Frontier estimation via kernel regression on high power-transformed data

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Abstract

We present a new method for estimating the frontier of a multidimensional sample. The estimator is based on a kernel regression on the power-transformed data. We assume that the exponent of the transformation goes to infinity while the bandwidth of the kernel goes to zero. We give conditions on these two parameters to obtain complete convergence and asymptotic normality. The good performance of the estimator is illustrated on some finite sample situations.

Keywords: kernel estimator, power-transform, frontier estimation.

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1 Introduction

Let (X_i, Y_i), $i = 1, \dots, n$ be independent copies of a random pair (X, Y) with support S defined by

$$S = \{(x, y) \in E \times \mathbb{R}; 0 \le y \le g(x)\}. \tag{1}$$

The unknown function $g : E \to \mathbb{R}$ is called the frontier. We address the problem of estimating g in the case $E = \mathbb{R}^d$. Our estimator of the frontier is based on a kernel regression on the power-transformed data. More precisely, the estimator of g is defined for all $x \in \mathbb{R}^d$ by

$$\hat{g}_n(x) = \left((p+1) \sum_{i=1}^n K_h(x - X_i) Y_i^p \Big/ \sum_{i=1}^n K_h(x - X_i) \right)^{1/p}, \tag{2}$$

where $p = p_n$ and $h = h_n$ are non random sequences such that $h \to 0$ and $p \to \infty$ as $n \to \infty$. This latter condition is the key so that the high power-transformed data "concentrate" along the frontier. We have also introduced $K_h(t) = K(t/h)/h^d$ where K is a probability density function (pdf) on $\mathbb{R}^d$. In this context, h is called the window-width.

From the practical point of view, note that, compared to the extreme value based estimators [9, 10, 13, 14, 16, 15], projection estimators [21] or piecewise polynomial estimators [25, 24, 20], this estimator does not require a partition of S and is thus not limited to bi-dimensional bounded supports. Moreover, it benefits from an explicit formulation which is not the case of estimators defined by optimization problems [12] such as local polynomial estimators [19, 18, 23]. From the theoretical point of view, this estimator reveals to be completely convergent to g without assumption neither on the distribution of X nor on the distribution of Y given $X = x$ (see Section 3). Note however that $(p+1)^{1/p} \to 1$ when $p \to \infty$. In fact, this correcting term is specially designed for the case where Y given $X = x$ is uniformly distributed on $[0, g(x)]$. In this latter situation, the estimator is asymptotically Gaussian with the rate of convergence $n^{-\alpha/(d+\alpha)}$ (see Section 4). This rate is proved to be minimax optimal for $\alpha-$ Lipschitzian $d-$ dimensional frontiers [25], Chapter 5. This result is generalized in [26] to boundaries of more general regions. Other extensions are provided in [17, 20] to densities of Y given $X = x$ decreasing as a power of the distance from the boundary. We refer to [5, 7, 11] for the estimation of frontier functions under monotonicity assumptions, and to [1, 3] for the definition of robust estimators in this context. We conclude this paper by an illustration of the behavior of our estimator on some finite sample situations in Section 5 and by describing our future work in Section 6. Technical lemmas are postponed to the appendix.

2 Notations and assumptions

To motivate the estimator (2), consider the random variable $Z = (p+1)Y^p$ and the conditional expectation $r_n(x) = \mathbb{E}(Z|X = x)$. Estimating the frontier g is often related to estimating the regression function r_n. For instance, if Y given $X = x$ is uniformly distributed on $[0, g(x)]$, we have $r_n^{1/p}(x) = g(x)$. A similar remark is done in [22] where regression estimators are modified to build estimators of the frontier, but the profound difference here is that $p \to \infty$. This condition allows to obtain $r_n^{1/p}(x) \to g(x)$ even when Y given $X = x$ is not uniformly distributed (see Lemma 1 below). We denote by f the pdf of the random vector X and we introduce

$$\hat{\varphi}_n(x) = \frac{1}{n} \sum_{i=1}^{n} K_h(x - X_i) Z_i, \tag{3}$$

where $Z_i = (p+1)Y_i^p$. Note that $\hat{\varphi}_n(x)$ can be seen as a classical kernel estimator of $\varphi_n(x) = f(x) r_n(x)$ but keep in mind that $p \to \infty$. Similarly,

$$\hat{f}_n(x) = \frac{1}{n} \sum_{i=1}^{n} K_h(x - X_i) \tag{4}$$

is an estimator of $f(x)$ and

$$\hat{r}_n(x) = \hat{\varphi}_n(x)/\hat{f}_n(x) \tag{5}$$

is an estimator of $r_n(x)$. Collecting (3), (4) and (5), our estimator (2) can be rewritten as

$$\hat{g}_n(x) = \hat{r}_n(x)^{1/p}.$$

To establish the asymptotic properties of $\hat{g}_n(x)$, the following assumptions are considered:

(A.1): g is α-Lipschitz, f is β-Lipschitz, with $0 < \alpha \leq \beta \leq 1$,

(A.2): $0 < g_{\min} \leq g(x)$, $\forall x \in \mathbb{R}^d$,

(A.3): $f(x) \leq f_{\max} < \infty$, $\forall x \in \mathbb{R}^d$,

(A.4): K is a Lipschitzian pdf on $\mathbb{R}^d$, with support included in B, the unit ball of $\mathbb{R}^d$.

Note that **(A.4)** implies that, for all $q \geq 1$, we have $0 < \int_B K^q(x)dx < +\infty$.

3 Complete convergence

In this section, the complete convergence of the frontier estimator toward the true frontier is established. The next lemma can be seen as the intuitive justification why no assumption on the conditional distribution of Y given X is required in the proof of Theorem 1.

Lemma 1 *Under* **(A.2)**, *for all $x \in B$, $r_n(x)^{1/p} \to g(x)$ as $n \to \infty$.*

Proof : Let $\varepsilon > 0$. Since (X, Y) has support S defined by (1), it follows that

$$r_n(x) = (p+1)\mathbb{E}\left(Y^p | X = x\right) \leq (p+1)g^p(x)$$

and thus, since $(p+1)^{1/p} \to 1$ as $p \to \infty$, for n large enough and all $x \in B$,

$$r_n^{1/p}(x) \leq (1+\varepsilon)g(x). \tag{6}$$

Moreover, we have,

$$\begin{aligned} r_n(x) &\geq (p+1)\mathbb{E}\left(Y^p \mathbf{1}\{Y > g(x) - \varepsilon\} | X = x\right) \\ &\geq (p+1)(g(x)-\varepsilon)^p \mathbb{P}(Y > g(x) - \varepsilon | X = x). \end{aligned}$$

Now, since (X, Y) has support S, one can assume without loss of generality that Y given $X = x$ has support $[0, g(x)]$ such that $\mathbb{P}(Y > g(x) - \varepsilon | X = x) > 0$. It follows that

$$\left[(p+1)\mathbb{P}(Y > g(x) - \varepsilon | X = x)\right]^{1/p} \to 1$$

as $p \to \infty$, and consequently, for n large enough,

$$r_n^{1/p}(x) \geq (1-\varepsilon)g(x). \tag{7}$$

Collecting (6) and (7) gives the result. ■

Theorem 1 *Suppose* **(A.1)**–**(A.4)** *hold and* $nh^d/\log n \to \infty$. *Then* $\widehat{g}_n(x)$ *converges completely to* $g(x)$ *for all* $x \in \mathbb{R}^d$ *such that* $f(x) > 0$.

