Two-stream instabilities in plasmas

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Dedicated to Cathleen S. Morawetz

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Abstract: One of the classical fluid models to describe plasma dynamics is so-called 'two-fluid' model, where electrons and ions are regarded as two compressible fluids. It is well-known that in many circumstances, two streams of charged steady fluids with different constant speeds are linearly unstable. It is shown in this article that they are indeed nonlinearly unstable in a dynamical setting.

1 Introduction

The two-stream instability is one of the most classical examples of velocity space instability that occurs in plasma physics. Some devices like traveling waves amplifiers are based on this phenomenon. On the other hand, it is also one of the reasons in the failure of some controlled thermonuclear reactions schemes (see [7] page 449).

We shall study the following three classical cases arising in the 'two-fluid' model in plasma physics. Let the spatial variable x belong to the periodic interval $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

We first consider two beams of cold *pressure-less* electrons with densities $n_i(t,x)$ and velocities $u_i(t,x)$, moving in a fixed ion background. In one space dimension, the Euler-Poisson equations are

$$\begin{aligned} &\partial_t n_i + \partial_x(n_i u_i) = 0, \\ &\partial_t u_i + u_i \partial_x u_i = -\frac{e}{m_e} E, \\ &\partial_x E = -4\pi(n_1 + n_2 - 1), \end{aligned} \tag{1}$$

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where $i = 1, 2$, $E(t, x)$ is the self-consistent electric field, and m_e is the mass of an electron. We also impose the neutral condition which is invariant for all time:

$$\int_{\mathbb{T}^d} [n_1(t, x) + n_2(t, x) - 1] dx = 0. \tag{2}$$

With the same notations, we next consider two streams of electrons with *pressure* moving along one direction in a motionless ion background. The Euler-Poisson system now takes the form:

$$\begin{aligned} &\partial_t n_i + \partial_x (n_i u_i) = 0, \\ &\partial_t u_i + u_i \partial_x u_i + \frac{1}{n_i} \partial_x p_i(n_i) = -\frac{e}{m_e} E, \\ &\partial_x E = -4\pi (n_1 + n_2 - 1), \end{aligned} \tag{3}$$

where $i = 1, 2$. Here the $p_i(n_i)$ are the corresponding partial pressures which are given strictly increasing functions, so called pressure laws, of the density. Typically, one consider adiabatic $p_i(n) = C_i n^\gamma$ for some $C_i > 0$ and $\gamma > 1$, or isothermal pressure law $p_i(n) = C_i \log n$, where C_i is a temperature. We also assume (2).

In the last case we consider two streams of moving ions and electrons. The Euler-Poisson system now consists of

$$\begin{aligned} &\partial_t n_j + \partial_x (u_j n_j) = 0, \\ &\partial_t u_j + u_j \partial_x u_j + \frac{1}{n_j} \partial_x p_j(n_j) = -\frac{q_j}{m_j} E, \\ &\partial_x E = n_1 - n_2, \end{aligned} \tag{4}$$

for $j = 1, 2$, where $x \in \mathbb{T}^d$ (periodic d-dimensional torus), E is the electric field, $n_j(t, x)$ are the densities of ions and electrons respectively, $u_j(t, x)$ are their velocities, m_i are their masses, $q_i = +1$, $q_e = -1$, and the partial pressure p_j are given functions as in system (3). We also assume neutral condition (4) as:

$$\int_{\mathbb{T}^d} (n_1(t, x) - n_2(t, x)) dx = 0. \tag{5}$$

For notational simplicity, we use $[\cdot, \cdot]$ to denote the transposition of a vector. In the study of Cauchy problems of all three systems (1), (3) and (4) with periodic boundary condition, clearly a constant vector $[n_1^0, n_2^0, u_1^0, u_2^0]$ with $n_1^0, n_2^0 > 0$ is a steady state solution if $n_1^0 + n_2^0 = 1$ in the cases of (1) and (3), or a solution to (4) if $n_1^0 = n_2^0$. In contrast to the case when

ions and electrons move at the same speed $u_1^0 = u_2^0$ (see Appendix A), if u_1^0 and u_2^0 are different, this simple equilibrium is *not* always linearly stable. This is a very well-known physical phenomenon, called two-stream instability in plasma literature, see p116-p169 in [9]. We shall discuss the conditions for linear instability in section 3. The main goal of this article is to show that this kind of linear exponential instability indeed implies the dynamical instability. Let $w(t) = [n_1(t) - n_1^0, n_2(t) - n_2^0, u_1(t) - u_1^0, u_2(t) - u_2^0]$.

Theorem 1.1 *In (1), (3) and (4), if the steady state* $[n_1^0, n_2^0, u_1^0, u_2^0]$ *is linearly unstable then it is nonlinearly unstable: for any* s *arbitrary large, there exists* $\epsilon_0 > 0$, *such that for any* $\delta > 0$ *arbitrary small, there exist solutions* $w^\delta(t,x)$ *to the corresponding evolution (1), (3) or (4), such that* $\|w^\delta(0,x)\|_{H^s(\mathbb{T}^d)} \leq \delta$ *but for some* $T^\delta = O(|\ln \delta|)$, *we have*

$$\|w^\delta(T^\delta, \cdot)\|_{L^1(\mathbb{T}^d)} \geq \epsilon_0. \tag{6}$$

and

$$\|w^\delta(T^\delta, \cdot)\|_{L^\infty(\mathbb{T}^d)} \geq \epsilon_0. \tag{7}$$

We remark that such instabilities occur before the possible break-down of the smooth solution (Theorem 2.1): that is, for all $\delta > 0$,

$$\sup_{0 \leq t \leq T^\delta} \|w^\delta(t, .)\|_{H^s(\mathbb{T}^d)} < +\infty.$$

Moreover, the escape time T^δ is determined by the exponential growth rate of the linearized system.

