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## A Finite Element Approximation of Grazing Collisions

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### ABSTRACT

In this article, we propose a finite element discretization of the Boltzmann-Lorentz operator for which it is possible to define a grazing collision limit. We illustrate this discretization by considering the evolution of a system of particles in a slab, subject to collisions both with the boundaries and with themselves. A comparison is made with isotropic collisions or with the Laplace-Beltrami operator. Moreover, in the case of multiplying boundary conditions, it is proven by numerical simulations the existence of a critical value for the absorption coefficient, which is independent on the grazing collision parameter. Finally, the focalization of a beam is studied and the numerical results are compared with previous simulations.

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## 1. INTRODUCTION

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45 Kinetic equations, as Boltzmann (or Fokker-Planck) equations,  
 46 model the evolution of a system of particles subject to the action of a force  
 47 field and to collisions. They are applied for example in semiconductor  
 48 physics (see Markowich et al. (1994)), in the modeling of rarefied gases  
 49 or plasmas (see Cercignani (1988)), in astrophysics (see Belleni-Morante  
 50 and Moro (1996) for problems relative to interstellar clouds) or in bio-  
 51 mathematics (see Arlotti et al. (2000) for the study of dynamics of popula-  
 52 tions). In the more general frame, the operator describing the scattering  
 53 event is nonlinear and due to its complexity, many other models have been  
 54 derived. Notably, the Lorentz operators, derived from the Boltzmann one  
 55 or from the Fokker-Planck one, model the elastic collisions of particles  
 56 against heaviest target ones (see Degond and Lucquin-Desreux (1996) and  
 57 Lucquin-Desreux (2000) for the disparate masses particles asymptotics).  
 58 In plasma physics for example, the Boltzmann-Lorentz operator repre-  
 59 sents (in first approximation with respect to the mass ratio) the effect  
 60 of collisions due to neutral particles on the electron distribution function.  
 61 On the other hand, the Fokker-Planck-Lorentz operator is the leading  
 62 order term (still with respect to the mass ratio) describing the collisions  
 63 of electrons against ions.

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65 The Fokker-Planck operator is usually considered as an approxi-  
 66 mation of the Boltzmann collision operator when collisions become  
 67 grazing, i.e., when the angle of deviation  $\theta$  of a particle during a collision  
 68 is very small. This essentially occurs for potentials of long range inter-  
 69 action, such as the Coulombian potential in the case of a plasma.  
 70 This grazing collision limit of the Boltzmann operator towards the  
 71 Fokker-Planck one has been proven in Degond and Lucquin-Desreux  
 72 (1992) and Desvilletes (1992). In Desvilletes (1992), the scattering cross  
 73 section is smooth; it depends on a small parameter  $\varepsilon$  and it is localized  
 74 around the value  $\theta = 0$  of the scattering angle when  $\varepsilon$  goes to zero. On the  
 75 other hand, the Coulomb case is studied in Degond (1992) by means of a  
 76 quite different asymptotics: the scattering cross section has a nonintegrable  
 77 singularity when the relative velocity of the colliding particles tends to  
 78 zero but also when the scattering angle  $\theta$  tends to zero. There is thus a  
 79 necessary truncation of this angle around the value zero ( $\theta \geq \varepsilon$ ) and the  
 80 small parameter  $\varepsilon$  has a precise physical meaning: it is clearly identified as  
 81 the plasma parameter. We also refer the reader to the works Goudon  
 82 (1997a, 1997b) concerning the grazing collision limit of the Boltzmann  
 83 operator when the scattering cross section corresponds to a inverse  
 84 power force (i.e.,  $1/r^s$ , with  $s > 1$ ) and without angular cut-off. More  
 recently, the convergence of Boltzmann-Lorentz operator towards the

85 Fokker-Planck-Lorentz (or Laplace-Beltrami) operator in the so-called  
86 grazing collision limit has been proven in Buet et al. (2001), both for  
87 the asymptotics developed in Degond and Lucquin-Desreux (1992) and  
88 Desvillettes (1992).

