DECOMPOSITION THEOREMS FOR HARDY SPACES ON CONVEX DOMAINS OF FINITE TYPE

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SANDRINE GRELLIER AND MARCO M. PELOSO

ABSTRACT. In this paper we study the holomorphic Hardy spaces $\mathcal{H}^p(\Omega)$, where Ω is a convex domain of finite type in $\mathbb{C}^n$. We show that for $0 < p \leq 1$, the space $\mathcal{H}^p(\Omega)$ admits an atomic decomposition. More precisely, we prove that each $f \in \mathcal{H}^p(\Omega)$ can be written as $f = P_S(\sum_{j=0}^{\infty} \nu_j a_j) = \sum_{j=0}^{\infty} \nu_j P_S(a_j)$, where P_S is the Szegö projection and the a_j's are real variable p-atoms on the boundary $\partial\Omega$ and $\sum_{j=0}^{\infty} |\nu_j|^p \lesssim \|f\|^p_{\mathcal{H}^p(\Omega)}$.

Moreover, we prove the following factorization theorem. Each $f \in \mathcal{H}^p(\Omega)$ can be written as $f = \sum_{j=0}^{\infty} f_j g_j$, where $f_j \in \mathcal{H}^{2p}$, $g_j \in \mathcal{H}^{2p}$, and $\sum_{j=0}^{\infty} \|f_j\|_{\mathcal{H}^{2p}} \|g_j\|_{\mathcal{H}^{2p}} \lesssim \|f\|_{\mathcal{H}^p(\Omega)}$.

Finally, we extend these theorems to a class of domains of finite type that includes the strongly pseudoconvex domains and the convex domains of finite type.

INTRODUCTION

Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. For $0 < p \leq \infty$, let $L^p(\Omega)$ denote the Lebesgue space with respect to the volume form, $L^p(\partial\Omega)$ be the Lebesgue spaces on $\partial\Omega$ with respect to the induced surface measure $d\sigma$ and, for $0 < p \leq 1$, $H^p(\partial\Omega)$ be the real-variable Hardy spaces on $\partial\Omega$.

We let $\mathcal{H}^p(\Omega)$ denote the Hardy space of holomorphic functions on Ω, with norm given by

$$\|f\|^p_{\mathcal{H}^p(\Omega)} := \sup_{0<\varepsilon<\varepsilon_0} \int_{\delta(w)=\varepsilon} |f(w)|^p \, d\sigma_\varepsilon(w),$$

where $\delta(w)$ is the distance from w to $\partial\Omega$ and $d\sigma_\varepsilon$ denotes the surface measure on the manifold $\{\delta(w) = \varepsilon\}$. To any $f \in \mathcal{H}^p(\Omega)$ corresponds a unique boundary function in $L^p(\partial\Omega)$, that we still denote by f, obtained as normal almost everywhere limit, [St1]. Thus, we may identify $\mathcal{H}^p(\Omega)$ with a closed subspace of $L^p(\partial\Omega)$.

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The Hilbert space orthogonal projection P_S of $L^2(\partial\Omega)$ onto $\mathcal{H}^2(\Omega)$ is given by the Szegö projection

$$P_S f(z) = \int_{\partial\Omega} f(\zeta) S_\Omega(z,\zeta) d\sigma(\zeta),$$

where $S_\Omega(z,\zeta)$ is the Szegö kernel.

When Ω is a smoothly bounded convex domain of finite type, there exists a natural pseudo-distance d_b on $\partial\Omega$ (see [Mc]) which makes $\partial\Omega$ into a space of homogenous type. The real-variable Hardy space $H^p(\partial\Omega)$, $0 < p \leq 1$, is defined as a space of distributions on $\partial\Omega$, in terms of atoms, in the following sense. Each distribution $f \in H^p(\partial\Omega)$ can be written as $f = \sum_{j=0}^{\infty} \nu_j a_j$, where $\{\nu_j\} \in \ell^p$, the a_j's are p-atoms and the series is assumed to converge in the sense of distributions (see Section 1 for the precise definition).

In this paper we prove that, for $0 < p \leq 1$, the Hardy space $\mathcal{H}^p(\Omega)$ continuously embeds into $H^p(\partial\Omega)$. In other words, every $f \in \mathcal{H}^p(\Omega)$ has boundary values that belong to $H^p(\partial\Omega)$, so that it admits an atomic decomposition.

To be more precise, for g a distribution on $\partial\Omega$, $P_S(g)$ is the holomorphic function in Ω defined for $z \in \Omega$ by

$$P_S(g)(z) = \langle g, \overline{S_\Omega(z,\cdot)} \rangle$$

where $S_\Omega(z,\zeta)$ denotes the Szegö kernel, which is $\mathcal{C}^\infty(\partial\Omega)$ in the second variable whenever $z \in \Omega$.

We prove that any $f \in \mathcal{H}^p(\Omega)$ can be written as $P_S(\sum \nu_j a_j)$ where $\sum \nu_j a_j$ is a distribution that belongs to $H^p(\partial\Omega)$. Moreover, it holds that $P_S(\sum \nu_j a_j) = \sum \nu_j P_S(a_j)$, with equality in the $\mathcal{H}^p(\Omega)$-sense, that is, the series converges to f in the $\mathcal{H}^p(\Omega)$-*norm*.

On the other hand, we prove that the Szegö projection P_S maps continuously $H^p(\partial\Omega)$ into $\mathcal{H}^p(\Omega)$, for $0 < p \leq 1$.

Moreover, we prove that on $\mathcal{H}^p(\Omega)$ a (weak) factorization theorem holds true. More precisely, we prove that, given any $f \in \mathcal{H}^p(\Omega)$ there exist $f_j, g_j \in \mathcal{H}^{2p}(\Omega)$ such that $f = \sum_{j=0}^{\infty} f_j g_j$ and $\sum_{j=0}^{\infty} \|f_j\|_{\mathcal{H}^{2p}} \|g_j\|_{\mathcal{H}^{2p}} \leq c\|f\|_{\mathcal{H}^p(\Omega)}$, with c independent of f.

Finally, in Section 8 we extend the above results to the H-domains, a class of smooth, bounded domains of finite type that includes the strongly pseudoconvex domains and the convex domains. Such domains are a natural extension of the mentioned domains, and were studied in [BPS3].

These results extend classical results to the case of convex domains of finite type, and more generally to the case of H-domains. The atomic decomposition of the (holomorphic) Hardy spaces was first proved in the case of the unit ball by Garnett and Latter [GL], and later extended to strongly pseudoconvex domains and domains of finite type in $\mathbb{C}^2$ by Krantz and Li [KL2], and independently by Dafni [D]. Related results about duality between $\mathcal{H}^1$ and *BMO* appear in [KL1] for strongly pseudoconvex domains and domains of finite type in $\mathbb{C}^2$ and in [KL3] for convex domains of finite type. The factorization theorem is classical in dimension 1, while in several

variables it was first proved by Coifman, Rochberg and Weiss [CRW], in the case of the unit ball, for the space $\mathcal{H}^1(\Omega)$. This theorem was extended to the case of strongly pseudoconvex domains in [KL2] for all $\mathcal{H}^p(\Omega)$, $0 < p \leq 1$ and to convex domains of finite type in [BPS2] for $\mathcal{H}^1(\Omega)$.

Applications of these results to the regularity properties of small Hankel operators appear, to name a few, in [CRW], [KL2] and [BPS2]. Another classical characterization of Hardy spaces, the area integral one, is proved to hold true in [KL4] in convex domains of finite type in $\mathbb{C}^n$. Finally, we mentioned that the optimal approach regions for a Fatou-type theorem for $\mathcal{H}^p$ functions are described in [DFi], in the case of a convex domain of finite type.

The geometry of convex domains of finite type was first described by McNeal [Mc]. Those results were later applied to the analysis of the mapping properties of the Bergman projection [McS1] and Szegö projection [McS2] by McNeal and Stein. In the case of strongly pseudoconvex domains and finite type domains in $\mathbb{C}^2$ the geometry was determined by canonical vector fields. The natural quasi-distance on $\partial\Omega$ was the control distance determined by these vector fields, i.e. the Carnot metric. The situation of convex domains of finite type is essentially more general. The "weight" of each vector field may vary from point to point, and one needs to take into consideration the different order of contact of complex lines with $\partial\Omega$. For these reasons it is natural to consider a *diameter function* $\tau(\zeta, \lambda, r)$, which gives the diameter of the largest one-dimensional disc in the direction of λ, that fits inside the region $\{z' : \varrho(z') < r\}$. Here ϱ denotes a fixed smooth defining function for Ω.

As a consequence, in order to define the cancellation property for p-atoms for small values of p, we need to consider the pairing between $f \in \mathcal{H}^p(\Omega)$ and smooth bump functions whose derivatives in *all tangential directions* can be controled in terms of the diameter function.

We remark that, althought the main lines of the proof of the atomic decomposition are standard, some of arguments have been greatly simplified with respect to the classical ones. On the other hand in order to prove the factorization theorem we adapt an idea from [BPS2], where the result was proved in the case $p = 1$. The proof does neither rely on the explicit expression of the Szegö kernel, as in [CRW], nor its asyptotic expansion, [KL4]. Instead, it based on a recent result by Diederich and Fornæss on the existence of support functions on convex domains of finite type.

We use the notation $A \lesssim B$ to indicate that $A \leq c \cdot B$ where the constant c does not depend on the important parameters on which the functions A and B depend. (Tipically, the constant c will only depend on the geometry of the domain Ω.) We use the symbols $\gtrsim$ and $\approx$ analogously.

1. Basic facts and notation

Let Ω be a smoothly bounded convex domain in $\mathbb{C}^n$. A point $\zeta \in \partial\Omega$ is said to be of finite type if the order of contact of complex lines with $\partial\Omega$ at this point is finite, see [BS] and references therein. The type of the point is the least upper bound of the various orders of contact. We say that Ω is of finite type M_Ω if every point on $\partial\Omega$ is of finite type $\leq M_\Omega$.

