

and, for the dual graph,

$$T(G^*, 1 + q \frac{p}{1-p}, \frac{1}{p}) = p^{n^* - m^* - 1} (1-p)^{1-n^*} \sum_{A^* \subseteq e(G^*)} p^{n_{A^*} - |A^*|} (1-p)^{|A^*|} q^{k(G^*, A^*, 1-p) - 1}.$$

Notice that $m^* = m$, $m^* - n^* + 1 = f^* = n - 1$ and $1 - n^* = -f = -m + n - 1$, so that the previous equation can be written as

$$T(G^*, 1 + q \frac{p}{1-p}, \frac{1}{p}) = p^{1-n} (1-p)^{n-1} \sum_{A^* \subseteq e(G^*)} p^{m - |A^*|} (1-p)^{|A^*| - m} q^{k(G^*, A^*, 1-p) - 1}. \quad (4)$$

Moreover, the following relation $T(G^*, x, y) = T(G, y, x)$ gives us an other way to calculate (4), namely :

$$\begin{aligned} T(G^*, 1 + q \frac{p}{1-p}, \frac{1}{p}) &= T(G, \frac{1}{p}, 1 + q \frac{p}{1-p}) \\ &= \frac{(1 + q \frac{p}{1-p})^m}{\frac{1-p}{p} \left(q \frac{p}{1-p} \right)^n} \sum_{A \subseteq e(G)} \left(\frac{q \frac{p}{1-p}}{1 + q \frac{p}{1-p}} \right)^{|A|} \left(\frac{1}{1 + q \frac{p}{1-p}} \right)^{m - |A|} q^{k(G, A, p)}. \end{aligned}$$

It becomes

$$T(G^*, 1 + q \frac{p}{1-p}, \frac{1}{p}) = p^{1-n} (1-p)^{n-1} \sum_{A \subseteq e(G)} p^{|A|} (1-p)^{-|A|} q^{(k(G, A, p) + |A| - n)} \quad (5)$$

Before, we made correspondence between the set A and the set A^* , namely an edge e ($e \in A$) is white in G when the corresponding edge e^* is black in G^* . So, we conclude $|A| + |A^*| = m$. A^* can be seen as the complementary of A in G^* . In this way, the equation (4) gives us

$$T(G^*, 1 + q \frac{p}{1-p}, \frac{1}{p}) = p^{1-n} (1-p)^{n-1} \sum_{A \subseteq e(G)} p^{|A|} (1-p)^{-|A|} q^{k(G^*, A^*, 1-p) - 1}.$$

Finally, if we compare this equality and the relation (4), it becomes

$$k(G^*, A^*, 1-p) - k(G, A, p) = |A| - n + 1.$$

3.3 Mean of connected components

The previous result yields a quick way to compare the means of $k(G^*, 1-p)$ and $k(G, p)$ under the uniform distribution :

Theorem 32

$$E(k(G^*, 1-p)) - E(k(G, p)) = mp - n + 1$$

In fact, by multiplying equality (3) by $p^{|A|}(1-p)^{m-|A|}$ and by summing on all subsets A of E , we get :

$$\sum_{A \subseteq e(G)} p^{|A|}(1-p)^{m-|A|} k(G^*, A^*, 1-p) - \sum_{A \subseteq e(G)} p^{|A|}(1-p)^{m-|A|} k(G, A, p) = \sum_{A \subseteq e(G)} p^{|A|}(1-p)^{m-|A|} (|A| - n + 1).$$

By the correspondence between the sets A and A^* and the relation $|A| + |A^*| = m$, this equality becomes :

$$\sum_{A^* \subseteq e(G^*)} p^{m-|A^*|}(1-p)^{|A^*|} k(G^*, A^*, 1-p) - \sum_{A \subseteq e(G)} p^{|A|}(1-p)^{m-|A|} k(G, A, p) = \sum_{l=0}^m C_m^l p^l (1-p)^{m-l} (l - n + 1).$$

which gives

$$E(k(G^*, 1-p)) - E(k(G, p)) = mp - n + 1.$$

Some remarks : (1) From the theorem 32, we can notice the particular status played by the value $p = (n-1)/m$. For this value, the two means are equal. We'll discuss on this value in the next section for different graphs.