Proof : Let $x \in \mathbb{R}^d$ such that $f(x) > 0$ and let ε such that $0 < \varepsilon < g(x)$. Define $0 < \eta < 1/4$ by $\eta = \varepsilon/(4g(x))$. Then, from Lemma 6,

$$\begin{aligned}
\{|\widehat{g}_n(x) - g(x)| > \varepsilon\} &= \left\{\left|\frac{\widehat{r}_n^{1/p}(x)}{g(x)} - 1\right| > 4\eta\right\} \\
&\subseteq \left\{\left|\left(\frac{\widehat{\varphi}_n(x)}{f(x)g^p(x)}\right)^{1/p} - 1\right| > \eta\right\} \cup \left\{\left|\left(\frac{\hat{f}_n(x)}{f(x)}\right)^{1/p} - 1\right| > \eta\right\}.
\end{aligned}$$

Since $\hat{f}_n(x)$ converges completely to $f(x)$, see e.g. [2], Chapter 4, Theorem III.3, it follows that $(\hat{f}_n(x)/f(x))^{1/p}$ converges completely to 1. Therefore, writing

$$\left(\frac{\widehat{\varphi}_n(x)}{f(x)g^p(x)}\right)^{1/p} = (p+1)^{1/p} T_n(x)$$

with

$$T_n(x) = \left[\frac{1}{n}\sum_{i=1}^n K_h(x - X_i)\left(\frac{Y_i}{g(x)}\right)^p \frac{1}{f(x)}\right]^{1/p}$$

and remarking that $(p+1)^{1/p} \to 1$ as $n \to \infty$, it suffices to consider

$$\{|T_n(x) - 1| > \eta\} \subseteq \{T_n(x) > 1 + \eta\} \cup \{T_n(x) < 1 - \eta\}.$$

The two events are studied separately. First, let $0 < \delta < \eta$. Then, $\|x - X_i\| \le h$ entails

$$Y_i - g(x)(1+\delta) \le g(X_i) - g(x) - \delta g(x) \le L_g h^\alpha - \delta g_{\min} < 0$$

for n large enough and where L_g is the Lipschitz constant associated to g. We thus have

$$\begin{aligned}
T_n(x) &= \left[\frac{1}{n}\sum_{i=1}^n K_h(x - X_i)\left(\frac{Y_i}{g(x)}\right)^p \mathbf{1}\{Y_i < g(x)(1+\delta)\}\frac{1}{f(x)}\right]^{1/p} \\
&\le (1+\delta)\left[\frac{1}{n}\sum_{i=1}^n K_h(x - X_i)\mathbf{1}\{Y_i < g(x)(1+\delta)\}\frac{1}{f(x)}\right]^{1/p},
\end{aligned}$$

and consequently,

$$\begin{aligned}
\{T_n(x) > 1 + \eta\} &\subseteq \left\{\frac{1}{n}\sum_{i=1}^n K_h(x - X_i)\mathbf{1}\{Y_i < g(x)(1+\delta)\}\frac{1}{f(x)} > \left(\frac{1+\eta}{1+\delta}\right)^p\right\} \\
&\subseteq \left\{\frac{1}{n}\sum_{i=1}^n K_h(x - X_i)\mathbf{1}\{Y_i < g(x)(1+\delta)\}\frac{1}{f(x)} > 2\right\},
\end{aligned}$$

since, for n large enough, $((1+\eta)/(1+\delta))^p > 2$. From [2], Chapter 5, Corollary II.4, the following complete convergence holds:

$$\frac{1}{n}\sum_{i=1}^n K_h(x - X_i)\mathbf{1}\{Y_i < g(x)(1+\delta)\}\frac{1}{f(x)} \xrightarrow{c} \mathbb{P}(Y < g(x)(1+\delta)|X = x) = 1,$$

and therefore

$$\sum_{n=1}^{\infty}\mathbb{P}(T_n(x) > 1+\eta) \leq \sum_{n=1}^{\infty}\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}K_h(x-X_i)\mathbf{1}\{Y_i < g(x)(1+\delta)\}\frac{1}{f(x)} - 1 > 1\right) < +\infty,$$

which concludes the first part of the proof. Second,

$$\begin{aligned} T_n(x) &\geq \left[\frac{1}{n}\sum_{i=1}^{n}K_h(x-X_i)\left(\frac{Y_i}{g(x)}\right)^p \mathbf{1}\{Y_i > g(x)(1-\delta)\}\frac{1}{f(x)}\right]^{1/p} \\ &\geq (1-\delta)\left[\frac{1}{n}\sum_{i=1}^{n}K_h(x-X_i)\mathbf{1}\{Y_i > g(x)(1-\delta)\}\frac{1}{f(x)}\right]^{1/p}, \end{aligned}$$

and consequently,

$$\{T_n(x) < 1-\eta\} \subseteq \left\{\frac{1}{n}\sum_{i=1}^{n}K_h(x-X_i)\mathbf{1}\{Y_i > g(x)(1-\delta)\}\frac{1}{f(x)} < \left(\frac{1-\eta}{1-\delta}\right)^p\right\}.$$

Now, since $\mathbb{P}(Y > g(x)(1-\delta)|X=x) > 0$, there exists $\gamma > 0$ such that, for n large enough,

$$\left(\frac{1-\eta}{1-\delta}\right)^p - \mathbb{P}(Y > g(x)(1-\delta)|X=x) < -\gamma,$$

entailing that, for n large enough,

$$\{T_n(x) < 1-\eta\} \subseteq \left\{\frac{1}{n}\sum_{i=1}^{n}K_h(x-X_i)\mathbf{1}\{Y_i > g(x)(1-\delta)\}\frac{1}{f(x)} - \mathbb{P}(Y/g(x) > 1-\delta|X=x) < -\gamma\right\}.$$

Taking into account of the following complete convergence

$$\frac{1}{n}\sum_{i=1}^{n}K_h(x-X_i)\mathbf{1}\{Y_i > g(x)(1-\delta)\}\frac{1}{f(x)} \xrightarrow{c} \mathbb{P}(Y > g(x)(1-\delta)|X=x),$$

it follows that

$$\begin{aligned} &\sum_{n=1}^{\infty}\mathbb{P}(T_n(x) < 1-\eta) \\ \leq\ &\sum_{n=1}^{\infty}\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}K_h(x-X_i)\mathbf{1}\{Y_i > g(x)(1-\delta)\}\frac{1}{f(x)} - \mathbb{P}(Y > g(x)(1-\delta)|X=x) < -\gamma\right) \\ <\ &+\infty, \end{aligned}$$

which concludes the second part of the proof. ■

4 Asymptotic normality

Second, the asymptotic normality of the frontier estimator centered on the true frontier is established. To this end, asymptotic expansions of the expectation and variance of $\hat{\varphi}_n(x)$ are needed. These calculations are done under the additional assumption

(A.5): Y given $X = x$ is uniformly distributed on $[0, g(x)]$.

The next two lemmas are similar to classical ones in kernel regression (see for instance [8], Theorem 6.11), but the dependence on n of the function $\varphi_n(x)$ induces technical difficulties. We first establish that $\widehat{\varphi}_n(x)$ is an asymptotically unbiased estimator of $\varphi_n(x)$ in the sense that $\mathbb{E}\widehat{\varphi}_n(x)/\varphi_n(x) \to 1$ as $n \to \infty$ provided that $ph^\alpha \to 0$.