Let $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$. There exists $\varepsilon_0 > 0$ such that for $|\varepsilon| \leq \varepsilon_0$ the sets $\Omega_\varepsilon = \{z \in \mathbb{C}^n : \rho(z) < \varepsilon\}$ are all convex, and the normal projection $\pi : \overline{U} \to \partial\Omega$ is well defined and smooth, where $U = \{z \in \mathbb{C}^n : \delta(z) < \varepsilon_0\}$.

The basic geometric facts about convex domains of finite type were first proved by McNeal [Mc], see also [McS1], [McS2] and [DFo]. By recalling the results that are involved in the present work we take the opportunity to review the main elements of the construction and set some notation.

For $z \in U$ and $\lambda \in \mathbb{C}^n$ a unit vector, we denote by $\tau(z, \lambda, r)$ the distance from z to the surface $\{z' : \varrho(z') = \varrho(z) + r\}$ along the complex line determined by λ.

For each $z \in U$ and $r < \varepsilon_0$ there exists a special set of coordinates $\{w_1^{z,r}, \dots, w_n^{z,r}\}$, that we call *$r$-extremal*. The first vector $v^{(1)}$ is given by the direction transversal to the boundary, in the sense that the shortest distance from z to the set $\{z' : \varrho(z') = \varrho(z) + r\}$ is realized in the complex line determined by $v^{(1)}$.

The vector $v^{(2)}$ is chosen among the vectors orthogonal to $v^{(1)}$ in such a way that $\tau(z, v^{(2)}, r)$ is maximal. We repeat the same process until we determine an orthonormal basis $\{v^{(1)}, \dots, v^{(n)}\}$. We denote by $(w_1, \dots, w_n)$ the coordinates with respect to this basis. Notice that these coordinates $(w_1, \dots, w_n) = (w_1^{z,r}, \dots, w_n^{z,r})$ depend on z and r. However, the transversal direction w_1 does not depend on r.

For $k = 1, \dots, n$ we set

$$\tau_k(z, r) = \tau(z, v^{(k)}, r), \tag{1}$$

and define the polydisc $Q(z, r)$

$$Q(z, r) = \{w : |w_k| < \tau_k(z, r),\ k = 1, \dots, n\}. \tag{2}$$

Basic relations among these quantities are the following, see [McS2] Prop. 1.1 and also [BPS2] Lemma 2.1.

Proposition 1.1. *There exists a constant $C > 0$ depending only on Ω such that for any unit vector $\lambda \in \mathbb{C}^n$, $0 < r \leq \varepsilon_0$, $z \in U$, and $0 < \eta < 1$ we have:*

(i) $\eta^{1/2}\tau(z, \lambda, r) \lesssim \tau(z, \lambda, \eta r) \lesssim \eta^{1/M_\Omega}\tau(z, \lambda, r)$;

(ii) $\eta^{1/2}Q(z, C^{-1}r) \subset Q(z, \eta r) \subset \eta^{1/M_\Omega}Q(z, Cr)$;

(iii) *if $w \in Q(z, \delta)$ then $\tau(z, \lambda, r) \approx \tau(w, \lambda, r)$.*

We define the quasi-distance $d_b : U \times U$ by setting

$$d_b(z, w) = \inf\{\delta : w \in Q(z, \delta)\}, \tag{3}$$

and the function d

$$d(z,w) = d_b(z,w) + \delta(z) + \delta(w). \tag{4}$$

Notice that d is initially defined on $U \times U$ and we extend it to $\mathbb{C}^n \times \mathbb{C}^n$ by setting

$$d(z,w) = \psi(\varrho(z))\psi(\varrho(w))d(z,w) + (1-\psi(\varrho(z)))(1-\psi(\varrho(w)))|z-w|,$$

where ψ is a smooth cut-off function on $\mathbb{R}$ such that $\psi(t) = 1$ for $|t| \leq \varepsilon_0/2$ and $\psi(t) = 0$ for $|t| \geq \varepsilon_0$.

On the boundary we will use a family of "balls" centered at $\zeta \in \partial\Omega$ of radius δ defined as

$$B(\zeta,\delta) = Q(\zeta,\delta) \cap \partial\Omega.$$

For any unit vector λ we introduce the differential operator

$$L_\lambda = (\partial_\lambda \varrho)\partial_{x_1} - (\partial_{x_1}\varrho)\partial_\lambda, \tag{5}$$

where $w_1 = x_1 + iy_1$ is the transversal direction fixed earlier. Here, ∂_λ is the standard vector field defined by λ as $\partial_\lambda f = \langle \lambda, df\rangle$, for a smooth function f, its real differential df, and where $\langle\,,\rangle$ denotes the usual pairing between a one-form and a vector.

Notice that L_λ is always a tangential vector field. If $\lambda \in S^{2n-1}$ is itself tangent to $\partial\Omega$, then L_λ is the directional derivative in the direction λ.

For $\Lambda = (\lambda_1, \ldots, \lambda_q)$ a q-list of vectors in S^{2n-1} and $\mu = (\mu_1, \ldots, \mu_q)$ a q-index we set $|\mu| = \mu_1 + \cdots + \mu_n$,

$$L^\mu_\Lambda = L^{\mu_1}_{\lambda_1} \cdots L^{\mu_q}_{\lambda_q}, \tag{6}$$

and

$$\tau^\mu(z,\Lambda,\delta) = \tau(z,\lambda_1,\delta)^{\mu_1} \cdots \tau(z,\lambda_q,\delta)^{\mu_q}. \tag{7}$$

Finally we recall the fundamental estimates for the Szegö kernel and its derivatives [McS2] (called interior estimates of S-type, see [McS2] Def. 4 and Thm. 3.6)

$$\left|L^\mu_{\Lambda,z} L^{\mu'}_{\Lambda',z'} S_\Omega(z,z')\right| \lesssim \frac{\tau^{-\mu}(z,\Lambda,\delta)\tau^{-\mu'}(z',\Lambda',\delta)}{\sigma\big(B(\pi(z),\delta)\big)}, \tag{8}$$

where $\delta = d(z,z')$, $z, z' \in \overline{\Omega} \times \overline{\Omega} \setminus \Delta_{\partial\Omega}$, $\Delta_{\partial\Omega}$ denotes the diagonal on $\partial\Omega$.

We conclude this section with the definition of the real-variable Hardy spaces. We need first to introduce the notion of p-atoms. Let $\zeta_0 \in \partial\Omega$, $r_0 < \varepsilon_0$ and N be a positive integer. On $\mathcal{C}^\infty(B(\zeta_0, r_0))$ we introduce the norm

$$\|\phi\|_{\mathcal{S}_N(B(\zeta_0,r_0))} = \sup_\Lambda \sum_{|\mu|=N} \|L^\mu_\Lambda \phi\|_{L^\infty(B(\zeta_0,r_0))} \tau^\mu(\zeta_0,\Lambda,r_0). \tag{9}$$

Definition 1.2. We set

$$\ell_0 = \big[(1-1/p)(M_\Omega + 2n - 2)\big], \quad \text{and} \quad N_p = \ell_0 + 1,$$

where $[x]$ denotes the integral part of x. A measurable function a on $\partial\Omega$ is called a p-atom if either it is the constant function on $\partial\Omega$ equal to $\sigma(\partial\Omega)^{-1/p}$, or if:

(i) $\operatorname{supp} a \subseteq B(\zeta_0, r_0)$;
(ii) $|a(\zeta)| \leq \sigma\big(B(\zeta_0, r_0)\big)^{-1/p}$;
(iii) $\int_{\partial\Omega} a(\zeta) d\sigma(\zeta) = 0$;
(iv) for all $\phi \in \mathcal{C}^\infty\big(B(\zeta_0, r_0)\big)$ we have

$$\Big| \int_{\partial\Omega} a(\zeta)\phi(\zeta) d\sigma(\zeta)\Big| \leq \|\phi\|_{\mathcal{S}_{N_p}(B(\zeta_0,r_0))} \sigma\big(B(\zeta_0, r_0)\big)^{1-1/p}.$$

Notice that the above condition (iv) replaces the classical *higher moment* condition. It is in the same spirit as the analog condition in [KL2]. The difference here is given by the choice of the norm $\|\cdot\|_{\mathcal{S}_{N_p}}$.

Real-variable Hardy spaces. Let $0 < p \leq 1$. The real Hardy space $H^p(\partial\Omega)$ is the space of distributions f on $\partial\Omega$ which can be written as

$$f = \sum_{j=0}^{\infty} \nu_j a_j, \tag{10}$$

where $\sum |\nu_j|^p < \infty$, the a_j's are p-atoms and the series is assumed to converge in the sense of distributions.

With a standard abuse of notation, the "norm" on $H^p(\partial\Omega)$ is defined as $\|f\|_{H^p}^p = \inf\{\sum_j |\nu_j|^p : f = \sum_j \nu_j a_j\}$. Setting $d(f,g) = \|f-g\|_{H^p}^p$ we see that $H^p(\partial\Omega)$ is a complete metric space. This implies that the series in (10) converges in *norm*. This is in fact obvious since $\|\sum_{j=m_1}^{m_2} \nu_j a_j\|_{H^p}^p \leq \sum_{j=m_1}^{m_2} |\nu_j|^p$ which tends to 0 as $m_1 \to \infty$. This implies that $\sum_{j=1}^{\infty} \nu_j a_j$ converges in norm, hence in the sense of distributions, necessarily to f[1].

We point out that this definition of $H^p(\partial\Omega)$ is consistent with the definition given in [CW] in the case of a space of homogenous type for values of p close to 1, see also [KL5].

2. Statement of the main results

The main results of the present work are the following.

