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Independent Component Analysis and estimation of a quadratic functional

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Abstract

Independent component analysis (ICA) is linked up with the problem of estimating a non-linear functional of a density for which optimal estimators are well known. The precision of ICA is analyzed from the viewpoint of functional spaces in the wavelet framework. In particular, it is shown that, under Besov smoothness conditions, parametric rate of convergence is achieved by a U-statistic estimator of the wavelet ICA contrast, while the previously introduced plug-in estimator $\hat{C}_j^2$, with moderate computational cost, has a rate in $n^{-\frac{4}{9}+\epsilon}$.

Keywords: density, ICA, quadratic functional estimation, wavelets.

1. Introduction

In signal processing, blind source separation consists in the identification of analogical, independent signals mixed by a black-box device. In psychometrics, one has the notion of structural latent variable whose mixed effects are only measurable through series of tests; an example are the Big Five identified from factorial analysis by researchers in the domain of personality evaluation (Roch, 1995). Other application fields such as digital imaging, bio-medicine, finance and econometrics also use models aiming to recover hidden independent factors from observation. Independent component analysis (ICA) is one such tool; it can be seen as an extension of principal component analysis, in that it goes beyond a simple linear decorrelation only satisfactory for a normal distribution; or as a complement, since its application is precisely pointless under the assumption of normality.

Papers on ICA are found in the fields of signal processing, neural networks, statistics and information theory. Comon (1994) defined the concept of ICA as maximizing the degree of statistical independence among outputs using contrast functions approximated by the Edgeworth expansion of the Kullback-Leibler divergence.

The model is usually stated as follows: let $X$ be a random variable on $\mathbb{R}^d$, $d \geq 2$; find pairs $(A,S)$, such that $X = AS$, where $A$ is a square invertible matrix and $S$ a latent random variable whose components are mutually independent. This is usually done by
minimizing some contrast function that cancels out if, and only if, the components of $WX$ are independent, where $W$ is a candidate for the inversion of $A$.

Matrix $A$ is identifiable up to a scaling matrix and a permutation matrix if and only if $S$ has at most one Gaussian component (Comon, 1994).


Let $f$ be the density of the latent variable $S$ relative to Lebesgue measure, assuming it exists. The observed variable $X = AS$ has the density $f_A$, given by

$$f_A(x) = |\det A^{-1}|f(A^{-1}x)$$

$$= |\det B|f_1(b_1x) \ldots f_d(b_dx),$$

where $b_\ell$ is the $\ell$th row of the matrix $B = A^{-1}$; this resulting from a change of variable if the latent density $f$ is equal to the product of its marginals $f_1 \ldots f_d$. In this regard, latent variable $S = (S_1, \ldots, S_d)$ having independent components means independence of the random variables $S_\ell \circ \pi_\ell$ defined on some product probability space $\Omega = \prod \Omega_\ell$, with $\pi_\ell$ the canonical projections. So $S$ can be defined as the compound of the unrelated $S_1, \ldots, S_d$ sources.

In the ICA model expressed this way, both $f$ and $A$ are unknown, and the data consists in a random sample of $f_A$. The semi-parametric case corresponds to $f$ left unspecified, except for general regularity assumptions.

In this paper, we consider the exact contrast provided by the factorization measure $\int (f_A - f_A^*)^2$, with $f_A^*$ the product of the marginals of $f_A$. Let’s mention that the idea of comparing in the $L_2$ norm a joint density with the product of its marginals, can be traced back to Rosenblatt (1975).

**Estimation of a quadratic functional**

The problem of estimating nonlinear functionals of a density has been widely studied. In estimating $\int f^2$ under Hölder smoothness conditions, Bickel and Ritov (1988) have shown that parametric rate is achievable for a regularity $s \geq 1/4$, whereas when $s \leq 1/4$, minimax rates of convergence under mean squared error are of the order of $n^{-8s/(1+4s)$. This result has been extended to general functionals of a density $\int \phi(f)$ by Birgé and Massart (1995). Laurent (1996) has built efficient estimates for $s > 1/4$.

Let $P_j$ be the projection operator on a multiresolution analysis (MRA) at level $j$, with $\alpha_{jk} = \int f \varphi_{jk}$ be the coordinate $k$ of $f$.

In the wavelet setting, given a sample $\tilde{X} = \{X_1, \ldots, X_n\}$ of a density $f$ defined on $\mathbb{R}$,
independent and identically distributed, the U-statistic
\[ \hat{B}_j^2(\tilde{X}) = \frac{2}{n(n-1)} \sum_{i_1 < i_2} \sum_{k \in \mathbb{Z}} \varphi_{jk}(X_{i_1})\varphi_{jk}(X_{i_2}) \]
with mean \( (\hat{P}_j)^2 \) is the usual optimal estimator of the quantity \( \int f^2 \); see Kerkyacharian and Picard (1996), and Tribouley (2000) for the white noise model with adaptive rules.

In what follows, this result is implicitly extended to \( d \) dimensions using a tensorial wavelet basis \( \Phi_{jk} \), with \( \Phi_{jk}(x) = \varphi_{jk}(x^1) \ldots \varphi_{jk}(x^d) \), \( k \in \mathbb{Z}^d, x \in \mathbb{R}^d \); that is to say with \( \tilde{X} \) an independent, identically distributed sample of a density \( f \) on \( \mathbb{R}^d \), the U-statistic \( \hat{B}_j^2(\tilde{X}) = \frac{2}{n(n-1)} \sum_{i_1 < i_2} \sum_{k \in \mathbb{Z}^d} \Phi_{jk}(X_{i_1})\Phi_{jk}(X_{i_2}) \) with mean \( (\hat{P}_j)^2 = \sum_{k \in \mathbb{Z}^d} \alpha_{jk}^2 \) is also optimal in estimating the quantity \( \int f^2 \).

In the case of a compactly supported density \( f \), \( \hat{B}_j^2 \) is computable with a Daubechies wavelet D2N and dyadic approximation of \( X \), but the computational cost is basically in \( O(n^2(2N-1)^d) \), which is generally too high in practice.

On the other hand, the plug-in, biased, estimator \( \hat{H}_j^2(f) = \sum_k \left[ \frac{1}{n} \sum \Phi_{jk}(X_i) \right]^2 = \sum_k \hat{\alpha}_{jk}^2 \)

enjoys both ease of computation and ease of transitions between resolutions through discrete wavelet transform (DWT), since it builds upon a preliminary estimation of all individual wavelet coordinates of \( f \) on the projection space at level \( j \), that is to say a full density estimation. In this setting it is just as easy to compute \( \sum_k |\hat{\alpha}_{jk}|^p \) for any \( p \geq 1 \) or even \( \sup |\hat{\alpha}_{jk}| \), with a fixed computational cost in \( O(n^2(2N-1)^d) \) plus sum total, or seek out the max, of a \( 2^n \) array.

Both estimators \( \hat{H}_j^2 \) and \( \hat{B}_j^2 \) build on the same kernel \( h_j(x,y) = \sum_{k \in \mathbb{Z}^d} \Phi_{jk}(x)\Phi_{jk}(y) \) since they are written
\[ \hat{H}_j^2(\tilde{X}) = (n^2)^{-1} \sum_{i \in \Omega^n} h_j(X_{i^1}, X_{i^2}) \quad \text{and} \quad \hat{B}_j^2(\tilde{X}) = (A^n_h)^{-1} \sum_{i \in \Omega^n} h_j(X_{i^1}, X_{i^2}), \]
where, here and in the sequel, \( \Omega^n_m = \{(i^1, \ldots, i^m) : i^j \in \mathbb{N}, 1 \leq i^j \leq n\}, I^n_m = \{i \in \Omega^n_m : \ell_1 \neq \ell_2 \Rightarrow i^{\ell_1} \neq i^{\ell_2}\} \) and \( A^n_h = n!/((n-p)!). \)

The plug-in estimator \( \hat{B}_j^2 \) is then identified as the Von Mises statistic associated to \( \hat{B}_j^2 \). In estimating \( \sum_k \alpha_{jk}^2 \), the mean squared error of unbiased \( \hat{B}_j^2 \) is merely its variance, while the mean squared error of \( \hat{H}_j^2 \) adds a squared component \( E(\hat{H}_j^2 - \hat{B}_j^2)^2 \) because of the inequality
\[ (\hat{H}_j^2 - \sum_k \alpha_{jk}^2)^2 \leq 2(\hat{H}_j^2 - \hat{B}_j^2)^2 + 2(\hat{B}_j^2 - \sum_k \alpha_{jk}^2)^2. \]

From general results, a U-statistic with finite second raw moment has a variance in \( Cn^{-1} \) and under similar conditions, the difference \( E|U - V|^r \) between the U-statistic and its associated Von Mises statistic is of the order of \( n^{-r} \) (See for instance Serfling, 1980).

In the wavelet case, the dependence of the statistics on the resolution \( j \) calls for special treatment in computing these two quantities. This special computation, taking \( j \) and other properties of wavelets into account, constitutes the main topic of the paper. In particular whether \( 2^{jd} \) is lower than \( n \) or not is a critical threshold for resolution parameter \( j \). Moreover, on the set \( \{j: 2^{jd} > n^2\} \), the statistic \( \hat{B}_j^2 \), and therefore also \( \hat{H}_j^2 \), have a mean squared error not converging to zero.

If \( \hat{B}_j^2 \) and \( \hat{H}_j^2 \) share some features in estimating \( \sum_k \alpha_{jk}^2 = (\hat{P}_j f)^2 \), they differ in an essential way: the kernel \( h_j \) is averaged in one case over \( \Omega^n_m \), the set of unconstrained indexes, and
in the other case over \( I^2_n \) the set of distinct indexes. As a consequence, it is shown in the sequel that \( \hat{H}^2_j \) has mean squared error of the order of \( 2^d n^{-1} \), which makes it inoperable as soon as \( 2^d \geq n \), while \( \hat{B}^2_j \) has mean squared error of the order of \( 2^d n^{-2} \), which is then parametric on the set \( \{ j : 2^d < n \} \). In a general way, this same parallel \( \Omega^m_n \) versus \( I^m_n \) is underpinning most of the proofs presented throughout the paper.

### Wavelet ICA

Let \( f \) be the latent density in the semi-parametric model introduced above. Let \( f_A \) be the mixed density and let \( f^*_A \) be the product of the marginals of \( f_A \).

Assume, as regularity condition, that \( f \) belongs to a Besov class \( B_{s2\infty} \). It has been checked in previous work (Barbedor, 2005) that \( f_A \) and \( f^*_A \), hence \( f_A - f^*_A \) belong to the same Besov space than \( f \).

As usual, the very definition of Besov spaces (here \( B_{s2\infty} \)) and an orthogonality property of the projection spaces \( V_j \) and \( W_j \) entails the relation

\[
0 \leq \int (f_A - f^*_A)^2 - \int [P_j(f_A - f^*_A)]^2 \leq C 2^{-2j^*}. 
\]

In this relation, the quantity \( \int [P_j(f_A - f^*_A)]^2 \) is recognized as the wavelet ICA contrast \( C^2_j (f_A - f^*_A) \), introduced in a preliminary paper (Barbedor, 2005).

The wavelet ICA contrast is then a factorization measure with bias, in the sense that a zero contrast implies independence of the projected densities, and that independence in projection transfers to original densities up to some bias \( 2^{-2j^*} \).

Assume for a moment that the difference \( f_A - f^*_A \) is a density and that we dispose of an independent, identically distributed sample \( \tilde{S} \) of this difference. Computing the estimators \( \hat{B}^2_j(\tilde{S}) \) or \( \hat{H}^2_j(\tilde{S}) \) provides an estimation of \( \int (f_A - f^*_A)^2 \), the exact ICA factorization measure. In this case, the \( j^* \) realizing the best compromise between the mean squared error in \( C^2_j \) estimation and the bias of the ICA wavelet contrast \( 2^{-2j^*} \), is exactly the same as the one to minimize the overall risk in estimating the quadratic functional \( \int (f_A - f^*_A)^2 \). It is found by balancing bias and variance, a standard procedure in nonparametric estimation. From what was said above \( \hat{B}^2_j(\tilde{S}) \) would be an optimal estimator of the exact factorization measure \( \int (f_A - f^*_A)^2 \).

The previous assumption being heuristic only, and since, in ICA, the data at hand is a random sample of \( f_A \) and not \( f_A - f^*_A \), we are lead to consider estimators different from \( \hat{B}^2_j \) and \( \hat{H}^2_j \), but still alike in some way.

Indeed, let \( \delta_{jk} = \int (f_A - f^*_A) \Phi_{jk} \) be the coordinate of the difference function \( f_A - f^*_A \). In the ICA context, \( \delta_{jk} \) is estimable only through the difference \( (\alpha_{jk} - \alpha_{jk^1} \ldots \alpha_{jk^d}) \) where \( \alpha_{jk} = \int f_A \Phi_{jk} \) is the coordinate of \( f_A \) and \( \alpha_{jk^1} = \int f^*_A \Phi_{jk^1} \) refers to the coordinate of marginal number \( \ell \) of \( f_A \), written \( f^*_{\ell} \).

To estimate \( \sum_k \delta^2_{jk} \), estimators of the type \( \hat{B}^2_j \) and \( \hat{H}^2_j \) are not alone enough. Instead we use the already introduced wavelet contrast estimator (plug-in), \( \hat{C}^2_j(X) = \sum_k (\hat{\delta}_{jk^1} \ldots k^d - \)
\(\hat{\alpha}_{jk} \ldots \hat{\alpha}_{jkd}\)^2, and the corresponding U-statistic estimator of order \(2d + 2\),

\[
\hat{D}_{jk}^2(\tilde{X}) = \frac{1}{A_{n}^{2d+2}} \sum_{i \in \mathbb{Z}^{d+2}} \sum_{k \in \mathbb{Z}^{d}} \left[ \Phi_{jk}(X_{i1}) - \varphi_{jk1}(X_{i1}^1) \ldots \varphi_{jkd}(X_{i1}^{d+1}) \right] \\
\left[ \Phi_{jk}(X_{i1+2}) - \varphi_{jk1}(X_{i1+2}^1) \ldots \varphi_{jkd}(X_{i1+2}^{d+2}) \right]
\]

with as above \(I_n^m = \{(i^1, \ldots, i^m) ; i^\ell \in \mathbb{N}, 1 \leq i^\ell \leq n, i^{\ell_1} \neq i^{\ell_2} \text{ if } \ell_1 \neq \ell_2\}\) and \(X^\ell\) referring to the dimension \(\ell\) of \(X \in \mathbb{R}^d\).

As it turns out, the U-statistic estimator \(\hat{D}_{jk}^2\) computed on the full sample \(\tilde{X}\) is slightly suboptimal, compared to the rate of a \(\hat{D}_{jk}^2\) in estimating a bare quadratic functional.

As an alternative to \(\hat{D}_{jk}^2(\tilde{X})\), we are then led to consider various U-statistic and plug-in estimators based on splits of the full sample, which seems the only way to find back the well-known optimal convergence rate of the estimation of quadratic functional, for reasons that will be explained in the course of the proofs.

These additional estimators and conditions of use, together with the full sample estimators \(\hat{C}_{jk}^2\) and \(D_{jk}^2\) are presented in section 3.

Section 2 of the paper recalls some essential definitions for the convenience of the reader not familiar with wavelets and Besov spaces, and may be skipped.

Section 4 is all devoted to the computation of a risk bound for the different estimators presented in section 3.

We refer the reader to a preliminary paper on ICA by wavelets (Barbedor, 2005) which contains numerical simulations, details on the implementation of the wavelet contrast estimator and other practical considerations not repeated here. Note that this paper gives an improved convergence rate in \(C^{2d}n^{-1}\) for the wavelet contrast estimator \(\hat{C}_{jk}^2\), already introduced in the preliminary paper.

1.1 Notations

We set here general notations and recall some definitions for the convenience of ICA specialists. The reader already familiar with wavelets and Besov spaces can skip this part.

- **Wavelets**

Let \(\varphi\) be some function of \(L_2(\mathbb{R})\) such that the family of translates \(\{\varphi(\cdot - k), k \in \mathbb{Z}\}\) is an orthonormal system; let \(V_j \subset L_2(\mathbb{R})\) be the subspace spanned by \(\{\varphi_{jk} = 2^{j/2}\varphi(2^j \cdot - k), k \in \mathbb{Z}\}\).

By definition, the sequence of spaces \((V_j)_j \in \mathbb{Z}\), is called a multiresolution analysis (MRA) of \(L_2(\mathbb{R})\) if \(V_j \subset V_{j+1}\) and \(\bigcup_{j \geq 0} V_j\) is dense in \(L_2(\mathbb{R})\); \(\varphi\) is called the father wavelet or scaling function.