89 In this article, we are concerned with the numerical approximation of  
90 the Boltzmann-Lorentz operator and with its grazing collision limit. Up  
91 to our knowledge, only few recent articles have been devoted to the  
92 study of the grazing collision limit at the discrete level: we can refer  
93 for example to Guérin and Méléard (2002) and Pareschi et al. (2002)  
94 concerning the full nonlinear Boltzmann operator discretized either by  
95 spectral methods (see Pareschi et al. (2002)) or by particle methods (see  
96 Guérin and Méléard (2002)). In our article, the considered scattering  
97 cross section is a simplified version of the one defined in Buet et al.  
98 (2001): it is assumed to be constant on a very small interval depending  
99 on the small parameter  $\varepsilon$  and zero outside this interval. Our goal is to give  
100 a numerical approximation of the Boltzmann-Lorentz operator for graz-  
101 ing collisions for which it is possible to pass to the limit  $\varepsilon \rightarrow 0$ . We first  
102 decided to apply a finite element method (FEM) because this method is  
103 particularly well adapted to the limit problem (we refer for example to  
104 Cordier et al. (2000) for 3D computations). Secondly, in the FEM  
105 described below, we decided to do exact computations of the collision  
106 terms (instead of doing quadrature formulas): denoting by  $h$  the velocity  
107 step, we can then choose  $\varepsilon \leq h$  and thus study the limit  $\varepsilon \rightarrow 0$  for a fixed  
108 discretization in velocity variable, i.e., for a fixed  $h$  (which would not be  
109 possible when using quadrature formulas).

110 In order to show the efficiency of the proposed FEM approximation,  
111 we will consider a transport equation in which the collision operator is the  
112 Boltzmann-Lorentz operator for grazing collisions defined on a bounded  
113 region. Thus, we equip the kinetic equation with some kind of boundary  
114 conditions: in a first approach, we only take into account specular reflec-  
115 tion boundary conditions. The numerical results obtained with the pro-  
116 posed FEM approximation are compared with those obtained considering  
117 the Laplace-Beltrami operator (i.e., its limit when  $\varepsilon$  goes to zero), with the  
118 isotropic collisions (i.e., the scattering cross section is constant), and also  
119 with the collisionless case. Successively, we consider multiplying reflection  
120 boundary conditions (see for instance Belleni-Morante and Totaro (1996)  
121 and Mancini and Totaro (1998)): each particle colliding on the boundary  
122 of the considered region is reflected and multiplied by a factor bigger than  
123 one. In this optic, as the region where the evolution of particles takes place  
124 is bounded, it seems physically necessary to consider also an absorption  
125 term in our transport equation. The existence and uniqueness of the  
126 solution of such a problem, with the absorption cross section  $\sigma$  bigger

127 than a theoretical value  $\sigma_t$ , has been proven in Belleni-Morante (1996) for  
 128 multiplying Maxwell boundary conditions, and in Mancini and Totaro  
 129 (1998) in the general case. We will numerically prove the existence of a  
 130 critical absorption cross section  $\sigma_n < \sigma_t$  for which the total number of  
 131 particles does not sharply increase nor vanishes with time. Moreover, we  
 132 also remark that both  $\sigma_n$  and the equilibrium profile of the particle density  
 133 do not depend on the initial data.

134 The article is organized as follows. In Sec. 2, we present the finite  
 135 element approximation for the Boltzmann-Lorentz operator for grazing  
 136 collisions and also for isotropic collisions. In Sec. 3, we present the test  
 137 problems used to validate our numerical approximation. Section 4 con-  
 138 cerns the results of the numerical simulations. Finally, in Sec. 5, we give  
 139 some concluding remarks.

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## 143 2. NUMERICAL SCHEMES

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145 The goal of this article is the numerical approximation of the  
 146 Boltzmann-Lorentz operator for grazing collisions. We recall that the  
 147 Boltzmann-Lorentz operator models for example the elastic collisions of  
 148 a heavy particle against a light one (see Degond and Lucquin-Desreux,  
 149 (1996) and Lucquin-Desreux (2000)). Thus, it acts only on the velocity  
 150 direction  $\theta$  of the particles, leaving unchanged the velocity modulus  $|v|$   
 151 (which we will consider normalized to one). In the bi-dimensional case,  
 152 the Boltzmann-Lorentz operator is given by:

$$153 \quad Q(f)(\theta) = \int_{S^1} B(\theta - \theta') (f(\theta') - f(\theta)) d\theta', \quad (1)$$

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156 where  $B(\theta - \theta')$  is the scattering cross section which represents the  
 157 probability for a particle entering a collision with velocity direction  $\theta'$   
 158 to out-go the scattering event with a velocity direction  $\theta$ . For instance,  
 159  $B(\theta - \theta')$  may be determined from the interaction force between the  
 160 particles (see Degond and Lucquin-Desreux (1992), Desvilletes (1992),  
 161 and Goudon (1997b)); in particular, we recall that, in the case of isotropic  
 162 collisions, it does not depend on  $\theta$  nor on  $\theta'$ .

163 We are interested in grazing collisions, i.e., we want to take into  
 164 account also the very small deviations in the trajectories of the particles.  
 165 This fact is emphasized by introducing the dependence of the scattering  
 166 cross section  $B$  on a small parameter  $\varepsilon$ . Recently, Buet, Cordier, and  
 167 Lucquin-Desreux (see Buet et al. (2001)) have proven the convergence,  
 168 when the small parameter  $\varepsilon$  tends to zero, of the Boltzmann-Lorentz

169 operator for grazing collisions towards the Fokker-Planck-Lorentz  
170 operator, which is simply the Laplace-Beltrami operator.

