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Independent Component Analysis and estimation of a quadratic functional

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Abstract

Independent component analysis (ICA) is linked up with the problem of estimating a non linear functional of a density, for which optimal estimators are well known. The precision of ICA is analyzed from the viewpoint of functional spaces in the wavelet framework. In particular, it is shown that, under Besov smoothness conditions, parametric rate of convergence is achieved by a U-statistic estimator of the wavelet ICA contrast, while the previously introduced plug-in estimator \hat{C}_j^2 , with moderate computational cost, has a rate in $n^{\frac{-4s}{4s+d}}$.

Keywords: non parametric density estimation, ICA, quadratic functional estimation, wavelets. ■

1. Introduction

In signal processing, blind source separation consists in the identification of analogical, independent signals mixed by a black-box device. In psychometry, one has the notion of structural latent variable whose mixed effects are only measurable through series of tests; an example are the Big Five identified from factorial analysis by researchers in the domain of personality evaluation (Roch, 1995). Other application fields such as digital imaging, biomedicine, finance and econometrics also use models aiming to recover hidden independent factors from observation. Independent component analysis (ICA) is one such tool; it can be seen as an extension of principal component analysis, in that it goes beyond a simple linear decorrelation only satisfactory for a normal distribution; or as a complement, since its application is precisely pointless under the assumption of normality.

Papers on ICA are found in the fields of signal processing, neural networks, statistics and information theory. Comon (1994) defined the concept of ICA as maximizing the degree of statistical independence among outputs using contrast functions approximated by the Edgeworth expansion of the Kullback-Leibler divergence.

The model is usually stated as follows: let X be a random variable on \mathbb{R}^d , $d \geq 2$; find pairs (A, S) , such that $X = AS$, where A is a square invertible matrix and S a latent random variable whose components are mutually independent. This is usually done by minimizing some contrast function that cancels out if, and only if, the components of WX are independent, where W is a candidate for the inversion of A .

Maximum-likelihood methods and contrast functions based on mutual information or other divergence measures between densities are commonly employed. Cardoso (1999) used higher-order cumulant tensors, which led to the Jade algorithm, Bell and Snejowski (1990s) published an approach based on the Infomax principle. Hyvarinen and Oja (1997) presented the fast ICA algorithm.

Let f be the density of the latent variable S relative to Lebesgue measure, assuming it exists. The observed variable $X = AS$ has the density f_A , given by

$$\begin{aligned} f_A(x) &= |\det A^{-1}| f(A^{-1}x) \\ &= |\det B| f^1(b_1x) \dots f^d(b_dx), \end{aligned}$$

where b_ℓ is the ℓ th row of the matrix $B = A^{-1}$; this resulting from a change of variable if the latent density f is equal to the product of its marginals $f^1 \dots f^d$. In this regard, latent variable $S = (S^1, \dots, S^d)$ having independent components means independence of the random variables $S^\ell \circ \pi^\ell$ defined on some product probability space $\Omega = \prod \Omega^\ell$, with π^ℓ the canonical projections. So S can be defined as the compound of the unrelated S^1, \dots, S^d sources.

In the ICA model expressed this way, both f and A are unknown, and the data consists in a random sample of f_A . The semi-parametric case corresponds to f left unspecified, except for general regularity assumptions.

In the semi-parametric case, Bach and Jordan (2002) proposed a contrast function based on canonical correlations in a reproducing kernel Hilbert space. Similarly, Gretton et al (2003) proposed kernel covariance and kernel mutual information contrast functions. Tsybakov and Samarov (2002) proposed a method of simultaneous estimation of the directions b_i , based on nonparametric estimates of matrix functionals using the gradient of f_A .

In this paper, we consider the semi-parametric case and an ICA contrast provided by the factorization measure $\int |f_A - f_A^*|^2$, with f_A^* the product of the marginals of f_A . Let's mention that the idea of comparing in the L_2 norm a joint density with the product of its marginals, can be traced back to Rosenblatt (1975).

Estimation of a quadratic functional

The problem of estimating nonlinear functionals of a density has been widely studied. In estimating $\int f^2$ under Hölder smoothness conditions, Bickel and Ritov (1988) have shown that parametric rate is achievable for a regularity $s \geq 1/4$, whereas when $s \leq 1/4$, minimax rates of convergence under mean squared error are of the order of $n^{-8s/(1+4s)}$. This result has been extended to general functionals of a density $\int \phi(f)$ by Birgé and Massart (1995). Laurent (1996) has built efficient estimates for $s > 1/4$.

Let P_j be the projection operator on a multiresolution analysis (MRA) at level j , with scaling function φ , and let $\alpha_{jk} = \int f \varphi_{jk}$ be the coordinate k of f .

In the wavelet setting, given an independent, identically distributed sample $\tilde{X} = \{X_1, \dots, X_n\}$ of a density f defined on \mathbb{R} , the U-statistic $\hat{B}_j^2(\tilde{X}) = \frac{2}{n(n-1)} \sum_{i_1 < i_2} \sum_{k \in \mathbb{Z}} \varphi_{jk}(X_{i_1}) \varphi_{jk}(X_{i_2})$ with mean $\int (P_j f)^2$ is the usual optimal estimator of the quantity $\int f^2$; see Kerkycharian and Picard (1996), and Tribouley (2000) for the white noise model with adaptive rules.

In what follows, this result is implicitly extended to d dimensions using a tensorial wavelet basis Φ_{jk} , with $\Phi_{jk}(x) = \varphi_{jk^1}(x^1) \dots \varphi_{jk^d}(x^d)$, $k \in \mathbb{Z}^d, x \in \mathbb{R}^d$, that is to say with \tilde{X} an independent, identically distributed sample of a density f on \mathbb{R}^d , the U-statistic $\hat{B}_j^2(\tilde{X}) =$

$\frac{2}{n(n-1)} \sum_{i_1 < i_2} \sum_{k \in \mathbb{Z}^d} \Phi_{jk}(X_{i_1}) \Phi_{jk}(X_{i_2})$ with mean $\int (P_j f)^2 = \sum_{k \in \mathbb{Z}^d} \alpha_{jk}^2$ is also optimal in estimating the quantity $\int_{\mathbb{R}^d} f^2$.

In the case of a compactly supported density f , \hat{B}_j^2 is computable with a Daubechies wavelet D2N and dyadic approximation of X , but the computational cost is basically in $O(n^2(2N-1)^d)$, which is generally too high in practice.

On the other hand, the plug-in, biased, estimator $\hat{H}_j^2(f) = \sum_k [\frac{1}{n} \sum \Phi_{jk}(X_i)]^2 = \sum_k \hat{\alpha}_{jk}^2$ enjoys both ease of computation and ease of transitions between resolutions through discrete wavelet transform (DWT), since it builds upon a preliminary estimation of all individual wavelet coordinates of f on the projection space at level j , that is to say a full density estimation. In this setting it is just as easy to compute $\sum_k |\hat{\alpha}_{jk}|^p$ for any $p \geq 1$ or even $\sup |\hat{\alpha}_{jk}|$, with a fixed computational cost in $O(n(2N-1)^d)$ plus sum total, or seek out the max, of a 2^{jd} array.

Both estimators \hat{H}_j^2 and \hat{B}_j^2 build on the same kernel $h_j(x, y) = \sum_{k \in \mathbb{Z}^d} \Phi_{jk}(x) \Phi_{jk}(y)$ since they are written

$$\hat{H}_j^2(\tilde{X}) = (n^2)^{-1} \sum_{i \in \Omega_n^2} h_j(X_{i^1}, X_{i^2}) \quad \text{and} \quad \hat{B}_j^2(\tilde{X}) = (A_n^2)^{-1} \sum_{i \in I_n^2} h_j(X_{i^1}, X_{i^2}),$$

where, here and in the sequel, $\Omega_n^m = \{(i^1, \dots, i^m) : i^\ell \in \mathbb{N}, 1 \leq i^\ell \leq n\}$, $I_n^m = \{i \in \Omega_n^m : \ell_1 \neq \ell_2 \Rightarrow i^{\ell_1} \neq i^{\ell_2}\}$ and $A_n^p = n!/(n-p)!$.

The plug-in estimator \hat{H}_j^2 is then identified as the Von Mises statistic associated to \hat{B}_j^2 . In estimating $\sum_k \alpha_{jk}^2$, the mean squared error of unbiased \hat{B}_j^2 is merely its variance, while the mean squared error of \hat{H}_j^2 adds a squared component $E(\hat{H}_j^2 - \hat{B}_j^2)^2$ because of the inequality $(\hat{H}_j^2 - \sum_k \alpha_{jk}^2)^2 \leq 2(\hat{H}_j^2 - \hat{B}_j^2)^2 + 2(\hat{B}_j^2 - \sum_k \alpha_{jk}^2)^2$.

From general results, a U-statistic with finite second raw moment has a variance in Cn^{-1} and under similar conditions, the difference $E|U - V|^r$ between the U-statistic and its associated Von Mises statistic is of the order of n^{-r} (See for instance Serfling, 1980).

In the wavelet case, the dependence of the statistics on the resolution j calls for special treatment in computing these two quantities. This special computation, taking j and other properties of wavelets into account, constitutes the main topic of the paper. In particular whether 2^{jd} is lower than n or not is a critical threshold for resolution parameter j . Moreover, on the set $\{j : 2^{jd} > n^2\}$, the statistic \hat{B}_j^2 , and therefore also \hat{H}_j^2 , have a mean squared error not converging to zero.

If \hat{B}_j^2 and \hat{H}_j^2 share some features in estimating $\sum_k \alpha_{jk}^2 = \int (P_j f)^2$, they differ in an essential way: the kernel h_j is averaged in one case over Ω_n^2 , the set of unconstrained indexes, and in the other case over I_n^2 the set of distinct indexes. As a consequence, it is shown in the sequel that \hat{H}_j^2 has mean squared error of the order of $2^{jd}n^{-1}$, which makes it inoperable as soon as $2^{jd} \geq n$, while \hat{B}_j^2 has mean squared error of the order of $2^{jd}n^2$, which is then parametric on the set $\{j : 2^{jd} < n\}$. In a general way, this same parallel Ω_n^m versus I_n^m is underpinning most of the proofs presented throughout the paper.

Wavelet ICA

Let f be the latent density in the semi-parametric model introduced above. Let f_A be the mixed density and let f_A^* be the product of the marginals of f_A .

Assume, as regularity condition, that f belongs to a Besov class $B_{s2\infty}$. It has been checked in previous work (Barbedor, 2005) that f_A and f_A^* , hence $f_A - f_A^*$ belong to the same Besov space than f .

As usual, the very definition of Besov spaces (here $B_{s2\infty}$) and an orthogonality property of the projection spaces V_j and W_j entails the relation

$$0 \leq \int (f_A - f_A^*)^2 - \int [P_j(f_A - f_A^*)]^2 \leq C2^{-2js}.$$

In this relation, the quantity $\int [P_j(f_A - f_A^*)]^2$ is recognized as the wavelet ICA contrast $C_j^2(f_A - f_A^*)$, introduced in a preliminary paper (Barbedor, 2005).

The wavelet ICA contrast is then a factorization measure with bias, in the sense that a zero contrast implies independence of the projected densities, and that independence in projection transfers to original densities up to some bias 2^{-2js} .

Assume for a moment that the difference $f_A - f_A^*$ is a density and that we dispose of an independent, identically distributed sample \tilde{S} of this difference. Computing the estimators $\hat{B}_j^2(\tilde{S})$ or $\hat{H}_j^2(\tilde{S})$ provides an estimation of $\int (f_A - f_A^*)^2$, the exact ICA factorization measure. In this case, the j^* realizing the best compromise between the mean squared error in C_j^2 estimation and the bias of the ICA wavelet contrast 2^{-2js} , is exactly the same as the one to minimize the overall risk in estimating the quadratic functional $\int (f_A - f_A^*)^2$. It is found by balancing bias and variance, a standard procedure in nonparametric estimation. From what was said above $\hat{B}_j^2(\tilde{S})$ would be an optimal estimator of the exact factorization measure $\int (f_A - f_A^*)^2$.

The previous assumption being heuristic only, and since, in ICA, the data at hand is a random sample of f_A and not $f_A - f_A^*$, we are lead to consider estimators different from \hat{B}_j^2 and \hat{H}_j^2 , but still alike in some way.

Indeed, let $\delta_{jk} = \int (f_A - f_A^*)\Phi_{jk}$ be the coordinate of the difference function $f_A - f_A^*$. In the ICA context, δ_{jk} is estimable only through the difference $(\alpha_{jk} - \alpha_{jk^1} \dots \alpha_{jk^d})$ where $\alpha_{jk} = \int f_A \Phi_{jk}$ is the coordinate of f_A and $\alpha_{jk^\ell} = \int f_A^{\star\ell} \varphi_{jk^\ell}$ refers to the coordinate of marginal number ℓ of f_A , written $f_A^{\star\ell}$.

To estimate $\sum_k \delta_{jk}^2$, estimators of the type \hat{B}_j^2 and \hat{H}_j^2 are not alone enough. Instead we use the already introduced wavelet contrast estimator (plug-in), $\hat{C}_j^2(\tilde{X}) = \sum_k (\hat{\alpha}_{jk^1} \dots \hat{\alpha}_{jk^d} - \hat{\alpha}_{jk^1} \dots \hat{\alpha}_{jk^d})^2$, and the corresponding U-statistic estimator of order $2d + 2$,

$$\hat{D}_j^2(\tilde{X}) = \frac{1}{A_n^{2d+2}} \sum_{i \in I_n^{2d+2}} \sum_{k \in \mathbb{Z}^d} [\Phi_{jk}(X_{i^1}) - \varphi_{jk^1}(X_{i^2}) \dots \varphi_{jk^d}(X_{i^{d+1}})] \\ [\Phi_{jk}(X_{i^{d+2}}) - \varphi_{jk^1}(X_{i^{d+3}}) \dots \varphi_{jk^d}(X_{i^{2d+2}})]$$

with as above $I_n^m = \{(i^1, \dots, i^m): i^\ell \in \mathbb{N}, 1 \leq i^\ell \leq n, i^{\ell_1} \neq i^{\ell_2} \text{ if } \ell_1 \neq \ell_2\}$ and X^ℓ referring to the dimension ℓ of $X \in \mathbb{R}^d$.

As it turns out, the U-statistic estimator \hat{D}_j^2 computed on the full sample \tilde{X} is slightly suboptimal, compared to the rate of a \hat{B}_j^2 in estimating a bare quadratic functional.

As an alternative to $\hat{D}_j^2(\tilde{X})$, we are then led to consider various U-statistic and plug-in estimators based on splits of the full sample, which seems the only way to find back the

well-known optimal convergence rate of the estimation of quadratic functional, for reasons that will be explained in the course of the proofs.

These additional estimators and conditions of use, together with the full sample estimators \hat{C}_j^2 and \hat{D}_j^2 are presented in section 3.

Section 2 of the paper recalls some essential definitions for the convenience of the reader not familiar with wavelets and Besov spaces, and may be skipped.

Section 4 is all devoted to the computation of a risk bound for the different estimators presented in section 3.

We refer the reader to a preliminary paper on ICA by wavelets (Barbedor, 2005) which contains numerical simulations, details on the implementation of the wavelet contrast estimator and other practical considerations not repeated here. Note that this paper gives a new proof of the convergence rate in $C2^{jd}n^{-1}$ of the wavelet contrast estimator \hat{C}_j^2 , already introduced in the preliminary paper.

2. Notations

We set here general notations and recall some definitions for the convenience of ICA specialists. The reader already familiar with wavelets and Besov spaces can skip this part.

- *Wavelets*

Let φ be some function of $L_2(\mathbb{R})$ such that the family of translates $\{\varphi(\cdot - k), k \in \mathbb{Z}\}$ is an orthonormal system; let $V_j \subset L_2(\mathbb{R})$ be the subspace spanned by $\{\varphi_{jk} = 2^{j/2}\varphi(2^j \cdot - k), k \in \mathbb{Z}\}$.