Let \((V_j)_{j \in \mathbb{Z}}\) be a multiresolution analysis of \(L_2(\mathbb{R})\), with \(V_j\) spanned by \(\{\varphi_{jk} = 2^{j/2}\varphi(2^j \cdot - k), k \in \mathbb{Z}\}\). Define \(W_j\) as the complement of \(V_j\) in \(V_{j+1}\), and let the families \(\{\psi_{jk}, k \in \mathbb{Z}\}\) be a basis for \(W_j\), with \(\psi_{jk}(x) = 2^{j/2}\varphi(2^j x - k)\). Let \(\alpha_{jk}(f) = \langle f, \varphi_{jk} \rangle\) and \(\beta_{jk}(f) = \langle f, \psi_{jk} \rangle\).
A function \( f \in L_2(\mathbb{R}) \) admits a wavelet expansion on \((V_j)_{j \in \mathbb{Z}}\) if the series

\[
\sum_k \alpha_{j,k}(f) \varphi_{jk} + \sum_{j=j_0}^{\infty} \sum_k \beta_{j,k}(f) \psi_{jk}
\]

is convergent to \( f \) in \( L_2(\mathbb{R}) \); \( \psi \) is called a mother wavelet.

A MRA in dimension one also induces an associated MRA in dimension \( d \), using the tensorial product procedure below.

Define \( V^d_j \) as the tensorial product of \( d \) copies of \( V_j \). The increasing sequence \((V^d_j)_{j \in \mathbb{Z}}\) defines a multiresolution analysis of \( L_2(\mathbb{R}^d) \) (Meyer, 1997):

– for \((i^1, \ldots, i^d) \in \{0, 1\}^d \) and \((i^1, \ldots, i^d) \neq (0, \ldots, 0)\), define

\[
\Psi(x_{i^1}, \ldots, x_{i^d}) = \prod_{\ell=1}^{d} \psi^{(i^\ell)}(x^\ell),
\]

with \( \psi^{(0)} = \varphi \), \( \psi^{(1)} = \psi \), so that \( \psi \) appears at least once in the product \( \Psi(x) \) (we now on omit \( i^1, \ldots, i^d \) in the notation for \( \Psi \), and in (2), although it is present each time);

– for \((i^1, \ldots, i^d) = (0, \ldots, 0)\), define \( \Phi(x) = \prod_{\ell=1}^{d} \varphi(x^\ell) \);

– for \( j \in \mathbb{Z}, k \in \mathbb{Z}^d, x \in \mathbb{R}^d \), let \( \Psi_{jk}(x) = 2^{jd} \psi(2^j x - k) \) and \( \Phi_{jk}(x) = 2^{jd} \varphi(2^j x - k) \);

– define \( W^d_j \) as the orthogonal complement of \( V^d_j \) in \( V^d_{j+1} \); it is an orthogonal sum of \( 2^d - 1 \) spaces having the form \( U_{1j} \otimes U_{dj} \), where \( U \) is a placeholder for \( V \) or \( W \); \( V \) or \( W \) are thus placed using up all permutations, but with \( W \) represented at least once, so that a fraction of the overall innovation brought by the finer resolution \( j + 1 \) is always present in the tensorial product.

A function \( f \) admits a wavelet expansion on the basis \((\Phi, \Psi)\) if the series

\[
\sum_{k \in \mathbb{Z}^d} \alpha_{j,k}(f) \Phi_{jk} + \sum_{j=j_0}^{\infty} \sum_{k \in \mathbb{Z}^d} \beta_{j,k}(f) \Psi_{jk}
\]

is convergent to \( f \) in \( L_2(\mathbb{R}^d) \).

In connection with function approximation, wavelets can be viewed as falling in the category of orthogonal series methods, or also in the category of kernel methods.

The approximation at level \( j \) of a function \( f \) that admits a multiresolution expansion is the orthogonal projection \( P_j f \) of \( f \) onto \( V_j \subset L_2(\mathbb{R}^d) \) defined by

\[
(P_j f)(x) = \sum_{k \in \mathbb{Z}^d} \alpha_{j,k} \Phi_{jk}(x),
\]

where \( \alpha_{j,k} = \alpha_{j,k^1, \ldots, k^d} = \int f(x) \Phi_{j,k}(x) \, dx \).
With a concentration condition verified for compactly supported wavelets, the projection operator can also be written

\[(P_j f)(x) = \int_{\mathbb{R}^d} K_j(x,y)f(y)dy,\]

with \(K_j(x,y) = 2^{jd} \sum_{k \in \mathbb{Z}^d} \Phi_{jk}(x)\Phi_{jk}(y)\). \(K_j\) is an orthogonal projection kernel with window \(2^{-jd}\) (which is not translation invariant).

### Besov spaces

Besov spaces admit a characterization in terms of wavelet coefficients, which makes them intrinsically connected to the analysis of curves via wavelet techniques.

\[f \in L^p(\mathbb{R}^d)\] belongs to the (inhomogeneous) Besov space \(B_{spq}(\mathbb{R}^d)\) if

\[J_{spq}(f) = \|a_0\|_{\ell^p} + \left[ \sum_{j \geq 0} \left(2^{js}2^{ps/2} + 1\right)^q \|b_j\|_{\ell^p}\right]^{1/q} < \infty,\]

with \(s > 0, 1 \leq p \leq \infty, 1 \leq q \leq \infty,\) and \(\varphi, \psi \in C^r, r > s\) (Meyer, 1997).

Let \(P_j\) be the projection operator on \(V_j\) and let \(D_j\) be the projection operator on \(W_j\). \(J_{spq}\) is equivalent to

\[J'_{spq}(f) = \|P_j f\|_p + \left[ \sum_{j \geq 0} \left(2^{js}\|D_j f\|_p\right)^q\right]^{1/q} < \infty.\]

A more complete presentation of wavelets linked with Sobolev and Besov approximation theorems and statistical applications can be found in the book from Härdle et al. (1998). General references about Besov spaces are Peetre (1975), Bergh & Löfström (1976), Triebel (1992), DeVore & Lorentz (1993).

### 1.2 Estimating the factorization measure \(\int (f_A - f_A^*)^2\)

We first recall the definition of the wavelet contrast already introduced in Barbedor (2005).

Let \(f\) and \(g\) be two functions on \(\mathbb{R}^d\) and let \(\Phi\) be the scaling function of a multiresolution analysis of \(L_2(\mathbb{R}^d)\) for which projections of \(f\) and \(g\) exist.

Define the approximate loss function

\[C^2_j(f - g) = \sum_{k \in \mathbb{Z}^d} \left(\int (f - g)\Phi_{jk}\right)^2 = \|P_j(f - g)\|_2^2.\]

It is clear that \(f = g\) implies \(C^2_j = 0\) and that \(C^2_j = 0\) implies \(P_j f = P_j g\) almost surely.

Let \(f\) be a density function on \(\mathbb{R}^d\); denote by \(f^{*\ell}\) the marginal distribution in dimension \(\ell\)

\[x^{\ell} \mapsto \int_{\mathbb{R}^{d-\ell}} f(x_1, \ldots, x^{\ell}, x^{d+1}, \ldots, x^d) dx_1 \ldots dx^{d-\ell} dx^{\ell+1} \ldots dx^d\]
and denote by $f^*$ the product of marginals $f^{s_1} \ldots f^{s_d}$. The functions $f$, $f^*$ and the $f^{s_\ell}$ admit a wavelet expansion on a compactly supported basis $(\varphi, \psi)$. Consider the projections up to order $j$, that is to say the projections of $f$, $f^*$ and $f^{s_\ell}$ on $V_j^d$ and $V_j$, namely

$$P_j f^* = \sum_{k \in \mathbb{Z}^d} \alpha_{jk}(f^*) \Phi_{jk}, \quad P_j f = \sum_{k \in \mathbb{Z}^d} \alpha_{jk}(f) \Phi_{jk} \quad \text{and} \quad P_j f^{s_\ell} = \sum_{k \in \mathbb{Z}^d} \alpha_{jk}(f^{s_\ell}) \varphi_{jk},$$

with $\alpha_{jk}(f^{s_\ell}) = \int f^{s_\ell} \varphi_{jk}$ and $\alpha_{jk}(f) = \int f \Phi_{jk}$. At least for compactly supported densities and compactly supported wavelets, it is clear that $P_j f^* = P_j f^{s_1} \ldots P_j f^{s_d}$.

**Proposition 1.1 (ICA wavelet contrast)**

Let $f$ be a compactly supported density function on $\mathbb{R}^d$ and let $\varphi$ be the scaling function of a compactly supported wavelet.

Define the wavelet ICA contrast as $C^2_j(f - f^*)$. Then,

$$f \text{ factorizes } \implies C^2_j(f - f^*) = 0$$

$$C^2_j(f - f^*) = 0 \implies P_j f = P_j f^{s_1} \ldots P_j f^{s_d} \ a.s.$$

**Proof** $f = f^1 \ldots f^d \implies f^{s_\ell} = f^\ell$, $\ell = 1, \ldots d$. □

**Wavelet contrast and quadratic functional**

Let $f = f_I$ be a density defined on $\mathbb{R}^d$ whose components are independent, that is to say $f$ is equal to the product of its marginals. Let $f_A$ be the mixed density given by $f_A(x) = \det A^{-1}/f(A^{-1}x)$, with $A$ a $d \times d$ invertible matrix. Let $f_A^\ell$ be the product of the marginals of $f_A$. Note that when $A = I$, $f_A^\ell = f_I^\ell = f_I = f$.

By definition of a Besov space $B_{spq}(\mathbb{R}^d)$ with a $r$-regular wavelet $\varphi$, $r > s$,

$$f \in B_{spq}(\mathbb{R}^d) \iff \|f - P_j f\|_p = 2^{-js} \epsilon_j, \quad \{\epsilon_j\} \in \ell_q(\mathbb{N}^d). \quad (3)$$

So, from the decomposition

$$\|f_A - f_A^\ell\|_2^2 = \int P_j(f_A - f_A^\ell)^2 + \int [f_A - f_A^\ell - P_j(f_A - f_A^\ell)]^2,$$

$$= C^2_j(f_A - f_A^\ell) + \int [f_A - f_A^\ell - P_j(f_A - f_A^\ell)]^2,$$

resulting from the orthogonality of $V_j$ and $W_j$, and assuming that $f_A$ and $f_A^\ell$ belong to $B_{s2\infty}(\mathbb{R}^d)$,

$$0 \leq \|f_A - f_A^\ell\|_2^2 - C^2_j(f_A - f_A^\ell) \leq C 2^{-2js}, \quad (4)$$

which gives an illustration of the shrinking (with $j$) distance between the wavelet contrast and the always bigger squared $L_2$ norm of $f_A - f_A^\ell$ representing the exact factorization measure. A side effect of (4) is that $C^2_j(f_A - f_A^\ell) = 0$ is implied by $A = I$.
Estimators under consideration

Let $S$ be the latent random variable with density $f$. Define the experiment $\mathcal{E} = (X^{\otimes n}, A^{\otimes n}, (X_1, \ldots, X_n), P^n_{f_A}, f_A \in B_{spn})$, where $X_1, \ldots, X_n$ is an iid sample of $X = AS$, and $P^n_{f_A} = P_{f_A} \circ \cdots \circ P_{f_A}$ is the joint distribution of $(X_1, \ldots, X_n)$.

Define the coordinates estimators

$$\hat{\alpha}_{jk} = \hat{\alpha}_{jk_1, \ldots, k_\ell} = \frac{1}{n} \sum_{i=1}^n \varphi_{jk_1}(X_i^1) \cdots \varphi_{jk_\ell}(X_i^{d_\ell})$$

and the full sample plug-in estimator

$$\hat{X} = \frac{1}{n} \sum_{i=1}^n \varphi_{jk}(X_i^d) \quad \text{and} \quad \hat{\alpha}_{jk} = \frac{1}{n} \sum_{i=1}^n \varphi_{jk}(X_i^d)$$

where $X^\ell$ is coordinate $\ell$ of $X \in \mathbb{R}^d$. Define also the shortcut $\hat{\lambda}_{jk} = \hat{\alpha}_{jk_1} \cdots \hat{\alpha}_{jk_\ell}$.

Define the full sample plug-in estimator

$$\hat{C}_j^2 = \hat{C}_j^2(X_1, \ldots, X_n) = \sum_{(k_1, \ldots, k_d) \in \mathbb{Z}^d} (\hat{\alpha}_{jk_1} - \hat{\alpha}_{jk_1})^2 = \sum_{k \in \mathbb{Z}^d} (\hat{\alpha}_{jk} - \hat{\lambda}_{jk})^2$$

and the full sample U-statistic estimator

$$\hat{D}_j^2 = \hat{D}_j^2(X_1, \ldots, X_n) = \frac{1}{A_n^{d_\ell + 2}} \sum_{i \in I_n^{d_\ell + 2}} \sum_{k \in \mathbb{Z}^d} \left[ \varphi_{jk}(X_{i_1}) - \varphi_{jk}(X_{i_2}) \cdots \varphi_{jk}(X_{i_{d_\ell + 2}}) \right]$$

where $I_n^m$ is the set of indices $\{(i_1, \ldots, i_m); i_\ell \in \mathbb{N}, 1 \leq i_\ell \leq n, i_{\ell_1} \neq i_{\ell_2} \text{ if } \ell_1 \neq \ell_2 \}$ and $A_n^m = \frac{n!}{(n-m)!} = |I_n^m|$.

Define also the U-statistic estimators

$$\hat{B}_j^2(\{X_1, \ldots, X_n\}) = \sum_k \frac{1}{A_n^d} \sum_{i \in I_n^d} \varphi_{jk}(X_{i_1}) \varphi_{jk}(X_{i_2})$$

$$\hat{B}_j^2(\{X_1^\ell, \ldots, X_n^\ell\}) = \sum_{k_\ell} \frac{1}{A_n^d} \sum_{i_\ell \in I_n^d} \varphi_{jk_\ell}(X_1^{i_\ell_1}) \varphi_{jk_\ell}(X_2^{i_\ell_2}).$$

Notational remark

Unless otherwise stated, superscripts designate coordinates of multi-dimensional entities while subscripts designate unrelated entities of the same set without reference to multi-dimensional unpacking. For instance, an index $k$ belonging to $\mathbb{Z}^d$ is also written $k = (k_1, \ldots, k_d)$, with $k^i \in \mathbb{Z}$. Likewise a multi-index $i$ is written $i = (i^1, \ldots, i^m)$ when belonging to some $\Omega_n^m = \{i = (i^1, \ldots, i^m); i^\ell \in \mathbb{N}, 1 \leq i^\ell \leq n\}$ or $I_n^m = \{i \in \Omega_n^m; i_1 \neq i_2 \Rightarrow i^1 \neq i^2\}$, for some $m \geq 1$; but $i_1, i_2$ would designate two different elements of $I_n^m$, so for instance $\left(\sum_{i^1=1}^{n} \Phi_{jk}(X_1^{i_1})\right)^2$ is written $\sum_{i_1, i_2} \sum_{k_1, k_2} \Phi_{jk_1}(X_{i_1}) \Phi_{jk_2}(X_{i_2})$. Finally $X^\ell$ is coordinate $\ell$ of observation $X \in \mathbb{R}^d$ and $\hat{X}$ refers to a sample $\{X_1, \ldots, X_n\}$.

As was said in the introduction and as is shown in proposition 1.6, the estimator $\hat{D}_j^2$ computed on the full sample is slightly suboptimal. We now review some possibilities to
split the sample so that various alternatives to $\hat{D}_j^2$ on the full sample could be computed in an attempt to regain optimality through block independence.

We need not consider $\hat{C}_j^2$ on independent sub samples because, as will be seen, the order of its risk upper bound is given by the order of the component $\sum_k \hat{\alpha}_{jk}^2 - \alpha_{jk}^2$, which is not improved by splitting the sample (contrary to $\sum_k \hat{\lambda}_{jk}^2 - \lambda_{jk}^2$ and $\sum_k \hat{\alpha}_{jk} \hat{\lambda}_{jk} - \alpha_{jk} \lambda_{jk}$). The rate of $\hat{C}_j^2$ is unchanged compared to what appeared in Barbedor (2005).

Sample split

- Split the full sample $\{X_1, \ldots, X_n\}$ in $d+1$ disjoint sub samples $\hat{R}^0, \hat{R}^1, \ldots, \hat{R}^d$ where the sample $\hat{R}^0$ refers to a plain section of the full sample, $\{X_1, \ldots, X_{n/d+1}\}$ say, and the samples $\hat{R}^1, \ldots, \hat{R}^d$ refer to dimension $\ell$ of their section of the full sample, for instance $\{X_{\ell n/d+1}^{\ell+1}, \ldots, X_{\ell n/d+1}(\ell+1)\}$.