Theorem 2.1. *Let Ω be a smoothly bounded domain of finite type. Let $0 < p \leq 1$. Then there exists a constant c depending only on p and Ω such that the following holds. Given any $f \in \mathcal{H}^p(\Omega)$ there exist constants ν_j and p-atoms a_j such that $\sum_{j=0}^{\infty} \nu_j a_j \in H^p(\partial\Omega)$ and*

$$f = P_S\Big(\sum_{j=0}^{\infty} \nu_j a_j\Big) = \sum_{j=0}^{\infty} \nu_j P_S(a_j),$$

[1]In order to clarify one point on which there is a little bit of confusion in the litterature, we remark that, in [MTW] p. 513 it is shown that a generic H^p-function f infinite sum of atoms cannot be written as finite sum of atoms $\sum_1^N \nu_j b_j$, with $\sum_{j=1}^N |\nu_j|^p \approx \|f\|_{H^p}^p$. This fact does not of course contradicts the norm convergence of the series in (10).

and moreover

$$\sum_{j=0}^{\infty} |\nu_j|^p \leq c\|f\|^p_{\mathcal{H}^p(\Omega)}.$$

Theorem 2.2. *Let Ω be a smoothly bounded domain of finite type. Let $0 < p \leq 1$ and $H^p(\partial\Omega)$ be the real-variable Hardy space on $\partial\Omega$. Then*

$$P_S : H^p(\partial\Omega) \to \mathcal{H}^p(\Omega)$$

is bounded.

Finally, we have the factorization theorem.

Theorem 2.3. *Let Ω be a smoothly bounded domain of finite type. Let $0 < p \leq 1$, $1 < q < \infty$ and q' be its conjugate exponent. There exists a constant c depending only on p, q and Ω such that the following holds. Given any $f \in \mathcal{H}^p(\Omega)$ there exist $f_j \in \mathcal{H}^{pq}$, $g_j \in \mathcal{H}^{pq'}$, $j = 1, 2, \ldots$ such that*

$$f = \sum_{j=0}^{\infty} f_j g_j$$

and

$$\sum_{j=0}^{\infty} \|f_j\|_{\mathcal{H}^{pq}} \|g_j\|_{\mathcal{H}^{pq'}} \leq c\|f\|_{\mathcal{H}^p(\Omega)}.$$

We remark that this theorem was proved in [BPS2] for $p = 1$, by a different, more indirect, method. Our proof relies on the atomic decomposition obtained in Theorem 2.1.

The above theorems are valid on a class of smoothly bounded finite type domains which includes the convex domains as well as the strongly pseudoconvex domains. These domains are introduced in [BPS3] and are called H-domains. We will discuss the extension of these theorems to the H-domains in the last section.

3. Proof of Theorem 2.2

Let $f = \sum_{j=0}^{\infty} \nu_j a_j \in H^p(\partial\Omega)$. By definition,

$$\begin{aligned} P_S\left(\sum_{j=0}^{\infty} \nu_j a_j\right)(z) &= \langle \sum_{j=0}^{\infty} \nu_j a_j, \overline{S_\Omega(z,\cdot)} \rangle \\ &= \sum_{j=0}^{\infty} \langle \nu_j a_j, \overline{S_\Omega(z,\cdot)} \rangle \\ &= \sum_{j=0}^{\infty} \nu_j P_S(a_j)(z), \end{aligned}$$

since the serie converges in the sense of distributions. It remains to prove that this last term belongs to $\mathcal{H}^p(\Omega)$, with norm controlled by $\|f\|_{H^p(\partial\Omega)}$.

We claim that there exists $C>0$ such that $\|P_S(a)\|_{\mathcal{H}^p(\Omega)}\leq C$ for any p-atom a. From this it then follows that, for any $m_1,m_2\in\mathbb{N}$, $m_1\leq m_2$,

$$\|\sum_{m_1}^{m_2}\nu_j P_S(a_j)\|_{\mathcal{H}^p}^p\leq C^p\sum_{m_1}^{m_2}|\nu_j|^p$$

so that, by the assumption on $\{\nu_j\}$ and the completeness of $\mathcal{H}^p(\Omega)$, one gets that $P_S(f)=\sum_{j=0}^{\infty}\nu_j P_S(a_j)$ belongs to $\mathcal{H}^p(\Omega)$. Moreover, $\|P_S(f)\|_{\mathcal{H}^p}^p\lesssim\sum_j|\nu_j|^p$ whenever $f=\sum_j\nu_j a_j$, which gives the desired estimate.

Thus, we only need to estimate $\|P_S(a)\|_{\mathcal{H}^p}$ for any p-atom a. Let $\varepsilon>0$. Then,

$$\begin{aligned}\int_{\partial\Omega_\varepsilon}|P_S(a)(\zeta)|^p d\sigma_\varepsilon(\zeta)&=\int_{\partial\Omega}|P_S(a)(\zeta-\varepsilon\nu(\zeta))|^p d\sigma(\zeta)\\&=\Big(\int_{B(\zeta_0,2\delta)}+\int_{{}^cB(\zeta_0,2\delta)}\Big)|P_S(a)(\zeta_\varepsilon)|^p d\sigma(\zeta)\\&=I+II,\end{aligned}$$

where $\nu(\zeta)$ denotes the outward unit normal at $\zeta\in\partial\Omega$ and we write $\zeta_\varepsilon=\zeta-\varepsilon\nu(\zeta)$.

Since $p<2$,

$$\begin{aligned}I&\leq\Big(\int_{B(\zeta_0,2\delta)}|P_S(a)(\zeta_\varepsilon)|^2 d\sigma(\zeta)\Big)^{p/2}\cdot\sigma\big(B(\zeta_0,2\delta)\big)^{1-p/2}\\&\leq c\|a\|_{L^2(\partial\Omega)}^p\sigma\big(B(\zeta_0,2\delta)\big)^{1-p/2}\\&\leq c,\end{aligned}$$

since $\|a\|_{L^2(\partial\Omega)}\leq\sigma\big(B(\zeta_0,2\delta)\big)^{1/2-1/p}$ and P_S maps $L^2(\partial\Omega_\varepsilon)$ into $L^2(\partial\Omega)$ with norm independent of ε, as a consequence of the $T(1)$-theorem of David and Journé and of the results in [McS2].

Next, set $E_k=\{\zeta\in\partial\Omega:\,2^k\delta\leq d(\zeta,\zeta_0)\leq 2^{k+1}\delta\}$. Then,

$$II=\sum_{k=0}^{\infty}\int_{E_k}|P_S(a)(\zeta-\varepsilon\nu(\zeta))|^p d\sigma_\varepsilon(\zeta),$$

and

$$\begin{aligned}|P_S(a)(\zeta_\varepsilon)|&=|\int_{\partial\Omega}S_\Omega(\zeta_\varepsilon,w)a(w)d\sigma(w)|\\&\leq\|S_\Omega(\zeta_\varepsilon,\cdot)\|_{\mathcal{S}_{N_p}(B(\zeta_0,\delta))}\cdot\sigma\big(B(\zeta_0,2\delta)\big)^{1-1/p}.\end{aligned}$$

Recall that the Szegö kernel satisfies the estimate (8), so that

$$|L_\Lambda^\mu S_\Omega(\zeta_\varepsilon,w)|\lesssim\frac{\tau^{-\mu}(w,\Lambda,d(\zeta_\varepsilon,w))}{\sigma(B(w,d(\zeta_\varepsilon,w)))},$$

and notice that for $\zeta \in E_k$ and $w \in B(\zeta_0, \delta)$,

$$\begin{aligned} d(\zeta_\varepsilon, w) = \varepsilon + d_b(\zeta, w) &\geq \varepsilon + d_b(\zeta, \zeta_0) - d_b(\zeta_0, w) \\ &\geq \varepsilon + 2^k\delta - \delta \geq \delta 2^{k-1}. \end{aligned}$$

Therefore, for $\zeta \in E_k$,

$$\begin{aligned} \|S_\Omega(\zeta_\varepsilon, \cdot)\|_{\mathcal{S}_{N_p}(B(\zeta_0,\delta))} &= \sup_\Lambda \sum_{|\mu|=N_p} \|L^\mu_\Lambda S_\Omega(\zeta_\varepsilon, \cdot)\|_{L^\infty(B(\zeta_0,\delta))} \cdot \tau^\mu(\zeta_0, \Lambda, \delta) \\ &= \sup_\Lambda \sum_{|\mu|=N_p} \sup_{w\in B(\zeta_0,\delta)} \frac{\tau^{-\mu}(w, \Lambda, d(\zeta_\varepsilon, w))}{\sigma(B(w, d(\zeta_\varepsilon, w)))} \cdot \tau^\mu(\zeta_0, \Lambda, \delta) \\ &\lesssim \sup_\Lambda \sum_{|\mu|=N_p} \sup_{w\in B(\zeta_0,\delta)} \frac{\tau^\mu(\zeta_0, \Lambda, \delta)}{\tau^\mu(w, \Lambda, \delta 2^{k-1})} \cdot \frac{1}{\sigma(B(w, \delta 2^{k-1}))} \\ &\lesssim \sup_\Lambda \sum_{|\mu|=N_p} \frac{\tau^\mu(\zeta_0, \Lambda, \delta)}{\tau^\mu(\zeta_0, \Lambda, \delta 2^{k-1})} \cdot \frac{1}{\sigma(B(w, \delta 2^{k-1}))} \\ &\lesssim \sum_{|\mu|=N_p} 2^{-k|\mu|/M_\Omega} \frac{1}{\sigma(B(w, \delta 2^{k-1}))}. \end{aligned}$$

Going back to the estimation of II,

$$\begin{aligned} II &= \sum_{k=1}^\infty \int_{E_k} |P_S(a)(\zeta)|^p d\sigma_\varepsilon(\zeta) \\ &\lesssim \sum_k \|S_\Omega(\zeta_\varepsilon, \cdot)\|^p_{\mathcal{S}_{N_p}(B(\zeta_0,\delta))} \sigma\big(B(\zeta_0, \delta)\big)^{p-1} \sigma(E_k) \\ &\lesssim \sum_k \sum_{|\mu|=N_p} 2^{-kp|\mu|/M_\Omega} \frac{\sigma(B(\zeta_0, \delta))^{p-1}}{\sigma(B(w, \delta 2^k))^{p-1}} \\ &\lesssim \sum_k \sum_{|\mu|=N_p} 2^{-k\left[(|\mu|/M_\Omega)p + (1+(2n-2)/M_\Omega)(p-1)\right]} \\ &\leq c, \end{aligned}$$

since the term in brackets in the exponent is positive, due to our choice of N_p. □

4. Maximal functions and a partition of unity

In the proof of the atomic decomposition of $\mathcal{H}^p(\Omega)$ we are going to use some maximal operators, that now we introduce. These operators are standard variants of classical ones, see [FS], [St2], and also [KL2].