By definition, the sequence of spaces $(V_j), j \in \mathbb{Z}$, is called a multiresolution analysis (MRA) of $L_2(\mathbb{R})$ if $V_j \subset V_{j+1}$ and $\bigcup_{j \geq 0} V_j$ is dense in $L_2(\mathbb{R})$; φ is called the father wavelet or scaling function.

Let $(V_j)_{j \in \mathbb{Z}}$ be a multiresolution analysis of $L_2(\mathbb{R})$, with V_j spanned by $\{\varphi_{jk} = 2^{j/2}\varphi(2^j \cdot - k), k \in \mathbb{Z}\}$. Define W_j as the complement of V_j in V_{j+1} , and let the families $\{\psi_{jk}, k \in \mathbb{Z}\}$ be a basis for W_j , with $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$. Let $\alpha_{jk}(f) = \langle f, \varphi_{jk} \rangle$ and $\beta_{jk}(f) = \langle f, \psi_{jk} \rangle$.

A function $f \in L_2(\mathbb{R})$ admits a wavelet expansion on $(V_j)_{j \in \mathbb{Z}}$ if the series

$$\sum_k \alpha_{j_0 k}(f) \varphi_{jk} + \sum_{j=j_0}^{\infty} \sum_k \beta_{jk}(f) \psi_{jk}$$

is convergent to f in $L_2(\mathbb{R})$; ψ is called a mother wavelet.

A MRA in dimension one also induces an associated MRA in dimension d , using the tensorial product procedure below.

Define V_j^d as the tensorial product of d copies of V_j . The increasing sequence $(V_j^d)_{j \in \mathbb{Z}}$ defines a multiresolution analysis of $L_2(\mathbb{R}^d)$ (Meyer, 1997):

– for $(i^1, \dots, i^d) \in \{0, 1\}^d$ and $(i^1, \dots, i^d) \neq (0, \dots, 0)$, define

$$\Psi(x)_{i^1, \dots, i^d} = \prod_{\ell=1}^d \psi^{(i^\ell)}(x^\ell), \quad (1)$$

with $\psi^{(0)} = \varphi$, $\psi^{(1)} = \psi$, so that ψ appears at least once in the product $\Psi(x)$ (we now omit i^1, \dots, i^d in the notation for Ψ , and in (2), although it is present each time);

– for $(i^1, \dots, i^d) = (0, \dots, 0)$, define $\Phi(x) = \prod_{\ell=1}^d \varphi(x^\ell)$;

– for $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$, let $\Psi_{jk}(x) = 2^{\frac{jd}{2}} \Psi(2^j x - k)$ and $\Phi_{jk}(x) = 2^{\frac{jd}{2}} \Phi(2^j x - k)$;

– define W_j^d as the orthogonal complement of V_j^d in V_{j+1}^d ; it is an orthogonal sum of $2^d - 1$ spaces having the form $U_{1j} \dots \otimes U_{dj}$, where U is a placeholder for V or W ; V or W are thus placed using up all permutations, but with W represented at least once, so that a fraction of the overall innovation brought by the finer resolution $j + 1$ is always present in the tensorial product.

A function f admits a wavelet expansion on the basis (Φ, Ψ) if the series

$$\sum_{k \in \mathbb{Z}^d} \alpha_{j_0 k}(f) \Phi_{j_0 k} + \sum_{j=j_0}^{\infty} \sum_{k \in \mathbb{Z}^d} \beta_{jk}(f) \Psi_{jk} \quad (2)$$

is convergent to f in $L_2(\mathbb{R}^d)$.

In connection with function approximation, wavelets can be viewed as falling in the category of orthogonal series methods, or also in the category of kernel methods.

The approximation at level j of a function f that admits a multiresolution expansion is the orthogonal projection $P_j f$ of f onto $V_j \subset L_2(\mathbb{R}^d)$ defined by

$$(P_j f)(x) = \sum_{k \in \mathbb{Z}^d} \alpha_{jk} \Phi_{jk}(x),$$

where $\alpha_{jk} = \alpha_{jk^1, \dots, k^d} = \int f(x) \Phi_{jk}(x) dx$.

With a concentration condition verified for compactly supported wavelets, the projection operator can also be written

$$(P_j f)(x) = \int_{\mathbb{R}^d} K_j(x, y) f(y) d(y),$$

with $K_j(x, y) = 2^{jd} \sum_{k \in \mathbb{Z}^d} \Phi_{jk}(x) \overline{\Phi_{jk}(y)}$. K_j is an orthogonal projection kernel with window 2^{-jd} (which is not translation invariant).

■ *Besov spaces*

Besov spaces admit a characterization in terms of wavelet coefficients, which makes them intrinsically connected to the analysis of curves via wavelet techniques.

$f \in L_p(\mathbb{R}^d)$ belongs to the (inhomogeneous) Besov space $B_{spq}(\mathbb{R}^d)$ if

$$J_{spq}(f) = \|\alpha_0\|_{\ell_p} + \left[\sum_{j \geq 0} \left[2^{js} 2^{dj(\frac{1}{2} - \frac{1}{p})} \|\beta_j\|_{\ell_p} \right]^q \right]^{\frac{1}{q}} < \infty,$$

with $s > 0$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, and $\varphi, \psi \in C^r$, $r > s$ (Meyer, 1997).

Let P_j be the projection operator on V_j and let D_j be the projection operator on W_j . J_{spq} is equivalent to

$$J'_{spq}(f) = \|P_j f\|_p + \left[\sum_{j \geq 0} [2^{js} \|D_j f\|_p]^q \right]^{\frac{1}{q}}$$

A more complete presentation of wavelets linked with Sobolev and Besov approximation theorems and statistical applications can be found in the book from Härdle et al. (1998). General references about Besov spaces are Peetre (1975), Bergh & Löfström (1976), Triebel (1992), DeVore & Lorentz (1993).

3. Estimating the factorization measure $\int (f_A - f_A^*)^2$

We first recall the definition of the wavelet contrast already introduced in Barbedor(2005).

Let f and g be two functions on \mathbb{R}^d and let Φ be the scaling function of a multiresolution analysis of $L_2(\mathbb{R}^d)$ for which projections of f and g exist.

Define the approximative loss function

$$C_j^2(f - g) = \sum_{k \in \mathbb{Z}^d} \left(\int (f - g) \Phi_{jk} \right)^2 = \|P_j(f - g)\|_2^2.$$

It is clear that $f = g$ implies $C_j^2 = 0$ and that $C_j^2 = 0$ implies $P_j f = P_j g$ almost surely.

Let f be a density function on \mathbb{R}^d ; denote by $f^{*\ell}$ the marginal distribution in dimension ℓ

$$x^\ell \mapsto \int_{\mathbb{R}^{d-1}} f(x^1, \dots, x^d) dx^1 \dots dx^{\ell-1} dx^{\ell+1} \dots dx^d$$

and denote by f^* the product of marginals $f^{*1} \dots f^{*d}$. The functions f , f^* and the $f^{*\ell}$ admit a wavelet expansion on a compactly supported basis (φ, ψ) . Consider the projections up to order j , that is to say the projections of f , f^* and $f^{*\ell}$ on V_j^d and V_j , namely

$$P_j f^* = \sum_{k \in \mathbb{Z}^d} \alpha_{jk}(f^*) \Phi_{jk}, \quad P_j f = \sum_{k \in \mathbb{Z}^d} \alpha_{jk}(f) \Phi_{jk} \quad \text{and} \quad P_j^\ell f^{*\ell} = \sum_{k \in \mathbb{Z}} \alpha_{jk}(f^{*\ell}) \varphi_{jk},$$

with $\alpha_{jk}(f^{*\ell}) = \int f^{*\ell} \varphi_{jk}$ and $\alpha_{jk}(f) = \int f \Phi_{jk}$. At least for compactly supported densities and compactly supported wavelets, it is clear that $P_j f^* = P_j^1 f^{*1} \dots P_j^d f^{*d}$.

Proposition 3.1 (ICA wavelet contrast)

Let f be a compactly supported density function on \mathbb{R}^d and let φ be the scaling function of a compactly supported wavelet.

Define the wavelet ICA contrast as $C_j^2(f - f^*)$. Then,

$$\begin{aligned} f \text{ factorizes} &\implies C_j^2(f - f^*) = 0 \\ C_j^2(f - f^*) = 0 &\implies P_j f = P_j f^{*1} \dots P_j f^{*d} \quad \text{a.s.} \end{aligned}$$

Proof $f = f^1 \dots f^d \implies f^{*\ell} = f^\ell, \ell = 1, \dots, d. \quad \square$

Wavelet contrast and quadratic functional

Let $f = f_I$ be a density defined on \mathbb{R}^d whose components are independent, that is to say f is equal to the product of its marginals. Let f_A be the mixed density given by $f_A(x) = |\det A^{-1}| f(A^{-1}x)$, with A a $d \times d$ invertible matrix. Let f_A^* be the product of the marginals of f_A . Note that when $A = I$, $f_A^* = f_I^* = f_I = f$.

By definition of a Besov space $B_{spq}(\mathbb{R}^d)$ with a r -regular wavelet $\varphi, r > s$,

$$f \in B_{spq}(\mathbb{R}^d) \iff \|f - P_j f\|_p = 2^{-js} \epsilon_j, \quad \{\epsilon_j\} \in \ell_q(\mathbb{N}^d). \quad (3)$$

So, from the decomposition

$$\begin{aligned} \|f_A - f_A^*\|_2^2 &= \int P_j (f_A - f_A^*)^2 + \int [f_A - f_A^* - P_j (f_A - f_A^*)]^2, \\ &= C_j^2(f_A - f_A^*) + \int [f_A - f_A^* - P_j (f_A - f_A^*)]^2, \end{aligned}$$

resulting from the orthogonality of V_j and W_j , and assuming that f_A and f_A^* belong to $B_{s2\infty}(\mathbb{R}^d)$,

$$0 \leq \|f_A - f_A^*\|_2^2 - C_j^2(f_A - f_A^*) \leq C 2^{-2js}, \quad (4)$$

which gives an illustration of the shrinking (with j) distance between the wavelet contrast and the always bigger squared L_2 norm of $f_A - f_A^*$ representing the exact factorization measure. A side effect of (4) is that $C_j^2(f_A - f_A^*) = 0$ is implied by $A = I$.

Estimators under consideration

Let S be the latent random variable with density f .

Define the experiment $\mathcal{E}^n = (\mathcal{X}^{\otimes n}, \mathcal{A}^{\otimes n}, (X_1, \dots, X_n), P_{f_A}^n, f_A \in B_{spq})$, where X_1, \dots, X_n is an iid sample of $X = AS$, and $P_{f_A}^n = P_{f_A} \dots \otimes P_{f_A}$ is the joint distribution of (X_1, \dots, X_n) .

Define the coordinates estimators

$$\hat{\alpha}_{jk} = \hat{\alpha}_{jk^1, \dots, k^d} = \frac{1}{n} \sum_{i=1}^n \varphi_{jk^1}(X_i^1) \dots \varphi_{jk^d}(X_i^d) \quad \text{and} \quad \hat{\alpha}_{jk^\ell} = \frac{1}{n} \sum_{i=1}^n \varphi_{jk^\ell}(X_i^\ell) \quad (5)$$

where X^ℓ is coordinate ℓ of $X \in \mathbb{R}^d$. Define also the shortcut $\hat{\lambda}_{jk} = \hat{\alpha}_{jk^1} \dots \hat{\alpha}_{jk^d}$.

Define the full sample plug-in estimator

$$\hat{C}_j^2 = \hat{C}_j^2(X_1, \dots, X_n) = \sum_{(k^1, \dots, k^d) \in \mathbb{Z}^d} (\hat{\alpha}_{j(k^1, \dots, k^d)} - \hat{\alpha}_{jk^1} \dots \hat{\alpha}_{jk^d})^2 = \sum_{k \in \mathbb{Z}^d} (\hat{\alpha}_{jk} - \hat{\lambda}_{jk})^2 \quad (6)$$

and the full sample U-statistic estimator

$$\hat{D}_j^2 = \hat{D}_j^2(X_1, \dots, X_n) = \frac{1}{A_n^{2d+2}} \sum_{i \in I_n^{2d+2}} \sum_{k \in \mathbb{Z}^d} [\Phi_{jk}(X_{i^1}) - \varphi_{jk^1}(X_{i^2}^1) \dots \varphi_{jk^d}(X_{i^{d+1}}^d)] \quad (7)$$

$$[\Phi_{jk}(X_{i^{d+2}}) - \varphi_{jk^1}(X_{i^{d+3}}^1) \dots \varphi_{jk^d}(X_{i^{2d+2}}^d)]$$

where I_n^m is the set of indices $\{(i^1, \dots, i^m) : i^\ell \in \mathbb{N}, 1 \leq i^\ell \leq n, i^{\ell_1} \neq i^{\ell_2} \text{ if } \ell_1 \neq \ell_2\}$ and $A_n^m = \frac{m!}{(n-m)!} = |I_n^m|$.

Define also the U-statistic estimators

$$\hat{B}_j^2(\{X_1, \dots, X_n\}) = \sum_k \frac{1}{A_n^2} \sum_{i \in I_n^2} \Phi_{jk}(X_{i^1}) \Phi_{jk}(X_{i^2}) \quad (8)$$

$$\hat{B}_j^2(\{X_1^\ell, \dots, X_n^\ell\}) = \sum_{k^\ell} \frac{1}{A_n^2} \sum_{i \in I_n^2} \varphi_{jk^\ell}(X_{i^1}^\ell) \varphi_{jk^\ell}(X_{i^2}^\ell).$$

Notational remark

Unless otherwise stated, superscripts designate coordinates of multi-dimensional entities while subscripts designate unrelated entities of the same set without reference to multi-dimensional unpacking. For instance, an index k belonging to \mathbb{Z}^d is also written $k = (k^1, \dots, k^d)$, with $k^\ell \in \mathbb{Z}$. Likewise a multi-index i is written $i = (i^1, \dots, i^m)$ when belonging to some $\Omega_n^m = \{i = (i^1, \dots, i^m) : i^\ell \in \mathbb{N}, 1 \leq i^\ell \leq n\}$ or $I_n^m = \{i \in \Omega_n^m : \ell_1 \neq \ell_2 \Rightarrow i^{\ell_1} \neq i^{\ell_2}\}$, for some $m \geq 1$; but i_1, i_2 would designate two different elements of I_n^m , so for instance $[\sum_{i=1}^n \sum_{k \in \mathbb{Z}^d} \Phi_{jk}(X_i)]^2$ is written $\sum_{i_1, i_2} \sum_{k_1, k_2} \Phi_{jk_1}(X_{i_1}) \Phi_{jk_2}(X_{i_2})$. Finally X^ℓ is coordinate ℓ of observation $X \in \mathbb{R}^d$ and \tilde{X} refers to a sample $\{X_1, \dots, X_n\}$.

■

As was said in the introduction and as is shown in proposition 4.6, the estimator \hat{D}_j^2 computed on the full sample is slightly suboptimal. We now review some possibilities to split the sample so that various alternatives to \hat{D}_j^2 on the full sample could be computed in an attempt to regain optimality through block independence.

We need not consider \hat{C}_j^2 on independent subsamples because, as will be seen, the order of its risk upper bound is given by the order of the component $\sum_k \hat{\alpha}_{jk}^2 - \alpha_{jk}^2$ which is not improved by splitting the sample (contrary to $\sum_k \hat{\lambda}_{jk}^2 - \lambda_{jk}^2$ and $\sum_k \hat{\alpha}_{jk} \hat{\lambda}_{jk} - \alpha_{jk} \lambda_{jk}$). The rate of \hat{C}_j^2 is unchanged compared to what appeared in Barbedor (2005).