Estimate each plug-in $\hat{\alpha}_{jk}(\hat{R}^0)$ and $\hat{\lambda}_{jk}(\hat{R}^0)$, and the U-statistics $\hat{B}_j^2(\hat{R}^0)$, $\hat{B}_j^2(\hat{R}^\ell)$, $\ell = 1, \ldots, d$ on each independent sub-sample. This leads to the definition of the $d+1$ samples mixed plug-in estimator

$$\hat{B}_j^2(\hat{R}^0, \hat{R}^1, \ldots, \hat{R}^d) = \hat{B}_j^2(\hat{R}^0) + \prod_{\ell=1}^d \hat{B}_j^2(\hat{R}^\ell) - 2 \sum_{k \in \mathbb{Z}^d} \hat{\alpha}_{jk}(\hat{R}^0) \hat{\alpha}_{jk}(\hat{R}^1) \ldots \hat{\alpha}_{jk}(\hat{R}^d).$$

(9)

to estimate the quantity $\sum_k \alpha_{jk}^2 + \prod_{\ell=1}^d \left( \sum_{k' \in \mathbb{Z}} \alpha_{jk'}^2 \right) - 2 \sum_k \alpha_{jk} \alpha_{jk^1} \ldots \alpha_{jk^d} = C_j^2$.

Using estimators $\hat{B}_j^2$ places us in the exact replication of the case $\hat{B}_j^2$ found in Kerkyacharian and Picard (1996), except for an estimation taking place in dimension $d$ in the case of $\hat{B}_j^2(\hat{R}^0)$. The risk of this procedure is given by proposition 1.3.

- Using the full sample $\{X_1, \ldots, X_n\}$ we can generate an identically distributed sample of $f_{x_A}$, namely $D\hat{S} = \cup_{i \in [\Omega_d]} \{X_i^1 \ldots X_i^d\}$, but is not constituted of independent observations when $A \neq I$.

But then using a Hoeffding like decomposition, we can pick from $D\hat{S}$, a sample of independent observations, $I\hat{S} = \cup_{k=1 \ldots [n/d]} \{X_{k-1}^1 \ldots X_{kd}^d\}$, although it leads to a somewhat arbitrary omission of a large part of the information available. Nevertheless we can assume that we dispose of two independent, identically distributed samples, one for $f_{x_A}$ labelled $\hat{R}$ and one for $f_{x_B}$ labelled $\hat{S}$, with $\hat{R}$ independent of $\hat{S}$. In this setting we define the mixed plug-in estimator

$$\hat{C}_j^2(\hat{R}, \hat{S}) = \hat{B}_j^2(\hat{R}) + \hat{B}_j^2(\hat{S}) - 2 \sum_{k \in \mathbb{Z}^d} \hat{\alpha}_{jk}(\hat{R}) \hat{\alpha}_{jk}(\hat{S})$$

(10)

and the two samples U-statistic estimator

$$\hat{\Delta}_j^2(\hat{R}, \hat{S}) = \frac{1}{A^n} \sum_{i \in I^d} \sum_{k \in \mathbb{Z}^d} \left[ \Phi_{jk}(R_i^\ell) - \Phi_{jk}(S_i^\ell) \right] \left[ \Phi_{jk}(R_i^\ell) - \Phi_{jk}(S_i^\ell) \right].$$

(11)

assuming for simplification that both samples have same size $n$ (that would be different from the size of the original sample). $\hat{\Delta}_j^2(R, S)$ is the exact replication (except for dimension $d$ instead of 1) of the optimal estimator of $\int (f - g)^2$ for unrelated $f$ and $g$ found in Butucea and Tribouley (2006). The risk of this optimal procedure is found in proposition 1.4.
Bias variance trade-off

Let an estimator \( \hat{T}_j \) be used in estimating the quadratic functional \( K_\star = \int (f_A - f_A')^2 \); using (4), an upper bound for the mean squared error of this procedure when \( f_A \in B_{2\infty}(\mathbb{R}^d) \) is given by

\[
E_{f_A}^n(\hat{T}_j - K_\star)^2 \leq 2E_{f_A}^n(\hat{T}_j - C_j^2)^2 + C2^{-4js},
\]

which shows that the key estimation is that of the wavelet contrast \( C_j^2(f_A - f_A') \) by the estimator \( \hat{T}_j \). Once an upper bound of the risk of \( \hat{T}_j \) in estimating \( C_j^2 \) is known, balancing the order of the bound with the squared bias \( 2^{-4js} \) gives the optimal resolution \( j \). This is a standard procedure in nonparametric estimation.

Before diving into the computation of risk bounds, we give a summary of the different convergence rates in proposition 1.2 below. The estimators based on splits of the full sample are optimal. \( \hat{D}_j^2 \) is almost parametric on \( \{2^j < n\} \) and is otherwise optimal.

**Proposition 1.2 (Minimal risk resolution in the class \( B_{2\infty} \) and convergence rates)**

Assume that \( f \) belongs to \( B_{2\infty}(\mathbb{R}^d) \), and that projection is based on a \( r \)-regular wavelet \( \varphi \), \( r > s \). Convergence rates for the estimators defined at the beginning of this section are the following:

<table>
<thead>
<tr>
<th>Convergence rates</th>
<th>( 2^jd &lt; n )</th>
<th>( 2^jd \geq n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>statistic</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \hat{\Delta}_j(R, \hat{S}) ), ( \hat{C}_j^2(R, \hat{S}) ), ( \hat{F}_j^2(R^0, \hat{R}^1, \ldots, \hat{R}^d) )</td>
<td>parametric</td>
<td>\n1^{-d/4+s}</td>
</tr>
<tr>
<td>( \hat{D}_j^2(\hat{X}) )</td>
<td>\n1^{-d/4+s}</td>
<td>\n1^{-d/4+s}</td>
</tr>
<tr>
<td>( \hat{C}_j^2(\hat{X}) )</td>
<td>\n1^{-d/4+s}</td>
<td>inoperable</td>
</tr>
</tbody>
</table>

Table 7. Convergence rates at optimal \( j_\star \)

The minimal risk resolution \( j_\star \) satisfies, \( 2^j \_d \approx (<)n \) for parametric cases; \( 2^j \_d \approx n1^{+ \frac{d}{4}+s} \) for \( \hat{D}_j^2 \), \( \hat{\Delta}_j^2 \), \( \hat{C}_j^2 \) or \( \hat{F}_j^2 \) when \( s \leq \frac{d}{4} \) and \( 2^j \_d \approx n \frac{1}{4} \) for \( \hat{C}_j^2 \).

Besov assumption about \( f \) transfers to \( f_A \) (see Barbedor, 2005). Using

\[
E_{f_A}^n(\hat{H}_j - K_\star)^2 \leq 2E_{f_A}^n(\hat{H}_j - C_j^2)^2 + C2^{-4js},
\]

and balancing bias \( 2^{-4js} \) and variance of the estimator \( \hat{H}_j \), yields the optimal resolution \( j \).

- from proposition 1.5, for estimator \( \hat{C}_j^2(\hat{X}) \), the bound is inoperable on \( \{2^j > n\} \). Otherwise equating \( 2^jdn^{-1} \) with \( 2^{-4js} \) yields \( 2^j = n \frac{1}{2} \) and a rate in \( n \frac{1}{2} \).

- from proposition 1.4, for estimators \( \hat{F}_j^2(R^0, R^1, \ldots, R^d) \), \( \hat{F}_j(R, S) \) and \( \hat{D}_j^2(R, S) \), on \( \{2^j > n\} \) equating \( 2^jdn^{-2} \) with \( 2^{-4js} \) yields \( 2^j = n \frac{1}{2} \) and a rate in \( n \frac{1}{2} \); on \( \{2^j < n\} \) the rate is parametric. Moreover \( 2^j < n \) implies that \( s \geq d/4 \) and \( 2^j > n \) implies that \( s \leq d/4 \).

- from proposition 1.6, for estimator \( \hat{D}_j^2(\hat{X}) \) on \( \{2^j > n\} \) equating \( 2^jdn^{-2} \) with \( 2^{-4js} \) yields \( 2^j = n \frac{1}{2} \) and a rate in \( n \frac{1}{2} \); on \( \{2^j < n\} \) the rate is found by equating \( 2^jn^{-1} \) with \( 2^{-4js} \).

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1.3 Risk upper bounds in estimating the wavelet contrast

In the forthcoming lines, we make the assumption that both the density and the wavelet
are compactly supported so that all sums in \(k\) are finite. For simplicity we further suppose
the density support to be the hypercube, so that \(\sum_{k\in\mathbb{Z}^d} 1 \approx 2^d\).

Proposition 1.3 (Risk upper bound, \(d + 1\) independent samples — \(f_A, f_A^1, \ldots, f_A^d\))

Let \(\{X_1, \ldots, X_n\}\) be an independent, identically distributed sample of \(f_A\). Let \(\{R_1^1, \ldots, R_n^d\}\) be an
independent, identically distributed sample of \(f_A^\ell\), \(\ell = 1, \ldots, d\). Assume that \(f\) is compactly supported
and that \(\varphi\) is a Daubechies D2N. Assume that the \(d + 1\) samples are independent. Let \(E_{f_A}^n\) be the
expectation relative to the joint distribution of the \(d+1\) samples. Then on \(\{2^id < n^2\}\),
\[
E_{f_A}^n \left( \hat{F}_j^2(\bar{X}, \bar{R}^1, \ldots, \bar{R}^d) - C_j^2 \right)^2 \leq Cn^{-1} + C2^{jd}n^{-2} \mathbb{I} \{2^id > n\}.
\]

For the U-statistic \(\hat{F}_j^2(\bar{X}, \bar{R}^1, \ldots, \bar{R}^d)\), with \(\hat{\alpha}_{jk} = \hat{\alpha}_{jk}(\bar{X}), \hat{\alpha}_{jk\ell} = \hat{\alpha}_{jk\ell}(\bar{R}^\ell)\) and \(\hat{\lambda}_k = \hat{\alpha}_{j_1k_1} \cdots \hat{\alpha}_{j_dk_d}\),
\[
(\hat{F}_j^2 - C_j^2)^2 \leq 3 \left[ \hat{B}_j^2(\bar{X}) - \sum_k \alpha_{jk}^2 \right]^2 + 3 \left[ \prod_\ell \hat{B}_{j_\ell}^2(\bar{R}^\ell) - \prod_\ell \sum_{k_\ell} \alpha_{jk_\ell}^2 \right]^2 + 6 \left[ \sum_k \hat{\alpha}_{jk} \hat{\lambda}_k - \sum_k \alpha_{jk} \lambda_k \right]^2.
\]

On \(\{2^id < n^2\}\), by proposition 1.9 for the term on the left, proposition 1.10 for the
middle term, and proposition 1.11 for the term on the right, the quantity is bounded by
\(Cn^{-1} + C2^{jd}n^{-2}\).

Proposition 1.4 (Risk upper bound, 2 independent samples — \(f_A, f_A^1\))

Let \(\bar{X} = \{X_1, \ldots, X_n\}\) be an independent, identically distributed sample of \(X\) with density \(f_A\). Let
\(\bar{R} = \{R_1, \ldots, R_n\}\) be an independent, identically distributed sample of \(R\) with density \(f_A^1\). Assume
that \(f\) is compactly supported and that \(\varphi\) is a Daubechies D2N. Assume that the two samples are
independent. Let \(E_{f_A}^n\) be the expectation relative to the joint distribution of the two samples.

Then
\[
E_{f_A}^n \left( \hat{G}_j^2(\bar{X}, \bar{R}) - C_j^2 \right)^2 \leq Cn^{-1} + C2^{jd}n^{-2} \mathbb{I} \{2^id > n\}
\]
\[
E_{f_A}^n \left( \hat{\Delta}_j(\bar{X}, \bar{R}) - C_j^2 \right)^2 \leq C^*n^{-1} + C2^{jd}n^{-2}.
\]

with \(C^* = 0\) at independence.

For the estimator \(\hat{G}_j^2(\bar{X}, \bar{R})\) the proof is identical to the proof of proposition 1.3, the
only difference being that \(\hat{\lambda}_k\) and \(\lambda_k\) no more designate a product of \(d\) one dimensional
coordinates but full fledged \(d\) dimensional coordinate equivalent to \(\hat{\alpha}_{jk}\) and \(\alpha_{jk}\).
The only new quantity to compute is then \( E_{fA}^n \left( \sum_k \hat{\alpha}_{jk}(\bar{X}) \hat{\lambda}_{jk}(\bar{R}) - \sum_k \alpha_{jk} \lambda_{jk} \right)^2 \), coming from the crossed term.

Let \( Q = E_{fA}^n \left( \sum_k \hat{\alpha}_{jk}(\bar{X}) \hat{\lambda}_{jk}(\bar{R}) \right)^2 \). Let \( \theta = \sum_k \alpha_{jk} \lambda_{jk} \). Recall that \( \Omega_n^m = \{(i^1, \ldots, i^m); i^\ell \in \mathbb{N}, 1 \leq i^\ell \leq m \} \).

Let \( \hat{i} \) be the set of distinct coordinates of \( i \in \Omega_n^m \). So that, estimators being plug-in, with a sum on \( \Omega_n^m \), with cardinality \( n^4 \),

\[
Q = E_{fA}^n \frac{1}{n^4} \sum_{i \in \Omega_n^m} \Phi_{jk_1}(X_{i^1}) \Phi_{jk_2}(R_{i^2}) \Phi_{jk_3}(X_{i^3}) \Phi_{jk_4}(R_{i^4})
\]

\[
\leq \frac{1}{n^4} \left[ \sum_{|i|=4} \theta^2 + \sum_{|i|=3} \theta^2 + (4N-3)^d \sum_k E_{fA}^n \Phi(X)^2 \lambda_{jk} + (4N-3)^d \sum_k E_{fA}^n \alpha_{jk} \Phi(R)^2 \right] + \sum_{|i|\leq2} (4N-3)^d \sum_k E_{fA}^n \Phi(X)^2 \Phi(R)^2
\]

with lines 2 and 3 expressing all possible matches between the coordinates of \( i \), and using lemma 1.7 to reduce double sums in \( k_1, k_2 \).

By independence of the samples, using lemma 1.8 and the fact that \(|\{i \in \Omega_n^m; i^\ell = c\}| = O(n^c)\) given by lemma 1.2,

\[
Q \leq \frac{A_0^4}{n^4} \theta^2 + Cn^{-1} \left( \theta^2 + C \sum_k \lambda_{jk}^2 + C \sum_k \alpha_{jk}^2 \right) + Cn^{-2} \theta^d.
\]

with \( A_0^4 = n!/(n-p)! \). So that, with \( A_0^4 n^{-4} = 1 - \frac{\alpha}{n} + Cn^{-2} \),

\[
Q - \theta^2 \leq Cn^{-2} + Cn^{-1} + Cn^{-2} \theta^d.
\]

The rate is thus unchanged for \( E_{fj}^n \) compared to the \( d+1 \) sample case in previous proposition.

**Case \( \hat{\Delta}_j^2(\bar{X}, \bar{R}) \)**

Recall that \( \iota_n^m = \{(i^1, \ldots, i^m); i^\ell \in \mathbb{N}, 1 \leq i^\ell \leq m, i^{\ell_1} \neq i^{\ell_2} \text{ if } \ell_1 \neq \ell_2 \} \).

For \( i \in \iota_n^m \), let \( h_{jk}(i) = [\Phi_{jk}(X_{i^1}) - \Phi_{jk}(R_{i^2})][\Phi_{jk}(X_{i^3}) - \Phi_{jk}(R_{i^4})] \) and let \( \theta = C^2 \); so that

\[
E_{fA}^n \left( \hat{\Delta}_j^2(\bar{X}, \bar{R}) - \theta \right)^2 = -\theta^2 + \frac{1}{(A_0^2)^2} \sum_{i_1, i_2, k_1, k_2} h_{jk_1}(i_1) h_{jk_2}(i_2)
\]

\[
= \left( \frac{\#\{i_1, i_2; |i_1 \cap i_2| = 0\}}{(A_0^2)^2} - 1 \right) \theta^2 + \frac{1}{(A_0^2)^2} \sum_{|i_1 \cap i_2| \geq 1} \sum_{k_1, k_2} E_{fA}^n h_{jk_1}(i_1) h_{jk_2}(i_2),
\]

and by lemma 1.3 the quantity in parenthesis on the left is of the order of \( Cn^{-2} \).

Label \( Q(i_1, i_2) \) the quantity \( E_{fA}^n \sum_{k_1, k_2} h_{jk_1}(i_1) h_{jk_2}(i_2) \). Let also \( \delta_{jk} = \alpha_{jk} - \lambda_{jk} \).