The passage from linear instability to nonlinear instability in PDE setting is subtle in general, especially when high-order perturbations in the full nonlinear equations consist of severe unbounded terms (usually with higher derivatives not controlled by linear estimates). In kinetic models for plasma, using a dominant linear growing mode, Strauss and the third author have developed methods to study weakly spatially inhomogeneous equilibria, [5], [6] (see Appendix B). In a different study of instabilities in perfect fluids [4], the second author has developed another approach in which higher-order growing modes are constructed. This method seems more general, particular for investigations of spatially inhomogeneous equilibria.

The abstract frame work is given in section 2. Notice that the key assumption enables us to prove the nonlinear instability is (A2), which is an estimate

of the spectral radius of the linearized operator in terms of its eigenvalues. In section 4, we verify (A2) in different Euler-Poisson systems to apply the abstract instability result. For the Euler-Poisson system (3) and (4) in presence of pressures, (A2) follows easily. On the other hand, the proof of (A2) for the pressure-less system (1) is more delicate. Because of the difference of the two speeds u_1^0 and u_2^0, certain compactness for solutions to the linearized system is obtained. An alternative proof for instability in the presence of pressures ((3) and (4)) is given in Appendix B.

2 Abstract Instability

Based on the paper of [4] we first establish the passage from the linear instability to nonlinear instability in a general L^2 framework.

Let $0 \in \Omega \subset \mathbb{R}^n$ be an open subset. For $\sigma > 0$ small enough, Ω contains a ball of center 0 with radius σ. Consider the system of equations

$$\partial_t w + \sum_{i=1}^{d} A_i(w)\partial_i w + L(w) = F(w) \tag{8}$$

where $w(t,x) \in \mathbb{R}^n$ for some $n \geq 1$, $x = (x_1, ..., x_d) \in \mathbb{T}^d$, $A_i(w)$ are $n \times n$ real matrices, $L(w)$ and $F(w)$ are n dimensional vector valued functions defined on Ω, depending in a C^∞ manner on w. We assume that L is linear with respect to w and the nonlinear part F satisfies $F(0) = 0$ and $F'(0) = 0$ (in applications, F is quadratic in w).

The main assumptions are

(A1) A_i are symmetrizable matrices : there exists a $n \times n$ positive definite matrix $S(w) \geq \alpha Id$ with $\alpha > 0$, for all $|w| \leq \sigma$, such that for every $1 \leq i \leq d$, SA_i is a symmetric matrix.

(A2) There exists a C^∞ eigenvector r of $-\sum_i A_i(0)\partial_i - L$ such that

$$\|e^{[-\sum_i A_i(0)\partial_i - L]t} v(\cdot)\|_{H^s} \leq C_s(\Lambda)\exp(\Lambda t)\|v(\cdot)\|_{H^s}, \tag{9}$$

for every $v \in H^s$, where $0 < \mathrm{Re}\lambda < \Lambda < 2\mathrm{Re}\lambda$, and λ is the eigenvalue of r.

The main theorem in this section is the dynamical instability of the zero solution to (8).

Theorem 2.1 *Assume (A1) and (A2). Then the stationary solution to (8) $w(t) \equiv 0$ is nonlinearly unstable: for any s arbitrary large, there exist $\epsilon_0 > 0$, such that for any $\delta > 0$ arbitrary small, there exists a solution $w^\delta(t)$ of (8) such that $\|w^\delta(0,.)\|_{H^s} \leq \delta$ but*

$$\sup_{0\leq t\leq T^\delta} \|w^\delta(t,\cdot)\|_{H^s(\mathbb{T}^d)} + \sup_{0\leq t\leq T^\delta} \|w^\delta(t,.)\|_{L^\infty(\mathbb{T}^d)} \leq \sigma/2; \tag{10}$$

$$\|w^\delta(T^\delta,\cdot)\|_{L^p(\mathbb{T}^d)} \geq \epsilon_0 > 0, \tag{11}$$

where $T^\delta = O(|\ln \delta|)$, for any $1 \leq p \leq \infty$.

The proof of this Theorem relies on two ingredients. First, we estimate the difference between the true solution and an approximate solution by basic energy estimates. Then, we construct an approximate solution with high-order accuracy. Notice that this Theorem can be extended to the whole space case (see [4] for more details).

We first estimate the error $v = w - \phi$, where ϕ is an approximate solution.

Lemma 2.2 *Assume (A1). Let $s > [d/2] + 1$, and let $\phi(t,x), R(t,x) \in L^\infty_{loc}(H^s)$. There exists a continuous function $g_s(\cdot,\cdot)$ such that if v satisfies*

$$\partial_t v + \sum_{i=1}^d A_i(\phi+v)\partial_i v + \sum_{i=1}^d [A_i(\phi+v) - A_i(\phi)]\partial_i\phi + L(v) = F(\phi+v) - F(\phi) + R, \tag{12}$$

then we have

$$\partial_t |||v|||_s^2 \leq g_s(|||\phi|||_s, |||v|||_s)|||v|||_s^2 + |||R|||_s^2, \tag{13}$$

where the norm $|||\cdot|||_s$ is defined by (with S in (A1))

$$|||v|||_s^2 = \sum_{|\alpha|\leq s} \int \partial_x^\alpha v S(\phi+v)\partial_x^\alpha v. \tag{14}$$