- Split the full sample $\{X_1, \dots, X_n\}$ in $d+1$ disjoint subsamples $\tilde{R}^0, \tilde{R}^1, \dots, \tilde{R}^d$ where the sample \tilde{R}^0 refers to a plain section of the full sample, $\{X_1, \dots, X_{[n/d+1]}\}$ say, and the samples $\tilde{R}^1, \dots, \tilde{R}^d$ refer to dimension ℓ of their section of the full sample, $\{X_{[n/d+1]\ell+1}^\ell, \dots, X_{[n/d+1](\ell+1)}^\ell\}$ for instance.

Estimate each plug-in $\hat{\alpha}_{jk}(\tilde{R}^0)$ and $\hat{\alpha}_{jk^\ell}(\tilde{R}^\ell)$, and the U-statistics $\hat{B}_j^2(\tilde{R}^0)$, $\hat{B}_j^2(\tilde{R}^\ell)$, $\ell = 1, \dots, d$ on each independent sub-sample. This leads to the definition of the $d + 1$ samples mixed plug-in estimator

$$\hat{F}_j^2(\tilde{R}^0, \tilde{R}^1, \dots, \tilde{R}^d) = \hat{B}_j^2(\tilde{R}^0) + \prod_{\ell=1}^d \hat{B}_j^2(\tilde{R}^\ell) - 2 \sum_{k \in \mathbb{Z}^d} \hat{\alpha}_{jk}(\tilde{R}^0) \hat{\alpha}_{jk^1}(\tilde{R}^1) \dots \hat{\alpha}_{jk^d}(\tilde{R}^d). \quad (9)$$

to estimate the quantity $\sum_k \alpha_{jk}^2 + \prod_{\ell=1}^d \left(\sum_{k^\ell \in \mathbb{Z}} \alpha_{jk^\ell}^2 \right) - 2 \sum_k \alpha_{jk} \alpha_{jk^1} \dots \alpha_{jk^d} = C_j^2$.

Using estimators \hat{B}_j^2 places us in the exact replication of the case \hat{B}_j^2 found in Kerkycharian and Picard (1996), except for an estimation taking place in dimension d in the case of $\hat{B}_j^2(R^0)$. The risk of this procedure is given by proposition 4.3.

- Using the full sample $\{X_1, \dots, X_n\}$ we can generate an identically distributed sample of f_A^* , namely $\tilde{D}S = \cup_{i \in \Omega_n^d} \{X_{i^1}^1 \dots X_{i^d}^d\}$, but is not constituted of independent observations when $A \neq I$.

But then using a Hoeffding like decomposition, we can pick from $\tilde{D}S$, a sample of independent observations, $\tilde{I}S = \cup_{k=1 \dots [n/d]} \{X_{(k-1)d+1}^1 \dots X_{kd}^d\}$, although it leads to a somewhat arbitrary omission of a large part of the information available. Nevertheless we can assume that we dispose of two independent, identically distributed samples, one for f_A labelled \tilde{R} and one for f_A^* labelled \tilde{S} , with \tilde{R} independent of \tilde{S} . In this setting we define the mixed plug-in estimator

$$\hat{G}_j^2(\tilde{R}, \tilde{S}) = \hat{B}_j^2(\tilde{R}) + \hat{B}_j^2(\tilde{S}) - 2 \sum_{k \in \mathbb{Z}^d} \hat{\alpha}_{jk}(\tilde{R}) \hat{\alpha}_{jk}(\tilde{S}) \quad (10)$$

and the two samples U-statistic estimator

$$\hat{\Delta}_j^2(\tilde{R}, \tilde{S}) = \frac{1}{A_n^2} \sum_{i \in I_n^2} \sum_{k \in \mathbb{Z}^d} [\Phi_{jk}(R_{i^1}) - \Phi_{jk}(S_{i^1})] [\Phi_{jk}(R_{i^2}) - \Phi_{jk}(S_{i^2})] \quad (11)$$

assuming for simplification that both samples have same size n (that would be different from the size of the original sample). $\hat{\Delta}_j^2(R, S)$ is the exact replication (except for dimension d instead of 1) of the optimal estimator of $\int (f - g)^2$ for unrelated f and g found in Butucea and Tribouley (2006). The risk of this optimal procedure is found in proposition 4.4.

■

Bias variance trade-off

Let an estimator \hat{T}_j be used in estimating the quadratic functional $K_\star = \int (f_A - f_A^*)^2$; using (4), an upper bound for the mean squared error of this procedure when $f_A \in B_{s2\infty}(\mathbb{R}^d)$ is given by

$$E_{f_A}^n (\hat{T}_j - K_\star)^2 \leq 2E_{f_A}^n (\hat{T}_j - C_j^2)^2 + C2^{-4js}, \quad (12)$$

which shows that the key estimation is that of the wavelet contrast $C_j^2(f_A - f_A^*)$ by the estimator \hat{T}_j . Once an upper bound of the risk of \hat{T}_j in estimating C_j^2 is known, balancing the order of the bound with the squared bias 2^{-4js} gives the optimal resolution j . This is a standard procedure in nonparametric estimation.

Before diving into the computation of risk bounds, we give a summary of the different convergence rates in proposition 3.2 below. The estimators based on splits of the full sample are optimal. \hat{D}_j^2 is almost parametric on $\{2^{jd} < n\}$ and is otherwise optimal.

Proposition 3.2 (Minimal risk resolution in the class $B_{s2\infty}$ and convergence rates)

Assume that f belongs to $B_{s2\infty}(\mathbb{R}^d)$, and that projection is based on a r -regular wavelet φ , $r > s$. Convergence rates for the estimators defined at the beginning of this section are the following:

Convergence rates		
statistic	$2^{jd} < n$	$2^{jd} \geq n$
$\hat{\Delta}_j^2(\tilde{R}, \tilde{S}), \hat{G}_j^2(\tilde{R}, \tilde{S}), \hat{F}_j^2(\tilde{R}^0, \tilde{R}^1, \dots, \tilde{R}^d)$	parametric	$n^{\frac{-8s}{4s+d}}$
$\hat{D}_j^2(\tilde{X})$	$n^{-1+\frac{1}{1+4s}}$	$n^{\frac{-8s}{4s+d}}$
$\hat{C}_j^2(\tilde{X})$	$n^{\frac{-4s}{4s+d}}$	inoperable

Table 7. Convergence rates at optimal j_\star

The minimal risk resolution j_\star satisfies, $2^{j_\star d} \approx (<)n$ for parametric cases ; $2^{j_\star d} \approx n^{1+\frac{d-4s}{d+4s}}$ for \hat{D}_j^2 , $\hat{\Delta}_j^2$, \hat{G}_j^2 or \hat{F}_j^2 when $s \leq \frac{d}{4}$ and $2^{j_\star d} \approx n^{\frac{d}{d+4s}}$ for \hat{C}_j^2 .

Proof

Besov assumption about f transfers to f_A (see Barbedor, 2005). Using

$$E_{f_A}^n(\hat{H}_j - K_\star)^2 \leq 2E_{f_A}^n(\hat{H}_j - C_j^2)^2 + C2^{-4js},$$

and balancing bias 2^{-4js} and variance of the estimator \hat{H}_j , yields the optimal resolution j .

- from proposition 4.5, for estimator $\hat{C}_j^2(\tilde{X})$, the bound is inoperable on $\{2^{jd} > n\}$. Otherwise equating $2^{jd}n^{-1}$ with 2^{-4js} yields $2^j = n^{\frac{1}{d+4s}}$ and a rate in $n^{\frac{-4s}{d+4s}}$.
- from proposition 4.4 and 4.3, for estimators $\hat{F}_j^2(R^0, R^1, \dots, R^d)$, $\hat{F}_j^2(R, S)$ and $\hat{D}_j^2(R, S)$, on $\{2^{jd} > n\}$ equating $2^{jd}n^{-2}$ with 2^{-4js} yields $2^j = n^{\frac{2}{d+4s}}$ and a rate in $n^{-\frac{8s}{d+4s}}$; on $\{2^{jd} < n\}$ the rate is parametric. Moreover $2^{jd} < n$ implies that $s \geq d/4$ and $2^{jd} > n$ implies that $s \leq d/4$.
- from proposition 4.6, for estimator $\hat{D}_j^2(\tilde{X})$ on $\{2^{jd} > n\}$ equating $2^{jd}n^{-2}$ with 2^{-4js} yields $2^j = n^{\frac{2}{d+4s}}$ and a rate in $n^{-\frac{8s}{d+4s}}$; on $\{2^{jd} < n\}$ the rate is found by equating 2^jn^{-1} with 2^{-4js} .

□

4. Risk upper bounds in estimating the wavelet contrast

In the forthcoming lines, we make the assumption that both the density and the wavelet are compactly supported so that all sums in k are finite. For simplicity we further suppose the density support to be the hypercube, so that $\sum_{k \in \mathbb{Z}^d} 1 \approx 2^{jd}$.

Proposition 4.3 (Risk upper bound, $d + 1$ independent samples — $f_A, f_A^{\star 1}, \dots, f_A^{\star d}$)

Let $\{X_1, \dots, X_n\}$ be an independent, identically distributed sample of f_A . Let $\{R_1^\ell, \dots, R_n^\ell\}$ be an independent, identically distributed sample of $f_A^{\star \ell}$, $\ell = 1, \dots, d$. Assume that f is compactly supported

and that φ is a Daubechies $D2N$. Assume that the $d + 1$ samples are independent. Let $E_{f_A}^n$ be the expectation relative to the joint distribution of the $d+1$ samples. Then on $\{2^{jd} < n^2\}$,

$$E_{f_A}^n \left(\hat{F}_j^2(\tilde{X}, \tilde{R}^1, \dots, \tilde{R}^d) - C_j^2 \right)^2 \leq Cn^{-1} + C2^{jd}n^{-2} \mathbb{I}\{2^{jd} > n\}.$$

Proof

For the U-statistic $\hat{F}_j^2(\tilde{X}, \tilde{R}^1, \dots, \tilde{R}^d)$, with $\hat{\alpha}_{jk} = \hat{\alpha}_{jk}(\tilde{X})$, $\hat{\alpha}_{jk^\ell} = \hat{\alpha}_{jk^\ell}(\tilde{R}^\ell)$ and $\hat{\lambda}_{jk} = \hat{\alpha}_{jk^1} \dots \hat{\alpha}_{jk^d}$,

$$(\hat{F}_j^2 - C_j^2)^2 \leq 3 \left[\hat{B}_j^2(\tilde{X}) - \sum_k \alpha_{jk}^2 \right]^2 + 3 \prod_\ell \left[\hat{B}_j^2(\tilde{R}^\ell) - \sum_{k^\ell} \alpha_{jk^\ell}^2 \right]^2 + 6 \left[\sum_k \hat{\alpha}_{jk} \hat{\lambda}_{jk} - \sum_k \alpha_{jk} \lambda_{jk} \right]^2.$$

By proposition 5.9 the term on the right is bounded by Cn^{-1} on $\{2^{jd} < n^2\}$.

Next,

$$E_{f_A}^n [\hat{B}_j^2(\tilde{X})]^2 = n^{-2}(n-1)^{-2} \sum_{i_1, i_2 \in I_n^2} \sum_{k_1, k_2} \Phi_{jk_1}(X_{i_1^1}) \Phi_{jk_1}(X_{i_1^2}) \Phi_{jk_2}(X_{i_2^1}) \Phi_{jk_2}(X_{i_2^2})$$

On $M_4 = \{i_1, i_2 \in I_n^2 : |i_1 \cup i_2| = 4\}$, *i.e.* with no match between the two indices, the kernel $h_{i_1} h_{i_2} = \sum_{k_1, k_2} \Phi_{jk_1}(X_{i_1^1}) \Phi_{jk_1}(X_{i_1^2}) \Phi_{jk_2}(X_{i_2^1}) \Phi_{jk_2}(X_{i_2^2})$ is unbiased, equal under expectation to $(\sum_k \alpha_{jk}^2)^2$.

On $M_c, c = 2, 3$, with at least one match between i_1 and i_2 lemma 6.7 is applicable to reduce the double sum in k_1, k_2 and,

$$\begin{aligned} E_{f_A}^n h_{i_1} h_{i_2} \mathbb{I}\{M_2 \cup M_3\} &= \sum_{i_1, i_2 \in I_n^2} \sum_{k_1, k_2} \Phi_{jk_1}(X_{i_1^1}) \Phi_{jk_1}(X_{i_1^2}) \Phi_{jk_2}(X_{i_2^1}) \Phi_{jk_2}(X_{i_2^2}) \mathbb{I}\{M_2 \cup M_3\} \\ &\leq \sum_{M_2, M_3} (4N-3)^d \sum_k |\Phi_{jk}(X_{i_1^1}) \Phi_{jk}(X_{i_1^2}) \Phi_{jk}(X_{i_2^1}) \Phi_{jk}(X_{i_2^2})| \\ &\leq \sum_{M_2, M_3} C \sum_k 2^{jd(2-|i_1 \cup i_2|)} = C \sum_{M_2, M_3} 2^{jd(3-|i_1 \cup i_2|)}, \end{aligned}$$

using lemma 6.4 with parameter $m = 2$ and $r = 2$ for line 3.

Next, by lemma 6.2, $|M_c| = O(n^c)$ and $|M_4|$ divided by $(A_n^2)^2$ is more precisely equal to $1 - 4n^{-1} + Cn^{-2}$. So that

$$E_{f_A}^n [\hat{B}_j^2(\tilde{X})]^2 \leq (1 + Cn^{-2}) \left(\sum_k \alpha_{jk}^2 \right)^2 + C \sum_{c=2}^3 n^c n^{-4} 2^{jd(3-c)} = \left(\sum_k \alpha_{jk}^2 \right)^2 + Cn^{-1} + C2^{jd}n^{-2}.$$

For $E_{f_A}^n [\hat{B}_j^2(\tilde{R}^\ell) - \sum_{k^\ell} \alpha_{jk^\ell}^2]^2$, likewise, the bound is $Cn^{-1} + C2^j n^{-2}$ and so the product of the d bounds has lower order than the one for multidimensional $\hat{B}_j^2(\tilde{X})$.

□

Proposition 4.4 (Risk upper bound, 2 independent samples — f_A, f_A^*)

Let $\tilde{X} = \{X_1, \dots, X_n\}$ be an independent, identically distributed sample of X with density f_A . Let $\tilde{R} = \{R_1, \dots, R_n\}$ be an independent, identically distributed sample of R with density f_A^* . Assume that f is compactly supported and that φ is a Daubechies D2N. Assume that the two samples are independent. Let $E_{f_A}^n$ be the expectation relative to the joint distribution of the two samples.

Then

$$\begin{aligned} E_{f_A}^n \left(\hat{G}_j^2(\tilde{X}, \tilde{R}) - C_j^2 \right)^2 &\leq Cn^{-1} + C2^{jd}n^{-2} \mathbb{I}\{2^{jd} > n\} \\ E_{f_A}^n \left(\hat{\Delta}_j^2(\tilde{X}, \tilde{R}) - C_j^2 \right)^2 &\leq C^*n^{-1} + C2^{jd}n^{-2}. \end{aligned}$$

with $C^* = 0$ at independence.

Proof

For the estimator $\hat{G}_j^2(\tilde{X}, \tilde{R})$ the proof is identical to the proof of proposition 4.3, the only difference being that $\hat{\lambda}_{jk}$ and λ_{jk} no more designate a product of d one dimensional coordinates but full fledged d dimensional coordinate equivalent to $\hat{\alpha}_{jk}$ and α_{jk} .

The only new quantity to compute is then $E_{f_A}^n \left(\sum_k \hat{\alpha}_{jk}(\tilde{X}) \hat{\lambda}_{jk}(\tilde{R}) - \sum_k \alpha_{jk} \lambda_{jk} \right)^2$, coming from the crossed term.