So that with only one matching coordinate between \( i_1 \) and \( i_2 \),

\[
Q(i_1, i_2) \mathbb{I}\{|i_1 \cap i_2| = 1\} = E_{fA}^n \sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} \left( \Phi_{jk_1}(X) \Phi_{jk_2}(X) + \Phi_{jk_1}(R) \Phi_{jk_2}(R) \right) - 2 \sum_k \delta_{jk} \alpha_{jk} \sum_k \delta_{jk} \lambda_{jk}
\]

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Again by lemma 1.7 and lemma 1.8, for $X$ or $R$

\[
E_{f,a}^n \sum_{k_1,k_2} \delta_{j,k_1,k_2} |\Phi_{j,k_1}(X)\Phi_{j,k_2}(X)| \leq (4N - 3)^d \sum_k \delta_{j,k}^2 E_{f,a}^n \Phi_{j,k}(X)^2 \leq C \sum_k \delta_{j,k}^2 \leq C
\]

and since all other terms are bounded by a constant not depending on $j$, by lemma 1.3

\[
(A_2^2)^{-2} \sum_{i_1,i_2} Q(i_1,i_2) \mathbb{I}\{|i_1 \cap i_2| = 1\} \leq Cn^{-1}.
\]

Likewise, the maximum order of $Q(i_1,i_2) \mathbb{I}\{|i_1 \cap i_2| = 2\}$ is $\sum_k (E_{f,a}^n \Phi_{j,k}(X))^2$, and the corresponding bound is $2^jd_n^{-1}$.

\[\square\]

**Proposition 1.5**  (Full sample $\hat{C}_j^2$ risk upper bound)

Let $\tilde{X} = X_1, \ldots, X_n$ be an independent, identically distributed sample of $f_a$. Assume that $f$ is compactly supported and that $\varphi$ is a Daubechies $D_2N$. Let $E_{f,a}^n$ be the the expectation relative to the joint distribution of the sample $\tilde{X}$. Let $\hat{C}_j^2$ be the plug-in estimator defined in (6), Then on $\{2^j < n^2\}$

\[
E_{f,a}^n \left( \hat{C}_j^2(\tilde{X}) - C_j^2 \right)^2 \leq C 2^j n^{-1}
\]

\[
E_{f,a}^n \left[ \hat{C}_j^2 - C_j^2 \right]^2 \leq E_{f,a}^n 3 \left( \sum_k \hat{\alpha}_{j,k}^2 - \alpha_{j,k}^2 \right)^2 + 3 \left( \sum_k \hat{\lambda}_{j,k}^2 - \lambda_{j,k}^2 \right)^2 + 3 \left( 4 \sum_k \hat{\alpha}_{j,k} \hat{\lambda}_{j,k} - \alpha_{j,k} \lambda_{j,k} \right)^2
\]

By proposition 1.7 the first term is of the order of $2^j n^{-1}$. By proposition 1.8 the two other terms are of the order of $C n^{-1} + 2^j n^{-1} \mathbb{I}\{2^j < n^2\}$.

\[\square\]

As is now shown, the rate of $\hat{D}_j^2(\tilde{X})$ computed on the full sample is slower than the one for $\hat{\Delta}_j^2(\tilde{R}, \tilde{S})$ in the two samples setting.

The reason is that we cannot always apply lemma 1.7 allowing to reduce double sums in $k_1, k_2$ to a sum on the diagonal $k_1 = k_2$ for translates of the same $\varphi$ functions. Indeed, when a match between multi indices $i_1$ and $i_2$ involves terms corresponding to margins, it is not guaranteed that a match on observation numbers also corresponds to a match on margin numbers ; that is to say, in the product $\varphi(X_{\ell_1} - k_1) \varphi(X_{\ell_2} - k_2)$, only once in a while $\ell_1 = \ell_2$ ; so most of the time we can say nothing about the support of the product, and the sum spans many more terms, hence the additional factor $2^j$ in the risk bound for $\hat{D}_j^2$ on the full sample.
Proposition 1.6 (Risk upper bound, full sample — $f_A$)

Let $X_1, \ldots, X_n$ be an independent, identically distributed sample of $f_A$. Assume that $f$ is compactly supported and that $\varphi$ is a Daubechies D2N. Let $\hat{D}_j^2$ be the U-statistic estimator defined in (7), then

\[
E_{f_A}^n \left( \hat{D}_j^2(\tilde{X}) - \sum_{k \in \mathbb{Z}^d} \delta_{jk}^2 \right)^2 \leq C 2^d n^{-2} + C^* 2^d n^{-1}
\]

with $\delta_{jk}$ the coordinate of $f_A - f_A^*$ and $C^* = 0$ at independence, when $f_A = f_A^*$.

\[
E_{f_A}^n \left[ \hat{D}_j^2(\tilde{X}) - \sum_{k \in \mathbb{Z}^d} \delta_{jk}^2 \right]^2 = E_{f_A}^n [\hat{D}_j^2(\tilde{X})]^2 - \left( \sum_{k \in \mathbb{Z}^d} \delta_{jk}^2 \right)^2.
\]

To make $\hat{D}_j^2(\tilde{X})$ look more like the usual U-estimator of $f(f - g)^2$ for unrelated $f$ and $g$, we define for $i \in \mathbb{I}_n^{2d+2}$, the dummy slice variables $Y_i = X_{ii}$, $V_i = (X_{i1}, \ldots, X_{i2d+1})$, $Z_i = X_{i2d+2}$, $T_i = (X_{i2d+3}, \ldots, X_{i2d+2+d})$; so that $Y_i$ and $Z_i$ have distribution $P_{f_A}$, $V_i$ and $T_i$ have distribution $P_{f_A}$, $P_{f_A}$, and $Z_i$, $T_i$ are independent variables under $P_{f_A}$. Next, for $k \in \mathbb{Z}^d$, define the function $\Lambda_{jk}$ as

\[
\begin{align*}
\Lambda_{jk}(X_1, \ldots, X_{2d+2}) &= \varphi_{jk^1}(X_1^1) \cdots \varphi_{jk^d}(X_1^d) \quad \forall i \in \Omega_n^d \\
\Lambda_{jk}(X_i) &= \varphi_{jk^1}(X_i^1) \cdots \varphi_{jk^d}(X_i^d) \quad \forall i \in \Omega_n^1 = \{1, \ldots, n\}
\end{align*}
\]

(13)

with second line taken as a convention.

So that $\hat{D}_j^2(\tilde{X})$ can be written under the more friendly form

\[
\hat{D}_j^2(\tilde{X}) = \frac{1}{A^{2d+2}_n} \sum_{i \in \mathbb{I}_n^{2d+2}} \sum_{k \in \mathbb{Z}^d} \left[ \Lambda_{jk}(Y_i) - \Lambda_{jk}(V_i) \right] \left[ \Lambda_{jk}(Z_i) - \Lambda_{jk}(T_i) \right],
\]

with $\mathbb{I}_n^m = \{(i^1, \ldots, i^m) \colon i^t \in \mathbb{N}, 1 \leq i^t \leq n, \ i^1 \neq i^{t_2} \text{ if } \ell_1 \neq \ell_2 \}$. 

Following the friendly notation, let $h_{jk} = [\Lambda_{jk}(Y_i) - \Lambda_{jk}(V_i)] [\Lambda_{jk}(Z_i) - \Lambda_{jk}(T_i)]$ be the kernel of $\hat{D}_j^2(\tilde{X})$ at fixed $k$. Then,

\[
[\hat{D}_j^2(\tilde{X})]^2 = \left| I_n^{2d+2} \right|^{-2} \sum_{i_1, i_2 \in \mathbb{I}_n^{2d+2}} \sum_{k_1, k_2 \in \mathbb{Z}^d} h_{i_1 k_1} h_{i_2 k_2}.
\]

Consider the partitioning sets $M_c = \{i_1, i_2 \in I_n^{2d+2} \times I_n^{2d+2}; |i_1 \cap i_2| = c, c = 0, \ldots, 2d + 2, \}$, that is to say the set of pairs with $c$ coordinates in common. Equivalently, $M_c$ can be defined as the set $\{i_1, i_2 \in I_n^{2d+2} \times I_n^{2d+2}; |i_1 \cup i_2| = 4d + 4 - c\}$.

According to the partitioning, with $h_i = \sum_k h_{jk},$

\[
E_{f_A}^n [\hat{D}_j^2(\tilde{X})]^2 = \left| I_n^{2d+2} \right|^{-2} \sum_{c=0}^{2d+2} \sum_{(i_1, i_2) \in M_c} E_{f_A}^n h_{i_1} h_{i_2}.
\]

Let $\lambda_{jk} = \alpha_{jk^1} \cdots \alpha_{jk^d}$ and $\delta_{jk} = \alpha_{jk} - \lambda_{jk}$.

- On $M_0$, no match,

\[
E_{f_A}^n h_{i_1} h_{i_2} I \{M_0\} = \sum_{k_1, k_2} (\alpha_{jk_1} - \lambda_{jk_1})^2 (\alpha_{jk_2} - \lambda_{jk_2})^2 = \left( \sum_k \delta_{jk}^2 \right)^2
\]

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By lemma 1.3, the ratio $|M_0|/|I_n^{2d+2}|$ is lower than $1 + Cn^{-2}$. So that

$$|I_n^{2d+2}|^{-2} \sum_{M_0} E^n_{f_A} h_{i_1} h_{i_2} - \left( \sum_k \delta^2_{jk} \right)^2 = |I_n^{2d+2}|^{-2} |M_0| E^n_{f_A} h_{i_1} h_{i_2} \ll |M_0| \leq Cn^{-2}.$$

**On $M_1$, assuming the match involves $Y_{i_1}$ and $Y_{i_2}$,**

$$E^n_{f_A} h_{i_1} h_{i_2} \ll |M_1| = \sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} E^n_{f_A} (\Phi_{jk_1}(Y_{i_1}) - \Lambda_{jk_1}(V_{i_1})) (\Phi_{jk_2}(V_{i_2}) - \Lambda_{jk_2}(V_{i_2}))$$

$$= \sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} (E^n_{f_A} \Phi_{jk_1}(X) \Phi_{jk_2}(X) - \lambda_{jk_1} \alpha_{jk_2} - \delta_{jk_1} \lambda_{jk_2})$$

$$= E^n_{f_A} \left( \sum_k \delta_{jk} \Phi_{jk}(X) \right)^2 - C^2 \sum_k \alpha_{jk} \delta_{jk} - \left( \sum_k \lambda_{jk} \delta_{jk} \right) \left( \sum_k \lambda_{jk} \delta_{jk} \right)$$

with $C^2 = \sum_k \delta^2_{jk}$.

Next by (17) in lemma 1.7 for the first line, the double sum in $k$ under expectation is bounded by a constant times the sum restricted to the diagonal $k_1 = k_2$ because of the limited overlapping of translates $\varphi_{jk}$; using also lemma 1.8,

$$E^n_{f_A} \left( \sum_k \delta_{jk} \Phi_{jk}(X) \right)^2 \leq (4N - 3)^d \sum_k \delta^2_{jk} E^n_{f_A} \Phi_{jk}(X)^2 \leq (4N - 3)^d \sum_k C^2 \delta_{jk}.$$

Since all other terms in (14) are clearly bounded by a constant not depending on $j$, we conclude by symmetry that $E^n_{f_A} h_{i_1} h_{i_2} \ll |M_1| \leq C$ for any match of cardinality 1 between narrow slices $(Y_{i_1}, Y_{i_2}$ or $Z_{i_1}, Z_{i_2}$ or $Y_{i_1}, Z_{i_2}$ or $Z_{i_1}, Y_{i_2})$. Moreover $C = 0$ when $f_A = f_A^*$ i.e. at independence, because of the omnipresence of $\delta_{jk}$, the coordinate of $f_A$, the coordinate of $f_A^*$.

**On $M_1$, if the match is between $Y_{i_1}$ and $Y_{i_2}$, a calculus as in (14) yields,**

$$E^n_{f_A} h_{i_1} h_{i_2} \ll |M_1| = - \sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} E^n_{f_A} \Phi_{jk_1}(Y_{i_1}) \Lambda_{jk_2}(V_{i_2}) + C^2 \sum_k \alpha_{jk} \delta_{jk} + \left( \sum_k \lambda_{jk} \delta_{jk} \right)^2 ;$$

which can also be found from line 2 of (14) using the swap $\Phi_{jk}(Y_{i_2}) \rightarrow -\Lambda_{jk}(V_{i_2})$ and $\alpha_{jk} \leftrightarrow -\lambda_{jk}$.

Next, for some $\ell \in \{1, \ldots, d\}$,

$$\sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} E^n_{f_A} \Phi_{jk_1}(Y_{i_1}) \Lambda_{jk_2}(V_{i_2}) = \sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} \lambda_{jk_1}^{(d-1)} E^n_{f_A} \Phi_{jk_1}(X) \varphi_{jk_2}(X^\ell)$$

with special notation $\lambda_{jk}^{(r)} = \lambda_{jk}^{(p_1)} \ldots \lambda_{jk}^{(p_d)}$ for some $p_i, 0 \leq p_i \leq r$, $\sum_i p_i = r$.

In the present case $\Phi_{jk_1}(X) \varphi_{jk_2}(X^\ell) = \Phi_{jk_1}(X) \varphi_{jk_2}(X^\ell) \ll |k_1 - k_2| < 2N - 1$ does not give any useful restriction of the double sum because the coefficient $\alpha_{jk}$ hidden in $\delta_{jk}$ is not guaranteed to factorize under any split of dimension unless $A = I$; and lemma 1.7 is useless. This is a difficulty that did not raise in propositions 1.3 and 1.4 because we could use the fact that these kind of terms were estimated over independent samples.

Instead write $E^n_{f_A} |\Phi_{jk_1}(X) \varphi_{jk_2}(X^\ell)| \leq 2^\ell \|u\|_\infty E^n_{f_A} |\Phi_{jk_1}(X)| \leq C 2^\ell 2^{-d\ell}$ using lemma 1.8. So that when multiplied by $\sum_k \delta_{jk} \sum_k \delta_{jk} \lambda_{jk}^{(d-1)}$, using Meyer’s lemma, the final order is $2^\ell$. 

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By symmetry, for any match of cardinality 1 between a narrow and a wide slice (Y or T or equivalent pairing), $E_{f_{t_a}}[d_{i_1}, d_{i_2}] |\{M_1\} \leq C2^j$, with $C = 0$ at independence.

- On $M_1$, if the match is between $V_i$ and $V_{i_2}$, by symmetry with (14) or using the swap defined above,

\[
E_{f_{t_a}}[d_{i_1}, d_{i_2}] |\{M_1\} = \sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} E_{f_{t_a}}[A_{jk}(V_i)A_{jk}(V_{i_2})] - C^2 \sum_k \alpha_{jk} \delta_{jk} - \left( \sum_k \lambda_{jk} \delta_{jk} \right) \left( \sum_k \alpha_{jk} \delta_{jk} \right),
\]

and for some not necessarily matching $\ell_1, \ell_2 \in \{1, \ldots, d\}$ (i.e., lemma 1.7 not applicable),

\[
\sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} E_{f_{t_a}}[A_{jk}(V_i)A_{jk}(V_{i_2})] = \sum_{k_1, k_2} \delta_{jk_1} \delta_{jk_2} \lambda_{jk_1}^{(d-1)} \lambda_{jk_2}^{(d-1)} E_{f_{t_a}}[\varphi_{jk_1}^{\ell_1}(X^{\ell_1})\varphi_{jk_2}^{\ell_2}(X^{\ell_2})] \leq \left( \sum_k \delta_{jk} \lambda_{jk}^{(d-1)} \right)^2 = C2^j
\]

with last line using Meyer’s lemma, and having reduced the term under expectation to a constant by Cauchy-Schwarz inequality and lemma 1.8.

And we conclude again that, for any match of cardinality 1 between two wide slices (V or T or equivalent), $E_{f_{t_a}}[d_{i_1}, d_{i_2}] |\{M_1\} \leq C2^j$, with $C = 0$ at independence.

By lemma 1.3, the ratio $|M_1|/|T_n^{2d+2} \times T_n^{2d+2}| \approx n^{-1}$, so in summary, the bound for $M_1$ has the order $C^*2^n n^{-1}$, with $C^* = 0$ at independence.

- On $M_c$, $c = 2, \ldots, 2d + 2$.