(2) Following [6], we can compare the expected values of the length of the minimal spanning tree of G denoted by $E(L_{MST}(G))$ and of G^* denoted by $E(L_{MST}(G^*))$. They prove the following relation

$$E(L_{MST}(G)) = \int_0^1 E(k(G, p)) dp - 1.$$

In our context, we obtain

$$E_{1-p}(L_{MST}(G^*)) - E_p(L_{MST}(G)) = \frac{m}{2} - n + 1.$$

In this equality, it appears the value $m = 2(n-1)$. This relation between number of edges and vertices of G is a necessary condition so that the Tutte polynomial may be symmetric in x and y .

3.4 An other approach

The result of theorem 32 can be presented under another form : if we go back to the equality (2), we infer that

$$E(q^{k(G,p)}) = \frac{qT(G, 1 + q\frac{(1-p)}{p}, \frac{1}{1-p})}{T(G, 1/p, 1/(1-p))}$$

and

$$E(q^{k(G^*, 1-p)}) = \frac{qT(G^*, 1 + q\frac{p}{1-p}, \frac{1}{p})}{T(G, 1/p, 1/(1-p))} = \frac{qT(G, \frac{1}{p}, 1 + q\frac{p}{1-p})}{T(G, 1/p, 1/(1-p))}.$$

These two expressions are differentiable in q so that

$$E(k(G, p)q^{k(G,p)-1}) = \frac{T(G, 1 + q\frac{(1-p)}{p}, \frac{1}{1-p})}{T(G, 1/p, 1/(1-p))} + q\frac{(1-p)}{p} \frac{\partial}{\partial x} T(G, 1 + q\frac{(1-p)}{p}, \frac{1}{1-p})}{T(G, 1/p, 1/(1-p))}$$

and

$$E(k(G^*, 1-p)q^{k(G^*, 1-p)-1}) = \frac{T(G, \frac{1}{p}, 1 + q\frac{p}{1-p})}{T(G, 1/p, 1/(1-p))} + q\frac{p}{1-p} \frac{\partial}{\partial y} T(G, \frac{1}{p}, 1 + q\frac{p}{1-p})}{T(G, 1/p, 1/(1-p))}.$$

Taking $q = 1$ for q , we get :

$$E(k(G, p)) - E(k(G^*, 1-p)) = p(1-p) \frac{\left[\frac{1}{p^2} \frac{\partial}{\partial x} T(G, \frac{1}{p}, \frac{1}{1-p}) - \frac{1}{(1-p)^2} \frac{\partial}{\partial y} T(G, \frac{1}{p}, \frac{1}{1-p}) \right]}{T(G, 1/p, 1/(1-p))}$$

which can be written as

Proposition 31

$$E(k(G, p)) - E(k(G^*, 1-p)) = -p(1-p) \frac{d}{dp} \log T(G, \frac{1}{p}, \frac{1}{1-p}). \quad (6)$$

By using the value (1), we recover the relation of the theorem.

Remarks :

1) For self dual graphs ($G = G^*$), it is natural to take $p = 1/2$ in (6). For this value the right term in (6) vanishes, thus $p = 1/2$ can be seen as a minimum of the Tutte polynomial $T(G, x, y)$ on the hyperbola $(x-1)(y-1) = q = 1$ with $x = 1/p$ and $y = 1/(1-p)$. We can see that $p = 1/2 = \sqrt{q}/(1 + \sqrt{q})$ for $q = 1$. For other real values of q , we have a similar result that $p = \sqrt{q}/(1 + \sqrt{q})$ is a minimum of the Tutte polynomial

on the hyperbola $(x-1)(y-1) = q$. Notice that this particular value is conjectured to be the critical probability for q-Potts model.