Let $Q = E_{f_A}^n \left(\sum_k \hat{\alpha}_{jk}(\tilde{X}) \hat{\lambda}_{jk}(\tilde{R}) \right)^2$. Let $\theta = \sum_k \alpha_{jk} \lambda_{jk}$. Recall that $\Omega_n^m = \{(i^1, \dots, i^m) : i^\ell \in \mathbb{N}, 1 \leq i^\ell \leq n\}$.

Let \tilde{i} be the set of distinct coordinates of $i \in \Omega_n^4$. So that, estimators being plug-in, with a sum on Ω_n^4 , with cardinality n^4 ,

$$\begin{aligned} Q &= E_{f_A}^n \frac{1}{n^4} \sum_{i \in \Omega_n^4} \sum_{k_1, k_2} \Phi_{jk_1}(X_{i^1}) \Phi_{jk_1}(R_{i^2}) \Phi_{jk_2}(X_{i^3}) \Phi_{jk_2}(R_{i^4}) \\ &\leq \frac{1}{n^4} \left[\sum_{|\tilde{i}|=4} \theta^2 + \sum_{|\tilde{i}|=3} \left[\theta^2 + (4N-3)^d \sum_k E_{f_A}^n \Phi(X)^2 \lambda_{jk}^2 + (4N-3)^d \sum_k E_{f_A}^n \alpha_{jk}^2 \Phi(R)^2 \right] + \right. \\ &\quad \left. + \sum_{|\tilde{i}| \leq 2} (4N-3)^d \sum_k E_{f_A}^n \Phi(X)^2 \Phi(R)^2 \right] \end{aligned}$$

with lines 2 and 3 expressing all possible matches between the coordinates of i , and using lemma 6.7 to reduce double sums in k_1, k_2 .

By independence of the samples, using lemma 6.8 and the fact that $|\{i \in \Omega_n^4 : |\tilde{i}| = c\}| = O(n^c)$ given by lemma 6.2,

$$Q \leq \frac{A_n^4}{n^4} \theta^2 + Cn^{-1} \left(\theta^2 + C \sum_k \lambda_{jk}^2 + C \sum_k \alpha_{jk}^2 \right) + Cn^{-2} 2^{jd}.$$

with $A_n^p = n!/(n-p)!$. So that, with $A_n^4 n^{-4} = 1 - \frac{6}{n} + Cn^{-2}$,

$$Q - \theta^2 \leq Cn^{-2} + Cn^{-1} + Cn^{-2} 2^{jd}.$$

The rate is thus unchanged for \hat{F}_j^2 compared to the $d+1$ sample case in previous proposition.

Case $\hat{\Delta}_j^2(\tilde{X}, \tilde{R})$

Recall that $I_n^m = \{(i^1, \dots, i^m) : i^\ell \in \mathbb{N}, 1 \leq i^\ell \leq n, i^{\ell_1} \neq i^{\ell_2} \text{ if } \ell_1 \neq \ell_2\}$.

For $i \in I_n^2$, let $h_{jk}(i) = [\Phi_{jk}(X_{i^1}) - \Phi_{jk}(R_{i^1})][\Phi_{jk}(X_{i^2}) - \Phi_{jk}(R_{i^2})]$ and let $\theta = C_j^2$; so that

$$\begin{aligned} E_{f_A}^n \left(\hat{\Delta}_j^2(\tilde{X}, \tilde{R}) - \theta \right)^2 &= -\theta^2 + E_{f_A}^n \frac{1}{(A_n^2)^2} \sum_{i_1, i_2} \sum_{k_1, k_2} h_{jk_1}(i_1) h_{jk_2}(i_2) \\ &= \left(\frac{\#\{i_1, i_2 : |i_1 \cap i_2| = 0\}}{(A_n^2)^2} - 1 \right) \theta^2 + \frac{1}{(A_n^2)^2} \sum_{|i_1 \cap i_2| \geq 1} \sum_{k_1, k_2} E_{f_A}^n h_{jk_1}(i_1) h_{jk_2}(i_2), \end{aligned}$$

and by lemma 6.3 the quantity in parenthesis on the left is of the order of Cn^{-2} .

Label $Q(i_1, i_2)$ the quantity $E_{f_A}^n \sum_{k_1, k_2} h_{jk_1}(i_1) h_{jk_2}(i_2)$. Let also $\delta_{jk} = \alpha_{jk} - \lambda_{jk}$.

So that with only one matching coordinate between i_1 and i_2 ,

$$\begin{aligned} Q(i_1, i_2) \mathbb{I}\{|i_1 \cap i_2| = 1\} &= E_{f_A}^n \sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} (\Phi_{jk_1}(X) \Phi_{jk_2}(X) + \Phi_{jk_1}(R) \Phi_{jk_2}(R)) \\ &\quad - 2 \sum_k \delta_{jk} \alpha_{jk} \sum_k \delta_{jk} \lambda_{jk} \end{aligned}$$

Again by lemma 6.7 and lemma 6.8, for X or R

$$E_{f_A}^n \sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} |\Phi_{jk_1}(X) \Phi_{jk_2}(X)| \leq (4N - 3)^d \sum_k \delta_{jk}^2 E_{f_A}^n \Phi_{jk}(X)^2 \leq C \sum_k \delta_{jk}^2 \leq C$$

and since all other terms are bounded by a constant not depending on j , by lemma 6.3 $(A_n^2)^{-2} \sum_{i_1, i_2} Q(i_1, i_2) \mathbb{I}\{|i_1 \cap i_2| = 1\} \leq Cn^{-1}$.

Likewise, the maximum order of $Q(i_1, i_2) \mathbb{I}\{|i_1 \cap i_2| = 2\}$ is $\sum_k [E_{f_A}^n \Phi_{jk}(X)^2]^2$, and the corresponding bound is $2^{jd} n^{-2}$.

□

Proposition 4.5 (Full sample \hat{C}_j^2 risk upper bound)

Let $\tilde{X} = X_1, \dots, X_n$ be an independent, identically distributed sample of f_A . Assume that f is compactly supported and that φ is a Daubechies $D2N$. Let $E_{f_A}^n$ be the the expectation relative to the joint distribution of the sample \tilde{X} . Let \hat{C}_j^2 be the plug-in estimator defined in (6), Then on $\{2^{jd} < n^2\}$

$$E_{f_A}^n \left(\hat{C}_j^2(\tilde{X}) - C_j^2 \right)^2 \leq C 2^{jd} n^{-1}$$

Proof

$$E_{f_A}^n [\hat{C}_j^2 - C_j^2]^2 \leq E_{f_A}^n 3 \left(\sum_k \hat{\alpha}_{jk}^2 - \alpha_{jk}^2 \right)^2 + 3 \left(\sum_k \hat{\lambda}_{jk}^2 - \lambda_{jk}^2 \right)^2 + 3 \left(4 \sum_k \hat{\alpha}_{jk} \hat{\lambda}_{jk} - \alpha_{jk} \lambda_{jk} \right)^2$$

By proposition 5.7 the first term is of the order of $2^{jd}n^{-1}$. By proposition 5.8 the two other terms are of the order of $Cn^{-1} + 2^jn^{-1} \mathbb{I}\{2^{jd} < n^2\}$.

□

As is now shown, the rate of $\hat{D}_j^2(\tilde{X})$ computed on the full sample is slower than the one for $\hat{\Delta}_j^2(\tilde{R}, \tilde{S})$ in the two samples setting.

The reason is that we cannot always apply lemma 6.7 allowing to reduce double sums in k_1, k_2 to a sum on the diagonal $k_1 = k_2$ for translates of the same φ functions. Indeed, when a match between multi indices i_1 and i_2 involves terms corresponding to margins, it is not guaranteed that a match on observation numbers also corresponds to a match on margin numbers; that is to say, in the product $\varphi(X^{\ell_1} - k_1)\varphi(X^{\ell_2} - k_2)$, only once in a while $\ell_1 = \ell_2$; so most of the time we can say nothing about the support of the product, and the sum spans many more terms, hence the additional factor 2^j in the risk bound for \hat{D}_j^2 on the full sample.

Proposition 4.6 (Risk upper bound, full sample — f_A)

Let X_1, \dots, X_n be an independent, identically distributed sample of f_A . Assume that f is compactly supported and that φ is a Daubechies D2N. Let \hat{D}_j^2 be the U-statistic estimator defined in (7), Then

$$E_{f_A}^n \left(\hat{D}_j^2(\tilde{X}) - \sum_{k \in \mathbb{Z}^d} \delta_{jk}^2 \right)^2 \leq C2^{jd}n^{-2} + C^*2^jn^{-1}$$

with δ_{jk} the coordinate of $f_A - f_A^*$ and $C^* = 0$ at independence, when $f_A = f_A^*$.

Proof

$$E_{f_A}^n \left[\hat{D}_j^2(\tilde{X}) - \sum_{k \in \mathbb{Z}^d} \delta_{jk}^2 \right]^2 = E_{f_A}^n [\hat{D}_j^2(\tilde{X})]^2 - \left(\sum_{k \in \mathbb{Z}^d} \delta_{jk}^2 \right)^2.$$

To make $\hat{D}_j^2(\tilde{X})$ look more like the usual U-estimator of $f(f - g)^2$ for unrelated f and g , we define for $i \in I_n^{2d+2}$, the dummy slice variables $Y_i = X_{i^1}$, $V_i = (X_{i^2}, \dots, X_{i^{d+1}})$, $Z_i = X_{i^{d+2}}$, $T_i = (X_{i^{d+3}}, \dots, X_{i^{2d+2}})$; so that Y_i and Z_i have distribution P_{f_A} , V_i and T_i have distribution $P_{f_A^*} = P_{f_A^{*1}} \dots P_{f_A^{*d}}$ (once canonically projected), and Y_i, V_i, Z_i, T_i are independent variables under $P_{f_A}^n$. Next, for $k \in \mathbb{Z}^d$, define the function Λ_{jk} as

$$\begin{aligned} \Lambda_{jk}(X_{i^1}, \dots, X_{i^d}) &= \varphi_{jk^1}(X_{i^1}^1) \dots \varphi_{jk^d}(X_{i^d}^d) \quad \forall i \in \Omega_n^d \\ \Lambda_{jk}(X_i) &= \Phi_{jk}(X_i) = \varphi_{jk^1}(X_i^1) \dots \varphi_{jk^d}(X_i^d) \quad \forall i \in \Omega_n^1 = \{1, \dots, n\} \end{aligned} \tag{13}$$

with second line taken as a convention.

So that $\hat{D}_j^2(\tilde{X})$ can be written under the more friendly form

$$\hat{D}_j^2(\tilde{X}) = \frac{1}{A_n^{2d+2}} \sum_{i \in I_n^{2d+2}} \sum_{k \in \mathbb{Z}^d} [\Lambda_{jk}(Y_i) - \Lambda_{jk}(V_i)] [\Lambda_{jk}(Z_i) - \Lambda_{jk}(T_i)],$$

with $I_n^m = \{(i^1, \dots, i^m) : i^\ell \in \mathbb{N}, 1 \leq i^\ell \leq n, i^{\ell_1} \neq i^{\ell_2} \text{ if } \ell_1 \neq \ell_2\}$.

Following the friendly notation, let $h_{ik} = [\Lambda_{jk}(Y_i) - \Lambda_{jk}(V_i)][\Lambda_{jk}(Z_i) - \Lambda_{jk}(T_i)]$ be the kernel of $\hat{D}_j^2(\tilde{X})$ at fixed k . Then,

$$[\hat{D}_j^2(\tilde{X})]^2 = |I_n^{2d+2}|^{-2} \sum_{i_1, i_2 \in I_n^{2d+2} \times I_n^{2d+2}} \sum_{k_1, k_2 \in \mathbb{Z}^d \times \mathbb{Z}^d} h_{i_1 k_1} h_{i_2 k_2}.$$

Consider the partitioning sets $M_c = \{i_1, i_2 \in I_n^{2d+2} \times I_n^{2d+2}: |i_1 \cap i_2| = c\}$, $c = 0 \dots, 2d+2$, that is to say the set of pairs with c coordinates in common. Equivalently, M_c can be defined as the set $\{i_1, i_2 \in I_n^{2d+2} \times I_n^{2d+2}: |i_1 \cup i_2| = 4d+4-c\}$.

According to the partitioning, with $h_i = \sum_k h_{ik}$,

$$E_{f_A}^n [\hat{D}_j^2(\tilde{X})]^2 = |I_n^{2d+2}|^{-2} \sum_{c=0}^{2d+2} \sum_{(i_1, i_2) \in M_c} E_{f_A}^n h_{i_1} h_{i_2}.$$

Let $\lambda_{jk} = \alpha_{jk^1} \dots \alpha_{jk^d}$ and $\delta_{jk} = \alpha_{jk} - \lambda_{jk}$.

- On M_0 , with no match,

$$E_{f_A}^n h_{i_1} h_{i_2} \mathbb{I}\{M_0\} = \sum_{k_1, k_2} (\alpha_{jk_1} - \lambda_{jk_1})^2 (\alpha_{jk_2} - \lambda_{jk_2})^2 = \left(\sum_k \delta_{jk}^2 \right)^2$$

By lemma 6.3, the ratio $|M_0|/|I_n^{2d+2}|$ is lower than $1 + Cn^{-2}$. So that

$$|I_n^{2d+2}|^{-2} \sum_{M_0} E_{f_A}^n h_{i_1} h_{i_2} - \left(\sum_k \delta_{jk}^2 \right)^2 = |I_n^{2d+2}|^{-2} |M_0| E_{f_A}^n h_{i_1} h_{i_2} \mathbb{I}\{M_0\} \leq Cn^{-2}.$$

- On M_1 , assuming the match involves Y_{i_1} and Y_{i_2} ,

$$\begin{aligned} E_{f_A}^n h_{i_1} h_{i_2} \mathbb{I}\{M_1\} &= \sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} E_{f_A}^n (\Phi_{jk_1}(Y_{i_1}) - \Lambda_{jk_1}(V_{i_1})) (\Phi_{jk_2}(Y_{i_2}) - \Lambda_{jk_2}(V_{i_2})) \\ &= \sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} (E_{f_A}^n \Phi_{jk_1}(X) \Phi_{jk_2}(X) - \lambda_{jk_1} \alpha_{jk_2} - \delta_{jk_1} \lambda_{jk_2}) \\ &= E_{f_A}^n \left(\sum_k \delta_{jk} \Phi_{jk}(X) \right)^2 - C_j^2 \sum_k \lambda_{jk} \delta_{jk} - \left(\sum_k \lambda_{jk} \delta_{jk} \right) \left(\sum_k \alpha_{jk} \delta_{jk} \right) \end{aligned} \quad (14)$$

with $C_j^2 = \sum_k \delta_{jk}^2$.

Next by (16) in lemma 6.7 for the first line, the double sum in k under expectation is bounded by a constant times the sum restricted to the diagonal $k_1 = k_2$ because of the limited overlapping of translates φ_{jk} ; using also lemma 6.8,

$$E_{f_A}^n \left(\sum_k \delta_{jk} \Phi_{jk}(X) \right)^2 \leq (4N-3)^d \sum_k \delta_{jk}^2 E_{f_A}^n \Phi_{jk}(X)^2 \leq (4N-3)^d \sum_k C \delta_{jk}^2.$$

Since all other terms in (14) are clearly bounded by a constant not depending on j , we conclude by symmetry that $E_{f_A}^n h_{i_1} h_{i_2} \mathbb{I}\{M_1\} \leq C$ for any match of cardinality 1 between

narrow slices ($Y_{i_1} Y_{i_2}$ or $Z_{i_1} Z_{i_2}$ or $Y_{i_1} Z_{i_2}$ or $Z_{i_1} Y_{i_2}$). Moreover $C = 0$ when $f_A = f_A^*$ i.e. at independence, because of the omnipresence of δ_{jk} , the coordinate of $f_A - f_A^*$.