Proof of Lemma 2.2. The proof is straightforward by classical energy methods as in [8], [3]. Notice that since $S \geq \alpha Id$,

$$\alpha\|v\|_{H^s} \leq |||v|||_s \leq C_s'(\|\phi\|_{H^s}, \|v\|_{H^s})\|v\|_{H^s} \tag{15}$$

provided $s > [d/2] + 1$, for some nondecreasing function C_s'. *QED*

We now construct an approximate solution. Let θ be a small positive constant (independent of δ). We define T^δ by

$$\theta = \delta e^{\mathrm{Re}\lambda T^\delta}. \tag{16}$$

Lemma 2.3 *Assume (A1) and (A2) and fix an integer $N > 0$. There is an approximate solution $w^a = \sum_{j=1}^N \delta^j r_j$ to (8) such that*

$$\partial_t w^a + \sum_{i=1}^d A_i(w^a)\partial_i w^a + L(w^a) = R_N^a + F(w^a). \tag{17}$$

Moreover, for every integer $s \geq 0$, there is θ sufficiently small, such that if $0 \leq t \leq T^\delta$ as in (16), r_j and R_N^a satisfy

$$\|r_j\|_{H^s} \leq C_{s,N} \exp(j\mathrm{Re}\ \lambda t), \text{ for } 1 \leq j \leq N, \tag{18}$$

$$\|R_N^a\|_{H^s} \leq C_{s,N}\delta^{N+1} \exp\Big((N+1)\mathrm{Re}\lambda t\Big). \tag{19}$$

Proof of Lemma 2.3: We shall construct r_j satisfying (18) inductively on j.

For $j = 1$, choose the eigenvector r in (A2) with its eigenvalue λ. We construct

$$r_1(t,x) = r\exp(\lambda t) + \bar{r}\exp(\bar{\lambda}t), \tag{20}$$

where $\bar{\cdot}$ denotes the complex conjugate. Clearly r_1 satisfies (18).

Assume that we have constructed r_j which satisfies (18) and (17) for $j < N$. We now construct r_{j+1}. Let

$$w_j = \sum_{k=1}^j \delta^k r_k(t,x). \tag{21}$$

We then define

$$h_{j+1}(\delta) = \sum_{i=1}^d [A_i(w_j) - A_i(0)]\partial_i w_j - F(w_j). \tag{22}$$

For $0 \leq t \leq T^\delta$ and with θ small, we can expand $h_{j+1}(\delta)$ in term of δ around $\delta = 0$. The coefficient of the $(j+1)$-th order term (which is a function of (t,x) still) is $\frac{h_{j+1}^{(j+1)}(0)}{(j+1)!}$. On the other hand, notice that for $0 \leq t \leq T^\delta$,

$$A_i(w_j) - A_i(0) = \sum_{l=1}^N \frac{A_i^l(0)}{l!} w_j^l + O\left(\frac{|A_i^{N+1}|_{L^\infty(\Omega)} w_j^{N+1}}{(N+1)!}\right), \tag{23}$$

$$F(w_j) = \sum_{l=2}^N \frac{F^l(0)}{l!} w_j^l + O\left(\frac{|F^{N+1}|_{L^\infty(\Omega)} w_j^{N+1}}{(N+1)!}\right). \tag{24}$$

Plugging (23) and (24) into (22), we obtain

$$\frac{h_{j+1}^{(j+1)}(0)}{(j+1)!} = \sum_{l_1+l_2+..+l_{N+1}=j+1} B_{j_1,j_2,...j_{N+1},i}^{l_1,l_2,...,l_{N+1}} r_{j_1}^{l_1} r_{j_2}^{l_2} \cdot\cdot r_{j_N}^{l_N} \partial_i r_{j_{N+1}}^{l_{N+1}} \tag{25}$$

where $l_k \geq 0, 1 \leq k \leq j$ and $B_{j_1,j_2,...j_{N+1},i}^{l_1,l_2,...,l_{N+1}}$ depends on A_i and F. By induction hypothesis (18) for r_k, $1 \leq k \leq j$, we obtain

$$\left\| \frac{h_{j+1}^{(j+1)}(0)}{(j+1)!} \right\|_{H^s} \leq C_{s,N} e^{(l_1+l_2+..+l_{N+1})\mathrm{Re}\lambda\, t} = C_{s,N} e^{(j+1)\mathrm{Re}\lambda\, t} \tag{26}$$

We now define the r_{j+1} as the solution of

$$\partial_t r_{j+1} + \sum_{i=1}^{d} A_i(0)\partial_i r_{j+1} + L(r_{j+1}) = -\frac{h_{j+1}^{(j+1)}(0)}{(j+1)!}$$

with initial data $r_{j+1}(0,x) = 0$. By (9), (26) and Duhamel principle,

$$\begin{aligned} \|r_{j+1}(t,.)\|_{H^s} &\leq C \int_0^t e^{\Lambda(t-\tau)} \left\| \frac{h_j^{(j+1)}(0)}{(j+1)!}(\tau) \right\|_s d\tau \leq C_N \int_0^t e^{\Lambda(t-\tau)} e^{(j+1)\mathrm{Re}\lambda\tau} d\tau \\ &\leq C_N e^{(j+1)\mathrm{Re}\lambda t} \end{aligned}$$

since $j+1 \geq 2$, and $\Lambda < 2\mathrm{Re}\lambda$. Hence (18) follows.

Having constructed all r_j, for $1 \leq j \leq N$, we now define $w^a = \sum_{j=1}^{N} \delta^j r_j$. Clearly

$$\partial_t w^a + \sum_{i=1}^{d} A_i(0)\partial_i w^a + L(w^a) = -\sum_{j=1}^{N} \frac{\delta^{j+1} h_j^{(j+1)}(0)}{(j+1)!}.$$

Let w_j be replaced by w^a in (22), we define

$$h(\delta) = \sum_{i=1}^{d} [A_i(w^a) - A_i(0)]\partial_i w^a - F(w^a).$$

We define

$$R_N^a = -\sum_{j=1}^{N} \frac{\delta^{j+1} h_{j+1}^{(j+1)}(0)}{(j+1)!} + h(\delta). \tag{27}$$

Replace w_j by w^a in expansions of (23) and (24). Notice that $h(\delta)$ is quadratic in w^a, its $j+1$-th order term is the same as in $h_{j+1}(\delta)$, for all $1 \leq j+1 \leq N$.