Lemma 2 *Under* **(A.1)**–**(A.5)***, if* $ph^\alpha \to 0$*, then for all* $x \in \mathbb{R}^d$

$$\mathbb{E}\widehat{\varphi}_n(x) = \varphi_n(x)\left[1 + O(ph^\alpha)\right].$$

Proof : From (3), it follows that

$$\mathbb{E}\widehat{\varphi}_n(x) = \mathbb{E}(K_h(x-X)Z) = \mathbb{E}(K_h(x-X)\mathbb{E}(Z|X)),$$

so that, by a straightforward calculation, and recalling that $\varphi_n(u) = g^p(u)f(u)$, we obtain

$$\begin{aligned} \mathbb{E}\widehat{\varphi}_n(x) &= \mathbb{E}(K_h(x-X)g^p(X)) = \int_{\mathbb{R}^d} \frac{1}{h^d} K\left(\frac{x-u}{h}\right)\varphi_n(u)du \\ &= \int_B K(y)\varphi_n(x-hy)dy, \end{aligned} \tag{8}$$

with a classical change of variable, and since K has a compact support. We thus can write:

$$\mathbb{E}\widehat{\varphi}_n(x) - \varphi_n(x) = \int_B K(y)\left[\varphi_n(x-hy) - \varphi_n(x)\right]dy.$$

Consider now the decomposition below:

$$|\varphi_n(x-hy) - \varphi_n(x)| \le f(x-hy)\left|g^p(x-hy) - g^p(x)\right| + g^p(x)\left|f(x-hy) - f(x)\right| := T_1 + T_2.$$

Following Lemma 5,

$$\begin{aligned} T_1 &= f(x-hy)g^p(x)\left|\frac{g^p(x-hy)}{g^p(x)} - 1\right| \le 2f_{\max}\frac{L_g}{g_{\min}}g^p(x)ph^\alpha = g^p(x)O(ph^\alpha), \\ T_2 &\le g^p(x)L_f h^\beta = g^p(x)O(h^\beta) = g^p(x)o(ph^\alpha), \end{aligned}$$

where L_f and L_g are the Lipschitz constants of the functions f and g. Finally,

$$\mathbb{E}\widehat{\varphi}_n(x) - \varphi_n(x) = g^p(x)O(ph^\alpha) = \varphi_n(x)O(ph^\alpha),$$

and the conclusion follows. ■

Similarly, we now provide an equivalent expression for $\mathbb{V}(\widehat{\varphi}_n(x)/\varphi_n(x))$ which appears to be of order $p/(nh^d)$. Thus, condition $p/(nh^d) \to 0$ will be necessary in Corollary 1 to obtain the weak consistency of $\widehat{\varphi}_n(x)$, i.e. to ensure that $\widehat{\varphi}_n(x)/\varphi_n(x) \xrightarrow{P} 1$.

Lemma 3 *Under* **(A.1)**–**(A.5)***, if* $ph^\alpha \to 0$ *then for all* $x \in \mathbb{R}^d$*,*

$$\mathbb{V}(\widehat{\varphi}_n(x)) = \frac{1}{nh^d}\frac{(p+1)^2}{2p+1}\int_B K^2(s)ds\frac{\varphi_n^2(x)}{f(x)}\left[1 + o(1)\right].$$

Proof : We have

$$\begin{aligned}\mathbb{V}(\widehat{\varphi}_n(x)) &= \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{V}\left(K_h(x-X_i)Z_i\right) = \frac{1}{n}\mathbb{V}\left(K_h(x-X)Z\right)\\ &= \frac{1}{nh^{2d}}\mathbb{E}\left(K^2\left(\frac{x-X}{h}\right)Z^2\right) - \frac{1}{n}\mathbb{E}^2\left(\widehat{\varphi}_n(x)\right) := T_3 + T_4.\end{aligned}$$

From Lemma 2, we immediately derive

$$T_4 = \frac{1}{n}\varphi_n^2(x)\left[1+o(1)\right].$$

We shall prove that

$$T_3 = \frac{1}{nh^d}\frac{(p+1)^2}{2p+1}\int_B K^2(s)ds\frac{\varphi_n^2(x)}{f(x)}\left[1+o(1)\right], \tag{9}$$

leading to $T_4/T_3 = O(h/p)$, and the announced result follows. To this end, remark that

$$\begin{aligned}T_3 &= \frac{1}{nh^{2d}}\mathbb{E}\left(K^2\left(\frac{x-X}{h}\right)\mathbb{E}(Z^2|X)\right)\\ &= \frac{1}{nh^{2d}}\frac{(p+1)^2}{2p+1}\mathbb{E}\left(K^2\left(\frac{x-X}{h}\right)g^{2p}(X)\right)\\ &= \frac{1}{nh^{d}}\frac{(p+1)^2}{2p+1}\int_B K^2(s)ds\int_{\mathbb{R}^d}\frac{1}{h^d}Q\left(\frac{x-u}{h}\right)g^{2p}(u)f(u)du,\end{aligned}$$

where we have introduced the kernel $Q = K^2/\int_B K^2(s)ds$. It is easily seen that the second integral is similar to this appearing in $\mathbb{E}\widehat{\varphi}_n(x)$, (see (8)), with K replaced by Q and p by $2p$. Thus, as in the proof of Lemma 2, we have

$$\int_{\mathbb{R}^d}\frac{1}{h^d}Q\left(\frac{x-u}{h}\right)g^{2p}(u)f(u)du = g^{2p}(x)f(x)\left[1+o(1)\right] = \frac{\varphi_n^2(x)}{f(x)}\left[1+o(1)\right],$$

and (9) is proved. ∎

As a simple consequence of Lemma 2 and Lemma 3, we have

Corollary 1 *Under* **(A.1)**–**(A.5)***, if* $ph^\alpha \to 0$ *and* $p/(nh^d) \to 0$*, then, for all* $x \in \mathbb{R}^d$*,*

$$\widehat{\varphi}_n(x)/\varphi_n(x) \xrightarrow{P} 1.$$

We can now turn to our main result.

Theorem 2 *Suppose that* $nph^{d+2\alpha} \to 0$ *and* $p/(nh^d) \to 0$*. Let us define*

$$\sigma_n^{-1}(x) = ((2p+1)nh^d)^{1/2}\left(\frac{f(x)}{\int_B K^2(t)dt}\right)^{1/2}.$$

Then, under **(A.1)**–**(A.5)***, for all* $x \in \mathbb{R}^d$*,*

$$\sigma_n^{-1}(x)\left(\frac{\widehat{g}_n(x)}{g(x)} - 1\right) \xrightarrow{d} N(0,1).$$

Proof : First, note that $nph^{d+2\alpha} \to 0$ and $p/(nh^d) \to 0$ imply $ph^\alpha \to 0$. From Lemma 9, it suffices to prove that

$$\xi_n := \frac{\sigma_n^{-1}(x)}{p}\left(\frac{\widehat{\varphi}_n(x)}{\varphi_n(x)} - \frac{\mathbb{E}\widehat{\varphi}_n(x)}{\varphi_n(x)}\right) \xrightarrow{d} N(0,1).$$

To this end, define

$$W_{i,n} = \frac{\sigma_n^{-1}(x)}{np}\frac{1}{\varphi_n(x)}K_h(x-X_i)Z_i$$

so that we can write

$$\xi_n = \sum_{i=1}^n \left(W_{i,n} - \mathbb{E}W_{i,n}\right).$$

Following Lemma 3, we have

$$\begin{aligned}
\mathbb{V}\left(\xi_n\right) &= n\mathbb{V}\left(W_{1,n}\right) = \frac{\sigma_n^{-2}(x)}{p^2}\frac{1}{\varphi_n^2(x)}\mathbb{V}(\widehat{\varphi}_n(x)) \\
&= \frac{(2p+1)\,nh^d f(x)}{p^2\int_B K^2(s)ds}\frac{1}{\varphi_n^2(x)}\frac{1}{nh^d}\frac{(p+1)^2}{2p+1}\int_B K^2(s)ds\frac{\varphi_n^2(x)}{f(x)}\left[1+o(1)\right] \\
&= 1+o(1).
\end{aligned}$$