In fact the Tutte polynomial for self dual graphs is symmetric and can be written as

$$T(G, x, y) = \sum_{\alpha=1}^{n-1} \sum_{\beta=0}^{\alpha} a_{\alpha,\beta} (xy)^{\beta} (x^{\alpha-\beta} + y^{\alpha-\beta})$$

where n is the number of vertices of G and $a_{\alpha,\beta}$ are the coefficients of $T(G, x, y)$. On the hyperbola $(x = 1 + q/v, y = 1 + v)$,

$$\frac{dx}{dv} = -q/v^2, \frac{dy}{dv} = 1, \frac{d^2x}{dv^2} = 2q/v^3, \frac{d^2y}{dv^2} = 0.$$

We can always write $T(G, x, y) = \sum_{\alpha=0}^{n_s-1} c_{\alpha} (x+y)^{\alpha} = T(G, v)$ thus $\frac{d}{dv}T(G, v)$ can be factorized by $1 - q/v^2$ and vanishes for $v^2 = q$ which corresponds to $x = y$. By showing that the Tutte polynomial is a convex function of v , $v \geq 0$ otherwise $p \in]0, 1[$, $v = \sqrt{q}$ reaches the minimum and is equal to $T(G, 1 + \sqrt{q}, 1 + \sqrt{q})$.

2) From the equation (2), we obtain

$$E(q^{k(G,p)-1}) = \frac{T(G, 1 + q \frac{(1-p)}{p}, \frac{1}{1-p})}{T(G, 1/p, 1/(1-p))}.$$

By i differentiations with respect to q , and taking $q = 1$, the factorial moments are

$$\forall i \in \{1, \dots, n\}, E[(k-1)(k-2)\dots(k-i)] = \left(\frac{1-p}{p}\right)^i \frac{\frac{\partial^i}{\partial x^i} T(G, \frac{1}{p}, \frac{1}{1-p})}{T(G, \frac{1}{p}, \frac{1}{1-p})} \quad (7)$$

where $\frac{\partial^i}{\partial x^i} T$ is the i th partial derivative with respect to x of Tutte polynomial.

4 Applications

4.1 Finite planar graphs

As example of finite planar graphs, we study the wheel which is a self dual graph. The wheel is the graph obtained from a cycle by adding a new vertex adjacent to all vertices in the cycle. This graph contains n vertices

and $2(n-1)$ edges. In this case, the equation in theorem 32 can be written as follows

$$E[k(G^*, 1-p)] - E[k(G, p)] = 2(n-1)(p - \frac{1}{2}) \quad (8)$$

The natural choice of p is 0.5 and so that for all graphs with $m = 2(n-1)$. This condition is necessary for having a symmetric Tutte polynomial and in particular this is satisfied for self dual graphs.

Now, by looking at the result of the theorem 32 and the previous example, we are led to study the particular value $p_e = \frac{n-1}{m}$ for general finite planar graphs. It comes

$$\sum_{i=1}^n E(|C_i|^{-1}) = E(k(G, p_e)) = E(k(G^*, 1-p_e))$$

where C_i is the cluster containing the site i .

In table 4.1, we present some graphs with their particular value p_e . For two firsts graphs, s denotes respectively the number of squares and triangles. The third graph is the square lattice at step $s = 2$. For the three following graphs, we just give the pattern associated with the square lattice, respectively : bow-tie lattice, crossed matching and octagonal. The two last graphs are respectively the honeycomb lattice and the kagome lattice at step $s = 2$. For all graphs, we compute, at each step s , the number of edges and vertices and the corresponding value p_e .

Remark that

$$p_e = \frac{n-1}{m} = \frac{2(n-1)}{\sum_{i=1}^n q_i}$$

where q_i denotes the number of neighbours of the site i .