Therefore, R_N^a consists of only those terms of orders at least of $N+1$. From the argument in (25), we deduce that R_N^a satisfy (19). *QED*

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1: Based on the approximate solution w^a in Lemma 2.3, for which N will be chosen later, we now construct a family of solutions w^δ which are unstable. We define w^δ to be the solution of (8) with initial data $w^a(0)$. We know that w^δ exists in small time since the matrices A_i are symmetrizable. We want to bound $w^\delta - w^a$ and estimate the existence time interval for w^δ. Let $v = w^\delta - w^a$ which satisfies $v(0,\cdot) = 0$ and

$$\partial_t v + \sum_{i=1}^d A_i(w^a+v)\partial_i v + \sum_{i=1}^d \Big[A_i(w^a+v) - A_i(w^a)\Big]\partial_i w^a + L(v) = \qquad (28)$$

$$-R_N^a + F(w^\delta) - F(w^a).$$

Using Lemma 2.2, with $s = 2d$,

$$\begin{aligned} \partial_t |||v|||_{2d}^2 &\le g_{2d}(|||w^a|||_{2d}, |||v|||_{2d})|||v|||_{2d}^2 + |||R_N^a|||_{2d}^2 \\ &\le g_{2d}(|||w^a|||_{2d}, |||v|||_{2d})|||v|||_{2d}^2 + C\delta^{2(N+1)}\exp\Big(2(N+1)\mathrm{Re}\lambda t\Big). \end{aligned}$$

Let T (depending on δ) be the first time t (possibly infinite) such that either $|||w^a|||_{2d} = \sigma/2$ or $|||v|||_{2d} = \sigma/2$:

$$T = \sup\{\, t:\ |||w^a(t)|||_{2d} \le \sigma/2, |||v(t)|||_{2d} \le \sigma/2.\}$$

As $v(0) = 0$ and as $|||w^a(0)|||_s = O(\delta)$, T is well-defined for δ small enough.

Recall (16). We claim that *for some θ small enough, $T^\delta \le T$.*

Proof of the claim: Suppose the opposite $T^\delta > T$. For $t \le T$, we have, using (15),

$$|||w^a|||_{2d} \le C\sum_{j=1}^N \delta^j \|r_j\|_{H^{2d}} \le \sum_{j=1}^N C_j\delta^j \exp(j\mathrm{Re}\lambda t) \le \sum_{j=1}^N C_j\theta^j \le \sigma/4$$

provided θ is small.

On the other hand, for $t \le T$, with $s = 2d$,

$$\partial_t |||v|||_{2d}^2 \le g_{2d}(\sigma/2,\sigma/2)|||v|||_{2d}^2 + C\delta^{2(N+1)}\exp\Big(2(N+1)\mathrm{Re}\lambda t\Big).$$

Using Gronwall inequality and choosing N such that

$$N > \frac{g_{2d}(\sigma/2, \sigma/2)}{2\mathrm{Re}\lambda} - 1, \tag{29}$$

and g_{2d} as defined in (13), this insures $2(N+1)\mathrm{Re}\lambda > g_{2d}(\sigma/2, \sigma/2)$ and we get

$$|||v|||_{2d} \leq C'\delta^{N+1} \exp\Big((N+1)\mathrm{Re}\lambda t\Big) = C'\theta^{N+1}. \tag{30}$$

But $\|v\|_{H^{2d}} \leq \alpha^{-1}|||v|||_{2d}$ and $\|v\|_{H^{2d}}$ controls $\|v\|_{L^\infty}$. Therefore $\|v\|_{H^{2d}} < \sigma/2$ provided θ is small enough.

Hence $T^\delta \leq T$ by their definitions. The claim is proved.

Notice for $0 \leq t = T^\delta$, by further choosing θ small enough, we have

$$\begin{aligned}
\|w^a\|_{L^1} &\geq \delta\|w_1\|_{L^1} - \sum_{j=2}^{N} \delta^j \|w_j\|_{L^1} \\
&\geq C\delta \exp(\mathrm{Re}\lambda T^\delta) - \sum_{j=2}^{N} C_j \delta^j \exp(j\mathrm{Re}\lambda T^\delta) \\
&= C\theta - \sum_{j=2}^{N} C_j \theta^j \geq \frac{C}{2}\theta.
\end{aligned}$$

Moreover, from (30), we conclude

$$\begin{aligned}
\|w^\delta\|_{L^1} &\geq \|w^a\|_{L^1} - \|w^\delta - w^a\|_{L^1} \geq \|w^a\|_{L^1} - C|||w^\delta - w^a|||_{2d} \\
&\geq \frac{C}{2}\theta - C\theta^{N+1} \geq \frac{C}{4}\theta \equiv \epsilon_0 > 0.
\end{aligned}$$

3 Dispersion relations and growing modes

In this section, we demonstrate that many equilibrium state $[n_j^0, u_j^0], j = 1, 2$ with $u_1^0 \neq u_2^0$ for (1), (3) and (4). Without loss of generality, we assume $u_1^0 < u_2^0$. The following computations are classical in plasma physics and can be found for instance in [7]. Let us first consider the 2 cold electrons beams without pressure, the Euler-Poisson system (1).