Thus, the condition of Lyapounov reduces to

$$\sum_{i=1}^n \mathbb{E}\left|W_{i,n} - \mathbb{E}W_{i,n}\right|^3 = n\mathbb{E}\left|W_{1,n} - \mathbb{E}W_{1,n}\right|^3 \to 0. \tag{10}$$

Taking into account that $W_{1,n}$ is a positive random variable, the triangular inequality together with Jensen's inequality yield

$$\mathbb{E}\left|W_{1,n} - \mathbb{E}W_{1,n}\right|^3 \leq 8\mathbb{E}\left(W_{1,n}^3\right).$$

Introducing the kernel $K^3/\int_B K^3(s)ds$, and mimicking the proof of Lemma 3, we obtain

$$\mathbb{E}(W_{1,n}^3) = \frac{n^{-3/2}h^{-d/2}p^{1/2}2^{3/2}}{3f(x)^{3/2}}\frac{\int_B K^3(s)ds}{\left(\int_B K^2(s)ds\right)^{3/2}}(1+o(1)) = \kappa n^{-3/2}h^{-d/2}p^{1/2}(1+o(1)), \tag{11}$$

where κ is a positive constant. Returning to (10), we have

$$\sum_{i=1}^n E\left|W_{i,n} - \mathbb{E}W_{i,n}\right|^3 \leq 8\kappa\left(\frac{p}{nh^d}\right)^{1/2}(1+o(1)) \to 0,$$

and the result is proved. ■

Remark 1 *Theorem 2 holds when $\sigma_n^{-1}(x)$ is replaced with*

$$\widehat{\sigma}_n^{-1}(x) = ((2p+1)nh^d)^{1/2}\left(\frac{\widehat{f}_n(x)}{\int_B K^2(t)dt}\right)^{1/2},$$

since in this context $\widehat{f}_n(x) \xrightarrow{P} f(x)$. This allows to produce pointwise confident intervals for the frontier.

Remark 2 *To fulfill the assumptions of Theorem 2, one can choose* $h = n^{-1/(d+\alpha)}$ *and* $p = \varepsilon_n n^{\alpha/(d+\alpha)}$*, where* (ε_n) *is a sequence tending to zero arbitrarily slowly. These choices yield*

$$\sigma_n^{-1}(x) = \varepsilon_n^{1/2} n^{\alpha/(d+\alpha)} \left(\frac{2f(x)}{\int_B K^2(t)dt} \right)^{1/2} (1 + o(1)),$$

which is the optimal speed (up to the ε_n *factor) for estimating* $\alpha-$ *Lipschitzian* $d-$ *dimensional frontiers, see [25], Chapter 5.*

The good performances of $\hat{g}_n(x)$ on finite sample situations are illustrated in the next section. Remark 2 will be of great help to choose p and h sequences.

5 Numerical experiments

Here, we limit ourselves to unidimensional random variables X $(p = 1)$ with compact support $E = [0, 1]$. Besides, Y given $X = x$ is distributed on $[0, g(x)]$ such that

$$\mathbb{P}(Y > y | X = x) = \left(1 - \frac{y}{g(x)} \right)^{\gamma}, \tag{12}$$

with $\gamma > 0$. This conditional survival distribution function belongs to the Weibull domain of attraction, with extreme value index $-\gamma$, see [6] for a review on this topic. The case $\gamma = 1$ corresponds to the situation where Y given $X = x$ is uniformly distributed on $[0, g(x)]$. The larger γ is, the smaller the probability (12) is, when y is close to the frontier $g(x)$. The behavior of the proposed frontier estimator is investigated on different situations:

- Two distributions are considered for X: a uniform distribution $U([0, 1])$ and a beta distribution $B(2, 2)$.
- Two frontiers are introduced. The first one

$$g_1(x) = \left| \begin{array}{ll} 1 + \exp\left(-60(x - 1/4)^2\right) & \text{if } 0 \le x \le 1/3, \\ 1 + \exp\left(-5/12\right) & \text{if } 1/3 < x \le 2/3, \\ 1 + 5\exp\left(-5/12\right) - 6\exp\left(-5/12\right)x & \text{if } 2/3 < x \le 5/6, \\ 6x - 4 & \text{if } 5/6 < x \le 1. \end{array} \right.$$

  is continuous but is not derivable at $x = 1/3$, $x = 2/3$ and $x = 5/6$. The second one

$$g_2(x) = (1/10 + \sin(\pi x)) \left(11/10 - \exp\left(-64(x - 1/2)^2\right)/2 \right)$$

  is C^∞.

- Four sample sizes are simulated $n \in \{200, 300, 500, 1000\}$.
- Three exponents are used $\gamma \in \{1, 2, 3\}$.

The following kernel is chosen

$$K(t) = \cos^2(\pi t/2)\mathbf{1}\{t \in [-1, 1]\},$$

with associated window width $h = 4\hat{\sigma}(X)n^{-1/2}$ and with $p = n^{1/2}$. The dependence of these sequences with respect to n is chosen according to Remark 2 with $\alpha = d = 1$. The multiplicative constant $4\hat{\sigma}(X)$ in h is chosen heuristically. The dependence with respect to the standard-deviation of X is inspired from the density estimation case. The scale factor 4 was chosen on the basis of intensive simulations.

Here, the experiment involves several steps:

- First, $m = 500$ replications of the sample are simulated.
- For each of the m previous set of points, the frontier estimator $\hat{g}_n$ is computed.
- The m associated L_1 distances to g are evaluated on a grid.
- The mean, smallest and largest L_1 errors are recorded.

Some results are depicted on Figure 1–3, where the best situation (i.e. the estimation corresponding to the smallest L_1 error) and the worst situation (i.e. the estimation corresponding to the largest L_1 error) are represented. Note that, even in the worst situations, the empirical choices of sequences h and p seem satisfying for all the considered frontiers and densities of X. In fact, the worst situations are obtained when no points were simulated at the boundaries of the support. This is specially the case on Figure 3(b) since the density of X decreases to 0 at the boundaries of the $[0, 1]$ interval and the density of $Y|X = x$ decreases to 0 in the neighborhood of $g(x)$.

Finally, the above estimator is compared to three other ones:

- The estimator $\hat{g}_n$ with $p = 1$, which reduces to a rescaling of the regression estimator, in a similar spirit as in [22].
- Geffroy's estimator [10], denoted by $\hat{g}_n^G$, which is a step function based on the extreme values of the sample.
- The kernel estimator $\hat{g}_n^K$ introduced in [15], which is a smoothed and bias-corrected version of Geffroy's estimator.

Results are summarized in Table 1. It appears that, when γ increases, performances of all estimators decrease, since the simulated points are getting more and more distant from the frontier function. In the case $p = 1$ and $\gamma = 3$, one can see that $\hat{g}_n$ does not converge to the true frontier when n increases. This shows that the condition $p \to \infty$ is necessary to obtain the convergence of the estimator. Finally, note that in all the situations considered in Table 1, $\hat{g}_n$ outperforms $\hat{g}_n^G$ and $\hat{g}_n^K$.

6 Conclusion and further work

To conclude, let us note that, even though $\hat{g}_n$ converges to the true frontier g in case of non uniform conditional distributions, it is possible to design new estimators dedicated to particular parametric models. For instance, in case of model (12), estimator $\hat{g}_n$ could be modified to obtain

$$\tilde{g}_{n,\gamma}(x) = \left(\frac{1}{\gamma B(1+p,\gamma)} \sum_{i=1}^{n} K_h(x-X_i) Y_i^p \Bigg/ \sum_{i=1}^{n} K_h(x-X_i) \right)^{1/p},$$

where B is the beta function defined by

$$B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1} du.$$

Of course, $\hat{g}_n$ corresponds to the particular case $\tilde{g}_{n,1}$. When γ is assumed to be known, the new multiplicative constant yields a very efficient bias correction, see Figure 4 for an illustration. A part of our future work will consist in defining an estimator of γ and plugging it into $\tilde{g}_{n,\gamma}$. New asymptotic results will be established. We also plan to investigate the asymptotic properties of local polynomial estimators based on the same ideas as those used for $\hat{g}_n$ and $\tilde{g}_{n,\gamma}$.