Given $\zeta \in \partial\Omega$ we define the *approach region* $\mathcal{A}_\gamma(\zeta)$ as the subset of Ω given by

$$\mathcal{A}_\gamma(\zeta) = \{z \in \Omega : \, d(\zeta, \pi(z)) < \gamma\delta(z)\}.$$

We define the *non-tangential maximal function* f^*_γ

$$f^*_\gamma(\zeta) = \sup_{z\in\mathcal{A}(\zeta)} |f(z)|, \tag{11}$$

and the tangential variant

$$f^{**}_N(\zeta) = \sup_{w\in\Omega}\left(\frac{\delta(w)}{\delta(w)+d(\zeta,\pi(w))}\right)^N |f(w)|. \tag{12}$$

We consider a space of *smooth bump functions at* ζ:

$$\mathcal{K}^M_\gamma(\zeta) = \{g\in\mathcal{C}^\infty(\partial\Omega) : \operatorname{supp} g \subseteq B(\zeta_0,t_0),\ \text{with } \zeta_0\in\mathcal{A}_\gamma(\zeta) \text{ and } \|g\|_{M,\zeta_0,t_0}\le 1\},$$

where

$$\|g\|_{M,\zeta_0,t_0} = \sup_{\Lambda,\,|\mu|\le M} \tau^\mu(\zeta_0,\Lambda,t_0)\sigma(B(\zeta_0,t_0))\|L^\mu_\Lambda g\|_{L^\infty(B(\zeta_0,t_0))}. \tag{13}$$

Following [McS2] Def. 2, we say that a function ψ is a smooth bump function of order N on $B(\zeta_0,t_0)\subseteq\partial\Omega$ if $\psi\in\mathcal{C}^\infty(B(\zeta_0,t_0))$ and

$$\sup_\Lambda |L^\mu_\Lambda\psi(z)|\tau^\mu(\zeta_0,\Lambda,t_0)\le C_\psi,$$

for all $z\in B(\zeta_0,t_0)$ and $|\mu|\le N$. If $C_\psi=1$, ψ is called a *normalized* smooth bump function of order N.

The *grand maximal function* is defined as

$$K_{\gamma,M}(f)(\zeta) = \sup_{g\in\mathcal{K}^M_\gamma(\zeta)} \Big|\int_{\partial\Omega} f(w)g(w)d\sigma(w)\Big|. \tag{14}$$

Lemma 4.1. *With the definitions above, there exist* $c=c(\Omega)$ *and* $N=N(\gamma,M)$ *such that*

$$K_{\gamma,M}f(\zeta) \lesssim f^*_{c\gamma}(\zeta) + f^{**}_N(\zeta).$$

Proof. We wish to estimate $|\int_{\partial\Omega} f(w)g(w)d\sigma(w)|$ for $g\in\mathcal{K}^M_\gamma(\zeta)$. Given such a g, there exist ζ_0 and t_0 such that $\operatorname{supp} g\subseteq B(\zeta_0,t_0)$ and $d(\zeta,\zeta_0)<\gamma t_0$, i.e. $\zeta_0 - t_0\nu(\zeta_0)\in\mathcal{A}_\gamma(\zeta)$. Using the holomorphicity of f by integration by parts, see Lemma 6.5 in [BPS1], we see that there exists a differential operator Y_{k+1} with smooth coefficients, of order $k+1$ such that

$$|\int_{\partial\Omega} f(w)g(w)d\sigma(w)| = |\int_\Omega f(w)Y_{k+1}\tilde g(w)\delta^k(w)dV(w)|,$$

where $\tilde g$ is a smooth extension of g, say

$$\tilde g(w) = \begin{cases} g(\pi(w))g_1(\varrho(w)) & \text{if } |\varrho(w)|<2t_0\\ 0 & \text{otherwise,}\end{cases}$$

and $g_1\in\mathcal{C}^\infty_0\big([-2t_0,2t_0]\big)$, $g_1(t)=1$ if $|t|\le t_0/2$. Here k is a positive integer to be selected later.

Notice that $|g_1^{(j)}(t)| \leq c_j t_0^{-j}$ and hence

$$\begin{aligned}
|Y_{k+1}\tilde{g}(w)| &\lesssim \sum_{j=0}^{k+1} \frac{1}{t_0^j} \sup_{\Lambda, |\mu|=k+1-j} \|L_\Lambda^\mu g\|_{L^\infty(B(\zeta_0,t_0))} \\
&\lesssim \sum_{j=0}^{k+1} \frac{1}{t_0^j} \sup_{\Lambda, |\mu|=k+1-j} \frac{1}{\tau^\mu(\zeta_0,\Lambda,t_0)} \cdot \frac{1}{\sigma(B(\zeta_0,t_0))} \\
&\lesssim \frac{1}{t_0^{k+1}} \cdot \frac{1}{\sigma(B(\zeta_0,t_0))}.
\end{aligned}$$

Now we write,

$$\begin{aligned}
|\int_{\partial\Omega} f(w)g(w)d\sigma(w)| &\leq \int_{\Omega_{t_0}} |f(w)Y_{k+1}\tilde{g}(w)|\delta^k(w)dV(w) \\
&\quad + \int_{\Omega\setminus\Omega_{t_0}} |f(w)Y_{k+1}\tilde{g}(w)|\delta^k(w)dV(w) \\
&= I + II.
\end{aligned}$$

Notice that $w \in \operatorname{supp}\tilde{g}$ implies that $d(\pi(w),\zeta) \leq c\gamma t_0$. So, if $w \in \Omega_{t_0} \cap \operatorname{supp}\tilde{g}$ then $w \in \mathcal{A}_{c\gamma}(\zeta)$. Hence,

$$\begin{aligned}
I &\lesssim f_{c\gamma}^*(\zeta) \int_{B(\zeta_0,t_0)} \int_{t_0}^{2t_0} \frac{t^k}{t_0^{k+1}\sigma(B(\zeta_0,t))} dt d\sigma(w) \\
&\lesssim f_{c\gamma}^*(\zeta).
\end{aligned}$$

Moreover, since

$$\begin{aligned}
|f(w)| &\leq f_N^{**}(\zeta)\left(1 + \frac{d(\pi(w),\zeta)}{\delta(w)}\right)^N \\
&\lesssim f_N^{**}(\zeta)\left(1 + \frac{t_0}{\delta(w)}\right)^N,
\end{aligned}$$

we have

$$\begin{aligned}
II &\lesssim \int_{B(\zeta_0,t_0)} \int_{t_0}^{\infty} |f(w)| \frac{t^k}{t_0^{k+1}\sigma(B(\zeta_0,t_0))} dt d\sigma(w) \\
&\lesssim \int_{t_0}^{\infty} f_N^{**}(\zeta)\left(1 + \frac{t_0}{t}\right)^N \frac{t^k}{t_0^{k+1}} dt \\
&\lesssim f_N^{**}(\zeta),
\end{aligned}$$

if $k - N < -1$. □

The next two lemmas are classical, and they hold on any smoothly bounded domain. The first one is due to Stein, see [St1], Sec. 9. The second one is a version of a result of Fefferman and Stein, [FS] Lemma VI.1.

Lemma 4.2. *Let Ω be any smoothly bounded domain, $0 < p \leq \infty$. Then*

$$\|f^*_\gamma\|_{L^p(\partial\Omega)} \lesssim \|f\|_{\mathcal{H}^p(\Omega)}.$$

Lemma 4.3. *Let Ω be any smoothly bounded domain, $0 < p \leq \infty$. Then, for N large enough,*

$$\|f^{**}_N\|_{L^p(\partial\Omega)} \lesssim \|f\|_{\mathcal{H}^p(\Omega)}.$$

We now introduce a smooth partition of unity on any open set $\mathcal{O} \subseteq \partial\Omega$.

Lemma 4.4. *Let $\mathcal{O} \subseteq \partial\Omega$ be an open set. Then there exist a collection of balls $B_i = B(\zeta_i, r_i)$, functions $\phi_i \in \mathcal{C}^\infty(\partial\Omega)$, $i = 0, 1, 2, \dots$, and constants $\alpha > 1 > \beta > 0$, depending only on Ω, such that the following conditions hold:*

(i) $0 \leq \phi_i \leq 1$;
(ii) $supp\, \phi_i \subseteq B_i$;
(iii) $\phi_i = 1$ *on* $\frac{1}{\alpha} B_i$;
(iv) $\sum_{i=0}^\infty \phi_i = \chi_{\mathcal{O}}$;
(v) *for each i there exists $\zeta_i \in {}^c\mathcal{O}$ such that, for any integer N we have $c_N \phi_i / \|\phi_i\|_{L^1}$ $\in \mathcal{K}^N_\alpha(\zeta_i)$, for some $c_N = c(N, \Omega)$.*

Proof. By [McS2] Prop. 1.9, given any $\zeta_0, \delta > 0$ and $C > 1$, there exist normalized smooth bump functions of order N on $B(\zeta_0, C\delta)$, that are identically equal to 1 on $B(\zeta_0, \delta)$.