Fix the pair of indexes $(i_1, i_2) \in T_n^{2d+2} \times T_n^{2d+2}$, we need to bound a term having the form

\[
Q(i_1, i_2) = E_{f_{t_a}}[\sum_{k_1, k_2} \Lambda_{jk}(R_{i_1}) \Lambda_{jk}(S_{i_1}) \Lambda_{jk}(R_{i_2}) \Lambda_{jk}(S_{i_2})]
\]

where both slices $R_{i_1} \neq S_{i_1}$ unrelated with both slices $R'_{i_2} \neq S'_{i_2}$ are chosen among any of the dummy Y, V, Z, T.

---

**Narrow slices only.** For a match spanning four narrow slices exclusively, that is to say $(Y_1 = Y_{i_1}) \cap (Z_1 = Z_{i_2})$ or $(Y_1 = Z_{i_2}) \cap (Z_1 = Y_{i_1})$, a case possible on $M_2$ only, the general term of higher order is written $\sum_{k_1, k_2} E_{f_{t_a}}[\varphi_{jk_1}(X)\varphi_{jk_2}(X)E_{f_{t_a}}[\varphi_{jk_1}(X)]^2]$, by lemma 1.7 this is again lower than $(4n - 3)^d \sum_k [E_{f_{t_a}}[\varphi_{jk}(X)]^2]^2$, that is $C2^{jd}$. By lemma 1.3, this case thus contributes to the general bound up to $C2^{jd} n^{-2}$.

Three narrow slices only is not possible and two narrow slices correspond to the case $M_1$ treated above.

---

**Wide slices only.** For a match spanning wide slices on $M_c$, $c = 2, \ldots, 2d$, a general term with higher order is written $\sum_{k_1, k_2} E_{f_{t_a}}[\Lambda_{jk_1}(V_{i_1})\Lambda_{jk_2}(T_{i_1})\Lambda_{jk_2}(V_{i_2})\Lambda_{jk_2}(T_{i_2})]$, with $|i_1 \cap i_2| = c$ (an equivalent is obtained by swapping one V with a T). Since the slices are wide, it is not possible to distribute expectation any further right now: if $V_{i_1}$ is always independent of $T_{i_1}$, both terms may depend on $V_{i_2}$, say. Also matching coordinates on $i_1, i_2$ do not necessarily correspond to matching dimensions $X^t$ of the observation, and then lemma 1.7 is not applicable. Instead write,

\[
Q(i_1, i_2) = \sum_{k_1, k_2} \lambda_{jk_1}^{(2d-c)} \lambda_{jk_2}^{(2d-c)} E_{f_{t_a}}[\Lambda_{jk_1}^{(c)}(V_{i_1})\Lambda_{jk_2}^{(c)}(V_{i_2})],
\]
with $\Lambda_{j_k}^{(c)}(V_i,T_i)$ a product of $c$ independent terms of the form $\varphi_{j_k}\ell(X^\ell)$ spanning at least one of the slices $V_i$, $T_i$.

By definition of $i_1$ and $i_2$, the product of $2c$ terms under expectation can be split into $c$ independent products of two terms. So, using $E^n_{f_A} |\varphi_{j_k}(X)|^2 \leq C$ on each bi-term, the order at the end is $C(\sum_k \lambda_{j_k}^{(2d-c)})^2$; and using Meyer’s lemma, the bound is then of the order of $C2^{jc}$.

Finally, using lemma 1.3 as above, the contribution of this kind of term to the general bound is $\sum_{c=1}^{2d} 2^{n-c}$.

On $\{2^j < n\} \supset \{2^{jd} < n^2\} \supset \{2^{jd} < n\}$, this quantity is bounded by $C2^{j}n^{-1} < C2^{jd}n^{-2}$ and on $\{2^j > n\}$ it is unbounded.

— Narrow and wide slices Reusing the general pattern above, with $c_w \leq 2d$ matching coordinates on wide slices and $c_r \leq 2$ on narrow slices

$$Q(i_1,i_2) = \sum_{k_i,k_2} \lambda_{j_k}^{(2d-c_w)} \lambda_{j_k}^{(2d-c_w)} \alpha_{j_k}^{2-c_r} \alpha_{j_k}^{2-c_r} E^n_{f_A} \left[ \Lambda_{j_k}^{(c)}(Y_{i_1}, V_{i_1}, Z_{i_1}, T_{i_1}) \Lambda_{j_k}^{(c)}(Y_{i_2}, V_{i_2}, Z_{i_2}, T_{i_2}) \right],$$

with $\Lambda_{j_k}^{(c)}(Y_i, V_i, Z_i, T_i)$ a product of $c$ independent terms of the form $\varphi_{j_k}\ell(X)$ or $\Phi_{j_k}(X)$ spanning at least one of the slices $V_i$, $T_i$ and one of the slices $Y_i$, $Z_i$. As above, the bracket is a product of independent bi-terms, each under expectation bounded by some constant $C$, by lemma 1.8, using Cauchy-schwarz inequality if needed. So this is bounded by

$$Q(i_1,i_2) \leq C \sum_{k_i,k_2} \lambda_{j_k}^{(2d-c_w)} \lambda_{j_k}^{(2d-c_w)} \alpha_{j_k}^{2-c_r} \alpha_{j_k}^{2-c_r} = C \left( \sum_k \lambda_{j_k}^{(2d-c_w)} \alpha_{j_k}^{2-c_r} \right)^2;$$

using Cauchy-Schwarz inequality and Meyer’s lemma this is bounded by $2^{2jd}n^{-2} 2^{2d}n^{-a} \| \{2^j < n^2\} \leq Cn^{-1}$

Finally on $\{2^{jd} < n^2\}$, $E^n_{f_A} \hat{\beta}_j \leq C 2^{j}n^{-1} + 2^{jd}n^{-2}$. □

Implementation issues

The statistic $C^2_j$ is a plug-in estimator; its evaluation uses in the first place the complete estimation of the density $f_A$ and margins; which takes a computing time of the order of $O(n(2N-1)^d)$ where $N$ is the order of the Daubechies wavelet, and $n$ the number of observations.

In the second place, the actual contrast is a simple function of the $2^{jd} + d2^j$ coefficients that estimate density $f_A$ and its margins; the additional computing time is then in $O(2^{jd})$.  


One can see here the main numerical drawback of the wavelet contrast in its total formulation — to be of exponential complexity in dimension $d$ of the problem; but this is by definition the cost of a condition that guarantees mutual independence of the components in full generality: $d$ sets $B_1, \ldots, B_d$ are mutually independent if $P(B_1 \cap \ldots \cap B_d) = PB_1 \ldots PB_d$ for each of the $2^d$ choices of indices in $\{1, \ldots, d\}$.

Complexity in $jd$ drops down to $O(d^2 2^d)$ if one concentrates on a pairwise independence, like in kernel ICA and related methods, and in the minimum marginal entropy type method of Miller and Fisher III (2003). Pairwise independence is in fact equivalent to mutual independence in the no noise ICA model and with at most one Gaussian component (Comon, 1994). The minimization used by the pairwise algorithm used by Miller et Fisher (2003) consists in searching for the minimum in each of the $C_d^2$ free plans of $\mathbb{R}^d$, applying Jacobi rotations to select a particular plan. A search in each plan is equivalent to the case $d = 2$, where the problem is to find the minimum in $\theta$ of a function on $\mathbb{R}$, for $\theta \in [0, \pi/2]$. To do so, the simplest could be to try out all points from $0$ to $\pi/2$ along a grid, or to use bisection type methods.

U-statistic estimators of $C_d^2$ have complexity at minimum in $O(n^2(2N - 1)^{2d})$, that is to say quadratic in $n$ as the method of Tsybakov and Samarov (2002) which also attains parametric rate of convergence; on the other hand the complexity in $jd$ is probably lowered since the contrast can be computed by accumulation, without it be necessary to keep all projection in memory, but only a window whose width depends upon the length of the Daubechies filter.

1.4 Appendix 1 – Propositions

Proposition 1.7 (2nd moment of $\sum_k \hat{\alpha}_{jk}^2$ about $\sum_k \alpha_{jk}^2$)

Let $X_1, \ldots, X_n$ be an independent, identically distributed sample of $f$, a compactly supported function defined on $\mathbb{R}^d$. Assume that $\varphi$ is a Daubechies $D2N$. Let $\hat{\alpha}_{jk} = \frac{1}{n} \sum_{i=1}^{n} \varphi_{jk}(X_i) \varphi_{jk}(X_i^*)$, $k \in \mathbb{Z}^d$.

Then $E_{f_A}^n \left( \sum_k \hat{\alpha}_{jk}^2 - \sum_k \alpha_{jk}^2 \right)^2 = C2^{jd}n^{-1} + C2^{2jd}n^{-2} \mathbb{I} \{2^d > n\}$

For the mean, using lemma 1.8,

$$E_{f_A}^n \sum_k \hat{\alpha}_{jk}^2 = \frac{1}{n^2} \sum_{i_1=i_2} \sum_k E_{f_A}^n \Phi_{jk}(X_{i_1}) \Phi_{jk}(X_{i_2}) + \frac{1}{n^2} \sum_{i_1 \neq i_2} \sum_k \alpha_{jk}^2$$

$$= \frac{1}{n} \sum_k \Phi_{jk}(X_i)^2 + \frac{n-1}{n} \sum_k \alpha_{jk}^2 = \sum_k \alpha_{jk}^2 + O\left(\frac{2^d}{n}\right).$$

For the second moment, let $M_c = \{i_1, i_2, i_3, i_4 \in \{1, \ldots, n\}; \{i_1\} \cup \ldots \cup \{i_4\} = c\}$.

$$E_{f_A}^n \left( \sum_k \hat{\alpha}_{jk}^2 \right)^2 = \frac{1}{n^2} \sum_{c=1}^{4} \sum_{i_1, \ldots, i_4} E_{f_A}^n \sum_{k_1, k_2} \Phi_{jk_1}(X_{i_1}) \Phi_{jk_1}(X_{i_2}) \Phi_{jk_2}(X_{i_3}) \Phi_{jk_2}(X_{i_4}) \mathbb{I} \{M_c\}$$
On \( c = 1 \), the kernel is equal to \( \sum_{k_1,k_2} \Phi_{jk_1}(X) \Phi_{jk_2}(X)^2 \leq (4N-3)^d \sum_k \Phi_{jk}(X)^4 \) by lemma 1.7. And by lemma 1.8, \( E^n f_A \sum_k \Phi_{jk}(X)^4 \leq \sum_k C 2^jd = C 2^jd \).

On \( c = 2 \), the kernel takes three generic forms: (a) \( \sum_{k_1,k_2} \Phi_{jk_1}(X) \Phi_{jk_2}(Y) \Phi_{jk_2}(X) \Phi_{jk_2}(Y) \) or (b) \( \sum_{k_1,k_2} \Phi_{jk_1}(X)^2 \Phi_{jk_2}(Y)^2 \) or (c) \( \sum_{k_1,k_2} \Phi_{jk_1}(X) \Phi_{jk_1}(Y) \Phi_{jk_2}(Y)^2 \). In cases (a) and (c), using lemma 1.7, the double sum can be reduced to the diagonal \( k_1 = k_2 \). So using also lemma 1.8,

\[
\begin{align*}
(a) & \quad E^n f_A \left| \sum_{k_1,k_2} \Phi_{jk_1}(X) \Phi_{jk_1}(Y) \Phi_{jk_2}(X) \Phi_{jk_2}(Y) \right| \leq E^n f_A (4N-3)^d \sum_k \Phi_{jk}(X)^2 \Phi_{jk}(Y)^2 \leq C 2^jd \\
(b) & \quad E^n f_A \sum_{k_1,k_2} \Phi_{jk_1}(X)^2 \Phi_{jk_2}(Y)^2 \leq C 2^jd \\
(c) & \quad E^n f_A \sum_{k_1,k_2} \Phi_{jk_1}(X) \Phi_{jk_1}(Y) \Phi_{jk_2}(Y)^2 \leq E^n f_A (4N-3)^d \sum_k [\Phi_{jk}(X) \Phi_{jk}(Y)^3] \leq C 2^jd.
\end{align*}
\]

On \( c = 3 \) the only representative form is

\[
E^n f_A \sum_{k_1,k_2} \Phi_{jk_1}(X) \Phi_{jk_1}(Y) \Phi_{jk_2}(Z)^2 = \sum_k \alpha_{jk}^2 \sum_k E^n f_A \Phi_{jk}(X)^2 \leq C 2^jd,
\]

and on \( c = 4 \) the statistic is unbiased equal to \( (\sum_k \alpha_{jk}^2)^2 \) under expectation.

Next, since \( |M_d| = A_n^4 \) and, using lemma 1.2, \( |M_i| = O(n^c) \),

\[
E^n f_A \left( \sum_k \alpha_{jk}^2 \right)^2 \leq A_n^4 n^{-4} \left( \sum_k \alpha_{jk}^2 \right)^2 + C 2^{2jd} n^{-3} + n^{-2} 2^j d + n^{-1} 2^j d
\]

\[
\leq \left( \sum_k \alpha_{jk}^2 \right)^2 + C n^{-2} + C n^{-1} 2^j d \mathbb{I} \{2^j d < n\} + C n^{-2} 2^j d \mathbb{I} \{2^j d > n\}
\]

with \( A_n^4 n^{-4} = 1 - \frac{6}{n} + C n^{-2} \).

Finally

\[
E^n f_A \left( \sum_k \alpha_{jk}^2 - \sum_k \alpha_{jk}^2 \right)^2 = E^n f_A \left( \sum_k \alpha_{jk}^2 \right)^2 + \left( \sum_k \alpha_{jk}^2 \right)^2 - 2 E^n f_A \sum_k \alpha_{jk}^2 \sum_k \alpha_{jk}^2
\]

\[
\leq C n^{-2} + C n^{-1} 2^j d + C n^{-2} 2^j d \mathbb{I} \{2^j d > n\}
\]

\[\blacksquare\]

**Proposition 1.8** (2nd moment of \( \sum_k \hat{\lambda}_{jk}^2 \) about \( \sum_k \lambda_{jk}^2 \) and of \( \sum_k \hat{\lambda}_{jk} \alpha_{jk} \) about \( \sum_k \lambda_{jk} \alpha_{jk} \))

Let \( X_1, \ldots, X_n \) be an independent, identically distributed sample of \( f \), a compactly supported function defined on \( \mathbb{R}^d \). Assume that \( \varphi \) is a Daubechies D2N. Let \( \hat{\lambda}_{jk} = \frac{1}{n^d} \sum_{i=1}^n \varphi_{jk}(X_i) \cdots \sum_{i=1}^n \varphi_{jk}(X_i), \) \( k \in \mathbb{Z}^d \).

Then on \( \{2^j d < n^2\} \),

\[
E^n f_A \left( \sum_k \hat{\lambda}_{jk} \alpha_{jk} - \sum_k \lambda_{jk} \alpha_{jk} \right)^2 \leq O(n^{-2}) + C \frac{2^j}{n}
\]

\[
E^n f_A \left( \sum_k \hat{\lambda}_{jk}^2 - \sum_k \lambda_{jk}^2 \right)^2 \leq O(n^{-2}) + C \frac{2^j}{n}
\]
\[ E^n_{f_A} \left( \sum_k \hat{\lambda}^2_{jk} - \lambda^2_{jk} \right)^2 = E^n_{f_A} \left[ \left( \sum_k \hat{\lambda}^2_{jk} \right)^2 - 2 \sum_k \hat{\lambda}^2_{jk} \sum_k \hat{\lambda}^2_{jk} + \left( \sum_k \hat{\lambda}^2_{jk} \right)^2 \right] \]

For \( i \in \Omega^d_n \), let \( V_i \) be the slice \((X^d_{i1}, X^d_{i2}, \ldots, X^d_{i_{d-1}}, X^d_{id})\). Let the coordinate-wise kernel function \( \Lambda_{jk} \) be given by \( \Lambda_{jk}(V_i) = \varphi_{j,k}(X^d_{i1})\varphi_{j,k}(X^d_{i2}) \ldots \varphi_{j,k}(X^d_{i_{d-1}})\varphi_{j,k}(X^d_{id}) \).

Let \( \{i\} \) be the shortcut notation for \( \{i^d_1 \cup \ldots \cup i^d_{2d}\} \). Let \( W^d_n = \{i \in \Omega^d_n : |i| < 2d\} \), that is to say the set of indices with at least one repeated coordinate.