7 Appendix: Auxiliary lemmas

The following lemma provides convenient bounds obtained by a specific study of the functions $u \to |(1+u)^p - 1| - 2p|u|$ and $u \to (1+u)^{1/p} - 1 - \frac{1}{p}u$. The study is left to the reader. Note that these bounds could not be directly derived from the Taylor formulas $|(1+u)^p - 1| = |pu + o(u)|$ and $\left|(1+u)^{1/p} - 1 - \frac{1}{p}u\right| = \left|\frac{1}{2p}(\frac{1}{p} - 1)u^2 + o(u^2)\right|$ where the dependence on p of $o(u)$ and $o(u^2)$ is not precised.

Lemma 4 *Suppose $p \geq 1$.*

(i) *Then, $p|u| \leq \ln 2$ entails $|(1+u)^p - 1| \leq 2p|u|$.*

(ii) *Let $C \geq 2$. Then, $|u| < 1/2$ entails $\left|(1+u)^{1/p} - 1 - \frac{1}{p}u\right| \leq \frac{C}{p}u^2$.*

The next lemma is dedicated to the control of the local variations of the frontier on a neighborhood of size h.

Lemma 5 *Suppose* **(A.1)**, **(A.2)** *hold. If $ph^\alpha \to 0$ and $\|x-y\| \leq h$, then for sufficiently large n,*

$$\left| \left(\frac{g(x)}{g(y)} \right)^p - 1 \right| \leq 2 \frac{L_g}{g_{\min}} ph^\alpha,$$

where L_g is the Lipschitz constant of the function g.

Proof : Take $u = \frac{g(x)}{g(y)} - 1$ and observe that $p|u| \leq p\frac{L_g}{g_{\min}}\|x-y\|^\alpha$, Thus, if $\|x-y\| \leq h$, and $ph^\alpha \to 0$, we have $p|u| \leq \ln 2$ for sufficiently large n. Then, following Lemma 4(i), for sufficiently large n, we obtain

$$|(1+u)^p - 1| = \left| \left(\frac{g(x)}{g(y)}\right)^p - 1 \right| \leq 2p|u| \leq 2\frac{L_g}{g_{\min}} ph^\alpha,$$

and the result is proved. ∎

Lemma 6 is used to establish the complete convergence of random variables ratio.

Lemma 6 *Let S, T be real random variables, a, b non zero real numbers, and $0 < \eta < 1/2$. Then,*

$$\left\{ \left| \frac{S}{T} - \frac{a}{b} \right| > 4\eta \left| \frac{a}{b} \right| \right\} \subseteq \left\{ \left| \frac{S}{a} - 1 \right| > \eta \right\} \cup \left\{ \left| \frac{T}{b} - 1 \right| > \eta \right\}.$$

Proof : Consider the following obvious equality:

$$\left(\frac{S}{T} - \frac{a}{b} \right) = \frac{a}{b}\left(\frac{S}{a} - 1 \right) + \frac{a}{b}\left(1 - \frac{T}{b} \right) + \left(\frac{S}{T} - \frac{a}{b} \right)\left(1 - \frac{T}{b} \right). \tag{13}$$

The triangular inequality yields for all $\eta > 0$:

$$\left\{ \left| \frac{S}{a} - 1 \right| \leq \eta \right\} \cap \left\{ \left| \frac{T}{b} - 1 \right| \leq \eta \right\} \subseteq \left\{ \left| \frac{S}{T} - \frac{a}{b} \right| \leq 2\eta \left| \frac{a}{b} \right| + \eta \left| \frac{S}{T} - \frac{a}{b} \right| \right\}.$$

Taking $0 < \eta < 1$, we obtain

$$\left\{ \left| \frac{S}{a} - 1 \right| \leq \eta \right\} \cap \left\{ \left| \frac{T}{b} - 1 \right| \leq \eta \right\} \subseteq \left\{ \left| \frac{S}{T} - \frac{a}{b} \right| \leq \frac{2\eta}{1-\eta} \left| \frac{a}{b} \right| \right\}.$$

Finally, note that $\frac{2\eta}{1-\eta} < 4\eta$ for $0 < \eta < 1/2$. ∎

The next three lemmas are of great use to deduce successively the asymptotic normality of $\widehat{g}_n(x)$ from $\widehat{r}_n(x)$ and the asymptotic normality of $\widehat{r}_n(x)$ from $\widehat{\varphi}_n(x)$.

Lemma 7 *Let $x \in \mathbb{R}^d$. If $\widehat{f}_n(x)/f(x) \xrightarrow{P} 1$ and $\widehat{\varphi}_n(x)/\varphi_n(x) \xrightarrow{P} 1$, then*

$$\left(\frac{\widehat{r}_n(x)}{r_n(x)} - 1 \right) = \left(\frac{\widehat{\varphi}_n(x)}{\varphi_n(x)} - 1 \right) - \left(\frac{\widehat{f}_n(x)}{f(x)} - 1 \right)(1 + o_p(1)).$$

Proof : The hypotheses yield $\frac{\widehat{r}_n(x)}{r_n(x)} = \frac{\widehat{\varphi}_n(x)}{\widehat{f}_n(x)} / \frac{\varphi_n(x)}{f(x)} \xrightarrow{P} 1$. Thus it suffices to consider $S = \frac{\widehat{\varphi}_n(x)}{\varphi_n(x)}$, $T = \frac{\widehat{f}_n(x)}{f(x)}$, and $a = b = 1$ in the equality (13). ∎

Lemma 8 *Let $x \in \mathbb{R}^d$. If $\widehat{f}_n(x)/f(x) \xrightarrow{P} 1$ and $\widehat{\varphi}_n(x)/\varphi_n(x) \xrightarrow{P} 1$, then*

$$\left(\frac{\widehat{g}_n(x)}{g(x)} - 1 \right) = \frac{1}{p}\left(\frac{\widehat{r}_n(x)}{r_n(x)} - 1 \right)(1 + o_p(1)).$$

Proof : From the hypotheses, $w_n(x) := \frac{\widehat{r}_n(x)}{r_n(x)} - 1 = o_p(1)$. Moreover, following Lemma 4(ii), on the event $\{|w_n(x)| < 1/2\}$ we have:

$$\Delta_n(x) := \left| \left(\frac{\widehat{g}_n(x)}{g(x)} - 1 \right) - \frac{1}{p}\left(\frac{\widehat{r}_n(x)}{r_n(x)} - 1 \right) \right| = \left| (1 + w_n(x))^{1/p} - 1 - \frac{w_n(x)}{p} \right| \leq C\frac{1}{p} w_n^2(x).$$