Given this, the proof now proceeds as the proof of Lemma 4.3 in [KL2]. We give the details for sake of completeness.

Let $\{B(\zeta_i, r_i)\}$ be a sequence of balls satisfying:

(a) $\cup_i \beta B(\zeta_i, r_i) = \mathcal{O}$;
(b) $B(\zeta_i, r_i) \subseteq \mathcal{O}$, $\alpha B(\zeta_i, r_i) \cap {}^c\mathcal{O} \neq \emptyset$;
(c) $\frac{1}{\alpha} B(\zeta_i, r_i)$ are pairwise disjoint;
(d) no point in $\mathcal{O}$ lies in more than N_Ω of the balls $B(\zeta_i, r_i)$,

for some $0 < \beta < 1 < \alpha$. Such a sequence exists, being a Whitney covering for $\mathcal{O}$.

Given $B_i = B(\zeta_i, r_i)$ let ψ_i be a normalized smooth bump function supported in B_i, that equals 1 on $\frac{1}{\alpha} B_i$. By (d) above,

$$1 \leq \sum_i \psi_i(\zeta) \leq N_\Omega \qquad \text{for all } \zeta \in \mathcal{O}.$$

Now set

$$\phi_i(\zeta) = \psi_i(\zeta) / \sum_j \psi_j(\zeta).$$

Then, (i)-(iv) are clearly satisfied. We only need to check (v).

By (b) above, let $\xi_i \in \alpha B(\zeta_i, r_i) \cap {}^c\mathcal{O}$. Then $\zeta_i - r_i \nu(\zeta_i) \in \mathcal{A}_\gamma(\xi_i)$, $\operatorname{supp} \phi_i \subseteq B_i$, with $\gamma > 1$, and

$$\begin{aligned} \sup_{\Lambda, |\mu| \leq N} \tau^\mu(\xi_i, \Lambda, r_i) \sigma\big(B(\xi_i, r_i)\big) \|L^\mu_\Lambda \phi_i\|_{L^\infty(\partial\Omega)} &\lesssim c_N \sigma\big(B(\xi_i, r_i)\big) \\ &\lesssim c_N \|\phi_i\|_{L^1(\partial\Omega)}. \quad \square \end{aligned}$$

5. Beginning of the proof of Theorem 2.1

We let k_0 to be the least integer such that

$$\|K_{\gamma,M}(f) + f^*_\gamma\|_{L^p(\partial\Omega)} \leq 2^{k_0}. \tag{15}$$

For a positive integer k define

$$\mathcal{O}_k = \{z \in \partial\Omega : \, K_{\gamma,M} f(z) + f^*_\gamma(z) > 2^{k_0+k}\}. \tag{16}$$

For each k we fix a Whitney covering and a partition of $\chi_{\mathcal{O}_k}$, $\{\phi^k_i\}$, as in Lemma 4.4.

Let $\zeta_0 \in \partial\Omega$ and $B(\zeta_0, r_0)$ be fixed. The ball $B(\zeta_0, r_0)$ is contained in the polydisc $Q(\zeta_0, r_0)$. By the results in [Mc] there exists an r_0-extremal (orthonormal) basis $\{v^{(1)}, \dots, v^{(n)}\}$ on $Q(\zeta_0, r_0)$, where $v^{(1)}$ is transversal to the boundary. We denote by $(w_1, \dots, w_n)$ the coordinates with respect to this basis and we define $V_\ell(\zeta_0, r_0)$ to be the space of polynomials of degree $\leq \ell$ in $\operatorname{Im} w_1, w_2, \overline{w_2}, \dots, w_n, \overline{w_n}$.

We remark that $V_\ell(\zeta_0, r_0)$ does not depend on r_0 since $v^{(1)}$ does not depend on r_0. Hence, we simply write $V_\ell(\zeta_0)$.

Let ϕ_0 be a smooth bump function supported on $B(\zeta_0, r_0)$. We denote by $L^2_{\phi_0}(d\sigma)$ the L^2-space with respect to the probability measure $(\phi_0/\|\phi_0\|_{L^1})d\sigma$. We let $\mathcal{P}_{\phi_0}$ denote the orthogonal projection of $L^2_{\phi_0}(d\sigma)$ onto $V_\ell(\zeta_0)$ (which is obviously contained in $L^2_{\phi_0}(d\sigma)$).

In what follows, we will write $w = s + it$, $s = (s_1, s')$, $t = (t_1, t')$, $s_1, t_1 \in \mathbb{R}$, $s', t' \in \mathbb{R}^{n-1}$, so that $V_\ell(\zeta_0)$ is the set of polynomials in s', t_1, t', of degree $\leq \ell$.

Let $\{\pi_J\}$ be an orthonormal basis for V_{ℓ_0}, where $\ell_0 = [(1 - 1/p)(M_\Omega - 2n - 2)]$ and $J = (j', j) = (0, j'_2, \dots, j'_n, j_1, \dots, j_n,)$, $|J| = j'_2 + \cdots + j'_n + j_1 + \cdots + j_n$.

Lemma 5.1. *Let π_J be as above. Then,*

(i) $|\pi_J(z)| \leq c_J$ *on the support of* ϕ_0*;*
(ii) $|L^\mu_\Lambda \pi_J(z)| \leq c_{J,\mu} \tau^{-\mu}(\zeta_0, \Lambda, r_0)$ *on the support of* ϕ_0.

Proof. We begin with (i). Recall that ζ_0 and r_0 are fixed. Then $B := B(\zeta_0, r_0) = Q(\zeta_0, r_0) \cap \partial\Omega$, where $Q(\zeta_0, r_0)$ is the polydisc centered at ζ_0 and polyradius $\tau_1(\zeta_0, r_0) = r_0$, $\tau_2(\zeta_0, r_0), \dots, \tau_n(\zeta_0, r_0)$.

Let

$$\pi_J(w) = \sum_{|\gamma'|+|\gamma| \leq \ell_0} a_{\gamma',\gamma} s^{\gamma'} t^\gamma,$$

where $\gamma' = (0, \gamma'_2, \dots, \gamma'_n)$, $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$. Then,

$$\begin{aligned} \|\pi_J\|_{L^\infty(B)} &\leq \sup_{(s',t)\in B} \Big| \sum_{|\gamma'|+|\gamma| \leq \ell_0} a_{\gamma',\gamma} s^{\gamma'} t^\gamma \Big| \\ &= \sup_{(\tilde{s}',\tilde{t})\in \tilde{B}} \Big| \sum_{|\gamma'|+|\gamma| \leq \ell_0} a_{\gamma',\gamma} \tilde{s}^{\gamma'} \tilde{t}^\gamma \tau^{\gamma'+\gamma}(\zeta_0, r_0) \Big|, \end{aligned}$$

where $\tilde{s} = (0, s_2'/\tau_2, \dots, s_n'/\tau_n)$, $\tilde{t} = (t_1/\tau_1, t_2/\tau_2, \dots, t_n/\tau_n)$, $\tau_j = \tau_j(\zeta_0, r_0)$, and $\tilde{B} = (-1,1)^{2n-1}$.

Hence, using the equivalence of all norms on finite dimensional vector spaces, we have

$$\begin{aligned}
\|\pi_J\|_{L^\infty(B)} &\le \sup_{(s',t)\in B} \Big| \sum_{|\gamma'|+|\gamma|\le \ell_0} a_{\gamma',\gamma} s^{\gamma'} t^{\gamma} \Big| \\
&\lesssim \Big\| \sum_{|\gamma'|+|\gamma|\le \ell_0} a_{\gamma',\gamma} \tau^{\gamma'+\gamma}(\zeta_0, r_0) \tilde{s}^{\gamma'} \tilde{t}^{\gamma} \Big\|_{L^2(\tilde{B})} \\
&= \sigma(B)^{-1/2} \Big\| \sum_{|\gamma'|+|\gamma|\le \ell_0} a_{\gamma',\gamma} s^{\gamma'} t^{\gamma} \Big\|_{L^2(B)} \\
&\approx \Big\| \sum_{|\gamma'|+|\gamma|\le \ell_0} a_{\gamma',\gamma} s^{\gamma'} t^{\gamma} \Big\|_{L^2_{\phi_0}(B)} \\
&= 1.
\end{aligned}$$

Here we use the fact that, since $\phi_0 = 1$ on $B(\zeta_0, r_0)$ and $\operatorname{supp}\phi_0 \subseteq \alpha B$, $\|\phi_0\|_{L^1} \approx \sigma(B(\zeta_0, r_0))$. This proves (i).

It remains to prove the estimates on the derivatives of π_J. We are going to show that, for any (γ', γ) such that $|\gamma'| + |\gamma| \le \ell_0$, there exists $c^J_{\gamma',\gamma}$ so that

$$|a_{\gamma',\gamma}| \le \frac{c^J_{\gamma',\gamma}}{\tau^{\gamma'+\gamma}(\zeta_0, r_0)}. \tag{17}$$

Assuming this estimate for the moment, it follows that, for any β', β with $|\beta'| + |\beta| \le \ell_0$,

$$\begin{aligned}
\big|\partial_s^{\beta'} \partial_t^{\beta} \pi_J(w)\big| &= \Big| \sum_{\gamma'\ge\beta', \gamma\ge\beta, |\gamma'|+|\gamma|\le\ell_0} c_{\beta',\beta} a_{\gamma',\gamma} s^{\gamma'-\beta'} t^{\gamma-\beta} \Big| \\
&\le \frac{c_{J,\gamma',\gamma}}{\tau^{\gamma'+\gamma}(\zeta_0, r_0)} \cdot \tau^{\gamma'-\beta'+\gamma-\beta}(\zeta_0, r_0) \\
&\le \frac{c_{J,\gamma',\gamma}}{\tau^{\beta'+\beta}(\zeta_0, r_0)}.
\end{aligned}$$

The estimate for any derivative L^μ_Λ follows from the fact that any L_λ is a linear combination fo the ∂_{s_j}'s and ∂_{t_k}'s, say $L_\lambda = \sum_{j=2}^n a_j \partial_{s_j} + \sum_{k=1}^n b_k \partial_{t_k}$, and that

$$\frac{1}{\tau(\zeta_0, \lambda, r_0)} \approx \sum_{j=1}^n \frac{|a_j| + |b_j|}{\tau_j(\zeta_0, r_0)},$$

(here $a_1 = 0$), see [McS2] Prop. 1.1.