- On M_1 , if the match is between Y_{i_1} and V_{i_2} , a calculus as in (14) yields,

$$E_{f_A}^n h_{i_1} h_{i_2} \mathbb{I}\{M_1\} = - \sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} E_{f_A}^n \Phi_{jk_1}(Y_{i_1}) \Lambda_{jk_2}(V_{i_2}) + C_j^2 \sum_k \alpha_{jk} \delta_{jk} + \left(\sum_k \lambda_{jk} \delta_{jk} \right)^2;$$

which can also be found from line 2 of (14) using the swap $\Phi_{jk}(Y_{i_2}) \longleftrightarrow -\Lambda_{jk}(V_{i_2})$ and $\alpha_{jk} \longleftrightarrow -\lambda_{jk}$.

Next, for some $\ell \in \{1, \dots, d\}$,

$$\sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} E_{f_A}^n \Phi_{jk_1}(Y_{i_1}) \Lambda_{jk_2}(V_{i_2}) = \sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} \lambda_{jk_2}^{\langle d-1 \rangle} E_{f_A}^n \Phi_{jk_1}(X) \varphi_{jk_2}^\ell(X^\ell)$$

with special notation $\lambda_{jk}^{\langle r \rangle} = \alpha_{jk_1}^{p_1} \dots \alpha_{jk_d}^{p_d}$ for some p_i , $0 \leq p_i \leq r$, $\sum_{i=1}^d p_i = r$.

In the present case $\Phi_{jk_1}(X) \varphi_{jk_2}^\ell(X^\ell) = \Phi_{jk_1}(X) \varphi_{jk_2}^\ell(X^\ell) \mathbb{I}\{|k_1^\ell - k_2^\ell| < 2N - 1\}$ does not give any useful restriction of the double sum because the coefficient α_{jk} hidden in δ_{jk} is not guaranteed to factorize under any split of dimension unless $A = I$; and lemma 6.7 is useless. This is a difficulty that did not raise in propositions 4.3 and 4.4 because we could use the fact that these kind of terms were estimated over independent samples.

Instead write $E_{f_A}^n |\Phi_{jk_1}(X) \varphi_{jk_2}^\ell(X^\ell)| \leq 2^{\frac{j}{2}} \|\varphi\|_\infty E_{f_A}^n |\Phi_{jk_1}(X)| \leq C 2^{\frac{j}{2}} 2^{-\frac{jd}{2}}$ using lemma 6.8. So that when multiplied by $\sum_k \delta_{jk} \sum_k \delta_{jk} \lambda_{jk}^{\langle d-1 \rangle}$, using Meyer's lemma, the final order is 2^j .

By symmetry, for any match of cardinality 1 between a narrow and a wide slice (Y or T or equivalent pairing), $E_{f_A}^n |h_{i_1} h_{i_2}| \mathbb{I}\{M_1\} \leq C 2^j$, with $C = 0$ at independence.

- On M_1 , if the match is between V_{i_1} and V_{i_2} , by symmetry with (14) or using the swap defined above,

$$E_{f_A}^n h_{i_1} h_{i_2} \mathbb{I}\{M_1\} = \sum_{k_1, k_2} \delta_{jk} \delta_{jk'} E_{f_A}^n \Lambda_{jk}(V_{i_1}) \Lambda_{jk'}(V_{i_2}) - C_j^2 \sum_k \alpha_{jk} \delta_{jk} - \left(\sum_k \lambda_{jk} \delta_{jk} \right) \left(\sum_k \alpha_{jk} \delta_{jk} \right),$$

and for some not necessarily matching $\ell_1, \ell_2 \in \{1, \dots, d\}$ (i.e. lemma 6.7 not applicable),

$$\begin{aligned} \sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} E_{f_A}^n \Lambda_{jk_1}(V_{i_1}) \Lambda_{jk_2}(V_{i_2}) &= \sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} \lambda_{jk_1}^{\langle d-1 \rangle} \lambda_{jk_2}^{\langle d-1 \rangle} E_{f_A}^n \varphi_{jk_1}^{\ell_1}(X^{\ell_1}) \varphi_{jk_2}^{\ell_2}(X^{\ell_2}) \\ &\leq \left(\sum_k \delta_{jk} \lambda_{jk}^{\langle d-1 \rangle} \right)^2 = C 2^j \end{aligned}$$

with last line using Meyer's lemma, and having reduced the term under expectation to a constant by Cauchy-Schwarz inequality and lemma 6.8.

And we conclude again that, for any match of cardinality 1 between two wide slices (V or T or equivalent), $E_{f_A}^n h_{i_1} h_{i_2} \mathbb{I}\{M_1\} \leq C 2^j$, with $C = 0$ at independence.

By lemma 6.3, the ratio $|M_1|/|I_n^{2d+2} \times I_n^{2d+2}| \approx n^{-1}$, so in summary, the bound for M_1 has the order $C^* 2^j n^{-1}$, with $C^* = 0$ at independence.

- On M_c , $c = 2 \dots 2d + 2$.

Fix the pair of indexes $(i_1, i_2) \in I_n^{2d+2} \times I_n^{2d+2}$, we need to bound a term having the form

$$Q(i_1, i_2) = E_{f_A}^n \sum_{k_1, k_2} \Lambda_{jk}(R_{i_1}) \Lambda_{jk}(S_{i_1}) \Lambda_{jk_2}(R'_{i_2}) \Lambda_{jk_2}(S'_{i_2})$$

where both slices $R_{i_1} \neq S_{i_1}$ unrelated with both slices $R'_{i_2} \neq S'_{i_2}$ are chosen among any of the dummy Y, V, Z, T .

- *Narrow slices only.* For a match spanning four narrow slices exclusively, that is to say $(Y_{i_1} = Y_{i_2}) \cap (Z_{i_1} = Z_{i_2})$ or $(Y_{i_1} = Z_{i_2}) \cap (Z_{i_1} = Y_{i_2})$, a case possible on M_2 only, the general term of higher order is written $\sum_{k_1, k_2} E_{f_A}^n \Phi_{jk_1}(X) \Phi_{jk_2}(X) E_{f_A}^n \Phi_{jk_1}(X) \Phi_{jk_2}(X)$. By lemma 6.7 this is again lower than $(4n - 3)^d \sum_k \left[E_{f_A}^n \Phi_{jk}(X)^2 \right]^2$, that is $C2^{jd}$. By lemma 6.3, this case thus contributes to the general bound up to $C2^{jd}n^{-2}$.

Three narrow slices only is not possible and two narrow slices correspond to the case M_1 treated above.

- *Wide slices only.* For a match spanning wide slices on M_c , $c = 2, \dots, 2d$, a general term with higher order is written $\sum_{k_1, k_2} E_{f_A}^n \Lambda_{jk_1}(V_{i_1}) \Lambda_{jk_1}(T_{i_1}) \Lambda_{jk_2}(V_{i_2}) \Lambda_{jk_2}(T_{i_2})$, with $|i_1 \cap i_2| = c$, (an equivalent is obtained by swapping one V with a T). Since the slices are wide, it is not possible to distribute expectation any further right now: if V_{i_1} is always independent of T_{i_1} , both terms may depend on V_{i_2} , say. Also matching coordinates on i_1, i_2 do not necessarily correspond to matching dimensions X^ℓ of the observation, and then lemma 6.7 is not applicable. Instead write,

$$Q(i_1, i_2) = \sum_{k_1, k_2} \lambda_{jk_1}^{\langle 2d-c \rangle} \lambda_{jk_2}^{\langle 2d-c \rangle} E_{f_A}^n \left[\Lambda_{jk_1}^{\langle c \rangle}(V_{i_1}, T_{i_1}) \Lambda_{jk_2}^{\langle c \rangle}(V_{i_2}, T_{i_2}) \right],$$

with $\Lambda_{jk}^{\langle c \rangle}(V_i, T_i)$ a product of c independent terms of the form $\varphi_{jk^\ell}(X^\ell)$ spanning at least one of the slices V_i, T_i .

By definition of i_1 and i_2 , the product of $2c$ terms under expectation can be split into c independent products of two terms. So, using $E_{f_A}^n |\varphi_{jk^\ell}(X)^2| \leq C$ on each bi-term, the order at the end is $C(\sum_k \lambda_{jk}^{\langle 2d-c \rangle})^2$; and using Meyer's lemma, the bound is then of the order of $C2^{jc}$.

Finally, using lemma 6.3 as above, the contribution of this kind of term to the general bound is $\sum_{c=1}^{2d} 2^{jc} n^{-c}$.

On $\{2^j < n\} \supset \{2^{jd} < n^2\} \supset \{2^{jd} < n\}$, this quantity is bounded by $C2^j n^{-1} < C2^{jd} n^{-2}$ and on $\{2^j > n\}$ it is unbounded.

- *Narrow and wide slices* Reusing the general pattern above, with $c_w \leq 2d$ matching coordinates on wide slices and $c_r \leq 2$ on narrow slices

$$Q(i_1, i_2) = \sum_{k_1, k_2} \lambda_{jk_1}^{\langle 2d-c_w \rangle} \lambda_{jk_2}^{\langle 2d-c_w \rangle} \alpha_{jk_1}^{2-c_r} \alpha_{jk_2}^{2-c_r} E_{f_A}^n \left[\Lambda_{jk_1}^{\langle c \rangle}(Y_{i_1}, V_{i_1}, Z_{i_1}, T_{i_1}) \Lambda_{jk_2}^{\langle c \rangle}(Y_{i_2}, V_{i_2}, Z_{i_2}, T_{i_2}) \right],$$

with $\Lambda_{jk}^{\langle c \rangle}(Y_i, V_i, Z_i, T_i)$ a product of c independent terms of the form $\varphi_{jk^\ell}(X)$ or $\Phi_{jk}(X)$ spanning at least one of the slices V_i, T_i and one of the slices Y_i, Z_i . As above, the bracket is a

product of independent bi-terms, each under expectation bounded by some constant C , by lemma 6.8, using Cauchy-schwarz inequality if needed. So this is bounded by

$$Q(i_1, i_2) \leq C \sum_{k_1, k_2} \lambda_{jk_1}^{\langle 2d-c_w \rangle} \lambda_{jk_2}^{\langle 2d-c_w \rangle} \alpha_{jk_1}^{2-c_r} \alpha_{jk_2}^{2-c_r} = C \left(\sum_k \lambda_{jk}^{\langle 2d-c_w \rangle} \alpha_{jk}^{2-c_r} \right)^2;$$

using Cauchy-Schwarz inequality and Meyer's lemma this is bounded by $2^{\frac{i}{2}(c_w-d)} 2^{\frac{jd}{2}(c_r-1)}$ and, with lemma 6.3, the contribution to the general bound on $\{2^j < n^2\} \supset \{2^{jd} < n^2\}$ is

$$2^{-jd} \sum_{a=1}^2 \sum_{b=1}^{2d} 2^{\frac{ib}{2}} n^{-b} 2^{\frac{jda}{2}} n^{-a} \mathbb{I}\{2^j < n^2\} \leq C n^{-1}$$

Finally on $\{2^{jd} < n^2\}$, $E_{f_A}^n \hat{B}_j^2 - \left(\sum_k \delta_{jk}^2 \right)^2 \leq C^* 2^j n^{-1} + 2^{jd} n^{-2}$. \square

5. Appendix 1 – Propositions

Proposition 5.7 (2nd moment of $\sum_k \hat{\alpha}_{jk}^2$ about $\sum_k \alpha_{jk}^2$)

Let X_1, \dots, X_n be an independent, identically distributed sample of f , a compactly supported function defined on \mathbb{R}^d . Assume that φ is a Daubechies $D2N$. Let $\hat{\alpha}_{jk} = \frac{1}{n} \sum_{i=1}^n \varphi_{jk^1}(X_i^1) \dots \varphi_{jk^d}(X_i^d)$, $k \in \mathbb{Z}^d$.

Then $E_{f_A}^n \left(\sum_k \hat{\alpha}_{jk}^2 - \sum_k \alpha_{jk}^2 \right)^2 = C 2^{jd} n^{-1} + C 2^{2jd} n^{-2} \mathbb{I}\{2^{jd} > n\}$

Proof

For the mean, using lemma 6.8,

$$\begin{aligned} E_{f_A}^n \sum_k \hat{\alpha}_{jk}^2 &= \frac{1}{n^2} \sum_{i_1=i_2} \sum_k E_{f_A}^n \Phi_{jk}(X_{i_1}) \Phi_{jk}(X_{i_2}) + \frac{1}{n^2} \sum_{i_1 \neq i_2} \sum_k \alpha_{jk}^2 \\ &= \frac{1}{n} \sum_k \Phi_{jk}(X_i)^2 + \frac{n-1}{n} \sum_k \alpha_{jk}^2 = \sum_k \alpha_{jk}^2 + O\left(\frac{2^{jd}}{n}\right). \end{aligned}$$

For the second moment, let $M_c = \{i_1, i_2, i_3, i_4 \in \{1, \dots, n\} : |\{i_1\} \cup \dots \cup \{i_4\}| = c\}$.

$$E_{f_A}^n \left(\sum_k \hat{\alpha}_{jk}^2 \right)^2 = \frac{1}{n^4} \sum_{c=1}^4 \sum_{i_1, \dots, i_4} E_{f_A}^n \sum_{k_1, k_2} \Phi_{jk_1}(X_{i_1}) \Phi_{jk_1}(X_{i_2}) \Phi_{jk_2}(X_{i_3}) \Phi_{jk_2}(X_{i_4}) \mathbb{I}\{M_c\}$$

On $c=1$, the kernel is equal to $\sum_{k_1, k_2} \Phi_{jk_1}(X)^2 \Phi_{jk_2}(X)^2 \leq (4N-3)^d \sum_k \Phi_{jk}(X)^4$ by lemma 6.7. And by lemma 6.8, $E_{f_A}^n \sum_k \Phi_{jk}(X)^4 \leq \sum_k C 2^{jd} = C 2^{2jd}$.

On $c = 2$, the kernel takes three generic forms: (a) $\sum_{k_1, k_2} \Phi_{jk_1}(X)\Phi_{jk_1}(Y)\Phi_{jk_2}(X)\Phi_{jk_2}(Y)$ or (b) $\sum_{k_1, k_2} \Phi_{jk_1}(X)^2\Phi_{jk_2}(Y)^2$ or (c) $\sum_{k_1, k_2} \Phi_{jk_1}(X)\Phi_{jk_1}(Y)\Phi_{jk_2}(Y)^2$. In cases (a) and (c), using lemma 6.7, the double sum can be reduced to the diagonal $k_1 = k_2$. So using also lemma 6.8,

$$\begin{aligned} (a) \quad E_{f_A}^n \left| \sum_{k_1, k_2} \Phi_{jk_1}(X)\Phi_{jk_1}(Y)\Phi_{jk_2}(X)\Phi_{jk_2}(Y) \right| &\leq E_{f_A}^n (4N - 3)^d \sum_k \Phi_{jk}(X)^2\Phi_{jk}(Y)^2 \leq C2^{jd} \\ (b) \quad E_{f_A}^n \sum_{k_1, k_2} \Phi_{jk_1}(X)^2\Phi_{jk_2}(Y)^2 &\leq C2^{2jd} \\ (c) \quad E_{f_A}^n \left| \sum_{k_1, k_2} \Phi_{jk_1}(X)\Phi_{jk_1}(Y)\Phi_{jk_2}(Y)^2 \right| &\leq E_{f_A}^n (4N - 3)^d \sum_k |\Phi_{jk}(X)\Phi_{jk}(Y)^3| \leq C2^{jd}. \end{aligned}$$

On $c = 3$ the only representative form is

$$E_{f_A}^n \sum_{k_1, k_2} \Phi_{jk_1}(X)\Phi_{jk_1}(Y)\Phi_{jk_2}(Z)^2 = \sum_k \alpha_{jk}^2 \sum_k E_{f_A}^n \Phi_{jk}(X)^2 \leq C2^{jd},$$

and on $c = 4$ the statistic is unbiased equal to $(\sum_k \alpha_{jk}^2)^2$ under expectation.