We thus have, on the one hand,

$$p\Delta_n(x)\mathbf{1}_{\{|w_n(x)|<1/2\}} = o_p(w_n(x)).$$

On the other hand, for all $\varepsilon > 0$,

$$\left\{p\frac{\Delta_n(x)}{w_n(x)}\mathbf{1}_{\{|w_n(x)|\geq 1/2\}} > \varepsilon\right\} \subseteq \{|w_n(x)| \geq 1/2\}$$

leading to

$$P\left\{p\frac{\Delta_n(x)}{w_n(x)}\mathbf{1}_{\{|w_n(x)|\geq 1/2\}} > \varepsilon\right\} \leq P\left\{|w_n(x)| \geq 1/2\right\} \to 0,$$

and thus

$$p\frac{\Delta_n(x)}{w_n(x)}\mathbf{1}_{\{|w_n(x)|\geq 1/2\}} = o_p(w_n(x)),$$

which completes the proof. ■

Lemma 9 *Suppose that $nph^{d+2\alpha} \to 0$ and $p/(nh^d) \to 0$. Let us define*

$$\sigma_n^{-1}(x) = ((2p+1)nh^d)^{1/2}\left(\frac{f(x)}{\int_B K^2(t)dt}\right)^{1/2},$$

and let Q be an arbitrary distribution. Then, under **(A.1)**–**(A.5)**,

$$\left\{\frac{\sigma_n^{-1}(x)}{p}\left(\frac{\widehat{\varphi}_n(x)}{\varphi_n(x)} - \frac{\mathbb{E}\widehat{\varphi}_n(x)}{\varphi_n(x)}\right) \xrightarrow{d} Q\right\} \Longrightarrow \left\{\sigma_n^{-1}(x)\left(\frac{\widehat{g}_n(x)}{g(x)} - 1\right) \xrightarrow{d} Q\right\}.$$

Proof : First, note that $nph^{d+2\alpha} \to 0$ and $p/(nh^d) \to 0$ imply $ph^\alpha \to 0$. Thus, from Corollary 1, $\widehat{\varphi}_n(x)/\varphi_n(x) \xrightarrow{P} 1$. Besides, $p/(nh^d) \to 0$ implies $nh^d \to \infty$, and thus, using a classical result on density estimation (see for instance [2], Chapter 4, Theorem II.1), we have $\widehat{f}_n(x)/f(x) \xrightarrow{P} 1$. Lemma 7 thus entails

$$\begin{aligned}
\frac{\sigma_n^{-1}(x)}{p}\left(\frac{\widehat{r}_n(x)}{r_n(x)} - 1\right) &= \frac{\sigma_n^{-1}(x)}{p}\left(\frac{\widehat{\varphi}_n(x)}{\varphi_n(x)} - 1\right) - \frac{\sigma_n^{-1}(x)}{p}\left(\frac{\widehat{f}_n(x)}{f(x)} - 1\right)(1 + o_p(1)) \\
&= \frac{\sigma_n^{-1}(x)}{p}\left(\frac{\widehat{\varphi}_n(x)}{\varphi_n(x)} - \frac{\mathbb{E}\widehat{\varphi}_n(x)}{\varphi_n(x)}\right) - \frac{\sigma_n^{-1}(x)}{p}\left(\frac{\widehat{f}_n(x)}{f(x)} - \frac{\mathbb{E}\widehat{f}_n(x)}{f(x)}\right)(1 + o_p(1)) \\
&+ \frac{\sigma_n^{-1}(x)}{p}\left(\frac{\mathbb{E}\widehat{\varphi}_n(x)}{\varphi_n(x)} - 1\right) - \frac{\sigma_n^{-1}(x)}{p}\left(\frac{\mathbb{E}\widehat{f}_n(x)}{f(x)} - 1\right)(1 + o_p(1)).
\end{aligned}$$

Following Lemma 2, we have,

$$\frac{\sigma_n^{-1}(x)}{p}\left(\frac{\mathbb{E}\widehat{\varphi}_n(x)}{\varphi_n(x)} - 1\right) = O\left(\left(\frac{nh^d}{p}\right)^{1/2}\right)O\left(ph^\alpha\right) = O\left(\left(nph^{d+2\alpha}\right)^{1/2}\right) = o(1),$$

and from a classical result on density estimation $\mathbb{E}\widehat{f}_n(x) - f(x) = O(h^\alpha)$, see [4], Proposition 2.1, we have

$$\frac{\sigma_n^{-1}(x)}{p}\left(\frac{\mathbb{E}\widehat{f}_n(x)}{f(x)} - 1\right) = O\left(\left(\frac{nh^d}{p}\right)^{1/2}\right)O\left(h^\alpha\right) = O\left(\left(np^{-1}h^{d+2\alpha}\right)^{1/2}\right) = o(1).$$

Consequently,

$$\frac{\sigma_n^{-1}(x)}{p}\left(\frac{\widehat{r}_n(x)}{r_n(x)}-1\right)=\frac{\sigma_n^{-1}(x)}{p}\left(\frac{\widehat{\varphi}_n(x)}{\varphi_n(x)}-1\right)-\frac{\sigma_n^{-1}(x)}{p}\left(\frac{\widehat{f}_n(x)}{f(x)}-1\right)(1+o_p(1))+o_p(1).$$

Again, using a classical result on density estimation, $\mathbb{V}(\widehat{f}_n(x))=O(1/(nh^d))$, see [4], Proposition 2.2, we have

$$\mathbb{V}\left(\frac{\sigma_n^{-1}(x)}{p}\frac{\widehat{f}_n(x)}{f(x)}\right)=O\left(\frac{nh^d}{p}\right)O\left(\frac{1}{nh^d}\right)=O(1/p)=o(1),$$

and thus

$$\frac{\sigma_n^{-1}(x)}{p}\left(\frac{\widehat{r}_n(x)}{r_n(x)}-1\right)=\frac{\sigma_n^{-1}(x)}{p}\left(\frac{\widehat{\varphi}_n(x)}{\varphi_n(x)}-1\right)+o_p(1). \tag{14}$$

Suppose now that there exists a probability distribution Q such that

$$\frac{\sigma_n^{-1}(x)}{p}\left(\frac{\widehat{\varphi}_n(x)}{\varphi_n(x)}-\frac{\mathbb{E}\widehat{\varphi}_n(x)}{\varphi_n(x)}\right)\xrightarrow{d}Q.$$

From (14), we deduce that

$$\frac{\sigma_n^{-1}(x)}{p}\left(\frac{\widehat{r}_n(x)}{r_n(x)}-1\right)\xrightarrow{d}Q.$$

Finally, from Lemma 8 we can conclude that

$$\sigma_n^{-1}(x)\left(\frac{\widehat{g}_n(x)}{g(x)}-1\right)\xrightarrow{d}Q,$$