It remains to prove (17). Since the π_J's are real-analytic they satisfy the mean-value property. In particular,

$$\tau^{\gamma'+\gamma}(\zeta_0, r_0) \big|\partial_s^{\gamma'} \partial_t^{\gamma} \pi_J(\zeta_0)\big| \le \frac{c}{\sigma(B(\zeta_0, r_0))} \int_{B(\zeta_0, r_0)} |\pi_J(w)| d\sigma(w) \le \tilde{c}_J,$$

so that

$$|a_{\gamma',\gamma}| = \left|\partial_s^{\gamma'}\partial_t^{\gamma}\pi_J(\zeta_0)\right| \leq \frac{\tilde{c}_J}{\tau^{\gamma'+\gamma}(\zeta_0, r_0)},$$

which proves (17) and finishes the proof. □

Lemma 5.2. *With the notation fixed above, there exists $c > 0$ such that for $f \in \mathcal{H}^p(\Omega)$*

$$|\mathcal{P}_{\phi_i^k}(f)(z)\phi_i^k(z)| \leq c2^k.$$

Furthermore,

$$\left|\mathcal{P}_{\phi_j^{k+1}}\left(\left(f - \mathcal{P}_{\phi_j^{k+1}}(f)\right)\phi_i^k\right)(z)\phi_j^{k+1}(z)\right| \leq c2^{k+1}.$$

Proof. We use Lemma 5.1 with $\zeta_i^k = \zeta_0$, $\phi_i^k = \phi_0$, and $B_i^k(\zeta_i^k, r_i^k) = B_0$. Then

$$\mathcal{P}_{\phi_i^k}(f)(z) = \sum_{|J|\leq \ell_0} c_J(f)\pi_J(z),$$

where

$$c_J(f) = \int_{\partial\Omega} f(w)\overline{\pi_J(w)}\phi_i^k(w)d\sigma(w).$$

The estimate on the π_J's and ϕ_i^k imply that $\pi_J \cdot \phi_i^k \in \mathcal{K}_\gamma^M(\zeta_i^k)$ so that

$$|c_J(f)| \leq K_{\gamma,M}(f)(\zeta_i^k) \leq c2^k.$$

This estimate, together with Lemma 5.1 again, imply the bound on $\mathcal{P}_{\phi_i^k}(f)\phi_i^k$.

Next we prove the second estimate in the statement. Set $h = \left(f - \mathcal{P}_{\phi_j^{k+1}}(f)\right)\phi_i^k$. Then,

$$\begin{aligned}
c_J^{(k+1)}(h) &= \int_{\partial\Omega}\left(f - \mathcal{P}_{\phi_j^{k+1}}(f)\right)(w)\phi_i^k(w)\overline{\pi_J^{(k+1)}(w)}d\sigma_{\phi_j^{k+1}}(w) \\
&= \int_{\partial\Omega} f(w)\phi_i^k(w)\overline{\pi_J^{(k+1)}(w)}\phi_j^{k+1}(w)\frac{d\sigma(w)}{\|\phi_j^{k+1}\|_{L^1}} \\
&\quad - \int_{\partial\Omega}\mathcal{P}_{\phi_j^{k+1}}(f)(w)\phi_i^k(w)\phi_j^{k+1}(w)\overline{\pi_J^{(k+1)}(w)}\frac{d\sigma(w)}{\|\phi_j^{k+1}\|_{L^1}} \\
&= I + II.
\end{aligned}$$

By condition (v) in Lemma 4.4, for some $\xi_j^{k+1} \in {}^c\mathcal{O}_{k+1}$,

$$|I| \leq K_{\gamma,M}(f)(\xi_j^{k+1}) \leq 2^{k+1},$$

by construction. On the other hand, by the previous majorization,

$$|\mathcal{P}_{\phi_j^{k+1}}(f)(w)\phi_j^{k+1}(w)| \leq c2^{k+1},$$

while $|\pi_J^{(k+1)}(w)| \leq c$, so that the desired conclusion follows. □

6. Atomic decomposition

In this section we define the atomic decomposition of a given function $f \in \mathcal{H}^p(\Omega)$, and finish the proof of Theorem 2.1. As noticed before, f admits boundary values defined a.e. on $\partial\Omega$, that we still denote by f.

We write,

$$\begin{aligned} f &= \big(f - \sum_{i=0}^{\infty} f\phi_i^k\big) + \sum_{i=0}^{\infty} f\phi_i^k \\ &= f_k + \sum_{i=0}^{\infty} f\phi_i^k \\ &= f_k + \sum_{i=0}^{\infty} \mathcal{P}_{\phi_i^k}(f)\phi_i^k + \sum_{i=0}^{\infty} \big(f - \mathcal{P}_{\phi_i^k}(f)\big)\phi_i^k \\ &= h_k + \sum_{i=0}^{\infty} \big(f - \mathcal{P}_{\phi_i^k}(f)\big)\phi_i^k, \end{aligned}$$

where $h_k = f_k + \sum_{i=0}^{\infty} \mathcal{P}_{\phi_i^k}(f)\phi_i^k$.

Notice that,

$$\big|\sum_{i=0}^{\infty} \mathcal{P}_{\phi_i^k}(f)(z)\phi_i^k(z)\big| \leq c_0 2^k, \tag{18}$$

since no point in ${}^c\mathcal{O}_k$ lies in more than N_Ω of the $B(\zeta_i^k, r_i^k)$'s. Moreover, notice that

$$\operatorname{supp}\left(\sum_{i=0}^{\infty} \big(f - \mathcal{P}_{\phi_i^k}(f)\big)\phi_i^k\right) \subseteq \mathcal{O}_k,$$

so that the function on the left hand-side above tends to 0 pointwise as $k \to \infty$. This implies that $h_k \to f$ pointwise a.e., so that the following equality holds a.e.

$$f = h_0 + \sum_{k=0}^{\infty} (h_{k+1} - h_k). \tag{19}$$

Now notice that

$$f - h_k = \sum_{i=0}^{\infty} \big(f - \mathcal{P}_{\phi_i^k}(f)\big)\phi_i^k \tag{20}$$

and that

$$\sum_{i=0}^{\infty} \mathcal{P}_{\phi_j^{k+1}}\left(\big(f - \mathcal{P}_{\phi_j^{k+1}}(f)\big)\phi_i^k\right) = \mathcal{P}_{\phi_j^{k+1}}\left(\big(f - \mathcal{P}_{\phi_j^{k+1}}(f)\big)\right) = 0. \tag{21}$$

Then, using (20) and (21) we write

$$\begin{aligned} h_{k+1}-h_k &= \sum_{i=0}^{\infty}\big(f-\mathcal{P}_{\phi_i^k}(f)\big)\phi_i^k-\sum_{j=0}^{\infty}\big(f-\mathcal{P}_{\phi_j^{k+1}}(f)\big)\phi_j^{k+1}\\ &= \sum_{i=0}^{\infty}\Bigg\{\big(f-\mathcal{P}_{\phi_i^k}(f)\big)\phi_i^k-\sum_{j=0}^{\infty}\Bigg(\big(f-\mathcal{P}_{\phi_j^{k+1}}(f)\big)\phi_j^k\\ &\qquad -\mathcal{P}_{\phi_j^{k+1}}\big((f-\mathcal{P}_{\phi_j^{k+1}}(f))\phi_i^k\big)\Bigg)\phi_j^{k+1}\Bigg\}\\ &= \sum_{i=0}^{\infty} b_i^k, \end{aligned}$$

i.e. we have set

$$b_i^k=\big(f-\mathcal{P}_{\phi_i^k}(f)\big)\phi_i^k-\sum_{j=0}^{\infty}\Bigg(\big(f-\mathcal{P}_{\phi_j^{k+1}}(f)\big)\phi_j^k-\mathcal{P}_{\phi_j^{k+1}}\big((f-\mathcal{P}_{\phi_j^{k+1}}(f))\phi_i^k\big)\Bigg)\phi_j^{k+1}. \tag{22}$$

Hence,

$$f=h_0+\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}b_i^k. \tag{23}$$

Now we let

$$a_0=\frac{1}{\nu_0}h_0,\qquad a_i^k=\frac{1}{\nu_i^k}b_i^k, \tag{24}$$

where

$$\nu_0=\|h_0\|_{L^\infty(\partial\Omega)}\sigma(\partial\Omega)^{1/p}\quad\text{and}\quad \nu_i^k=2^{k_0+k+1}\sigma(B_i^k)^{1/p}.$$

Then, equation (23) becomes

$$f=\nu_0 a_0+\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\nu_i^k a_i^k.$$

The remaining of this section is devoted to proving that the above is the desired atomic decomposition of f.

Estimate for h_0. By definition,

$$h_0=f_0+\sum_{i=0}^{\infty}\mathcal{P}_{\phi_i^0}(f)\phi_i^0.$$

By (18) we have

$$\Big|\sum_{i=0}^{\infty}\mathcal{P}_{\phi_i^0}(f)\phi_i^0\Big|\le c,$$

and, by definition,

$$|f_0|\le f^*|_{^c\mathcal{O}_0}\le 2^{k_0}.$$

Then, $\|h_0\|_{L^\infty} \leq c2^{k_0}$, so that a_0 is an atom supported in $\partial\Omega$.