Then the mean term is written
\[
E^n_{f_A} \sum_k \hat{\lambda}^2_{jk} = n^{-2d} \sum_{i \in \Omega^d_n} \sum_k \Lambda_{jk}(V_i)
\]
\[
= n^{-2d} \sum_{W^d_n} \sum_k E^n_{f_A} \Lambda_{jk}(V_i) + A^{2d} n^{-2d} \sum_k \lambda^2_{jk}
\]
\[
= Q_1 + A^{2d} n^{-2d} \theta^2
\]

Let \( M_c = \{i \in \Omega^d_n : |i| = c\} \) be the set indices with \( c \) common coordinates. So that \( Q_1 \) is written
\[
Q_1 = n^{-2d} \sum_{c=1}^{2d-1} 1 \{M_c\} \sum_{M_c} \sum_k E^n_{f_A} \Lambda_{jk}(V_i) = \sum_{c=1}^{2d-1} Q_{1jk}
\]

By lemma 1.4 with lemma parameters \((d = 1, m = 2d, r = 1)\), \( E^n_{f_A} |\Lambda_{jk}(V_i)| 1 \{M_c\} \leq C2^{(2d-2c)} \) and by lemma 1.2, \( |M_c| = O(n^c) \). Hence,
\[
Q_{1jk} \leq \sum_{c=1}^{2d-1} n^{-2d+c} C2^{(d-c)} = 2^{-jd} \sum_{c=1}^{2d-1} C \left( \frac{2^j}{n} \right)^{(2d-c)}
\]

which on \( \{2^d < n\} \) has maximum order \( 2^{(1-d)n-1} \) when \( d - c \) is minimum i.e. \( c = 2d - 1 \). Finally \( |Q_1| \leq \sum_{c} C2^{(1-d)n-1} \leq C2^{d} n^{-1} \).

Next, the second moment about zero is written
\[
E^n_{f_A} \left( \sum_k \lambda^2_{jk} \right)^2 = n^{-4d} \sum_{i_1, i_2 \in \Omega^d_n} \sum_{k_1, k_2} \Lambda_{jk_1}(V_{i_1})\Lambda_{jk_2}(V_{i_2})
\]
\[
= n^{-4d} \sum_{W^{4d}_n} \sum_{k_1, k_2} E^n_{f_A} \Lambda_{jk_1}(V_{i_1})\Lambda_{jk_2}(V_{i_2}) + A^{4d} n^{-4d} \left( \sum_k \lambda^2_{jk} \right)^2
\]
\[
= Q_2 + A^{4d} n^{-4d} \theta^2
\]

with \( W^{4d}_n = \{i_1, i_2 \in \Omega^d_n : |i_1 \cup i_2| < 4d\} \), that is to say the set of indices with at least one repeated coordinate somewhere.

Let this time \( M_c = \{i_1, i_2 \in \Omega^d_n : |i_1 \cup i_2| = c\} \) be the set indices with overall \( c \) common coordinates in \( i_1 \) and \( i_2 \). So that \( Q_2 \) is written
\[
Q_2 = n^{-4d} \sum_{c=1}^{4d-1} 1 \{M_c\} \sum_{M_c} \sum_{k_1, k_2} E^n_{f_A} \Lambda_{jk_1}(V_{i_1})\Lambda_{jk_2}(V_{i_2}) = \sum_{k_1, k_2} Q_{2j_1, k_1 j_2, k_2}
\]
By lemma 1.6, unless \( c = 1 \), it is always possible to find indices \( i_1, i_2 \) with no match between the observations falling under \( k_1 \) and those falling under \( k_2 \), so that there is no way to reduce the double sum in \( k_1, k_2 \) to a sum on the diagonal using lemma 1.7. Note that if \( c = 1 \), \( E^n_{J_A} \Lambda_{jk}(V_i^k) \Lambda_{jk}(V_i) = E^n_{J_A} \Phi_{jk}(X)^d \) has order \( C2^{3d} \).

So coping with the double sum, by lemma 1.4 with lemma parameters \( (d = 1, m = 2d, r = 2) \), \( E^n_{J_A} |\Lambda_{jk}(V_i^k)\Lambda_{jk}(V_i)\| \leq C2^{4(d^2 - 2c)} \), and again by lemma 1.2 \( |M_c| = O(n^c) \), so \( E^n_{J_A} |Q_{2j}, k_1, k_2| \geq \sum_{c=1}^{2d-1} \sum_{\binom{n-c}{2}^d} C2^{4(d^2 - 2c)} \), which on \( \binom{2d}{d} < n \) has maximum order \( 2^{(1-2d)n-1} \) when \( c = 4d - 1 \).

Finally, \( E^n_{J_A} Q_2 \leq \sum_{k_1, k_2} C2^{(1-2d)n-1} \leq C2^{d} n^{-1} \).

Putting all together, and since \( A^n n^{-p} = 1 - \frac{(d+1)(d+2)}{4n} + O(n^{-2}) \),

\[
E^n_{J_A} \left( \sum_k \lambda^2_{jk} - \Lambda^2_{jk} \right)^2 = Q_2 + A_{n}^{4d} n^{-4d} \theta^2 - 2\theta(Q_1 + A_{n}^{2d} n^{-2d} \theta) + \theta^2
= Q_2 - 2\theta Q_1 + \theta^2(1 + A_{n}^{4d} n^{-4d} - 2A_{n}^{2d} n^{-2d}) \leq |Q_2| + 2\theta |Q_1| + O(n^{-2})
\leq C2^{d} n^{-1}
\]

For the cross product,

As above, for \( i \in \Omega_{n}^{d+1} \), let \( V_i \) be the slice \((X^0_i, X_1^i, \ldots, X^d_i)\). Let the coordinate-wise kernel function \( \Lambda_{jk} \) be given by \( \Lambda_{jk}(V_i) = \Psi_j(X^0_i) \Psi_j(X^1_i) \cdots \Psi_j(X^d_i) \). Let \( \theta = \sum_k \alpha_{jk} \lambda_{jk} \).

Let \( W^d n = \{ i \in \Omega_{n}^{d+1} \mid |i| < d+1 \} \), that is to say the set of indices with at least one repeated coordinate.

So that, \( E^n_{J_A} \sum_k \alpha_{jk} \lambda_{jk} = Q_1 + A_{n}^{d+1} n^{-d-1} \theta \) with \( Q_1 = n^{-d-1} \sum_{W^d n} \sum_k E^n_{J_A} \Lambda_{jk}(V_i) \) and likewise

\[
E^n_{J_A} \left( \sum_k \alpha_{jk} \lambda_{jk} \right)^2 = Q_2 + A_{n}^{2d+2} n^{-2d-2} \theta^2
\]

with \( Q_2 = n^{-2d-2} \sum_{W^d n} \sum_{k_1, k_2} E^n_{J_A} \Lambda_{jk_1}(V_i) \Lambda_{jk_2}(V_i) \). And we obtain in the same way,

\[
E^n_{J_A} \left( \sum_k \alpha_{jk} \lambda_{jk} - \alpha_{jk} \lambda_{jk} \right)^2 \leq |Q_2| + 2\theta |Q_1| + O(n^{-2})
\]

Let \( M_c = \{ i \in \Omega_{n}^{d+1} \mid |i| = c \} \) be the set indices with \( c \) common coordinates. So that \( Q_1 \) is written

\[
Q_1 = n^{-d-1} \sum_{c=1}^{d} \sum_{M_c} \sum_k E^n_{J_A} \Lambda_{jk}(V_i) = \sum_k Q_{1jk}
\]

By lemma 1.4 with lemma parameters \( (m_1 = 1, m_2 = d, r = 1) \),

\[
E^n_{J_A} |\Lambda_{jk}(V_i)| \| \{ M_c \} \leq C2^{4(d^2 - 2c_1)} \frac{1}{2} \binom{2d}{d-2c_1}
\]

with \( c_1 + c_d = c, 0 \leq c_1 \leq d, 1 \leq c_d \leq 1 \) and by lemma 1.2, \( |M_c| = O(n^c) \). Hence,

\[
Q_{1jk} \leq \sum_{c=1}^{d} n^{-d-1+c} C2^{4(d^2 - 2c_1)} 2^j \binom{2d}{d-2c_1} \sum_{c=1}^{d} C \left( \frac{2^d}{n} \right)^{(d+1-c)}
\]

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which on \( \{2^d < n\} \) has maximum order \( C2^{(1-d)n^{-1}} \) when \( d + 1 - c \) is minimum \( i.e. \ c = d. \)

Finally \( |Q_1| \leq \sum_k C2^{(1-d)n^{-1}} \leq C2^{|n|}. \)

Next, as above \( Q_2 = \sum_{k_1,k_2} Q_{2;k_1,k_2} \), and again by lemma 1.6, unless \( c = 1 \), it is always possible to find indices \( i_1,i_2 \) with no matching coordinates corresponding also to matching dimension number, so that there is no way to reduce the double sum in \( k_1,k_2 \) to a sum on the diagonal using lemma 1.7.

So coping once more with the double sum, by lemma 1.4 with lemma parameters \( (md = 1,m_1 = d,r = 2) \), \( \sum_{i_1 \in I_{h_1}^2} \sum_{i_2 \in I_{h_2}^2} \Phi_{j_1 k_1}(X_{i_1}) \Phi_{j_2 k_2}(X_{i_2}) \leq C2^{d/2-\epsilon}2^{d/2}2^{(d-2c-1)} \), with \( c_1 + c_2 = c \leq c_2 \leq 2 \), \( 0 \leq c_1 \leq 2d \), and again by lemma 1.2 \( |M_c| = O(n^\epsilon) \), so

\[
E_{f_A}^n |Q_{2;k_1,k_2}| \leq \sum_{c=1}^{2d+1} n^{c-2d+2} C 2^{d-2d+c-1} = 2^{d-2+1} \left( \frac{2d+1}{n} \right)^{2d+2-c},
\]

which on \( \{2^d < n\} \) has maximum order \( C2^{-d^2}n^{-1} \) when \( c = 2d+1 \). Then either \( c_2 = 1 \), which means that the two terms \( \Phi_{j_1 k_1}(X_{i_1}) \Phi_{j_2 k_2}(X_{i_2}) \) match on the observation number, in which case the sum in \( k_1,k_2 \) can be reduced; either \( c_2 = 2 \). In the first case the order is \( E_{f_A}^n Q_2 \leq (4N-3)^d \sum k C2^{-d}n^{-1} \leq Cn^{-1} \) and in the second case \( E_{f_A}^n Q_2 \leq \sum k_1,k_2 C2^{1-2d}n^{-1} \leq C2^|n|. \)

\[\square\]

**Proposition 1.9 (Variance of \( \sum_k \bar{B}^2_j \))**

Let \( \{X_1,\ldots,X_n\} \) be an i.i.d. sample with density \( f \). Assume that \( f \) is compactly supported and that \( \varphi \) is a Daubechies \( D2N. \)

Let \( \bar{B}^2_j = \sum_k \frac{1}{n} \sum_{i \in I_n^2} \Phi_{jk}(X_{i_1}) \Phi_{jk}(X_{i_2}) \) be the U-statistic estimator of \( \sum_k \alpha_{jk}^2 \).

Then on \( \{2^d < n^2\} \),

\[
E_{f_A}^n \left( \bar{B}^2_j - \sum_k \alpha_{jk}^2 \right)^2 \leq Cn^{-1} + 2^d n^{-2}
\]

Write that,

\[
E_{f_A}^n \left[ \bar{B}^2_j(X) \right]^2 = n^{-2}(n-1)^{-2} \sum_{i_1,i_2 \in I_n^2} \sum_{k_1,k_2} \Phi_{jk_1 k_1}(X_{i_1}) \Phi_{jk_2 k_2}(X_{i_2}) \Phi_{jk_1 k_2}(X_{i_1}) \Phi_{jk_2 k_2}(X_{i_2})
\]

On \( M_4 = \{i_1,i_2 \in I_n^2 : |i_1 \cup i_2| = 4\} \), i.e. with no match between the two indices, the kernel \( h_{i_1,i_2} = \sum_{k_1,k_2} \Phi_{jk_1 k_1}(X_{i_1}) \Phi_{jk_2 k_2}(X_{i_2}) \Phi_{jk_1 k_2}(X_{i_1}) \Phi_{jk_2 k_2}(X_{i_2}) \) is unbiased, equal under expectation to \( (\sum_k \alpha_{jk}^2)^2 \).

On \( M_c, c = 2,3 \), with at least one match between \( i_1 \) and \( i_2 \) lemma 1.7 is applicable to reduce the double sum in \( k_1,k_2 \) and,

\[
E_{f_A}^n h_{i_1,i_2} \{ M_2 \cup M_3 \} \leq \sum_{i_1,i_2 \in I_n^2} \sum_{k_1,k_2} \Phi_{jk_1 k_1}(X_{i_1}) \Phi_{jk_2 k_2}(X_{i_2}) \Phi_{jk_1 k_2}(X_{i_1}) \Phi_{jk_2 k_2}(X_{i_2}) \{ M_2 \cup M_3 \}
\]

\[
\leq \sum_{M_2,M_3} (4N-3)^d \sum_k \{ \Phi_{jk_1 k_1}(X_{i_1}) \Phi_{jk_2 k_2}(X_{i_2}) \Phi_{jk_1 k_2}(X_{i_1}) \Phi_{jk_2 k_2}(X_{i_2}) \}
\]

\[
\leq \sum_{M_2,M_3} C \sum_k 2^d (2d-|i_1 \cup i_2|) = C \sum_{M_2,M_3} 2^d (3d-|i_1 \cup i_2|),
\]

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using lemma 1.4 with parameter \( m = 2 \) and \( r = 2 \) for line 3.

Next, by lemma 1.2, \( |M_c| = O(n^c) \) and \( |M_d| \) divided by \((A_n^2)^2\) is more precisely equal to \(1 - 4n^{-1} + Cn^{-2}\). So that

\[
E_{f_n}^n \left[ B_n^2(\tilde{X}) \right]^2 \leq (1 + Cn^{-2}) \left( \sum_k \alpha_{jk}^2 \right)^2 + C \sum_{c=2}^3 n^c n^{-4} 2^{jd(3-c)} = \left( \sum_k \alpha_{jk}^2 \right)^2 + Cn^{-1} + C 2^{jd} n^{-2}.
\]

\( \Box \)

**Proposition 1.10** (Variance of multisample \( \prod \sum_k B_n^2(\tilde{R}) \))

Let \( \{R_1^n, \ldots, R_d^n\} \) be an i.i.d. sample of \( f^\ell \), \( \ell = 1, \ldots, d \). Assume that \( f \) is compactly supported and that \( \varphi \) is a Daubechies D2N. Assume that the \( d \) samples are independent.

Let \( \tilde{B}_n^2(R) = \sum_k \sum_{i \in I_n^\ell} \phi_{jk} R_i \) be the U-statistic estimator of \( \sum_k \phi_{jk}^2 \), \( \ell = 1, \ldots, d \). Then on \( \{2^{jd} < n^2\} \),

\[
E_{f_n}^n \left( \prod_{\ell=1}^d \tilde{B}_n^2(R^\ell) \right) - \sum_{k_1, \ldots, k_d} \phi_{j_1 k_1} \cdots \phi_{j_d k_d} \right)^2 \leq Cn^{-1} + C \frac{2^d}{n^2}.
\]

Successive application of \( ab - cd = (a - c)b + (b - d)c \) leads to

\[
a_1 \cdots a_d - b_1 \cdots b_d = \sum_{\ell=1}^d (a_\ell - b_\ell)b_1 \cdots b_{\ell-1}a_{\ell+1} \cdots a_d. \tag{15}
\]

So applying (15),

\[
\sum_k \lambda_{jk}^2 - \lambda_{jk}^2 = \sum_{k_1, \ldots, k_d} \hat{\alpha}_{j_1 k_1}^2 \cdots \hat{\alpha}_{j_d k_d}^2 = \sum_{k_1, \ldots, k_d} \sum_{\ell=1}^d (\hat{\alpha}_{j_\ell k_\ell}^2 - \alpha_{j_\ell k_\ell}^2) \alpha_{j_{\ell-1} k_{\ell-1}}^2 \cdots \alpha_{j_{d} k_{d}^2}
\]

\[
= \sum_{\ell=1}^d C \left( \sum_{k_\ell} \hat{\alpha}_{j_\ell k_\ell}^2 - \alpha_{j_\ell k_\ell}^2 \right) \sum_{k_{\ell+1}} \hat{\alpha}_{j_{\ell+1} k_{\ell+1}}^2 \cdots \sum_{k_d} \hat{\alpha}_{j_d k_d}^2
\]

And

\[
\left( \sum_k \lambda_{jk}^2 - \lambda_{jk}^2 \right)^2 \leq d \sum_{\ell=1}^d C \left( \sum_{k_\ell} \hat{\alpha}_{j_\ell k_\ell}^2 - \alpha_{j_\ell k_\ell}^2 \right) \sum_{k_{\ell+1}} \hat{\alpha}_{j_{\ell+1} k_{\ell+1}}^2 \cdots \sum_{k_d} \hat{\alpha}_{j_d k_d}^2)^2
\]

Label \( Q = E_{f_n}^n \left( \sum_k \lambda_{jk}^2 - \lambda_{jk}^2 \right)^2 \).
If the \( d \) samples are independent, if \( 2^{2d} < n^2 \), and by proposition 1.9 with parameter \( d = 1 \),
\[
Q \leq \sum_{\ell=1}^{d-1} \left[ C(Cn^{-1} + \frac{2j}{n^2}) \prod_{\ell=\ell+1}^{d-1} (C + Cn^{-1} + \frac{2j}{n^2}) \right] + C(Cn^{-1} + \frac{2j}{n^2})
\]
\[
\leq Cn^{-1} + C\frac{2j}{n^2}
\]
\[
\]

**Proposition 1.11 (Variance of multi sample \( \sum_k \hat{a}_{jk} \hat{\lambda}_{jk} \))**

Let \( \{X_1, \ldots, X_n\} \) be an independent, identically distributed sample of \( f_A \). Let \( \{R_1^\ell, \ldots, R_n^\ell\} \) be an independent, identically distributed sample of \( f^{\ell t} \), \( \ell = 1, \ldots, d \). Assume that \( f \) is compactly supported and that \( \varphi \) is a Daubechies D2N. Assume that the \( d + 1 \) samples are independent. Let \( E_{f_A} \) be the expectation relative to the joint samples.