and the result is proved. ■

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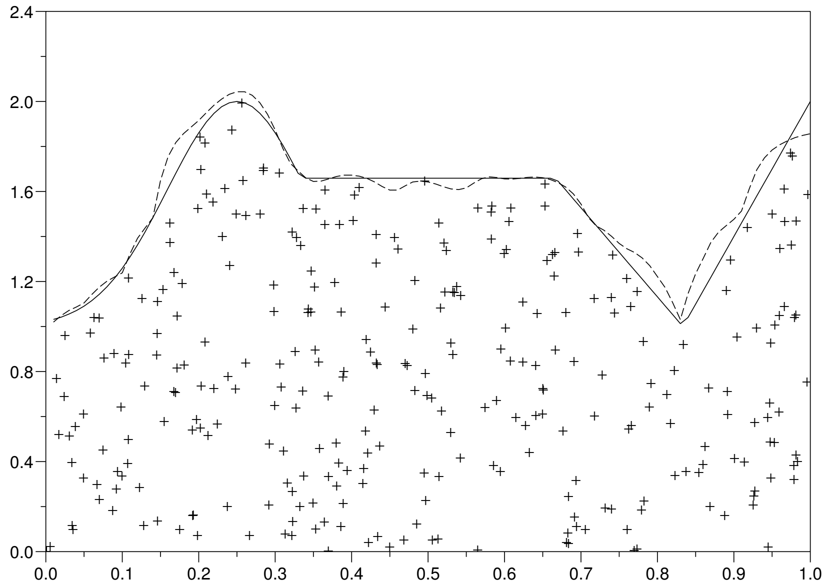
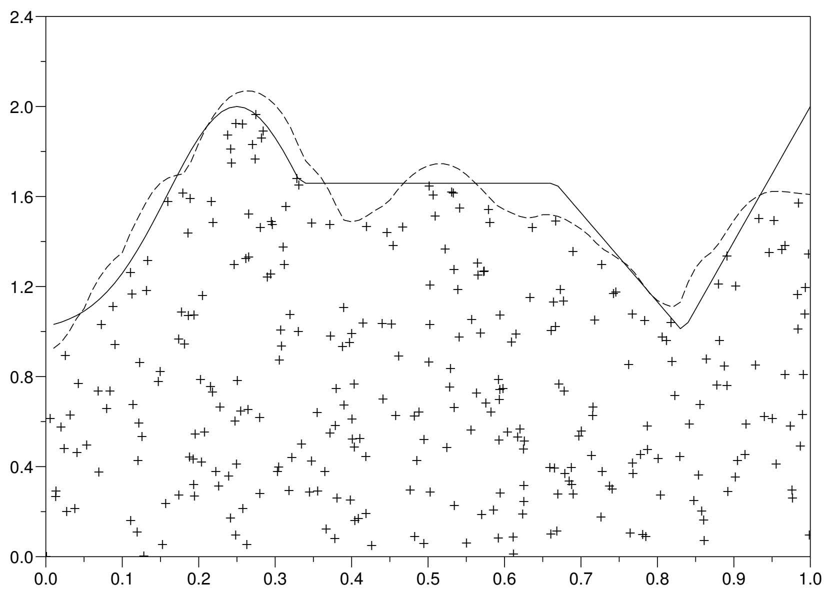
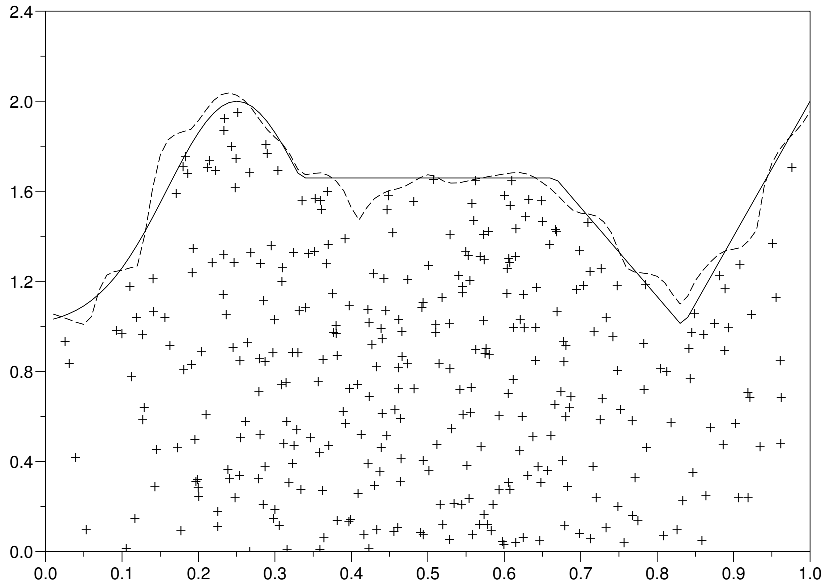


(a) Best situation

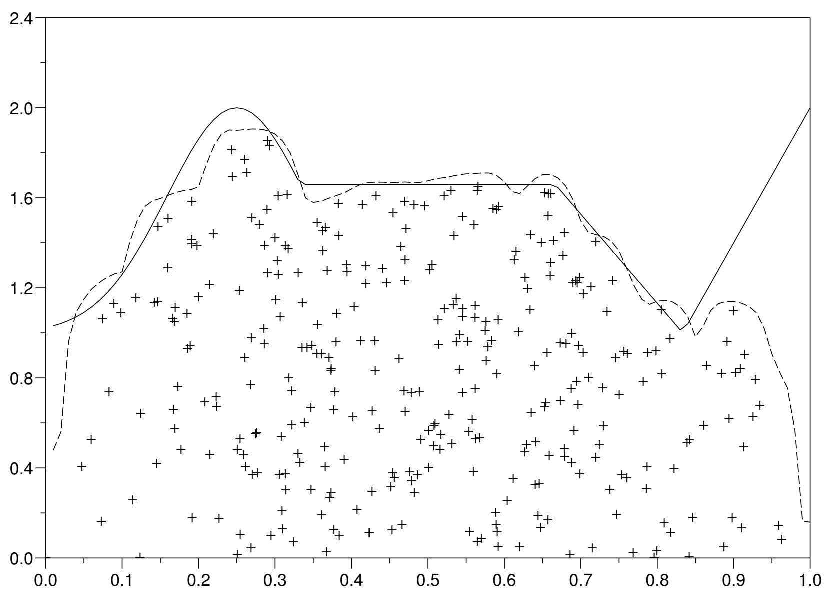


(b) Worst situation

Figure 1: The frontier g_1 (continuous line) and its estimation (dashed line). The sample size is $n = 300$, X is uniformly distributed on $[0, 1]$ and $\gamma = 1$.

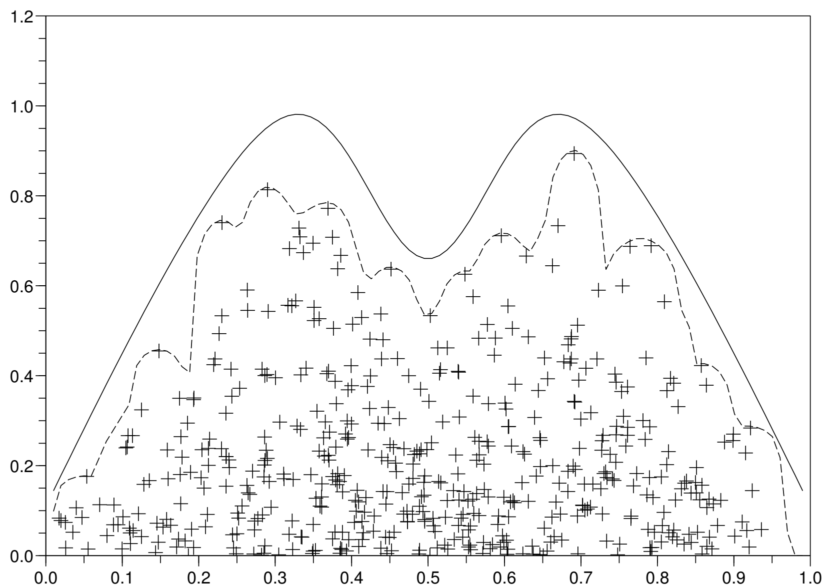


(a) Best situation

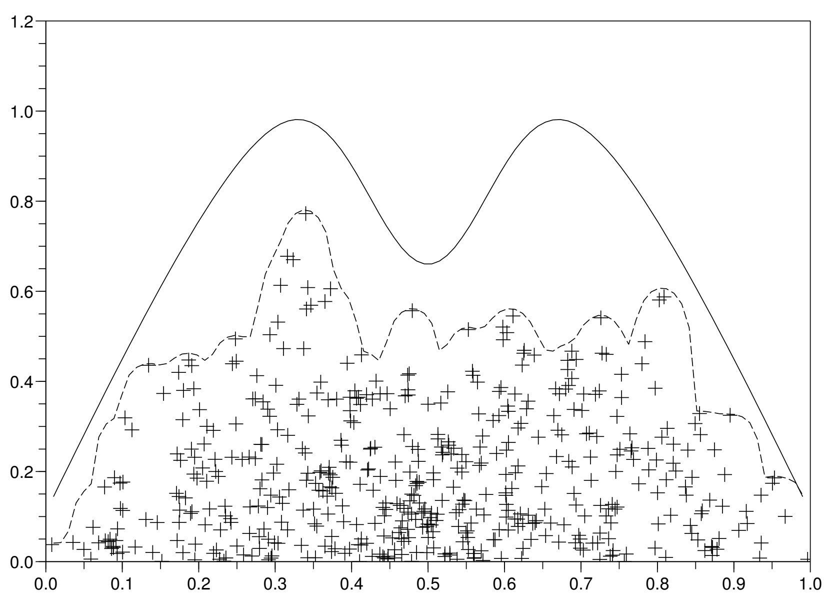


(b) Worst situation

Figure 2: The frontier g_1 (continuous line) and its estimation (dashed line). The sample size is $n = 300$, X is $B(2, 2)$ distributed on $[0, 1]$ and $\gamma = 1$.