The mass conservation for the perturbation density $\rho_j = \bar{\rho}_j \exp(ikx - i\omega t)$ and velocity $u_j = \bar{u}_j \exp(ikx - i\omega t)$ gives

$$-\omega\bar{\rho}_j + k(n_j^0\bar{u}_j + \bar{\rho}_j u_j^0) = 0, \quad j = 1, 2,$$

The momentum equation for such cold electron beams leads to

$$-i\omega\bar{u}_j + iku_j^0\bar{u}_j = -\frac{e}{m_e}\bar{E},$$

where $\bar{E}$ is the perturbation of the electric field which is related to the perturbation of densities through the Poisson equation

$$ik\bar{E} = -4\pi e(\bar{\rho}_1 + \bar{\rho}_2).$$

Eliminating the densities, we get the dispersion relation:

$$ik\bar{E}\left(1 - \frac{\omega_{p1}^2}{(\omega - ku_1^0)^2} - \frac{\omega_{p2}^2}{(\omega - ku_2^0)^2}\right) = 0 \tag{31}$$

where $\omega_{pj}^2 = 4\pi n_j^0 e^2/m_e$ is the squared plasma frequency of the i-th beam.

We want to compute the roots of (31) in the variable $y = \omega/k$, where the wave number $k = 2\pi l$, with an integer l. In other words, for a given equilibrium state $[n_j^0, u_j^0], j = 1, 2$, the dispersion relation

$$k^2 = \frac{\omega_{p1}^2}{(y - u_1^0)^2} + \frac{\omega_{p2}^2}{(y - u_2^0)^2} \equiv g(y)$$

determines the possible values of y in terms of the wave number k. There are always at least two real solutions (out of the the interval bounded by $[u_1^0, u_2^0]$). The instability with wave number k relies on the existence of complex roots. Notice that the minimal value of $g(y)$ on the interval $[u_1^0, u_2^0]$ is reached for the unique root (in the interval $[u_1^0, u_2^0]$) of the third order polynomial (in variable y)

$$w_{p1}^2(y - u_2^0)^3 + w_{p2}^2(y - u_1^0)^3.$$

A simple solution can be found in the special case

$$n_1^0 = n_2^0, \quad u_1^0 = -u_2^0.$$

In this case, the plasma frequencies are equal $\omega_{p1} = \omega_{p2} = \omega_p$ and the minimal value of $g(y)$ is $2\omega_p^2/|u_1^0|^2$, obtained at $y = (u_1^0 + u_2^0)/2 = 0$. Since the spatial variable satisfies $0 \leq x \leq 1$, the wave number k has to be greater than 2π. Finally, this proves that the linearized system is unstable (with complex y) if $\omega_p > \sqrt{2}\pi u_1^0$. From the physical point of view, this means that the plasma

frequency is greater than the typical frequency ($1/u_0^1$ being the time for the electrons beams to cross the domain).

In the general case, the minimum in $[u_1^0, u_2^0]$ (for $y = \omega/k$) is obtained for

$$y = \frac{u_2^0 \omega_{p1}^{2/3} + u_1^0 \omega_{p2}^{2/3}}{\omega_{p1}^{2/3} + \omega_{p2}^{2/3}},$$

and there exists complex root y provided that

$$\frac{u_2^0 - u_1^0}{(\omega_{p1}^{2/3} + \omega_{p2}^{2/3})^{3/2}} > 2\pi. \tag{32}$$

We have proved

Lemma 3.1 *The system (1) is linearly unstable around* $[n_1^0, u_1^0, n_2^0, u_2^0]$ *if and only if (32) holds.*

Let us now consider the case (3) with pressure p.

The momentum and mass linearized equations in this case are modified as

$$-i\left(\frac{(\omega - ku_j^0)^2}{k} - kc_j^2\right)\rho_j = -\frac{en_j^0}{m_e}E,$$

where $c_j^2 = p_j'(n_j^0)/n_j^0$ is the squared thermal velocity associated to the j-th beam of electrons, $j = 1, 2$. The thermal velocities measure the velocity dispersion of the electrons. The two-stream instability occurs for cold plasma for which $c_j \ll \|u_2^0 - u_1^0\|$. Notice that, for neutral particles, there is no electric field and the above relation gives the velocities of the fluid acoustic waves

$$\omega/k = u_j^0 \pm c_j.$$

Then, using again the Poisson equation, we obtain the following dispersion relation :

$$1 = \frac{\omega_{p1}^2}{(\omega - ku_1^0)^2 - c_1^2 k^2} + \frac{\omega_{p2}^2}{(\omega - ku_2^0)^2 - c_2^2 k^2}.$$

Let us assume k fixed. The r.h.s. is a function of ω with 4 real poles $ku_j^0 \pm c_j k$ $(j = 1, 2)$. When the two poles associated with different species are entrelaced, there are four real solutions in ω (for example if $u_2^0 - c_2 \in [u_1^0 - c_1, u_1^0 + c_1]$). When the relative speed is so large that the roots are not interlaced, some roots become complex. In the physically relevant case

i.e. for cold plasma, the relative speed is greater than the thermal velocities. Using classical result on the perturbation of the roots of polynomials, we obtain:

Lemma 3.2 *The system (3) is linearly unstable (resp. stable) around* $[n_1^0, u_1^0, n_2^0, u_2^0]$ *for sufficiently small value of* $c_j, j = 1, 2$ *if (32) is true (resp. false).*

A more precise criterion may be found in some particular cases. Note that the above analysis could be extended to the case of electrons and ions drifting with respect to each other, which includes the linearized Euler-Poisson system (4) with $d = 1$. See for instance p166 of [9]. Another way to get such dispersion relation for (4) relies on the Vlasov theory. Assume a plasma with an equilibrium distribution of cold electrons $f_e(v) = \delta(v)$ and ions with a drifting speed u_1^0 i.e. $f_i(v)\delta(v-u_1^0)$. This equilibrium state is electrostatically perturbed with an oscillation of frequency ω. The dispersion relation reads

$$1 = \frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{pi}^2}{(\omega - ku_1^0)^2}.$$

For any fixed value of k the two-stream instability condition

$$\|ku_1^0\| < \omega_{pe}\left\{1 + (\frac{\omega_{pi}}{\omega_{pe}})^{2/3}\right\}^{3/2}$$

insures the existence of complex roots. The unstable wave has velocity somewhere in between the velocity of the two streams. Since the sign of the charge does not appear in the above dispersion relation, the same analysis can be carried out. We refer to [7] for a physical presentation of this phenonenon and the related Penrose and Gardner criteria.