Size estimates for the b_i^k's. We have,

$$\begin{aligned}
|b_i^k| &= \left| \big(f - \mathcal{P}_{\phi_i^k}(f)\big)\phi_i^k - \sum_{j=0}^{\infty} \left(\big(f - \mathcal{P}_{\phi_j^{k+1}}(f)\big)\phi_i^k - \mathcal{P}_{\phi_j^{k+1}}\big(f - \mathcal{P}_{\phi_j^{k+1}}(f)\big)\phi_i^k \right) \phi_j^{k+1} \right| \\
&\leq \left| \big(f - \mathcal{P}_{\phi_i^k}(f)\big)\phi_i^k - \sum_{j=0}^{\infty} \big(f - \mathcal{P}_{\phi_j^{k+1}}(f)\big)\phi_i^k\phi_j^{k+1} \right| \\
&\qquad + \left| \sum_{j=0}^{\infty} \left(\mathcal{P}_{\phi_j^{k+1}}\big(f - \mathcal{P}_{\phi_j^{k+1}}(f)\big)\phi_i^k \right) \phi_j^{k+1} \right|.
\end{aligned}$$

The second term on the right hand-side above is bounded by $c_0 c 2^{k+1}$ by Lemma 5.2, while the first one is bounded by

$$\begin{aligned}
&\left| \sum_{j=0}^{\infty} \left(\big(f - \mathcal{P}_{\phi_i^k}(f)\big) - \big(f - \mathcal{P}_{\phi_j^{k+1}}(f)\big)\phi_i^k \right) \phi_j^{k+1} \right| + \left| \big(f - \mathcal{P}_{\phi_i^k}(f)\big)\phi_i^k \chi_{\mathcal{C}\mathcal{O}_{k+1}} \right| \\
&\quad \leq \left| \sum_{j=0}^{\infty} \big(\mathcal{P}_{\phi_j^{k+1}}(f) - \mathcal{P}_{\phi_i^k}(f)\big)\phi_i^k\phi_j^{k+1} \right| + \left| \big(f - \mathcal{P}_{\phi_i^k}(f)\big)\phi_i^k \chi_{\mathcal{O}_k \setminus \mathcal{O}_{k+1}} \right| \\
&\quad \leq \left| \sum_{j=0}^{\infty} \mathcal{P}_{\phi_j^{k+1}}(f)\phi_j^{k+1} \right| + |\mathcal{P}_{\phi_i^k}(f)\phi_i^k| + |f\chi_{\mathcal{O}_k \setminus \mathcal{O}_{k+1}}| + |\mathcal{P}_{\phi_i^k}(f)\phi_i^k| \\
&\quad \leq c_0 c 2^k + f^* \chi_{\mathcal{O}_k \setminus \mathcal{O}_{k+1}} \\
&\quad \leq c_0 c 2^k,
\end{aligned}$$

where we have used Lemma 5.2.

Support of the b_i^k's. The first term in (22) is supported in B_i^k. To unsure that the terms in the series not to be identically 0, the condition $B_i^k \cap B_j^{k+1} \neq \emptyset$ must be satisfied for some j.

We claim that if $B_i^k \cap B_j^{k+1} \neq \emptyset$, then $r_j^{k+1} \leq c\alpha r_i^k$. For, let $w \in B_i^k \cap B_j^{k+1}$. Since $\mathcal{O}_{k+1} \subseteq \mathcal{O}_k$,

$$\begin{aligned}
C r_j^{k+1} &\leq d(B_j^{k+1}, \partial\mathcal{O}_{k+1}) \leq d(B_j^{k+1}, \partial\mathcal{O}_k) \\
&\leq d(w, \partial\mathcal{O}_k) \leq d(B_i^k, \partial\mathcal{O}_k) + 2d(w, \zeta_i^k) \\
&\leq c\alpha r_i^k,
\end{aligned}$$

since, by the Whitney property, one has $C r_i^k \leq d(B_i^k, \partial\mathcal{O}_k) \leq \alpha r_i^k$.

Moment condition. We wish to estimate

$$\left| \int_{\partial\Omega} b_i^k(w)\phi(w)d\sigma(w) \right|,$$

for $\phi \in \mathcal{S}_{N_p}(B_i^k)$.

On B_i^k we work in local coordinates and write the Taylor expansion of ϕ around ζ_i^k, up to order ℓ_0. We denote by $S_\phi^{\ell_0}(\zeta_i^k)$ the corresponding Taylor polynomial. Notice that $S_\phi^{\ell_0}(\zeta_i^k) \in V_{\ell_0}(\zeta_i^k)$.

By definition of $\mathcal{S}_{N_p}(B_i^k)$ we have

$$\|\phi - S_\phi^{\ell_0}(\zeta_i^k)\|_{L^\infty(\partial\Omega)} \leq \|\phi\|_{\mathcal{S}_{N_p}(B(\zeta_0,r_0))}.$$

By construction, the first term in b_i^k is orthogonal to $V_{\ell_0}(\zeta_i^k)$ and each non-vanishing term in the series is orthogonal to some $V_{\ell_0}(\zeta_j^{k+1})$, so that $B_i^k \cap B_j^{k+1} \neq \emptyset$ or when $r_j^{k+1} \leq C\alpha r_i^k$ and $B_j^{k+1} \subseteq C'\alpha B_i^k$. In that case it follows that $V_{\ell_0}(\zeta_j^{k+1}) \subseteq V_{\ell_0}(\zeta_i^k)$, since the same coordinate system works, and that

$$\begin{aligned}\Big|\int_{\partial\Omega} b_i^k(w)\phi(w)d\sigma(w)\Big| &= \Big|\int_{\partial\Omega} b_i^k(w)\big(\phi(w) - S_\phi^{\ell_0}(\zeta_i^k)\big)d\sigma(w)\Big| \\ &\lesssim \|\phi\|_{\mathcal{S}_{N_p}(B_i^k)} \cdot \sigma(B_i^k) \cdot 2^{k+k_0},\end{aligned}$$

by the size estimate of the b_i^k's.

Coefficients in ℓ^p. It remains to prove that $\{\nu_i^k\} \in \ell^p$. We have

$$\begin{aligned}\sum_{k=0}^\infty \sum_{i=0}^\infty |\nu_i^k|^p &\leq c^p \sum_{k=0}^\infty \sum_{i=0}^\infty 2^{k+k_0}\sigma(B_i^k) \\ &\leq c^p \sum_{k=0}^\infty 2^{k+k_0}\sigma(\mathcal{O}_k) \\ &\leq c^p \int_1^\infty t^{p-1}\sigma\big(\{\zeta \in \partial\Omega :\, K_\alpha^M f(\zeta) + f_\alpha^*(\zeta) \geq t\}\big)dt \\ &\leq c^p \|f\|_{\mathcal{H}^p}^p.\end{aligned}$$

Convergence in $\mathcal{H}^p(\Omega)$-norm. We proved that the boundary value of $f \in \mathcal{H}^p(\Omega)$, that we still denote by f, admits a decomposition $f = \sum_j \nu_j a_j$, for p-atoms a_j and constants ν_j such that $\sum_j |\nu_j|^p < \infty$.

Such equality, that stems from (23), holds a.e., see (19). We now show that it also valid in the distribution sense.

¿From this fact and Theorem 2.2, it converges to f in $\mathcal{H}^p(\Omega)$-norm.

We now show that the equality

$$f = h_0 + \sum_{k=0}^\infty h_{k+1} - h_k,$$

that holds a.e., also holds in the distribution sense.

Recall that $h_{k+1} - h_k = \sum_{i=0}^{\infty} b_i^k$, so it suffices to show that $\sum_{k=0}^{m} h_{k+1} - h_k$ converges in the sense of distributions. We show that

$$
\begin{aligned}
(25) \qquad \Big| \int_{\partial\Omega} \big(\sum_{k=\ell}^{m} h_{k+1} - h_k\big)(z)\psi(z)d\sigma(z)\Big| &= \Big| \sum_{k=\ell}^{m} \int_{\partial\Omega} \sum_{i=0}^{\infty} b_i^k \psi(z) d\sigma(z)\Big| \\
&= \Big| \sum_{k=\ell}^{m} \sum_{i=0}^{\infty} \int_{\partial\Omega} b_i^k \psi(z) d\sigma(z)\Big| \\
&\lesssim \sum_{k=\ell}^{m} \sum_{i=0}^{\infty} 2^{k+k_0} \sigma(B_i^k) \|\psi\|_{\mathcal{S}_{N_p}(B_i^k)} \\
&\lesssim \sum_{k=\ell}^{m} \sum_{i=0}^{\infty} 2^{k+k_0} \sigma(B_i^k) \|\psi\|_{\mathcal{C}^{\ell_0+1}(\partial\Omega)} (r_i^k)^{[(1/p-1)(M_\Omega+2n-2)]+1} \\
&\lesssim \sum_{k=\ell}^{m} \sum_{i=0}^{\infty} 2^{k+k_0} \sigma(B_i^k) \|\psi\|_{\mathcal{C}^{\ell_0+1}(\partial\Omega)} (r_i^k)^{(1/p-1)(2n-1)},
\end{aligned}
$$

since $[(1/p-1)(M_\Omega+2n-2)]+1 \geq (1/p-1)(M_\Omega+2n-1) \geq (1/p-1)(2n-1)$.