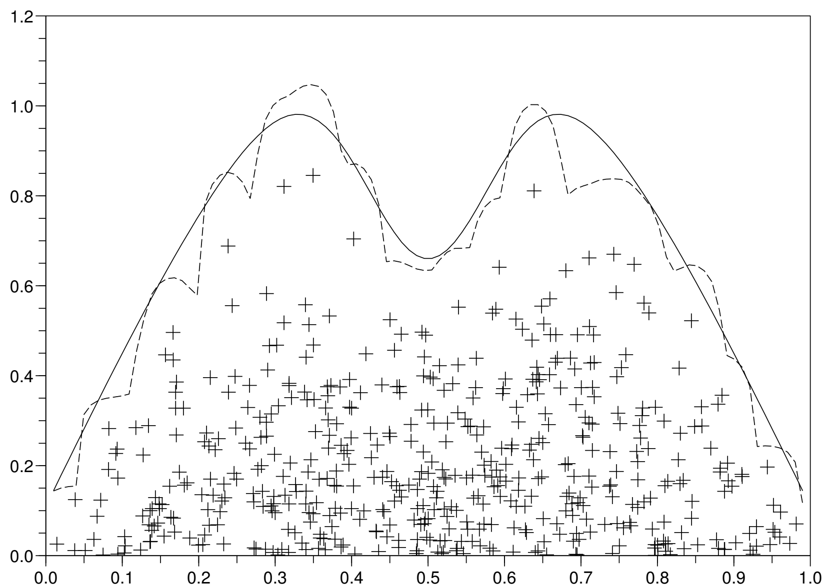


(a) Best situation

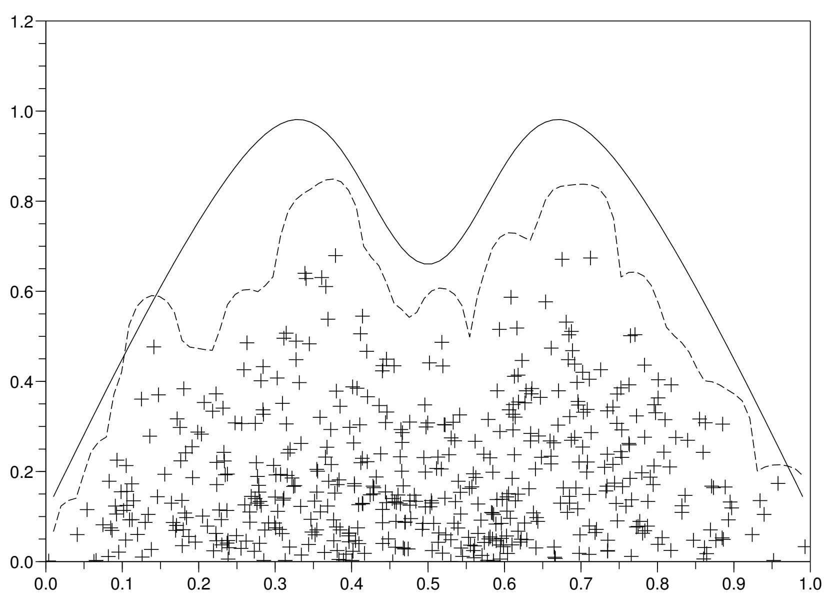


(b) Worst situation

Figure 3: The frontier g_2 (continuous line) and its estimation (dashed line). The sample size is $n = 500$, X is $B(2, 2)$ distributed on $[0, 1]$ and $\gamma = 3$.



(a) Best situation



(b) Worst situation

Figure 4: The frontier g_2 (continuous line) and the $\tilde{g}_{n,3}$ estimate (dashed line). The sample size is $n = 500$, X is $B(2,2)$ distributed on $[0,1]$ and $\gamma = 3$.

$\gamma = 1$				
n	$\hat{g}_n$ with $p \to \infty$	$\hat{g}_n$ with $p = 1$	$\hat{g}_n^K$	$\hat{g}_n^G$
200	0.121 *[0.051, 0.237]*	0.651 *[0.407, 0.907]*	0.134 *[0.056, 0.261]*	0.183 *[0.080, 0.334]*
300	0.100 *[0.049, 0.184]*	0.636 *[0.445, 0.831]*	0.111 *[0.061, 0.219]*	0.157 *[0.073, 0.300]*
500	0.078 *[0.042, 0.138]*	0.627 *[0.441, 0.813]*	0.087 *[0.046, 0.168]*	0.128 *[0.064, 0.234]*
1000	0.057 *[0.028, 0.112]*	0.616 *[0.486, 0.752]*	0.062 *[0.033, 0.117]*	0.093 *[0.049, 0.158]*
$\gamma = 2$				
n	$\hat{g}_n$ with $p \to \infty$	$\hat{g}_n$ with $p = 1$	$\hat{g}_n^K$	$\hat{g}_n^G$
200	0.321 *[0.197, 0.496]*	0.575 *[0.415, 0.759]*	0.337 *[0.180, 0.519]*	0.426 *[0.269, 0.591]*
300	0.297 *[0.194, 0.457]*	0.562 *[0.399, 0.755]*	0.311 *[0.171, 0.490]*	0.393 *[0.255, 0.569]*
500	0.262 *[0.169, 0.379]*	0.545 *[0.429, 0.667]*	0.275 *[0.172, 0.380]*	0.347 *[0.251, 0.452]*
1000	0.226 *[0.153, 0.303]*	0.533 *[0.463, 0.623]*	0.240 *[0.152, 0.336]*	0.293 *[0.200, 0.388]*
$\gamma = 3$				
n	$\hat{g}_n$ with $p \to \infty$	$\hat{g}_n$ with $p = 1$	$\hat{g}_n^K$	$\hat{g}_n^G$
200	0.526 *[0.331, 0.709]*	0.749 *[0.627, 0.888]*	0.550 *[0.340, 0.724]*	0.624 *[0.410, 0.780]*
300	0.496 *[0.363, 0.669]*	0.744 *[0.632, 0.865]*	0.523 *[0.371, 0.687]*	0.591 *[0.452, 0.739]*
500	0.457 *[0.366, 0.590]*	0.741 *[0.649, 0.817]*	0.486 *[0.375, 0.620]*	0.545 *[0.434, 0.668]*
1000	0.410 *[0.315, 0.505]*	0.742 *[0.685, 0.817]*	0.442 *[0.327, 0.531]*	0.486 *[0.375, 0.573]*

Table 1: Comparison between L_1 errors obtained with four different estimators. The mean error is indicated as well as the range between the minimum and the maximum error. The experiments are conducted on a $B(2,2)$ covariate, with frontier function g_2.