Then
\[
E_{f_A}^n \left( \sum_k \hat{a}_{jk}(\hat{X})\hat{\lambda}_{jk}(\hat{R}_1^\ell, \ldots, \hat{R}_d^\ell) - \sum_k a_{jk}\lambda_{jk} \right)^2 \leq Cn^{-1} \{2^j < n\} + C2^{jd}n^{-d-1} \{2^j > n\}
\]

Let \( Q = E_{f_A}^n \left( \sum_{k \in \mathbb{Z}^d} \hat{a}_{jk}(\hat{X})\hat{\lambda}_{jk} \right)^2 \); expanding the statistic,
\[
Q = E_{f_A}^n \left( \sum_{k_1, k_2} \frac{1}{n^{2d+2}} \sum_{i \in \Omega_n^{d+2}} \Phi_{jk_1}(X_i)\Phi_{jk_2}(X_i)\varphi_{jk_1}(R_1^\ell)\varphi_{jk_2}(R_2^\ell)\cdots\varphi_{jk_d}(R_d^\ell) \right)
\]

By independence of the samples, we only need to consider local constraints on the coordinates of \( i \in \Omega_n^{d+2} \).

Let \( a \) be a subset of \( \{0, 1, \ldots, d\} \). Let \( J_a = \{i \in \Omega_n^{d+2} : \ell \in a \Rightarrow i^{2\ell+1} = i^{2\ell+2} ; \ell \notin a \Rightarrow i^{2\ell+1} \neq i^{2\ell+2} \} \). It is clear that \( |J_a| = (n(n-1))^{d+1-|a|}n^{|a|} \) and that the \( J_a \)’s define a partition of \( \Omega_n^{d+2} \) when \( a \) describes the \( 2^{d+1} \) subsets of \( \{0, 1, \ldots, d\} \). One can check that there are \( C_{d+1} \) distinct sets \( a \) such that \(|a| = c\), and that \( \sum_{c=0}^{d+1} C_{d+1}^c(n(n-1))^{d+1-c} = n^{d+1} \sum_{c=0}^{d+1} C_{d+1}^c(n-1)^{d+1-c} = n^{2d+2} \).

On \( J_\emptyset \) the kernel is unbiased. On \( J_a \), \( 0 \in a \), with the first two coordinates matching, the sum in \( k_1, k_2 \) can be reduced to a sum on the diagonal by lemma 1.7. If \( 0 \notin a \), but some \( \ell \in a \) the sum can be reduced only on dimension \( \ell \), \( k_1^\ell = k_2^\ell \), but to no purpose as will be seen below.

So \( Q \) is written \( Q = n^{-2d-2} \sum_{a \in \mathcal{P}(\{0, \ldots, d\})} Q_{0a} + Q_{1a} \), with
\[
Q_{0a} = C_1 \sum_{i \in J_a, 0 \in a} \sum_{k \in \mathbb{Z}^d} E_{f_A}^n \Phi_{jk}(X)^2 E_{f_A}^n \varphi_{jk_1}(R_1^\ell)^2 \cdots E_{f_A}^n \varphi_{jk_d}(R_d^\ell)^2 \alpha_{jk_1}^2 \cdots \alpha_{jk_d}^2
\]

and
\[
Q_{1a} = \sum_{i \in J_a, 0 \notin a} \sum_{k_1, k_2} \alpha_{jk_1} \alpha_{jk_2} E_{f_A}^n \varphi_{jk_1}(R_1^\ell) \varphi_{jk_2}(R_2^\ell) \cdots E_{f_A}^n \varphi_{jk_d}(R_d^\ell) \varphi_{jk_1}(R_1^\ell) \varphi_{jk_2}(R_2^\ell) \cdots \varphi_{jk_d}(R_d^\ell)
\]
for some all distinct $\ell_1, \ldots, \ell_{|a|-1}$ and $l_1, \ldots, l_{d-|a|+1}$ whose union is $\{1, \ldots, d\}$ and with $C_1 = (4N - 3)^d$. The bound for $Q_{0a}$ is also written

$$(4N - 3)^d \sum_{i \in J_n, 0 \not\in a} \sum_{k \in \mathbb{Z}^d} C \lambda_{jk}^{(2d - 2|a| + 2)}$$

with special notation $\lambda_{jk}^{(r)} = \alpha_{jk_1}^{p_1} \cdots \alpha_{jk_r}^{p_r}$ for some integers $p_1, \ldots, p_r$, $0 \leq p_i \leq r$ with $\sum_1^d p_i = r$. And so, by Meyer’s lemma this is also bounded by $\sum_{i \in J_n, 0 \not\in a} C 2^j(|a| - 1)$.

For $Q_{1a}$ with $|a| \geq 1$, the sum in $k_1, k_2$ could be split in $k_1^d \ldots k_1^{l_{d-|a|+1}}, k_2^d \ldots k_2^{l_{d-|a|+1}}$ where no concentration on the diagonal is ensured, and $k_1^{d \ldots k_1^{l_{d-|a|+1}}}$ where lemma 1.7 is applicable, but precisely the multidimensional coefficient $\alpha_{jk} = \alpha_{jk_1 \cdots k_d}$ is not guaranteed factorisable under any split, unless $A = I$. So we simply fall back to

$$Q_{1a} \leq \sum_{i \in J_n, 0 \not\in a} \sum_k [\alpha_{jk_1} \alpha_{jk_2}] [\alpha_{jk_1} \alpha_{jk_2}^2 \cdots \alpha_{jk_1} \alpha_{jk_2}^2 \cdots \alpha_{jk_1} \alpha_{jk_2}^2] [C 2^j E^m |\varphi_{jk_1} (R'O)|]^{|a| - 1}.$$ 

This is also written, using Meyer’s lemma at the end,

$$Q_{1a} \leq \sum_{i \in J_n, 0 \not\in a} \left( \sum_k \alpha_{jk} \lambda_{jk}^{(d - |a| + 1)} \right)^2 \leq \sum_{i \in J_n, 0 \not\in a} C 2^j(|a| - 1)$$

Finally, with $\sum_{i \in J_n} 1 = |J_n|$ given above, the general bound is written,

$$Q \leq n^{-2d - 2} \left[ \sum_{a \not= \emptyset} C 2^j(|a| - 1)n^{d+1}(n - 1)^{d+1-|a|} + n^{d+1}(n - 1)^{d+1} \left( \sum_k \alpha_{jk} \lambda_{jk} \right)^2 \right]$$

and so

$$Q - \left( \sum_k \alpha_{jk} \lambda_{jk} \right)^2 \leq 2^{-j} \sum_{c=1}^{d+1} 2^{jc}(n - 1)^{-c} + Cn^{-2}$$

$$\leq Cn^{-1} \{2^j < n\} + 2^jd n^{-d-1} \{2^j > n\}$$

\[ \square \]

1.5 Appendix 2 – Lemmas

Lemma 1.1 (Property set)

Let $A_1, \ldots, A_r$ be $r$ non empty subsets of a finite set $\Omega$. Let $J$ be a subset of $\{1, \ldots, r\}$.

Define the property set $B_J = \{ x \in \cup_{j \in J} A_j : x \not\in \cup_{j \in J'} A_j \}$, that is to say the set of elements belonging exclusively to the sets listed through $J$. Let $b_J = |B_J|$ and $b = \sum_{|J| = n} b_J$.

Then $\sum_{n=0}^r \sum_{|J| = n} B_J = \Omega$, and

$$|A_1| \vee \ldots |A_r| \leq \sum_{n=1}^r b_n = |A_1 \cup \ldots A_r| \leq |A_1| + \ldots |A_r| = \sum_{n=1}^r b_n$$

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with equality for the right part only if \( b_k = 0, \kappa = 2, \ldots, r \) i.e. if all sets are disjoint, and equality for the left part if one set \( A_i \) contains all the others.

It follows from the definition that no two different property sets intersect and that the union of property sets defines a partition of \( \cup A_i \), hence a partition of \( \Omega \) with the addition of the missing complementary \( \Omega - \cup A_i \), denoted by \( B_0 \). The \( B_J \) are also the atoms of the Boolean algebra generated by \( \{ A_1, \ldots, A_r, \Omega - \cup A_i \} \) with usual set operations.

With \( B_0 \), an overlapping of \( r \) sets defines a partition of \( \Omega \) with cardinality at most \( 2^r \); there are \( C_r^m \) property sets satisfying \( |J| = \kappa \), with \( \sum_{\kappa=0}^r C_r^\kappa = 2^r \).

\[ \square \]

**Lemma 1.2** (Many sets matching indices)

Let \( m \in \mathbb{N}, m \geq 1. \) Let \( \Omega_m^n \) be the set of indices \( \{(i^1, \ldots, i^m) : i^j \in \mathbb{N}, 1 \leq i^j \leq n \} \). Let \( r \in \mathbb{N}, r \geq 1. \) Let \( I_m^r = \{ i \in \Omega_m^n : i_1 \neq i_2 \Rightarrow i^1 \neq i^{i^2} \} \).

For \( i = (i_1^1, \ldots, i_1^m) \in \Omega_m^n \), let \( i = \cup_{j=1}^m \{ i^j \} \subset \{1, \ldots, n\} \) be the set of distinct integers in \( i \).

Then, for some constant \( C \) depending on \( m \),

\[
\# \left\{ (i_1, \ldots, i_r) \in (\Omega_m^n)^r : |i_1 \cup \ldots \cup i_r| = a \right\} = O(n^a) I \{ |i_1| \lor \ldots \lor |i_r| \leq a \leq mr \}
\]

and in corollary \( \# \left\{ (i_1, \ldots, i_r) \in (I_m^r)^r : |i_1 \cup \ldots \cup i_r| = a \right\} = O(n^a) I \{ m \leq a \leq mr \} \).

In the setting introduced by lemma 1.1, building the compound \( \hat{i}_1, \ldots, \hat{i}_r \) while keeping track of matching indices is achieved by drawing \( b_{1(1)} = |\hat{i}_1| \) integers in the \( 2^0 \)-partition \( b_0 = \{1, \ldots, n\} \) thus constituting \( \hat{i}_1 \), then \( b_{2(1,2)} + b_{2(2)} = |\hat{i}_2| \) integers in the \( 2^1 \)-partition \( \{b_{1(1)}, b_{0}\} \)

thus constituting two subindexes from which to build \( \hat{i}_2 \), then \( b_{3(1,2,3)} + b_{3(2,3)} + b_{3(1,3)} + b_{3(3)} = |\hat{i}_3| \) integers in the \( 2^2 \)-partition \( \{b_{2(1,2)}, b_{2(2)}, b_{2(2)}, b_{2(3)}\} \) thus constituting 2\(^r\) subindexes from which to build \( \hat{i}_3 \), and so on, up to \( b_{r(1, \ldots, r)} + \ldots + b_{r(r)} = |\hat{i}_r| \) integers in the cardinality \( 2^{r-1} \) partition \( \{b_{r(1, \ldots, r-1)} \ldots, b_0\} \) thus constituting \( 2^{r-1} \) subindexes from which to build \( \hat{i}_r \).

The number of ways to draw the subindexes composing the \( r \) indexes is then

\[
A_{b_{0(1)}^{(1)}} \cdot A_{b_{1(2)}^{(1)}} \cdot A_{b_{0(2)}^{(2)}} \cdot \ldots \cdot A_{b_{r(1, \ldots, r)}^{(r)}} \cdot \ldots \cdot A_{b_{r-1}^{(r)}}^{(r)}
\]

\[ = A_{b_{0}^{(r)}}^{(r)} \]

with the nesting property \( b_{j}^{(j)} = b_{j}^{(j+1)} + b_{j}^{(j+1)} + b_{j+1}^{(j+1)} \) (provided \( J \) exists at step \( j \)) and \( A_{b_{0}^{(r)}}^{(r)} = \frac{m!}{(n-m)!} \).

At step \( j \), the only property set with cardinality equivalent to \( n \), is \( B_{j-1}^{(j)} \), while all others have cardinalities lower than \( m \); so picking integers inside these light property sets involve cardinalities at most in \( m! \) that go in the constants, while the pick in \( B_{0}^{(j)} \) entails a cardinality \( A_{b_{0}^{(r)}}^{(r)} \approx n b_{0}^{(r)} \).

Note that, at step \( j - 1, \) \( b_{0}^{(j)} = n - |\hat{i}_1 \cup \ldots \cup \hat{i}_{j-1}| \), because, at step \( j, \) \( b_{0}^{(j)} \) designates the number of integers in \( \hat{i}_j \) not matching any previous index \( \hat{i}_1, \ldots, \hat{i}_{j-1} \); so that also \( \sum_{j=1}^{r} b_{j}^{(j)} = |\hat{i}_1 \cup \ldots \cup \hat{i}_r| \); and incidentally \( \sum_{j=m}^{r} b_{j}^{(j)} = |\hat{i}_m| \).

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The number of integers picked from the big property set at each step is

\[
A_{b_0}^{k_1} A_{b_0}^{k_2} \ldots A_{b_0}^{k_r} \sum_{i=1}^{n-1} b_i^j - n - |i_1 \cup \ldots \cup i_{j-1}|, \quad b_0 = n \text{ and } \sum_{j=1}^{r} b_i^j = |i_1 \cup \ldots \cup i_r|.
\]

For large \( n \) this is equivalent to \( n^{|i_1 \cup \ldots \cup i_r|} \).

Having drawn the subindexes, building the indexes effectively is a matter of iteratively intermixing two sets of \( a \) and \( b \) elements; an operation equivalent to highlighting \( b \) cells in a line of \( a+b \) cells, which can be done in \( C_n^b \) ways, with \( C_n^b = A_n^b / b! \).

Intermixing the subindexes thus involve cardinalities at most in \( m! \), that go in the constant \( C \).

Likewise, passing from \( i \) to \( i \) involve cardinalities at most in \( C_m^i \) and no dependence on \( n \).

For the corollary, if \( i \in I_m^m \) then \( i = i \) and \( |i| = m \). If moreover \( i^1 < \ldots < i^r \), the number of ways to draw the subindexes is given by replacing occurrences of 'A' by 'C' in (16), with \( C_n^m = \binom{n!}{m!(n-m)!} \), which does not change the order in \( n \). Also there is only one way to intermix subindexes, because of the ordering constraint.

\[ \blacksquare \]

**Lemma 1.3** (Two sets matching indices [Corollary and complement])

Let \( I_m^m \) be the set of indices \( \{(i^1, \ldots, i^m) \} \), \( i^j \in \mathbb{N}, 1 \leq i^j \leq n, i^j \neq i^\ell \) if \( i \neq \ell \), and let \( I_n^m \) be the subset of \( I_m^m \) such that \( \{i^1 < \ldots < i^m \} \).

Then for \( 0 \leq b \leq m \),

\[
\begin{align*}
\# \left\{(i_1, i_2) \in I_m^m \times I_m^m : |i_1 \cap i_2| = b \right\} &= A_m^b A_m^{m-b} C_m^b = O(n^{2m-b}) \\
\# \left\{(i_1, i_2) \in I_n^m \times I_n^m : |i_1 \cap i_2| = b \right\} &= C_n^b C_{n-m}^{m-b} = O(n^{2m-b})
\end{align*}
\]

In corollary, with \( P \) (resp. \( P' \)) the mass probability on \( (I_m^m)^2 \) (resp. \( (I_n^m)^2 \)), \( P(|i_1 \cap i_2| = b) \approx P'(|i_1 \cap i_2| = b) = O(n^{-b}) \) and \( P(|i_1 \cap i_2| = 0) = P'(|i_1 \cap i_2| = 0) \leq 1 - m^2 n^{-1} + C n^{-2} \).