4.2 Infinite planar graphs

Taking the thermodynamic limit in p_e , we obtain $p_l = \frac{2}{\bar{q}}$ where \bar{q} is the coordination number of the graph. Obviously from Euler relation

$$p_e + p_e^* = \frac{n-1}{m} + \frac{n^*-1}{m} = 1.$$

Of course, for self dual graph, $p_l = 0.5$: it corresponds to the critical bond percolation threshold of the square lattice.

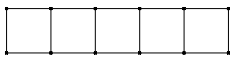
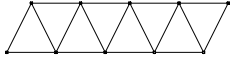
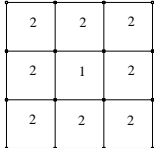


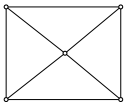
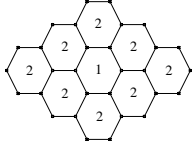
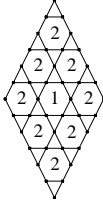
Name	Graph	$ E $	$ V $	$p_e = \frac{ V -1}{ E }$	$\lim_{s \rightarrow +\infty} p_e$
s squares		$3s + 1$	$2s + 2$	$\frac{2s + 1}{3s + 1}$	$2/3$
s triangles		$2s + 1$	$s + 2$	$\frac{s + 1}{2s + 1}$	$1/2$
Square Lattice		$4s(2s - 1)$	$(2s)^2$	$\frac{2s + 1}{4s}$	$1/2$
Bow-tie Lattice	$C2 =$ 	$(4s - 2)(10s - 3)$	$(4s - 1)^2$	$\frac{4s}{10s - 3}$	$2/5$
Crossed Matching	$C3 =$ 	$4(4s - 2)(3s - 1)$	$(4s - 1)^2$	$\frac{s}{3s - 1}$	$1/3$
Octagonal		$4(2s - 1)(3s - 1)$	$8s^2 - 4s + 1$	$\frac{s}{3s - 1}$	$1/3$
Honeycomb Lattice		$12s^2 - 4s - 2$	$8s^2 - 2$	$\frac{8s^2 - 3}{12s^2 - 4s - 2}$	$2/3$
kagome lattice		$2(2s - 1)(6s - 1)$	$4s(3s - 1)$	$\frac{(6s + 1)}{2(6s - 1)}$	$1/2$

Table 1 Some graphs with their particular value p_e .

In order to understand the last column of the table 4.1, we write down the equation of theorem 32 as follows

$$E(k(G^*, 1 - p_e)) - E(k(G, p_e)) = mp_e - n + 1.$$

By taking $p_e = \frac{n-1}{m}$, we obtain

$$E(k(G^*, 1 - \frac{n-1}{m})) = E(k(G, \frac{n-1}{m})).$$

When n tends to infinity with m , m function of n , it becomes

$$E(k(G^*, 1 - \lim_{n \rightarrow +\infty} p_e)) = E(k(G, \lim_{n \rightarrow +\infty} p_e)).$$

The last column of the table 4.1 gives value of these limits for our particular graphs. For honeycomb lattice G and its dual, the triangular lattice, we find

$$E(k(G, 2/3)) = E(k(G^*, 1/3)).$$

How to understand for example this $2/3$ associated with honeycomb lattice ? We can break up the expression of p_e as

$$p_e = \frac{2(n-1)}{\sum_{i=1}^{n_{int}} q_i + \sum_{i=1}^{n_{ext}} q_i} = \frac{2}{\frac{1}{n-1} \sum_{i=1}^{n_{int}} q_i + \frac{1}{n-1} \sum_{i=1}^{n_{ext}} q_i}$$

with $n_{int} + n_{ext} = n$. n_{ext} is the number of vertices of the convex hull associated with honeycomb lattice at step s . In this case and for all regular lattice with bounded degree, $\lim_{n \rightarrow +\infty} \frac{n_{ext}}{n} = 0$, which gives

$$\lim_{n \rightarrow +\infty} p_e = \lim_{n \rightarrow +\infty} \frac{2(n-1)}{\sum_{i=1}^{n_{int}} q_i} = \frac{2}{\bar{q}}.$$