Next, since $|M_4| = A_n^4$ and, using lemma 6.2, $|M_c| = O(n^c)$,

$$\begin{aligned} E_{f_A}^n \left(\sum_k \hat{\alpha}_{jk}^2 \right)^2 &\leq A_n^4 n^{-4} \left(\sum_k \alpha_{jk}^2 \right)^2 + C2^{2jd} n^{-3} + n^{-2} 2^{2jd} + n^{-1} 2^{jd} \\ &\leq \left(\sum_k \alpha_{jk}^2 \right)^2 + Cn^{-2} + Cn^{-1} 2^{jd} \mathbb{I} \{2^{jd} < n\} + Cn^{-2} 2^{2jd} \mathbb{I} \{2^{jd} > n\} \end{aligned}$$

with $A_n^4 n^{-4} = 1 - \frac{6}{n} + Cn^{-2}$.

Finally

$$\begin{aligned} E_{f_A}^n \left(\sum_k \hat{\alpha}_{jk}^2 - \sum_k \alpha_{jk}^2 \right)^2 &= E_{f_A}^n \left(\sum_k \hat{\alpha}_{jk}^2 \right)^2 + \left(\sum_k \alpha_{jk}^2 \right)^2 - 2E_{f_A}^n \sum_k \hat{\alpha}_{jk}^2 \sum_k \alpha_{jk}^2 \\ &\leq Cn^{-2} + Cn^{-1} 2^{jd} + Cn^{-2} 2^{2jd} \mathbb{I} \{2^{jd} > n\} \end{aligned}$$

□

Proposition 5.8 (2nd moment of $\sum_k \hat{\lambda}_{jk}^2$ about $\sum_k \lambda_{jk}^2$ and of $\sum_k \hat{\lambda}_{jk} \hat{\alpha}_{jk}$ about $\sum_k \lambda_{jk} \alpha_{jk}$)

Let X_1, \dots, X_n be an independent, identically distributed sample of f , a compactly supported function defined on \mathbb{R}^d . Assume that φ is a Daubechies $D2N$. Let $\hat{\lambda}_{jk} = \frac{1}{n^d} \sum_{i=1}^n \varphi_{jk^1}(X_i^1) \dots \sum_{i=1}^n \varphi_{jk^d}(X_i^d)$, $k \in \mathbb{Z}^d$.

Then on $\{2^{jd} < n^2\}$

$$\begin{aligned} E_{f_A}^n \left(\sum_k \hat{\lambda}_{jk} \hat{\alpha}_{jk} - \sum_k \lambda_{jk} \alpha_{jk} \right)^2 &\leq O(n^{-2}) + C \frac{2^j}{n} \\ E_{f_A}^n \left(\sum_k \hat{\lambda}_{jk}^2 - \sum_k \lambda_{jk}^2 \right)^2 &\leq O(n^{-2}) + C \frac{2^j}{n} \end{aligned}$$

Proof

$$E_{f_A}^n \left(\sum_k \hat{\lambda}_{jk}^2 - \lambda_{jk}^2 \right)^2 = E_{f_A}^n \left[\left(\sum_k \hat{\lambda}_{jk}^2 \right)^2 - 2 \sum_k \lambda_{jk}^2 \sum_k \hat{\lambda}_{jk}^2 + \left(\sum_k \lambda_{jk}^2 \right)^2 \right]$$

For $i \in \Omega_n^{2d}$, let V_i be the slice $(X_{i_1}^1, X_{i_2}^1, \dots, X_{i_{2d-1}}^d, X_{i_{2d}}^d)$. Let the coordinate-wise kernel function Λ_{jk} be given by $\Lambda_{jk}(V_i) = \varphi_{jk^1}(X_{i_1}^1) \varphi_{jk^1}(X_{i_2}^1) \dots \varphi_{jk^d}(X_{i_{2d-1}}^d) \varphi_{jk^d}(X_{i_{2d}}^d)$.

Let $|i|$ be the shortcut notation for $|\{i^1\} \cup \dots \cup \{i^{2d}\}|$. Let $W_n^{2d} = \{i \in \Omega_n^{2d}: |i| < 2d\}$, that is to say the set of indices with at least one repeated coordinate.

Then the mean term is written

$$\begin{aligned} E_{f_A}^n \sum_k \hat{\lambda}_{jk}^2 &= n^{-2d} \sum_{i \in \Omega_n^{2d}} \sum_k \Lambda_{jk}(V_i) \\ &= n^{-2d} \sum_{W_n^{2d}} \sum_k E_{f_A}^n \Lambda_{jk}(V_i) + A_n^{2d} n^{-2d} \sum_k \lambda_{jk}^2 \\ &= Q_1 + A_n^{2d} n^{-2d} \theta \end{aligned}$$

Let $M_c = \{i \in \Omega_n^{2d}: |i| = c\}$ be the set indices with c common coordinates. So that Q_1 is written

$$Q_1 = n^{-2d} \sum_{c=1}^{2d-1} \mathbb{I}\{M_c\} \sum_{M_c} \sum_k E_{f_A}^n \Lambda_{jk}(V_i) = \sum_k Q_{1jk}$$

By lemma 6.4 with lemma parameters $(d=1, m=2d, r=1)$, $E_{f_A}^n |\Lambda_{jk}(V_i)| \mathbb{I}\{M_c\} \leq C 2^{\frac{d}{2}(2d-2c)}$ and by lemma 6.2, $|M_c| = O(n^c)$. Hence,

$$Q_{1jk} \leq \sum_{c=1}^{2d-1} n^{-2d+c} C 2^{j(d-c)} = 2^{-jd} \sum_{c=1}^{2d-1} C \left(\frac{2^j}{n} \right)^{(2d-c)}$$

which on $\{2^{jd} < n\}$ has maximum order $2^{j(1-d)} n^{-1}$ when $d-c$ is minimum *i.e.* $c = 2d-1$. Finally $|Q_1| \leq \sum_k C 2^{j(1-d)} n^{-1} \leq C 2^j n^{-1}$.

Next, the second moment about zero is written

$$\begin{aligned} E_{f_A}^n \left(\sum_k \hat{\lambda}_{jk}^2 \right)^2 &= n^{-4d} \sum_{i_1, i_2 \in (\Omega_n^{2d})^2} \sum_{k_1, k_2} \Lambda_{jk_1}(V_{i_1}) \Lambda_{jk_2}(V_{i_2}) \\ &= n^{-4d} \sum_{W_n^{4d}} \sum_{k_1, k_2} E_{f_A}^n \Lambda_{jk_1}(V_{i_1}) \Lambda_{jk_2}(V_{i_2}) + A_n^{4d} n^{-4d} \left(\sum_k \lambda_{jk}^2 \right)^2 \\ &= Q_2 + A_n^{4d} n^{-4d} \theta^2 \end{aligned}$$

with $W_n^{4d} = \{i_1, i_2 \in (\Omega_n^{2d})^2: |i_1 \cup i_2| < 4d\}$, that is to say the set of indices with at least one repeated coordinate somewhere.

Let this time $M_c = \{i_1, i_2 \in (\Omega_n^{2d})^2: |i_1 \cup i_2| = c\}$ be the set indices with overall c common coordinates in i_1 and i_2 . So that Q_2 is written

$$Q_2 = n^{-4d} \sum_{c=1}^{4d-1} \mathbb{I}\{M_c\} \sum_{M_c} \sum_{k_1, k_2} E_{f_A}^n \Lambda_{jk_1}(V_{i_1}) \Lambda_{jk_2}(V_{i_2}) = \sum_{k_1, k_2} Q_{2j_1 k_1 j_2 k_2}$$

By lemma 6.6, unless $c = 1$, it is always possible to find indices i_1, i_2 with no match between the observations falling under k_1 and those falling under k_2 , so that there is no way to reduce the double sum in k_1, k_2 to a sum on the diagonal using lemma 6.7. Note that if $c = 1$, $E_{f_A}^n \Lambda_{jk}(V_i) \Lambda_{jk}(V_i) = E_{f_A}^n \Phi_{jk}(X)^4$ has order $C2^{jd}$.

So coping with the double sum, by lemma 6.4 with lemma parameters ($d = 1, m = 2d, r = 2$), $E_{f_A}^n |\Lambda_{jk}(V_{i_1}) \Lambda_{jk}(V_{i_2})| \leq C2^{\frac{j}{2}(4d-2c)}$, and again by lemma 6.2 $|M_c| = O(n^c)$, so $E_{f_A}^n |Q_{2j_1 k_1 j_2 k_2}| \leq \sum_{c=1}^{4d-1} n^{c-4d} C2^{\frac{j}{2}(4d-2c)}$, which on $\{2^{jd} < n\}$ has maximum order $2^{j(1-2d)} n^{-1}$ when $c = 4d - 1$. Finally, $E_{f_A}^n Q_2 \leq \sum_{k_1, k_2} C2^{j(1-2d)} n^{-1} \leq C2^j n^{-1}$.

Putting all together, and since $A_n^p n^{-p} = 1 - \frac{(d+1)(d+2)}{2n} + O(n^{-2})$,

$$\begin{aligned} E_{f_A}^n \left(\sum_k \hat{\lambda}_{jk}^2 - \lambda_{jk}^2 \right)^2 &= Q_2 + A_n^{4d} n^{-4d} \theta^2 - 2\theta(Q_1 + A_n^{2d} n^{-2d} \theta) + \theta^2 \\ &= Q_2 - 2\theta Q_1 + \theta^2(1 + A_n^{4d} n^{-4d} - 2A_n^{2d} n^{-2d}) \leq |Q_2| + 2\theta|Q_1| + O(n^{-2}) \\ &\leq C2^j n^{-1} \end{aligned}$$

For the cross product,

As above, for $i \in \Omega_n^{d+1}$, let V_i be the slice $(X_{i^0}, X_{i^1}^1, \dots, X_{i^d}^d)$. Let the coordinate-wise kernel function Λ_{jk} be given by $\Lambda_{jk}(V_i) = \Psi_{jk}(X_{i^0}) \psi_{jk^1}(X_{i^1}^1) \dots \psi_{jk^d}(X_{i^d}^d)$. Let $\theta = \sum_k \alpha_{jk} \lambda_{jk}$.

Let $W_n^{d+1} = \{i \in \Omega_n^{d+1}: |i| < d+1\}$, that is to say the set of indices with at least one repeated coordinate.

So that, $E_{f_A}^n \sum_k \hat{\alpha}_{jk} \hat{\lambda}_{jk} = Q_1 + A_n^{d+1} n^{-d-1} \theta$ with $Q_1 = n^{-d-1} \sum_{W_n^{d+1}} \sum_k E_{f_A}^n \Lambda_{jk}(V_i)$ and likewise $E_{f_A}^n \left(\sum_k \hat{\alpha}_{jk} \hat{\lambda}_{jk} \right)^2 = Q_2 + A_n^{2d+2} n^{-2d-2} \theta^2$ with $Q_2 = n^{-2d-2} \sum_{W_n^{2d+2}} \sum_{k_1, k_2} E_{f_A}^n \Lambda_{jk_1}(V_{i_1}) \Lambda_{jk_2}(V_{i_2})$. And we obtain in the same way,

$$E_{f_A}^n \left(\sum_k \hat{\alpha}_{jk} \hat{\lambda}_{jk} - \alpha_{jk} \lambda_{jk} \right)^2 \leq |Q_2| + 2\theta|Q_1| + O(n^{-2})$$

Let $M_c = \{i \in \Omega_n^{d+1}: |i| = c\}$ be the set indices with c common coordinates. So that Q_1 is written

$$Q_1 = n^{-d-1} \sum_{c=1}^d \mathbb{I}\{M_c\} \sum_{M_c} \sum_k E_{f_A}^n \Lambda_{jk}(V_i) = \sum_k Q_{1jk}$$

By lemma 6.4 with lemma parameters ($m_d = 1, m_1 = d, r = 1$),

$$E_{f_A}^n |\Lambda_{jk}(V_i)| \mathbb{I}\{M_c\} \leq C2^{\frac{jd}{2}(1-2c_d)} 2^{\frac{j}{2}(d-2c_1)}$$

with $c_1 + c_d = c$, $0 \leq c_1 \leq d$, $1 \leq c_d \leq 1$ and by lemma 6.2, $|M_c| = O(n^c)$. Hence,

$$Q_{1jk} \leq \sum_{c=1}^d n^{-d-1+c} C2^{j(d-dc_d-c_1)} = 2^{j(-1+(1-d)c_d)} \sum_{c=1}^d C \left(\frac{2^j}{n} \right)^{(d+1-c)}$$

which on $\{2^{jd} < n\}$ has maximum order $C2^{j(1-d)}n^{-1}$ when $d+1-c$ is minimum *i.e.* $c = d$. Finally $|Q_1| \leq \sum_k C2^{j(1-d)}n^{-1} \leq C2^jn^{-1}$.

Next, as above $Q_2 = \sum_{k_1, k_2} Q_{2jk_1jk_2}$, and again by lemma 6.6, unless $c = 1$, it is always possible to find indices i_1, i_2 with no matching coordinates corresponding also to matching dimension number, so that there is no way to reduce the double sum in k_1, k_2 to a sum on the diagonal using lemma 6.7.

So coping once more with the double sum, by lemma 6.4 with lemma parameters ($m_d = 1, m_1 = d, r = 2$), $E_{f_A}^n |\Lambda_{jk_1}(V_{i_1})\Lambda_{jk_2}(V_{i_2})| \leq C2^{\frac{jd}{2}(2-2c_d)}2^{\frac{j}{2}(2d-2c_1)}$, with $c_1 + c_d = c$, $1 \leq c_d \leq 2$, $0 \leq c_1 \leq 2d$, and again by lemma 6.2 $|M_c| = O(n^c)$, so

$$E_{f_A}^n |Q_{2j_1k_1j_2k_2}| \leq \sum_{c=1}^{2d+1} n^{c-2d-2} C2^{j(d-dc_d+d-c_1)} = 2^{j(-2+(1-d)c_d)} \sum_{c=1}^{2d+1} C \left(\frac{2^j}{n} \right)^{(2d+2-c)},$$

which on $\{2^{jd} < n\}$ has maximum order $C2^{-jd}n^{-1}$ when $c = 2d+1$. Then either $c_d = 1$, which means that the two terms $\Phi_{jk_1}(X_{i_1})\Phi_{jk_2}(X_{i_2})$ match on the observation number, in which case the sum in k_1, k_2 can be reduced; either $c_d = 2$. In the first case the order is $E_{f_A}^n Q_2 \leq (4N-3)^d \sum_k C2^{-jd}n^{-1} \leq Cn^{-1}$ and in the second case $E_{f_A}^n Q_2 \leq \sum_{k_1, k_2} C2^{1-2jd}n^{-1} \leq C2^jn^{-1}$. \square

Proposition 5.9 (Variance of multi sample $\sum_k \hat{\alpha}_{jk} \hat{\lambda}_{jk}$)

Let $\{X_1, \dots, X_n\}$ be an independent, identically distributed sample of f_A . Let $\{R_1^\ell, \dots, R_n^\ell\}$ be an independent, identically distributed sample of $f^{\star\ell}$, $\ell = 1, \dots, d$. Assume that f is compactly supported and that φ is a Daubechies D2N. Assume that the $d+1$ samples are independent. Let $E_{f_A}^n$ be the expectation relative to the joint samples.