These results have also been confirmed by computer simulation experiments (by computing the orbits of all charged particles in a plasma using the actual forces between particles) [1]. This two-stream growth mechanism was demonstrated experimentally in 1949 by Pierce and Heibenstreit in [10, 7].

4 Two-stream instability

We now prove Theorem 1.1 by applying the abstract Theorem 2.1. We first normalize all physical constants to be for notational simplicity.

Proof of Theorem 1.1 for (3) and (4): It suffices to just consider (4), the case of (3) is similar.

The Euler-Poisson system (4) takes the form:

$$\partial_t w + A(w)\partial_x w + L(w) = 0, \tag{33}$$

with

$$w = [n_1 - n_1^0, n_2 - n_2^0, u_1 - u_1^0, u_2 - u_2^0].$$

Here

$$A(w) = \begin{pmatrix} u_i^0 + w_3 & 0 & n_i^0 + w_1 & 0 \\ 0 & u_e^0 + w_4 & 0 & n_e^0 + w_2 \\ p_i'(n_i^0 + w_1)/n_i & 0 & u_i^0 + w_3 & 0 \\ 0 & p_e'(n_e^0 + w_2)/n_e & 0 & u_e^0 + w_4 \end{pmatrix} \tag{34}$$

and

$$L(v) = \begin{pmatrix} 0 \\ 0 \\ m_i^{-1}E \\ -m_e^{-1}E \end{pmatrix}. \tag{35}$$

Let us check assumptions (A1), (A2) and (A3) of Theorem 2.1. We take

$$\Omega = \Big\{ w \big| |w_1| < \frac{n_i^0}{2}, |w_2| < \frac{n_e^0}{2} \Big\}.$$

Let

$$S = \begin{pmatrix} p_i'(n_i^0 + w_1)/n_i & 0 & 0 & 0 \\ 0 & p_e'(n_e^0 + w_2)/n_e & 0 & 0 \\ 0 & 0 & n_i^0 + w_1 & 0 \\ 0 & 0 & 0 & n_e^0 + w_2 \end{pmatrix}. \tag{36}$$

S is positive definite and SA is a symmetric matrix, therefore (A1) is true. Since L is compact from L^2 to L^2, (A2) is straightforward by Lemma 4.1 with $K = L$ and $T = A(0)\partial_x$. As $A(0)$ and L have constant coefficients, taking Fourier series, we see that for a given eigenvalues λ we can take an associate eigenvector which is a plane wave, and which is therefore smooth. By the neutral condition (2), L commute with spatial derivatives. This proves (A2). Theorem 1.1 is then a consequence of Theorem 2.1.

Proof of Theorem 1.1 for (1): In order to use Theorem 1.1, we first take x derivative in the momentum equation to rewrite (1) as:

$$\begin{aligned}
&\partial_t n_i + u_i \partial_x n_i = -n_i \partial_x u_i, \\
&\partial_t u_i + E = -u_i \partial_x u_i \\
&\partial_t(\partial_x u_i) + u_i \partial_x(\partial_x u_i) + \partial_x E = -\partial_x u_i \partial_x u_i,
\end{aligned}$$

where $i = 1, 2$, and the electric field is still given by the Poisson equation. By letting

$$w = [n_1 - n_1^0, n_2 - n_2^0, u_1 - u_1^0, u_2 - u_2^0, \partial_x u_1, \partial_x u_2],$$

we reformulate (1) as

$$\partial_t w + A(w)\partial_x w + L(w) = F(w) \tag{37}$$

where $A(w) = \mathrm{diag}(u_1, u_2, 0, 0, u_1, u_2)$ with $u_1 = w_3 + u_1^0$,$u_2 = w_4 + u_2^0$, and

$$\begin{aligned}
L(w) &= [n_1^0 \partial_x u_1, n_2^0 \partial_x u_2, E + u_1^0 \partial_x u_1, E + u_2^0 \partial_x u_2, \partial_x E, \partial_x E] \\
&= [n_1^0 w_5, n_2^0 w_6, E + u_1^0 w_5, E + u_2^0 w_6, -w_1 - w_2, -w_1 - w_2].
\end{aligned}$$

We also have

$$\begin{aligned}
F(w) = \Big[&-(n_1 - n_1^0)\partial_x u_1, -(n_2 - n_2^0)\partial_x u_2, -(u_1 - u_1^0)\partial_x u_1, -(u_2 - u_2^0)\partial_x u_2, \\
&-(\partial_x u_1)^2, -(\partial_x u_2)^2\Big] \\
= [&-w_1 w_5, -w_2 w_6, -w_3 w_5, -w_4 w_6, -w_5^2, -w_6^2].
\end{aligned}$$

Clearly, (A1) is satisfied.

We now check (A2). In order to verify the estimate for the spectral radius in 2.1, we now state a lemma which was essentially proven by Vidav [11].