So that to calculate this limit, we avoid boundary effects. We only need of the coordination number of the graph : 3 for the honeycomb lattice. So we can state for Archimedean and Laves lattices the following relation

$$E(k(G, \frac{2}{\bar{q}})) = E(k(G^*, 1 - \frac{2}{\bar{q}})).$$

Concluding remarks

- 1) The arithmetic and harmonic means of the number of neighbours of each vertex of the graph are the same for Archimedean lattices and also for Laves lattices with different weights as explained in [1]. In this paper, he built a new mean with other weights in order to classify graphs with different bond percolation thresholds. Otherwise, results about ordering critical site and bond values with different connective constant are presented for some Archimedean lattices in [14].

-
- 2) In the preceding decomposition of \bar{q} , one may ask what is the role of part concerning the vertices of the convex hull, it remembers of course conjecture connecting isoperimetric constant to percolation threshold of a graph. For example, it could be instructive to compare the speed of convergence of p_e to $\frac{2}{\bar{q}}$ for different graphs.
 - 3) Chang and Shrock [4] remark that for strips of graphs such as square, triangular, kagome, that complex singularity of the number of clusters per bond are precisely on the boundary of the accumulation set of the zeros of the partition function.
 - 4) The present study shows the major role of self-dual graphs. An interesting perspective will be the location of complex zeros of their Tutte polynomial and a discussion on the thermodynamic questions.

References

1. S. E. Alm. On measures of average degree for lattices. Technical report, Uppsala University, 2003. U.U.D.M. Report 2003:6 ISSN 1101-3591. To appear in *Combinatorics, Probability and Computing*.
2. K. Appel and W. Haken. Every planar map is 4-colourable. part i discharging. *Illinois J. Math.*, 21:429–490, 1977.
3. K. Appel and W. Haken. Every planar map is 4-colourable. part ii reductibility. *Illinois J. Math.*, 21:491–567, 1977.
4. S.-C. Chang and R. Shrock. Exact results for average cluster numbers in bond percolation on lattice strips. *Phys. Rev. E*, 70(056130), 2004.
5. A. de Mier. *Graph and matroids determined by their Tutte polynomials*. PhD thesis, Universitat Politècnica de Catalunya, Barcelone, 2003.
6. J.A. Fill and J.M. Steele. Exact expectations of minimal spanning trees for graphs with random edge weights. In National University of Singapore Lecture Notes Series, Institute for Mathematical Sciences, editor, *Stein's Method and Applications*, (A. Barbour and L. Chen, eds.), pages 169–180, 2005.
7. B. Jackson. Zeros of chromatic and flow polynomials of graphs. *J. Geom.*, pages 95–109, 2003.
8. H. Kesten. The critical probability of bond percolation on the square lattice equal 1/2. *Commun. Math. Phys.*, 74(1):41–59, 1980.
9. C. King and F. Y. Wu. New correlation duality relations for the planar potts model. *J. of Statist. Physics*, 107(3-4):919–940, 2002.
10. B. Procacci, A. and Scoppola. Infinite graphs with a nontrivial bond percolation threshold: some sufficient conditions. *J. of Statist. Physics*, 115(3/4):1113–1127, 2004.
11. A. Sokal. Chromatic roots are dense in the whole complex plane. *Combin. Probab. Comput.*, 13:221–261, 2004.
12. M.F. Sykes and J.W. Essam. Exact critical percolation probabilities for site and bond problems in two dimensions. *Journal of Mathematical Physics*, 5:1117–1127, 1964.
13. J.C. Wierman. Bond percolation on honeycomb and triangular lattices. *Adv. Appl. Prob.*, 13:298–313, 1981.
14. J.C. Wierman. Bond percolation critical probability bounds for three archimedean lattices. *Random Structures and Algorithms*, 20(4):507–518, 2002.