Then

$$E_{f_A}^n \left(\sum_k \hat{\alpha}_{jk}(\tilde{X}) \hat{\lambda}_{jk}(\tilde{R}^1, \dots, \tilde{R}^d) - \sum_k \alpha_{jk} \lambda_{jk} \right)^2 \leq Cn^{-1} \mathbb{I}\{2^j < n\} + C2^{jd}n^{-d-1} \mathbb{I}\{2^j > n\}$$

Proof

Let $Q = E_{f_A}^n \left(\sum_{k \in \mathbb{Z}^d} \hat{\alpha}_{jk} \hat{\lambda}_{jk} \right)^2$; expanding the statistic,

$$Q = E_{f_A}^n \sum_{k_1, k_2 \in \mathbb{Z}^d} \frac{1}{n^{2d+2}} \sum_{i \in \Omega_n^{2d+2}} \Phi_{jk_1}(X_{i^1}) \Phi_{jk_2}(X_{i^2}) \varphi_{jk_1^1}(R_{i^3}^1) \varphi_{jk_2^1}(R_{i^4}^1) \dots \varphi_{jk_1^d}(R_{i^{2d+1}}^d) \varphi_{jk_2^d}(R_{i^{2d+2}}^d).$$

By independence of the samples, we only need to consider local constraints on the coordinates of $i \in \Omega_n^{2d+2}$.

Let a be a subset of $\{0, 1, \dots, d\}$. Let $J_a = \{i \in \Omega_n^{2d+2} : \ell \in a \Rightarrow i^{2\ell+1} = i^{2\ell+2}; \ell \notin a \Rightarrow i^{2\ell+1} \neq i^{2\ell+2}\}$. It is clear that $|J_a| = (n(n-1))^{d+1-|a|} n^{|a|}$ and that the J_a s define a partition of Ω_n^{2d+2} when a describes the 2^{d+1} subsets of $\{0, 1, \dots, d\}$. One can check that there are C_{d+1}^c distinct sets a such that $|a| = c$, and that $\sum_{c=0}^{d+1} C_{d+1}^c n^c (n(n-1))^{d+1-c} = n^{d+1} \sum_{c=0}^{d+1} C_{d+1}^c (n-1)^{d+1-c} = n^{2d+2}$.

On J_\emptyset the kernel is unbiased. On J_a , $0 \in a$, with the first two coordinates matching, the sum in k_1, k_2 can be reduced to a sum on the diagonal by lemma 6.7. If $0 \notin a$, but some $\ell \in a$ the sum can be reduced only on dimension ℓ , $k_1^\ell = k_2^\ell$, but to no purpose as will be seen below.

So Q is written $Q = n^{-2d-2} \sum_{a \in \mathcal{P}(\{0, \dots, d\})} Q_{0a} + Q_{1a}$, with

$$Q_{0a} \leq C_1 \sum_{i \in J_a, 0 \in a} \sum_{k \in \mathbb{Z}^d} E_{f_A}^n \Phi_{jk}(X)^2 E_{f_A}^n \varphi_{jk^{\ell_1}}(R^{\ell_1})^2 \dots E_{f_A}^n \varphi_{jk^{\ell_{|a|-1}}}(R^{\ell_{|a|-1}})^2 \alpha_{jk^{\ell_1}}^2 \dots \alpha_{jk^{\ell_{d-|a|+1}}}^2$$

and

$$Q_{1a} = \sum_{i \in J_a, 0 \notin a} \sum_{k_1, k_2} \alpha_{jk_1} \alpha_{jk_2} E_{f_A}^n \varphi_{jk_1^{\ell_1}}(R^{\ell_1}) \varphi_{jk_2^{\ell_1}}(R^{\ell_1}) \dots E_{f_A}^n \varphi_{jk_1^{\ell_{|a|-1}}}(R^{\ell_{|a|-1}}) \varphi_{jk_2^{\ell_{|a|-1}}}(R^{\ell_{|a|-1}}) \\ \alpha_{jk_1^{\ell_1}} \alpha_{jk_2^{\ell_1}} \dots \alpha_{jk_1^{\ell_{d-|a|+1}}} \alpha_{jk_2^{\ell_{d-|a|+1}}}$$

for some all distinct $\ell_1, \dots, \ell_{|a|-1}$ and $\ell_1, \dots, \ell_{d-|a|+1}$ whose union is $\{1, \dots, d\}$ and with $C_1 = (4N-3)^d$. The bound for Q_{0a} is also written

$$(4N-3)^d \sum_{i \in J_a, 0 \in a} \sum_{k \in \mathbb{Z}^d} C \lambda_{jk}^{\langle 2d-2|a|+2 \rangle}$$

with special notation $\lambda_{jk}^{\langle r \rangle} = \alpha_{jk^1}^{p_1} \dots \alpha_{jk^d}^{p_d}$ for some integers p_1, \dots, p_d , $0 \leq p_i \leq r$ with $\sum_{i=1}^d p_i = r$. And so, by Meyer's lemma this is also bounded by $\sum_{i \in J_a, 0 \in a} C 2^{j \langle |a|-1 \rangle}$.

For Q_{1a} with $|a| \geq 1$, the sum in k_1, k_2 could be split in $k_1^{\ell_1} \dots k_1^{\ell_{d-|a|+1}}, k_2^{\ell_1} \dots k_2^{\ell_{d-|a|+1}}$ where no concentration on the diagonal is ensured, and $k^{\ell_1} \dots k^{\ell_{|a|-1}}$ where lemma 6.7 is applicable, but precisely the multidimensional coefficient $\alpha_{jk} = \alpha_{jk^1} \dots k^d$ is not guaranteed factorizable under any split, unless $A = I$. So we simply fall back to

$$Q_{1a} \leq \sum_{i \in J_a, 0 \notin a} \sum_{k_1, k_2} [\alpha_{jk_1} \alpha_{jk_2}] [\alpha_{jk_1^{\ell_1}} \alpha_{jk_2^{\ell_1}} \dots \alpha_{jk_1^{\ell_{d-|a|+1}}} \alpha_{jk_2^{\ell_{d-|a|+1}}}] [C 2^{\frac{j}{2}} E_{f_A}^n |\varphi_{jk_1^{\ell_1}}(R^{\ell_1})|]^{|a|-1}.$$

This is also written, using Meyer's lemma at the end,

$$Q_{1a} \leq \sum_{i \in J_a, 0 \notin a} \left(\sum_k \alpha_{jk} \lambda_{jk}^{\langle d-|a|+1 \rangle} \right)^2 \leq \sum_{i \in J_a, 0 \notin a} C 2^{j \langle |a|-1 \rangle}$$

Finally, with $\sum_{i \in J_a} 1 = |J_a|$ given above, the general bound is written,

$$Q \leq n^{-2d-2} \left[\sum_{a \neq \emptyset} C 2^{j \langle |a|-1 \rangle} n^{d+1} (n-1)^{d+1-|a|} + n^{d+1} (n-1)^{d+1} \left(\sum_k \alpha_{jk} \lambda_{jk} \right)^2 \right]$$

and so

$$Q - \left(\sum_k \alpha_{jk} \lambda_{jk} \right)^2 \leq 2^{-j} \sum_{c=1}^{d+1} 2^{jc} (n-1)^{-c} + C n^{-2} \\ \leq C n^{-1} \mathbb{I} \{2^j < n\} + 2^{jd} n^{-d-1} \mathbb{I} \{2^j > n\}$$

□

6. Appendix 2 – Lemmas

Lemma 6.1 (Property set)

Let A_1, \dots, A_r be r non empty subsets of a finite set Ω . Let J be a subset of $\{1, \dots, r\}$.

Define the property set $B_J = \{x \in \cup A_j; x \in \cap_{j \in J} A_j; x \notin \cup_{j \in J^c} A_j\}$, that is to say the set of elements belonging exclusively to the sets listed through J . Let $b_J = |B_J|$ and $b_\kappa = \sum_{|J|=\kappa} b_J$.

Then $\sum_{\kappa=0}^r \sum_{|J|=\kappa} B_J = \Omega$, and

$$|A_1| \vee \dots \vee |A_r| \leq \sum_{\kappa=1}^r b_\kappa = |A_1 \cup \dots \cup A_r| \leq |A_1| + \dots + |A_r| = \sum_{\kappa=1}^r \kappa b_\kappa$$

with equality for the right part only if $b_\kappa = 0$, $\kappa = 2, \dots, r$ i.e. if all sets are disjoint, and equality for the left part if one set A_i contains all the others.

Proof

It follows from the definition that no two different property sets intersect and that the union of property sets defines a partition of $\cup A_i$, hence a partition of Ω with the adjunction of the missing complementary $\Omega - \cup A_i$ denoted by B_\emptyset .

With B_\emptyset , an overlapping of r sets defines a partition of Ω with cardinality at most 2^r ; there are C_r^κ property sets satisfying $|J| = \kappa$, with $\sum_{\kappa=0}^r C_r^\kappa = 2^r$.

□

Lemma 6.2 (Many sets matching indices)

Let $m \in \mathbb{N}$, $m \geq 1$. Let Ω_n^m be the set of indices $\{(i^1, \dots, i^m): i^j \in \mathbb{N}, 1 \leq i^j \leq n\}$. Let $r \in \mathbb{N}$, $r \geq 1$. Let $I_n^m = \{i \in \Omega_n^m: \ell_1 \neq \ell_2 \Rightarrow i^{\ell_1} \neq i^{\ell_2}\}$.

For $i = (i^1, \dots, i^m) \in \Omega_n^m$, let $\tilde{i} = \cup_{j=1}^m \{i^j\} \subset \{1, \dots, n\}$ be the set of distinct integers in i .

Then, for some constant C depending on m ,

$$\#\{(i_1, \dots, i_r) \in (\Omega_n^m)^r, : |\tilde{i}_1 \cup \dots \cup \tilde{i}_r| = a\} = O(n^a) I \{|\tilde{i}_1| \vee \dots \vee |\tilde{i}_r| \leq a \leq mr\}$$

and in corollary $\#\{(i_1, \dots, i_r) \in (I_n^m)^r : |i_1 \cup \dots \cup i_r| = a\} = O(n^a) I \{m \leq a \leq mr\}$.

Proof

In the setting introduced by lemma 6.1, building the compound $(\tilde{i}_1, \dots, \tilde{i}_r)$ while keeping track of matching indices is achieved by drawing $b_{\{1\}}^1 = |\tilde{i}_1|$ integers in the 2^0 -partition $b_\emptyset^0 = \{1, \dots, n\}$ thus constituting \tilde{i}_1 , then $b_{\{1,2\}}^2 + b_{\{2\}}^2 = |\tilde{i}_2|$ integers in the 2^1 -partition $\{b_{\{1\}}^1, b_\emptyset^1\}$ thus constituting two subindexes from which to build \tilde{i}_2 , then $b_{\{1,2,3\}}^3 + b_{\{2,3\}}^3 + b_{\{1,3\}}^3 + b_{\{3\}}^3 = |\tilde{i}_3|$ integers in the 2^2 -partition $\{b_{\{1,2\}}^2, b_{\{1\}}^2, b_{\{2\}}^2, b_\emptyset^2\}$ thus constituting 2^2 subindexes from which

to build \tilde{i}_3 , and so on, up to $b_{\{1,\dots,r\}}^r + \dots + b_{\{r\}}^r = |\tilde{i}_r|$ integers in the cardinality 2^{r-1} partition $\{b_{\{1,\dots,r-1\}}^{r-1}, \dots, b_{\emptyset}^{r-1}\}$ thus constituting 2^{r-1} subindexes from which to build \tilde{i}_r .

The number of ways to draw the subindexes composing the r indexes is then

$$A_{b_{\emptyset}^0}^{b_{\{1\}}^1} A_{b_{\{1\}}^1}^{b_{\{1,2\}}^2} A_{b_{\{1\}}^1}^{b_{\{2\}}^2} \dots A_{b_{\{1,\dots,r-1\}}^{r-1}}^{b_{\{1,\dots,r\}}^r} \dots A_{b_{\emptyset}^{r-1}}^{b_{\{r\}}^r} \quad (15)$$

with the nesting property $b_J^j = b_J^{j+1} + b_{J \cup \{j+1\}}^{j+1}$ (provided J exists at step j) and $A_n^m = \frac{n!}{(n-m)!}$.

At step j , the only property set with cardinality equivalent to n , is B_{\emptyset}^{j-1} , while all others have cardinalities lower than m ; so picking integers inside these light property sets involve cardinalities at most in $m!$ that go in the constants, while the pick in B_{\emptyset}^{j-1} entails a cardinality

$$A_{b_{\emptyset}^{j-1}}^{b_{\{r\}}^j} = A_{n-|\tilde{i}_1 \cup \dots \cup \tilde{i}_{j-1}|}^{b_{\{r\}}^j} \approx n^{b_{\{r\}}^j}.$$

Note that, at step $j-1$, $b_{\emptyset}^{j-1} = n - |\tilde{i}_1 \cup \dots \cup \tilde{i}_{j-1}|$, because, at step j , $b_{\{j\}}^j$ designates the number of integers in \tilde{i}_j not matching any previous index $\tilde{i}_1, \dots, \tilde{i}_{j-1}$; so that also $\sum_{j=1}^r b_{\{j\}}^j = |\tilde{i}_1 \cup \dots \cup \tilde{i}_r|$; and accessorially $\sum_{J \ni j_0} b_J^{j_0} = |\tilde{i}_{j_0}|$.

The number of integers picked from the big property set at each step is

$$A_{b_{\emptyset}^0}^{b_{\{1\}}^1} A_{b_{\{1\}}^1}^{b_{\{2\}}^2} \dots A_{b_{\emptyset}^{r-1}}^{b_{\{r\}}^r}$$

with $b_{\emptyset}^j = n - |\tilde{i}_1 \cup \dots \cup \tilde{i}_{j-1}|$, $b_{\emptyset}^0 = n$ and $\sum_{j=1}^r b_{\{j\}}^j = |\tilde{i}_1 \cup \dots \cup \tilde{i}_r|$.

For large n this is equivalent to $n^{|\tilde{i}_1 \cup \dots \cup \tilde{i}_r|}$.

Having drawn the subindexes, building the indexes effectively is a matter of iteratively intermixing two sets of a and b elements; an operation equivalent to highlighting b cells in a line of $a+b$ cells, which can be done in C_{a+b}^b ways, with $C_n^p = A_n^p/p!$.

Intermixing the subindexes thus involve cardinalities at most in $m!$, that go in the constant C .

Likewise, passing from \tilde{i} to i involve cardinalities at most in $C_m^{|\tilde{i}|}$ and no dependence on n .

For the corollary, if $i \in I_n^m$ then $\tilde{i} = i$ and $|\tilde{i}| = m$. If moreover $i^1 < \dots < i^r$, the number of ways to draw the subindexes is given by replacing occurrences of 'A' by 'C' in (15), with $C_n^m = \frac{n!}{m!(n-m)!}$, which does not change the order in n . Also there is only one way to intermix subindexes, because of the ordering constraint.

□

Lemma 6.3 (Two sets matching indices [Corollary and complement])

Let I_n^m be the set of indices $\{(i^1, \dots, i^m): i^j \in \mathbb{N}, 1 \leq i^j \leq n, i^j \neq i^\ell \text{ if } i \neq \ell\}$, and let $I'_n{}^m$ be the subset of I_n^m such that $\{i^1 < \dots < i^m\}$.

Then for $0 \leq b \leq m$,

$$\begin{aligned} \#\{(i_1, i_2) \in I_n^m \times I_n^m : |i_1 \cap i_2| = b\} &= A_n^m A_m^b A_{n-m}^{m-b} C_m^b = O(n^{2m-b}) \\ \#\{(i_1, i_2) \in I'_n{}^m \times I'_n{}^m : |i_1 \cap i_2| = b\} &= C_n^m C_m^b C_{n-m}^{m-b} = O(n^{2m-b}) \end{aligned}$$

In corollary, with P (resp. P') the mass probability on $(I_n^m)^2$ (resp. $(I_n^{m'})^2$), $P(|i_1 \cap i_2| = b) \approx P'(|i_1 \cap i_2| = b) = O(n^{-b})$ and $P(|i_1 \cap i_2| = 0) = P'(|i_1 \cap i_2| = 0) \leq 1 - m^2 n^{-1} + Cn^{-2}$.