For \( i_1, i_2 \in I_m^m \), the equivalence \( |i_1 \cap i_2| = b \iff |i_1 \cup i_2| = 2m - b \) gives the link with the general case of lemma 1.2.

Reusing the pattern of lemma 1.2 in a particular case: there are \( A_m^m \) ways to constitute \( i_1 \), there are \( A_m^{m-b} \) ways to draw \( b \) unordered integers from \( i_1 \) and \( A_m^{m-b} \) ways to draw \( m-b \) unordered integers from \{1, \ldots, n \} - i_1.

To constitute \( i_2 \), intermixing both subindexes of \( b \) and \( m-b \) integers is equivalent to highlighting \( b \) cells in a line of \( m \) cells: there are \( C_m^b \) ways to do so. On \( I_n^m \), by definition, having drawn the \( b \) then \( m-b \) ordered distinct integers, intermixing is uniquely determined.

Incidentally, one can check that \( \sum_{b=0}^{m} A_m^b A_m^{m-b} C_m^b = A_m^m \), and that \( \sum_{b=0}^{m} C_m^b C_m^{m-b} = C_m^m \).
Dividing by 
\[ \frac{A_n^m}{A_{n-c}^m} = (1 - \frac{c}{n}) \ldots (1 - \frac{c}{n-m+1}) \leq (1 - \frac{c}{n})^m \]

\[ \Box \]

**Lemma 1.4 (Product of \( r \) kernels of degree \( m \))**

Let \( r \in \mathbb{N}^+ \). Let \( m \geq 1 \). Let \( (X_1, \ldots, X_n) \) be an independent, identically distributed sample of a random variable on \( \mathbb{R}^d \). Let \( \Omega_n^m \) be the set of indices \( \{\{i_1, \ldots, i^m\} : i^m \in \mathbb{N}, 1 \leq i \leq n\} \).

For \( i \in \Omega_n^m \), define

\[ a_{ik} = \Phi_{jk}(X_{i_1}) \ldots \Phi_{jk}(X_{i_m}) \]

\[ b_{ik} = \varphi_{jk}(X_{i_1}) \ldots \varphi_{jk}(X_{i_m}) \Phi_{jk}(X_{i_{m+1}}) \ldots \Phi_{jk}(X_{i_{m+m}}). \]

Let \( \tilde{i} \) be the set of distinct coordinates in \( i \) and let \( c = c(\tilde{i}_1, \ldots, \tilde{i}_r) = |\tilde{i}_1 \cup \ldots \cup \tilde{i}_r| \) be the overall number of distinct coordinates in \( r \) indices \( (i_1, \ldots, \tilde{i}_r) \in (\Omega_n^m)^r \).

Then

\[ E^m_j[a_{i_1,k_1} \ldots a_{i_r,k_r}] \leq C 2^{d(m-r-2c)} \]

\[ E^m_j[b_{i_1,k_1} \ldots b_{i_r,k_r}] \leq C 2^{d(m-r-2c)} 2^{d(m_r - 2c + 2r)} \]

with \( c_d = c_d(\tilde{i}_1, \ldots, \tilde{i}_r) \leq c \) the fraction of \( c \) corresponding to products with at least one \( \Phi(X) \) term and \( 1 \leq c_d \leq m_dr \), \( 0 \leq c - c_d \leq m_r \), \( 1 \leq c \leq (m_1 + m_2)r \).

Using lemma 1.1, one can see that the product \( a_{i_1,k_1} \ldots a_{i_r,k_r} \), made of \( mr \) terms, can always be split into \( |\tilde{i}_1 \cup \ldots \cup \tilde{i}_r| \) independent products of \( c(l) \) dependent terms, \( 1 \leq l \leq |\tilde{i}_1 \cup \ldots \cup \tilde{i}_r| \), with \( c(l) \) in the range from \( |\tilde{i}_1| \cup \ldots \cup |\tilde{i}_r| \) to \( m_r \) and \( \sum_l c(l) = m_r \).

Using lemma 1.8, a product of \( c(l) \) dependent terms, is bounded under expectation by \( C 2^{d(c(l)-2)} \). Accumulating all independent products, the overall order is \( C 2^{d(m-r-2|\tilde{i}_1 \cup \ldots \cup \tilde{i}_r|)} \).

For \( b_{i_1,k_1} \ldots b_{i_r,k_r} \) make the distinction between groups containing at least one \( \Phi(X) \) term and the others containing only \( \varphi(X) \) terms. This splits the number \( |\tilde{i}_1 \cup \ldots \cup \tilde{i}_r| \) into \( g_{\Phi,\varphi} + g_{\varphi} \). Let \( c_{\varphi}(l) \) be the number of \( \varphi \) terms in a product of \( c(l) \) terms, mixed or not.

On the \( g_{\Phi,\varphi} \) groups containing \( \Phi \) terms, first bound the product of \( c_{\varphi}(l) \) terms by \( C 2^{d(c_{\varphi}(l))} \), and the remaining terms by \( C 2^{d(c(l)-c_{\varphi}(l)-2)} \). On the \( g_{\varphi} \) groups with only \( \varphi \) terms, bound the product by \( C 2^{d(c_{\varphi}(l)-2)} \).

The overall order is then

\[ C 2^{d(\sum_{l=1}^{g_{\varphi}} c_{\varphi}(l) - c_{\varphi}(l)) - 2g_{\Phi,\varphi}} = 2^{d(l)} 2^{d(l)} 2^{d(l)} \sum_{l=1}^{g_{\varphi}} c_{\varphi}(l) - 2g_{\Phi,\varphi}. \]

The final bound is found using \( \sum_{l=1}^{g_{\varphi}} c_{\varphi}(l) + \sum_{l=1}^{g_{\varphi}} c_{\varphi}(l) = m_1 r \) and \( \sum_{l=1}^{g_{\varphi}} c(l) - c_{\varphi}(l) = m_d r \).
Rename $c_d = g_{\Phi,\phi}$ and $c - c_d = g_{\phi}$.

As for the constraints, in the product of $(m_1 + m_d)r$ terms, it is clear that $\Phi$ terms have to be found somewhere, so $c_d \geq 1$, which also implies that $c - c_d = 0$ when $c = 1$ (in this case there are no independent group with only $\phi$ terms, but only one big group with all indices equal). Otherwise $c_d \leq m_d r$ and $c - c_d \leq m_1 r$ since there are no more that this numbers of $\Phi$ and $\phi$ terms in the overall product.

Lemma 1.5 (Meyer)

Let $V_j$, $j \in \mathbb{Z}$ an $r$-regular multiresolution analysis of $L_2(\mathbb{R}^n)$ and let $\varphi \in V_0$ be the father wavelet.

There exist two constant $c_2 > c_1 > 0$ such that for all $p \in [1, +\infty]$ and for all finite sum $f(x) = \sum_k \alpha(k) \varphi_{jk}(x)$ one has,

$$c_1 \|f\|_p \leq 2^{jd(\frac{1}{2} - \frac{1}{p})} \left( \sum_k |\alpha(k)|^p \right)^{\frac{1}{p}} \leq c_2 \|f\|_p$$

See Meyer (1997)

We use the bound under a special form.

First note that if $f \in B_{sp\infty}$, $\|f\|_{sp\infty} = \|P_j f\|_p + \sup_{s \geq 0} 2^s \|f - P_j f\|_p$ so that $\|f - P_j f\|_p \leq C \|f\|_{sp\infty} 2^{-js}$. So using (3),

$$\sum_k |\alpha_{jk}|^p \leq C 2^{jd(1-p/2)} \|P_j f\|_p \leq C 2^{jd(1-p/2)} 2^{p-1} (\|f\|_p + \|f - P_j f\|_p)$$

$$\leq C 2^{jd(1-p/2)} 2^{p-1} (\|f\|_p + C \|f\|_{sp\infty} 2^{-js})$$

$$\leq C 2^{jd(1-p/2)} \|f\|_p$$

When applying the lemma to special coefficient $\lambda_{jk}^{(r)} = \alpha_{j k_1}^{p_1} \ldots \alpha_{j k_d}^{p_d}$ for some integers $p_1, \ldots, p_d$, $0 \leq p_i \leq r$ with $\sum_{i=1}^d p_i = r$, we use

$$\sum_{k \in \mathbb{Z}^d} |\lambda_{jk}^{(r)}| = \sum_{k_1 \in \mathbb{Z}} |\alpha_{j k_1}^{p_1}| \ldots \sum_{k_d \in \mathbb{Z}} |\alpha_{j k_d}^{p_d}|$$

$$\leq C 2^{j (2 - p_1)} \|f^{p_1}_{sp1\infty} \ldots 2^{j (2 - p_d)} \|f^{p_d}_{spd\infty}$$

$$\leq C 2^{j (2d - r)} \max_{\ell} f^{\ell \|r_{sp\infty}}$$

so that even if some $p_\ell$ was zero, the result is a $2^j$, which returns the effect of $\sum_k 1$. 

$\square$
Lemma 1.6 (Path of non matching dimension numbers)

Let \( r \in \mathbb{N}, r \geq 2 \). Let \( \Omega_{n}^{d} = \{(i^{1}, \ldots, i^{m}); i^{\ell} \in \mathbb{N}, 1 \leq \ell \leq n\} \). For \( i \in \Omega_{n}^{d} \), let \( \Lambda_{j,k}(V_{i}) = \varphi_{jk}(X_{i}^{1}) \ldots \varphi_{jk}(X_{i}^{d}) \). Let \( i \) be the set of distinct coordinates of \( i \).

In the product

\[
\left( \sum_{j} \sum_{k} \frac{1}{n^{d}} \sum_{i \in \Omega_{n}^{d}} \Lambda_{j,k}(V_{i}) \right)^{r} = \frac{1}{n^{dr}} \sum_{i_{1}, \ldots, i_{r} \in \Omega_{n}^{d}, j_{1} \ldots j_{r}, k_{1} \ldots k_{r}} \Lambda_{j_{1}k_{1}}(V_{i_{1}}) \ldots \Lambda_{j_{r}k_{r}}(V_{i_{r}})
\]

unless \( |\tilde{i}_{1} \cup \ldots \cup \tilde{i}_{r}| < r \), it is always possible to find indices \( (i_{1}, \ldots, i_{r}) \) such that no two functions \( \varphi_{j_{k}} \varphi_{j_{k'}} \) match on observation number.

Let \( c = |\tilde{i}_{1} \cup \ldots \cup \tilde{i}_{r}| \). For \( 1 \leq \ell \leq n \), let \( \ell^{\otimes d} = (\ell, \ldots, \ell) \in \Omega_{n}^{d} \).

With \( r \) buckets of width \( d \) defined by the extent of each index \( k_{1}, \ldots, k_{r} \), and only \( c < r \) distinct observation numbers, once \( c \) buckets have been stuffed with terms \( V_{\ell^{\otimes d}} \), some already used observation number must be reused in order to fill in the remaining \( r - c \) buckets. So that \( r - c \) buckets will match on dimension and observation number allowing to reduce the sum to only \( c \) distinct buckets.

Once \( c > r \), starting with a configuration using \( V_{\ell^{\otimes d}} \), we can always use additional observation numbers to fragment further the \( \ell^{\otimes d} \) terms, which preserves the empty intersection between buckets.

Lemma 1.7 (Daubechies wavelet concentration property)

Let \( r \in \mathbb{N}, r \geq 1 \). Let \( \varphi \) be the scaling function of a Daubechies wavelet \( D2N \). Let \( h_{k} \) be the function on \( \mathbb{R}^{m} \) defined as a product of translations of \( \varphi \)

\[
h_{k}(x_{1}, \ldots, x_{m}) = \varphi(x_{1} - k^{1}) \ldots \varphi(x_{m} - k^{m}),
\]

with \( k = (k^{1}, \ldots, k^{m}) \in \mathbb{Z}^{m} \).

Then for a Haar wavelet \( \left[ \sum_{k} h_{k}(x_{1}, \ldots, x_{m}) \right]^{r} = \sum_{k} h_{k}(x_{1}, \ldots, x_{m})^{r} \).

For any \( D2N \),

\[
\left( \sum_{k} |h_{k}(x_{1}, \ldots, x_{m})| \right)^{r} \leq (4N - 3)^{m(r-1)} \sum_{k} |h_{k}(x_{1}, \ldots, x_{m})|^{r}
\]

(17)

With a Daubechies Wavelet \( D2N \), whose support is \([0, 2N - 1]\) with \( \varphi(0) = \varphi(2N - 1) = 0 \) (except for Haar where \( \varphi(0) = 1 \)), one has the relation

\[
x \mapsto \varphi(x - k)\varphi(x - \ell) = 0, \quad \text{for } |\ell - k| \geq 2N - 1;
\]

when \( k \) is fixed, the cardinal of the set \(|\ell - k| < 2N - 1\) is equal to \((4N - 3)\).
So that, with \(k_1, \ldots, k_r\) denoting \(r\) independent multi-index,

\[
(\sum_k h_k)^r = \sum_{k_1} \sum_{k_2} \ldots \sum_{k_r} h_{k_1} \ldots h_{k_r} I(\Delta)
\]

with \(\Delta = \{k_{i_1}^\ell_1 - k_{i_2}^\ell_2 < (2N - 1)~;~ i_1, i_2 = 1 \ldots r;~ \ell_1, \ell_2 = 1 \ldots m\}\). Once \(k_1\) say, is fixed, the cardinal of \(\Delta\) is not greater than \((4N - 3)^m(r-1)\) and is exactly equal to 1 for Haar, when all \(k_1 = \ldots = k_r\).

For any Daubechies wavelet, and \(r \geq 1\), using the inequality \((|h_{k_1}|^r \ldots |h_{k_r}|^r)^{\frac{1}{r}} \leq \frac{1}{r} \sum_i |h_{k_i}|^r,\)

\[
(\sum_k |h_k|)^r \leq \sum_{k_1} \sum_{k_2} \ldots \sum_{k_r} \frac{1}{r} (|h_{k_1}|^r + \ldots + |h_{k_r}|^r) I(\Delta)
\]

\[
= \frac{1}{r} \left[ \sum_{k_1} \sum_{k_2} \ldots \sum_{k_r} |h_{k_1}|^r I(\Delta) + \ldots + \sum_{k_1} \sum_{k_2} \ldots \sum_{k_r} |h_{k_r}|^r I(\Delta) \right]
\]

\[
\leq (4N - 3)^m(r-1) \sum_k |h_k|^r,
\]

\[\blacksquare\]

Lemma 1.8 \((r\)th order moment of \(\Phi_{jk}\))

Let \(X\) be random variables on \(\mathbb{R}^d\) with density \(f\). Let \(\Phi\) be the tensorial scaling function of an MRA of \(L_2(\mathbb{R}^d)\). Let \(\alpha_{jk} = E_f \Phi_{jk}(X)\). Then for \(r \in \mathbb{N}^+\),

\[
E_f |\Phi_{jk}(X) - \alpha_{jk}|^r \leq 2^r E_f |\Phi_{jk}(X)|^r \leq 2^r 2^{jd(d-\frac{1}{2})} \|f\|_\infty \|\Phi\|_r^r.
\]

If \(\Phi\) is the Haar tensorial wavelet then also \(E_f \Phi_{jk}(X)^r \leq 2^{jd(d-\frac{1}{2})} \alpha_{jk}\).

For the left part of the inequality, \(\left( E_f |\Phi_{jk}(X) - \alpha_{jk}|^r \right)^{\frac{1}{r}} \leq \left( E_f |\Phi_{jk}(X)|^r \right)^{\frac{1}{r}} + E_f |\Phi_{jk}(X)|\), and also \(E_f |\Phi_{jk}(X)| \leq \left( E_f |\Phi_{jk}(X)|^r \right)^{\frac{1}{r}} \left( E_f 1 \right)^{\frac{1}{r}}\).

For the right part, \(E_f |\Phi_{jk}(X)|^r = 2^{jd/2} \int |\Phi(2^j x - k)|^r f(x)dx \leq 2^{jd(d-\frac{1}{2})} \|f\|_\infty \|\Phi\|_r^r\).

Or also if \(\Phi\) is positive,

\[
E_f \Phi_{jk}(X)^r = 2^{jd(r-1)} \int \Phi(2^j x - k)^{r-1} \Phi_{jk}(x) f(x)dx
\]

\[
\leq 2^{jd(r-1)} \|\Phi\|_\infty^{r-1} \alpha_{jk}.
\]

\[\blacksquare\]
1.6 References


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