Lemma 4.1 *Let Y be a Banach space and T be a linear operator that generates a strongly continuous semigroup on Y such that $\|\exp(-tT)\| \leq M$. Consider*

$$\begin{aligned}
&\frac{dv}{dt} + Tv + Kv = 0, \\
&v(0) = v_0
\end{aligned}$$

where K is a bounded operator from Y to Y and $e^{-t[T+K]} - e^{-tT}$ is compact from Y to Y for every t. Then, $(T+K)$ generates a strongly continuous semigroup $\exp(-t(T+K))$ and the spectrum of $(-T-K)$ consists of a finite number of eigenvalues of finite multiplicities in $\{\mathrm{Re}\lambda > \delta\}$ for all $\delta > 0$. These eigenvalues can be labeled by

$$\mathrm{Re}\lambda \geq \mathrm{Re}\lambda_1 \geq ... \geq \mathrm{Re}\lambda_n \geq \delta.$$

Furthermore, for every $\Lambda > \mathrm{Re}\lambda$, there is a constant C_Λ such that

$$\|\exp(-t(T+K))\|_{L(Y,Y)} \leq C_\Lambda \exp(\Lambda t).$$

We now apply Lemma 4.1 to the pressure-less case (37). We shall study its linear operator near $w = 0$. Notice that the components equations for w_3 and w_4 in (37) are decoupled from other unknowns. We therefore only need to study the following reduced linear system to verify (A2). We decompose the reduced (ignoring w_3, w_4) linearized operator as:

$$\frac{dv}{dt} + Tv + Kv = 0 \tag{38}$$

where $v = (w_1, w_2, w_5, w_6)$, acting on $Y = L^2$, where

$$Tv = T_1 v + T_2 v \equiv \begin{pmatrix} u_1^0 & 0 & 0 & \\ 0 & u_2^0 & 0 & 0 \\ 0 & 0 & u_1^0 & 0 \\ 0 & 0 & 0 & u_2^0 \end{pmatrix} \partial_x v + \begin{pmatrix} 0 & 0 & n_1^0 & 0 \\ 0 & 0 & 0 & n_2^0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} v \tag{39}$$

and

$$Kv = \begin{pmatrix} 0 \\ 0 \\ -v_2 \\ -v_1 \end{pmatrix}.$$

Notice that for the unperturbed semigroup e^{-Tt}, we have the following conservation law:

$$\|v_1(t)\|_{L^2}^2 + \|v_2(t)\|_{L^2}^2 + n_1^0\|v_3(t)\|_{L^2}^2 + n_2^0\|v_4(t)\|_{L^2}^2 \equiv \text{constant}.$$

Hence we deduce that

$$\|e^{-Tt}v_0\|_{L^2} \leq M\|v_0\|_{L^2}.$$

In order to get (A2), by lemma 4.1, it suffices to show that for every t, $\{e^{-t[T+K]} - e^{tT}\}$ is compact in H^s. We first show this is true for $s = 0$. For the case $s > 0$ we just repeat the same argument by taking more spatial derivatives.

From (38), we can express $e^{-[T+K]t}v_0$ in term of $e^{-Tt}v_0$ plus the source term $-Kv$ as:

$$\frac{dv}{dt} + Tv = -Kv.$$

Therefore, from Duhamel's principle,

$$v(t) = e^{-[T+K]t}v_0 = e^{-Tt}v_0 - \int_0^t e^{-T(t-\tau)}(Kv)(\tau)d\tau,$$

where $v(t)$ is the solution to (38). Notice that by (39), $e^{-T_1 t}$ is a system of transport equations with constant speeds u_1^0 and u_2^0:

$$e^{-T_1 s}g(x) = [g_1(x - u_1^0 s), g_2(x - u_2^0 s), g_3(x - u_1^0 s), g_4(x - u_2^0 s)]$$

and

$$e^{-Ts}g(x) = e^{-T_1 s - T_2 s}g(x) = e^{-T_2 s}[g_1(x-u_1^0 s), g_2(x-u_2^0 s), g_3(x-u_1^0 s), g_4(x-u_2^0 s)]$$

for a four vector function g, since T_1 and T_2 commute. Hence,

$$\begin{aligned} [e^{-[T+K]t} - e^{-Tt}]v_0 &= -\int_0^t e^{-T(t-\tau)}(Kv)(\tau)d\tau \qquad (40) \\ &= \int_0^t e^{-T_2(t-\tau)} \begin{pmatrix} 0 \\ 0 \\ v_2(\tau, x - u_1^0(t-\tau)) \\ v_1(\tau, x - u_2^0(t-\tau)) \end{pmatrix} d\tau \end{aligned}$$

where $v(t) \in L^2$ is a solution to (38) and T_2 is the constant matrix defined in (39). The key observation is that the right-hand side of (40) indeed is compact for solution $v(t)$ to (38) due to different propagation speeds. We define $\partial_i = \partial_t + u_i^0 \partial_x$ for $i = 1, 2$. We represent

$$\partial_x = \frac{1}{u_1^0 - u_2^0}[\partial_1 - \partial_2]. \qquad (41)$$

Using (41), we take x derivative in (40) to get

$$\partial_x[e^{-[T+K]t} - e^{-Tt}]v_0 = \frac{1}{u_1^0 - u_2^0}\int_0^t e^{-T_2(t-\tau)} \begin{pmatrix} 0 \\ 0 \\ [\partial_1 - \partial_2]v_2(\tau, x - u_1^0(t-\tau)) \\ [\partial_1 - \partial_2]v_1(\tau, x - u_2^0(t-\tau)) \end{pmatrix} d\tau$$

Since $v(t) \in L^2$ is a solution to the equation (38), we know that both $\partial_2 v_2$ and $\partial_1 v_1$ are in L^2. On the other hand, notice that $\partial_1 v_2(\tau, x - u_1^0(t-\tau))$ and $\partial_2 v_1(\tau, x - u_2^0(t-\tau))$ are exactly derivatives of v over two different characteristics. We now integrate over τ to get (v_1 being the same):

$$\begin{aligned}
&\int_0^t e^{-T_2(t-\tau)} \partial_1 v_2(\tau, x - u_1^0(t-\tau)) d\tau \\
&= \int_0^t e^{-T_2(t-\tau)} \frac{d}{d\tau} v_2(\tau, x - u_1^0(t-\tau)) d\tau \\
&= v_2(t,x) - e^{-T_2 t} v_2(0, x - u_1^0 t) - \int_0^t e^{-T_2(t-\tau)} v_2(\tau, x - u_1^0(t-\tau)) d\tau
\end{aligned}$$

which again is in L^2. Therefore, we verify that $e^{-[T+K]t} - e^{-Tt}$ is compact and (A2) is valid from Lemma 4.1. *QED*