Hence, using the fact that $1/p \geq 1$ twice, the left hand-side in (25) is less or equal to a constant times

$$
\begin{aligned}
\sum_{k=\ell}^{m} \sum_{i=0}^{\infty} 2^{k+k_0} \sigma(B_i^k)^{1/p} \|\psi\|_{\mathcal{C}^{\ell_0+1}(\partial\Omega)} &\lesssim \sum_{k=\ell}^{m} 2^k \left(\sum_{i=0}^{\infty} \sigma(B_i^k)^{1/p} \right)^{1/p} \\
&\lesssim \left(\sum_{k=\ell}^{m} 2^{kp} \sigma(\mathcal{O}_k) \right)^{1/p} \\
&\lesssim \left(\sum_{k=\ell}^{\infty} \int_{2^{k-1}}^{2^k} t^{p-1} \sigma\big(\{f^* + K_\alpha^M f \geq t\}\big) dt \right)^{1/p} \\
&\lesssim \left(\int_{2^{\ell-1}}^{\infty} t^{p-1} \int_{\{f^* + K_\alpha^M f \geq t\}} d\sigma dt \right)^{1/p} \\
&\lesssim \left(\int_{\mathcal{O}_{\ell-1}} |f^* + K_\alpha^M f|^p d\sigma \right)^{1/p},
\end{aligned}
$$

which tends to 0 as $\ell \to \infty$.

This finishes the proof of the atomic decomposition. □

7. Proof of Theorem 2.3

The method used in the proof of the atomic decomposition allows one to prove the same result using atoms with order of cancellation arbitrarly high. Hence, we

call an atom a (k,p)-atom if $k \geq N_p$, a is a p-atom and moreover

$$|\int_{\partial\Omega} a(\zeta)\phi(\zeta)d\sigma(\zeta)| \leq \|\phi\|_{\mathcal{S}_k(B(\zeta_0,r_0))}\sigma\big(B(\zeta_0,r_0)\big)^{1-1/p},$$

for all $\phi \in \mathcal{C}^\infty((B(\zeta_0,r_0))$, $B(\zeta_0,r_0)$ being the support of a.

Lemma 7.1. *Let a be a (k,p)-atom, having support in $B = B(\zeta_0,r_0)$, and let $A = P_S(a)$. Then A satisfies the estimates:*

(i) $\|A\|_{\mathcal{H}^{2p}} \leq c\sigma(B)^{-1/2p}$;

(ii) *for $C > 1$ and $d(\zeta,\zeta_0) \geq Cr_0$,*

$$|A(\zeta)| \leq c\bigg(\frac{r_0}{d(\zeta,\zeta_0)}\bigg)^{1+(k-2+2n)/M_\Omega}\sigma\big(B(\zeta_0,r_0)\big)^{-1/p};$$

(iii) *for $C > 1$, ℓ a positive integer and $d(\zeta,\zeta_0) \geq Cr_0$,*

$$|\nabla^\ell A(\zeta)| \leq c\frac{{r_0}^{1+(k-2+2n)/M_\Omega}}{d(\zeta,\zeta_0)^{1-\ell+(k-2+2n)/M_\Omega}}\sigma\big(B(\zeta_0,r_0)\big)^{-1/p}.$$

In the following we will denote by $\beta = 1 + (k-2+2n)/M_\Omega$.

Proof. The proof follows the same lines as the ones in [BPS2] Lemma 4.7.

The estimate in (i) is the same as the estimate for $\|A\|_{\mathcal{H}^p}$.

Denote by S_k the Taylor polynomial of $w \mapsto S_\Omega(\zeta,w)$ around ζ_0 of order k. Then $\|S_\Omega(\zeta,w) - S_k(w)\|_{L^\infty} \leq \|S_\Omega(\zeta,\cdot)\|_{\mathcal{S}_k(B(\zeta_0,r_0))}$.

Hence,

$$\begin{aligned}|A(\zeta)| &= \big|\int_{\partial\Omega} a(w)S_\Omega(\zeta,w)d\sigma(w)\big| \\ &= \big|\int_{\partial\Omega} a(w)\big(S_\Omega(\zeta,w) - S_k(w)\big)d\sigma(w)\big| \\ &\leq \|S_\Omega(\zeta,\cdot)\|_{\mathcal{S}_k(B(\zeta_0,r_0))}\sigma\big(B(\zeta_0,r_0)\big)^{1-1/p}.\end{aligned}$$

Using the estimate (8) we have

$$\begin{aligned}\|S_\Omega(\zeta,\cdot)\|_{\mathcal{S}_k(B(\zeta_0,r_0))} &= \sum_{|\mu|=k}\sup_\Lambda \|L^\mu_{\Lambda,w}S_\Omega(\zeta,\cdot)\|_{L^\infty(B)}\tau^\mu(\zeta_0,\Lambda,r_0) \\ &\lesssim \sum_{|\mu|=k}\sup_{w\in B}\sup_\Lambda \frac{\tau^\mu(\zeta_0,\Lambda,r_0)}{\tau^\mu(w,\Lambda,d(\zeta,w))}\cdot\frac{1}{\sigma(B(w,d(\zeta,w)))}.\end{aligned}$$

Recall that, when $d(\zeta,\zeta_0) \geq Cr_0$, for $w \in B(\zeta_0,r_0) \subseteq B(\zeta_0,d(\zeta_0,\zeta)/C)$, $d(\zeta_0,\zeta) \approx d(\zeta,w)$. Then,

$$\|S_\Omega(\zeta,\cdot)\|_{\mathcal{S}_k(B(\zeta_0,r_0))} \lesssim \bigg(\frac{r_0}{d(\zeta,\zeta_0)}\bigg)^{k/M_\Omega}\cdot\sigma\big(B(\zeta_0,d(\zeta,\zeta_0)\big)^{-1}$$

so that

$$|A(\zeta)| \lesssim \left(\frac{r_0}{d(\zeta,\zeta_0)}\right)^{\beta} \cdot \sigma\big(B(\zeta_0,r_0)\big)^{-1/p}$$

The estimates for the the derivatives of A follow in the same fashion. □

End of the proof of Theorem 2.3. By the atomic decomposition, it suffices to factorize each holomorphic atom $A = P_S(a)$, for a (k,p)-atom with k large enough, a having support in some ball $B(\zeta_0, r_0)$.

For this factorization, we rely on a recent result of Diederich and Fornæss [DFo] on the existence of support functions on convex domains of finite type. More precisely, there exists a neighborhood U of $\partial\Omega$ and a function $H : \Omega \times U \to \mathbb{C}$ such that $H \in \mathcal{C}^\infty(\Omega \times U)$, $H(\cdot, w)$ is holomorphic for each $w \in U$, and

$$d(z,w) \lesssim |H(z,w)| \lesssim d(z,w),$$

on $\Omega \times U$. This function H was used in [BPS2] to prove the factorization theorem for $\mathcal{H}^1$.

We set $H_0 = H(\cdot, \tilde{\zeta}_0)$, where $\tilde{\zeta}_0 = \zeta_0 - r_0\nu(\zeta_0)$. We write $A = AH_0^\ell H_0^{-\ell}$, with ℓ to be selected later.

We prove the result for $q = q' = 2$, the general case being completely analogous. Writing $\zeta_\varepsilon = \zeta - \varepsilon\nu_\zeta$, we now have

$$\begin{aligned}\|H_0^{-\ell}\|_{\mathcal{H}^{2p}}^{2p} &\lesssim \sup_{\varepsilon>0} \int_{\partial\Omega} d(\zeta_\varepsilon, \tilde{\zeta}_0)^{-2p\ell} d\sigma(\zeta) \\ &\lesssim r_0^{-2pl}\sigma(B),\end{aligned}$$

by a standard integration result, see [BPS2] Lemma 2.2.

On the other hand, for a fixed constant $C > 1$,

$$\begin{aligned}\|AH_0^\ell\|_{\mathcal{H}^{2p}}^{2p} &\lesssim \sup_{\varepsilon_0>\varepsilon>0} \int_{B(\zeta_0,Cr_0)} |A(\zeta_\varepsilon)|^{2p} d(\zeta_\varepsilon,\tilde{\zeta}_0)^{2p\ell} d\sigma(\zeta) \\ &\qquad + \frac{1}{\sigma(B)} \int_{cB(\zeta_0,Cr_0)} \frac{r_0^{2p\beta}}{d(\zeta,\zeta_0)^{2p\beta}} d(\zeta,\tilde{\zeta}_0)^{2p\ell} d\sigma(\zeta) \\ &\lesssim (Cr_0)^{2p\ell}\|A\|_{\mathcal{H}^p}^{2p} + \frac{Cr_0^{2p\ell}}{\sigma(B)} \int_{cB(\zeta_0,Cr_0)} \frac{r_0^{2p\beta}}{d(\zeta,\zeta_0)^{2p\beta}} \left(1 + \frac{d(\zeta,\zeta_0)}{r_0}\right)^{2p\ell} d\sigma(\zeta) \\ &\lesssim \frac{r_0^{2p\ell}}{\sigma(B)},\end{aligned}$$

for k large enough, where we have used a standard integral estimate, see Lemma 2.2 in [BPS2].

This finishes the proof of the factorization theorem. □

8. Final remarks

We now introduce the class of H-domains.

Definition 8.1. *We say that Ω is an* H-domain *if it is a smooth bounded domain of finite type and the following condition hold: For each $\zeta \in \partial\Omega$ there exist a neighborhood V_ζ and a biholomorphic map Φ_ζ defined on V_ζ such that $\Phi_\zeta(\Omega \cap V_\zeta)$ is geometrically convex.*

Such class of domains, that obviously includes the strongly pseudoconvex domains and the convex domains of finite type, has been analyzed in [BPS3]. In that paper it is proved that the Szegö and Bergman kernel satisfy the same local estimates (8), as in the case of the convex domains of finite type (or, more in particular, of the strongly convex domains). Therefore, Theorems 2.1-2.3 immediately extend to the case of H-domains, since all the arguments are based on the local estimates for the kernels, see [BPS3].

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