Proof

For $i_1, i_2 \in I_n^m$, the equivalence $|i_1 \cap i_2| = b \iff |i_1 \cup i_2| = 2m - b$ gives the link with the general case of lemma 6.2.

Reusing the pattern of lemma 6.2 in a particular case: there are A_n^m ways to constitute i_1 , there are A_m^b ways to draw b unordered integers from i_1 and A_{n-m}^{m-b} ways to draw $m - b$ unordered integers from $\{1, \dots, n\} - i_1$.

To constitute i_2 , intermixing both subindexes of b and $m - b$ integers is equivalent to highlighting b cells in a line of m cells; there are C_m^b ways to do so. On I_n^m , by definition, having drawn the b then $m - b$ ordered distinct integers, intermixing is uniquely determined.

Accessorily, one can check that $\sum_{b=0}^m A_m^b A_{n-m}^{m-b} C_m^b = A_n^m$, and that $\sum_{b=0}^m C_m^b C_{n-m}^{m-b} = C_n^m$.

Dividing by $(A_n^m)^2$ or $(C_n^m)^2$, both equivalent to n^{2m} , gives the probabilities. Finally for the special case $b = 0$, use the fact that

$$\frac{A_n^m}{A_{n-c}^m} = \left(1 - \frac{c}{n}\right) \dots \left(1 - \frac{c}{n - m + 1}\right) \leq \left(1 - \frac{c}{n}\right)^m$$

□

Lemma 6.4 (Product of r kernels of degree m)

Let $r \in \mathbb{N}^*$. Let $m \geq 1$. Let (X_1, \dots, X_n) be an independent, identically distributed sample of a random variable on \mathbb{R}^d . Let Ω_n^m be the set of indices $\{(i^1, \dots, i^m) : i^j \in \mathbb{N}, 1 \leq i^j \leq n\}$.

For $i \in \Omega_n^m$, define

$$\begin{aligned} a_{ik} &= \Phi_{jk}(X_{i^1}) \dots \Phi_{jk}(X_{i^m}) \\ b_{ik} &= \varphi_{jk}(X_{i^1}^{\ell_1}) \dots \varphi_{jk}(X_{i^{\ell_{m_1}}}^{\ell_{m_1}}) \Phi_{jk}(X_{i^{m_1+1}}) \dots \Phi_{jk}(X_{i^{m_1+m_d}}). \end{aligned}$$

Let \tilde{i} be the set of distinct coordinates in i and let $c = c(\tilde{i}_1, \dots, \tilde{i}_r) = |\tilde{i}_1 \cup \dots \cup \tilde{i}_r|$ be the overall number of distinct coordinates in r indices $(i_1, \dots, i_r) \in (\Omega_n^m)^r$.

Then

$$\begin{aligned} E_f^n |a_{i_1 k_1} \dots a_{i_r k_r}| &\leq C 2^{\frac{jd}{2}(mr-2c)} \\ E_f^n |b_{i_1 k_1} \dots b_{i_r k_r}| &\leq C 2^{\frac{jd}{2}(m_d r - 2c_d)} 2^{\frac{j}{2}(m_1 r - 2c + 2c_d)} \end{aligned}$$

with $c_d = c_d(\tilde{i}_1, \dots, \tilde{i}_r) \leq c$ the fraction of c corresponding to products with at least one $\Phi(X)$ term and $1 \leq c_d \leq m_d r$, $0 \leq c - c_d \leq m_1 r$, $1 \leq c \leq (m_1 + m_2)r$.

Proof

Using lemma 6.1, one can see that the product $a_{i_1 k_1} \dots a_{i_r k_r}$, made of mr terms, can always be split into $|\tilde{i}_1 \cup \dots \cup \tilde{i}_r|$ independent products of $c(l)$ dependent terms, $1 \leq l \leq |\tilde{i}_1 \cup \dots \cup \tilde{i}_r|$, with $c(l)$ in the range from $|\tilde{i}_1| \vee \dots \vee |\tilde{i}_r|$ to mr and $\sum_l c(l) = mr$.

Using lemma 6.8, a product of $c(l)$ dependent terms, is bounded under expectation by $C2^{\frac{jd}{2}(c(l)-2)}$. Accumulating all independent products, the overall order is $C2^{\frac{jd}{2}(mr-2|\tilde{i}_1 \cup \dots \cup \tilde{i}_r|)}$.

For $b_{i_1 k_1} \dots b_{i_r k_r}$ make the distinction between groups containing at least one $\Phi(X)$ term and the others containing only $\varphi(X^\ell)$ terms. This splits the number $|\tilde{i}_1 \cup \dots \cup \tilde{i}_d|$ into $g_{\Phi, \varphi} + g_\varphi$. Let $c_\varphi(l)$ be the number of φ terms in a product of $c(l)$ terms, mixed or not.

On the $g_{\Phi, \varphi}$ groups containing Φ terms, first bound the product of $c_\varphi(l)$ terms by $C2^{\frac{jd}{2}c_\varphi(l)}$, and the remaining terms by $C2^{\frac{jd}{2}(c(l)-c_\varphi(l)-2)}$. On the g_φ groups with only φ terms, bound the product by $C2^{\frac{jd}{2}(c_\varphi(l)-2)}$.

The overall order is then

$$C2^{\frac{jd}{2}[(\sum_{l=1}^{g_{\Phi, \varphi}} c(l) - c_\varphi(l)) - 2g_{\Phi, \varphi}]} 2^{\frac{jd}{2} \sum_{l=1}^{g_{\Phi, \varphi}} c_\varphi(l)} 2^{\frac{jd}{2}[(\sum_{l=1}^{g_\varphi} c_\varphi(l)) - 2g_\varphi]}.$$

The final bound is found using $\sum_{l=1}^{g_\varphi} c_\varphi(l) + \sum_{l=1}^{g_{\Phi, \varphi}} c_\varphi(l) = m_1 r$ and $\sum_{l=1}^{g_{\Phi, \varphi}} c(l) - c_\varphi(l) = m_d r$.

Rename $c_d = g_{\Phi, \varphi}$ and $c - c_d = g_\varphi$.

As for the constraints, in the product of $(m_1 + m_d)r$ terms, it is clear that Φ terms have to be found somewhere, so $c_d \geq 1$, which also implies that $c - c_d = 0$ when $c = 1$ (in this case there are no independent group with only ϕ terms, but only one big group with all indices equal). Otherwise $c_d \leq m_d r$ and $c - c_d \leq m_1 r$ since there are no more that this numbers of Φ and ϕ terms in the overall product.

□

Lemma 6.5 (Meyer)

Let $V_j, j \in \mathbb{Z}$ an r -regular multiresolution analysis of $L_2(\mathbb{R}^n)$ and let $\varphi \in V_0$ be the father wavelet.

There exist two constant $c_2 > c_1 > 0$ such that for all $p \in [1, +\infty]$ and for all finite sum $f(x) = \sum_k \alpha(k) \varphi_{jk}(x)$ one has,

$$c_1 \|f\|_p \leq 2^{jd(\frac{1}{2} - \frac{1}{p})} \left(\sum_k |\alpha(k)|^p \right)^{\frac{1}{p}} \leq c_2 \|f\|_p$$

Proof See Meyer (1997)

We use the bound under a special form.

First note that if $f \in B_{sp\infty}$, $\|f\|_{sp\infty} = \|P_j f\|_p + \sup_j 2^{js} \|f - P_j f\|_p$ so that $\|f - P_j f\|_p \leq C \|f\|_{sp\infty} 2^{-js}$. So using (3),

$$\begin{aligned} \sum_k |\alpha_{jk}|^p &\leq C 2^{jd(1-p/2)} \|P_j f\|_p^p \leq C 2^{jd(1-p/2)} 2^{p-1} (\|f\|_p^p + \|f - P_j f\|_p^p) \\ &\leq C 2^{jd(1-p/2)} 2^{p-1} (\|f\|_p^p + C \|f\|_{sp\infty}^p 2^{-jps}) \\ &\leq C 2^{jd(1-p/2)} \|f\|_{sp\infty}^p. \end{aligned}$$

When applying the lemma to special coefficient $\lambda_{jk}^{(r)} = \alpha_{jk^1}^{p_1} \dots \alpha_{jk^d}^{p_d}$ for some integers p_1, \dots, p_d , $0 \leq p_i \leq r$ with $\sum_{i=1}^d p_i = r$, we use

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} |\lambda_{jk}^{(r)}| &= \sum_{k^1 \in \mathbb{Z}} |\alpha_{jk^1}^{p_1}| \dots \sum_{k^d \in \mathbb{Z}} |\alpha_{jk^d}^{p_d}| \\ &\leq C 2^{\frac{j}{2}(2-p_1)} \|f^{*1}\|_{sp_1\infty}^{p_1} \dots 2^{\frac{j}{2}(2-p_d)} \|f^{*d}\|_{sp_d\infty}^{p_d} \\ &\leq C 2^{\frac{j}{2}(2d-r)} \max_{\ell} \|f^{*\ell}\|_{sr\infty}^r \end{aligned}$$

so that even if some p_ℓ was zero, the result is a 2^j , which restitutes the effect of $\sum_{k^\ell} 1$.
□

Lemma 6.6 (Path of non matching dimension numbers)

Let $r \in \mathbb{N}$, $r \geq 2$. Let $\Omega_n^m = \{(i^1, \dots, i^m) : i^\ell \in \mathbb{N}, 1 \leq i^\ell \leq n\}$. For $i \in \Omega_n^d$, let $\Lambda_{jk}(V_i) = \varphi_{jk}(X_{i^1}^1) \dots \varphi_{jk}(X_{i^d}^d)$. Let \tilde{i} be the set of distinct coordinates of i .

In the product

$$\left(\sum_j \sum_k \frac{1}{n^d} \sum_{i \in \Omega_n^d} \Lambda_{jk}(V_i) \right)^r = \frac{1}{n^{dr}} \sum_{i_1, \dots, i_r \in (\Omega_n^d)^r} \sum_{j_1, \dots, j_r} \sum_{k_1, \dots, k_r} \Lambda_{j_1 k_1}(V_{i_1}) \dots \Lambda_{j_r k_r}(V_{i_r})$$

unless $|\tilde{i}_1 \cup \dots \cup \tilde{i}_r| < r$, it is always possible to find indices (i_1, \dots, i_r) such that no two functions φ_{jk} match on observation number.

Proof

Let $c = |\tilde{i}_1 \cup \dots \cup \tilde{i}_r|$. For $1 \leq \ell \leq n$, let $\ell^{\otimes d} = (\ell, \dots, \ell) \in \Omega_n^d$.

With r buckets of width d defined by the extent of each index $k_1 \dots, k_r$, and only $c < r$ distinct observation numbers, once c buckets have been stuffed with terms $V_{\ell^{\otimes d}}$, some already used observation number must be reused in order to fill in the remaining $r - c$ buckets. So that $r - c$ buckets will match on dimension and observation number allowing to reduce the sum to only c distinct buckets.

Once $c > r$, starting with a configuration using $V_{\ell_1^{\otimes d}}, \dots, V_{\ell_r^{\otimes d}}$ we can always use additional observation numbers to fragment further the $\ell^{\otimes d}$ terms, which preserves the empty intersection between buckets.
□

Lemma 6.7 (Daubechies wavelet concentration property)

Let $r \in \mathbb{N}$, $r \geq 1$. Let φ be the scaling function of a Daubechies wavelet $D2N$. Let h_k be the function on \mathbb{R}^m defined as a product of translations of φ

$$h_k(x_1, \dots, x_m) = \varphi(x_1 - k^1) \dots \varphi(x_m - k^m),$$

with $k = (k^1, \dots, k^m) \in \mathbb{Z}^m$.

Then for a Haar wavelet $[\sum_k h_k(x_1, \dots, x_m)]^r = \sum_k h_k(x_1, \dots, x_m)^r$.

For any $D2N$,

$$\left(\sum_k |h_k(x_1, \dots, x_m)| \right)^r \leq (4N-3)^{m(r-1)} \sum_k |h_k(x_1, \dots, x_m)|^r \quad (16)$$

Proof

With a Daubechies Wavelet $D2N$, whose support is $[0, 2N-1]$ with $\varphi(0) = \varphi(2N-1) = 0$ (except for Haar where $\varphi(0) = 1$), one has the relation

$$x \mapsto \varphi(x-k)\varphi(x-\ell) = 0, \quad \text{for } |\ell-k| \geq 2N-1;$$

when k is fixed, the cardinal of the set $|\ell-k| < 2N-1$ is equal to $(4N-3)$.

So that, with k_1, \dots, k_r denoting r independent multi-index,

$$\left(\sum_k h_k \right)^r = \sum_{k_1} \sum_{k_2 \dots k_r} h_{k_1} \dots h_{k_r} I(\Delta)$$

with $\Delta = \{|k_{i_1}^{\ell_1} - k_{i_2}^{\ell_2}| < (2N-1); i_1, i_2 = 1 \dots r; \ell_1, \ell_2 = 1 \dots m\}$. Once k_1 say, is fixed, the cardinal of Δ is not greater than $(4N-3)^{m(r-1)}$ and is exactly equal to 1 for Haar, when all $k_1 = \dots = k_r$.

For any Daubechies wavelet, and $r \geq 1$, using the inequality $(|h_{k_1}|^r \dots |h_{k_r}|^r)^{\frac{1}{r}} \leq \frac{1}{r} \sum_i |h_{k_i}|^r$,

$$\begin{aligned} \left(\sum_k |h_k| \right)^r &\leq \sum_{k_1, \dots, k_r} \frac{1}{r} (|h_{k_1}|^r + \dots + |h_{k_r}|^r) \mathbb{I}\{\Delta\} \\ &= \frac{1}{r} \left[\sum_{k_1, \dots, k_r} |h_{k_1}|^r \mathbb{I}\{\Delta\} + \dots + \sum_{k_1, \dots, k_r} |h_{k_r}|^r \mathbb{I}\{\Delta\} \right] \\ &\leq (4N-3)^{m(r-1)} \sum_k |h_k|^r, \end{aligned}$$

□

Lemma 6.8 (*r*th order moment of Φ_{jk})

Let X be random variables on \mathbb{R}^d with density f . Let Φ be the tensorial scaling function of an MRA of $L_2(\mathbb{R}^d)$. Let $\alpha_{jk} = E_f \Phi_{jk}(X)$. Then for $r \in \mathbb{N}^*$,

$$E_f |\Phi_{jk}(X) - \alpha_{jk}|^r \leq 2^r E_f |\Phi_{jk}(X)|^r \leq 2^r 2^{jd(\frac{r}{2}-1)} \|f\|_\infty \|\Phi\|_r^r.$$

If Φ is the Haar tensorial wavelet then also $E_f |\Phi_{jk}(X)|^r \leq 2^{jd(\frac{r}{2}-\frac{1}{2})} \alpha_{jk}$.

Proof

For the left part of the inequality, $(E_f |\Phi_{jk}(X) - \alpha_{jk}|^r)^{\frac{1}{r}} \leq (E_f |\Phi_{jk}(X)|^r)^{\frac{1}{r}} + E_f |\Phi_{jk}(X)|$, and also $E_f |\Phi_{jk}(X)| \leq (E_f |\Phi_{jk}(X)|^r)^{\frac{1}{r}} (E_f 1)^{\frac{r-1}{r}}$.

For the right part, $E_f |\Phi_{jk}(X)|^r = 2^{jd(r-1)} \int |\Phi(2^j x - k)|^r f(x) dx \leq 2^{jd(\frac{r}{2}-1)} \|f\|_\infty \|\Phi\|_r^r$.

Or also if Φ is positive,

$$\begin{aligned} E_f \Phi_{jk}(X)^r &= 2^{\frac{jd}{2}(r-1)} \int \Phi(2^j x - k)^{r-1} \Phi_{jk}(x) f(x) dx \\ &\leq 2^{\frac{jd}{2}(r-1)} \|\Phi\|_\infty^{r-1} \alpha_{jk}. \end{aligned}$$

□

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