5 Appendix A: Stability of One-fluid Plasma

We show that if $u_i^0 = u_e^0$, i.e. if the ions and the electrons have the same speed, then w^0 is dynamically stable. The proof is classical and relies on the construction of a Lyapounov functional, following the 'energy-Casimir' method. We first notice, using Galilean invariance, that we can reduce to the case $u_i^0 = u_e^0 = 0$. We then define the energy as

$$\mathcal{E}(w(t)) = \sum_{\alpha=i,e} \int_0^1 \{\frac{1}{2} m_\alpha n_\alpha u_\alpha^2 + P_\alpha(n_\alpha)\} + \frac{1}{2} \int_0^1 |E|^2 \tag{42}$$

Here P_α is a strictly convex function (typically, $P_\alpha = \frac{C_\alpha}{\gamma - 1} n_\alpha^\gamma$ if $\gamma > 1$, and $P_\alpha(n_\alpha) = C_\alpha n_\alpha \ln n_\alpha$ for $\gamma = 1$).

We then observe that

$$\mathcal{H}(w(t)) = \mathcal{E}(w(t)) - \sum_{\alpha=i,e} \int_0^1 \{P_\alpha(n_\alpha^0) + P_\alpha'(n_\alpha^0)(n_\alpha - n_\alpha^0)\}$$

is constant in time, positive, convex near w^0 and that $\mathcal{H}(w^0) = 0$. We therefore have stability (in a norm linked with $\mathcal{H}$).

6 Appendix B: Instabilities for (3) and (4)

In the section, we give a brief sketch of another proof of Theorem for $p \geq 2$ for the Euler-Poisson system (3) and (4) with pressures by using only the

linear growing mode r_1 in (20), Lemma 2.3. As in Lemma 2.3, we choose $w^\delta(0,x) = \delta w_1(0,x)$ as in (20). We define

$$T = \sup\{t : \|w^\delta - \delta w_1(t,x)\|_2 \le \|r\|_2 \delta e^{\mathrm{Re}\lambda t}/2\}.$$

We recall T^δ as defined in (16) with θ a small constant, and recall that the linear operator L as in (35) is *compact* operator from L^2 to H^1.

We now estimate the growth of $\|w^\delta(t)\|_{H^{2d}}$ in terms of $\|w^\delta(t)\|_2$ on $0 \le t \le \min[T, T^\delta]$. Notice that from the compactness of L, for any $\epsilon > 0$,

$$|||L(w^\delta)|||_{2d} \le \epsilon |||w^\delta|||_{2d} + C_\epsilon \|w^\delta\|_2.$$

By the standard energy estimate (with $\phi \equiv 0$ as in Lemma 2.2), we obtain

$$\frac{d}{dt}|||w^\delta(t)|||_{2d} \le g(|||w^\delta(t)|||_{2d})|||w^\delta(t)|||_{2d} + C\|w^\delta(t)\|_2,$$

where g is continuous and $g(0) = \epsilon$. By definition of T, $\|w^\delta(t)\|_2 \le 3/2\|r\|_2 e^{\mathrm{Re}\lambda t}$ for $0 \le t \le T$. By a standard bootstrap argument, we have

$$\|w^\delta(t)\|_{H^{2d}} \le C_0 \delta e^{\mathrm{Re}\lambda t} \tag{43}$$

for $0 \le t \le \min[T, T^\delta]$ and with θ in (16) sufficiently small. We can further choose

$$\theta < \theta_0 = \frac{2\|r\|_2 \mathrm{Re}\lambda}{C_0 \Lambda}.$$

In particular, $w^\delta(t)$ is a classical solution for $0 \le t \le \min[T, T^\delta]$. By the Duhamel's principle, we have

$$w^\delta(t) = \delta w_1 + \int_0^t e^{\mathcal{L}(t-\tau)} A(w^\delta(\tau)) \partial_j w^\delta(\tau) d\tau.$$

Here $\mathcal{L} = -A - L$. By Lemma 4.1 and (43), for $0 \le t \le \min[T, T^\delta]$, we have

$$\begin{aligned}
\|w^\delta(t) - \delta w_1\|_2 &\le \int_0^t e^{\Lambda(t-\tau)} \|A(w^\delta(\tau)) \partial_j w^\delta(\tau)\|_{2d} d\tau \\
&\le \int_0^t e^{\Lambda(t-\tau)} \|A(w^\delta(\tau))\|_{2d} \|w^\delta(\tau)\|_{2d} d\tau \\
&\le \frac{C_0}{2\mathrm{Re}\lambda - \Lambda} \left(\delta e^{\mathrm{Re}\lambda t}\right)^2 = \frac{C_0 \theta}{2\mathrm{Re}\lambda - \Lambda} \delta^2 e^{2\mathrm{Re}\lambda t} < \frac{\|r\|_2}{2} \delta e^{\mathrm{Re}\lambda t}
\end{aligned}$$

since $\theta < \theta_0$. Hence $T^\delta \leq T$ and we deduce the theorem by:

$$\begin{aligned} \sup_{0\leq t\leq T^\delta} \|w^\delta(t)\|_{2d} &\leq C_0\delta e^{\mathrm{Re}\lambda t} \leq C_0\theta \equiv \sigma/2; \\ \|w^\delta(T^\delta)\|_{L^2} &\geq \delta\|w_1\|_{L^2} - \|w^\delta(t) - \delta w_1\|_{L^2} \\ &= \theta\|r\|_2 - \frac{\theta}{2}\|r\|_2 = \frac{\theta}{2}\|r\|_2 \equiv \epsilon_0. \end{aligned}$$

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