171 Our goal here is to give a numerical approximation of the  
172 Boltzmann-Lorentz operator for which it is possible to pass to the limit  
173  $\varepsilon \rightarrow 0$  for a fixed velocity discretization. As shown in the sequel, this is  
174 possible when applying for example a finite element approach, where the  
175 integral terms are all exactly computed. The numerical results we obtain  
176 are then compared with the collisionless case, the case of isotropic  
177 collisions, and those obtained when considering the Laplace-Beltrami  
178 operator.

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### 2.1. An Implicit Finite Element Approximation

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We want to approximate by means of a finite element method the  
following Cauchy problem:

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$$\begin{cases} \frac{\partial f}{\partial t} = Q(f), \\ f(t=0) = f^0, \end{cases} \quad (2)$$

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where  $Q(f)$  is defined by Eq. (1). We note that the dependence of the  
density  $f$  on the space variable is not underlined in this paragraph  
because the position of the particles does not play any role during a  
collision. The angle variable  $\theta$  belongs to the unit circle  $S^1$ , i.e.,  
 $\theta \in [-\pi, \pi]$  and the distribution function  $f$  satisfies periodic boundary  
conditions.

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We first consider the following piecewise affine approximation  $f^{N_\theta}$  of  
the function  $f$ :

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$$f(t, \theta) \simeq f^{N_\theta}(t, \theta) = \sum_{j=0}^{N_\theta-1} \varphi_j(\theta) f_j(t),$$

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where  $\varphi_j$ ,  $j \in \{0, \dots, N_\theta - 1\}$ , are the classical ‘‘hat functions’’ of the  $P^1$   
finite element approximation. More precisely, denoting by  $\theta_j = -\pi + hj$ ,  
with  $h = 2\pi/N_\theta$ , we have  $\varphi_i(\theta_j) = \delta_{ij}$  ( $= 1$  if  $i=j$  and  $0$  otherwise), so that  
 $f_j(t) = f^{N_\theta}(t, \theta_j)$ .

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The weak discretized form of Eq. (2) then reads:

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$$\begin{aligned} & \sum_{j=0}^{N_\theta-1} \frac{\partial f_j}{\partial t} \int_{S^1} \varphi_j(\theta) \varphi_i(\theta) d\theta \\ & = \sum_{j=0}^{N_\theta-1} f_j \int_{S^1} \int_{S^1} B(\theta - \theta') [\varphi_j(\theta') \varphi_i(\theta) - \varphi_j(\theta) \varphi_i(\theta')] d\theta' d\theta. \end{aligned}$$

211 Setting:

$$\begin{cases}
 212 & A_{ij} = \int_{S^1} \varphi_i(\theta) \varphi_j(\theta) d\theta, \\
 213 & \\
 214 & \\
 215 & Q_{ij} = \int_{S^1} \int_{S^1} B(\theta - \theta') [\varphi_i(\theta) \varphi_j(\theta') - \varphi_i(\theta') \varphi_j(\theta)] d\theta d\theta', \\
 216 &
 \end{cases} \quad (3)$$

217 we write the discretized problem more compactly as follows:

$$\sum_{j=0}^{N_\theta-1} \frac{\partial f_j}{\partial t} A_{ij} = \sum_{j=0}^{N_\theta-1} f_j Q_{ij}. \quad (4)$$

222 The matrix  $A$  with entries  $A_{ij}$  is the so called ‘‘mass matrix,’’ while the  
 223 matrix  $Q$  with entries  $Q_{ij}$  represents the collision matrix. In the next  
 224 paragraph, we give the expression of these two matrices in the context  
 225 of grazing collisions; the detailed computations are performed in  
 226 the Appendix A1. Before this, let us precise the time discretization of  
 227 Eq. (4). In order to avoid a limitation of the time step  $\Delta t$  in terms of the  
 228 angular step  $\Delta\theta = h$ , we will use a fully implicit scheme. We introduce  
 229 some notations. Let us denote by  $F^n$  the  $N_\theta$  dimensional vector with  
 230 entries  $F_j^n = f(n\Delta t, \theta_j)$ , for  $j \in \{0, \dots, N_\theta - 1\}$ . The implicit finite element  
 231 scheme we consider reads:

$$\begin{cases}
 232 & F^{n+1} = (A - \Delta t Q)^{-1} A F^n, \quad n \geq 0 \\
 233 & \\
 234 & F^0 = (F_i^0)_{0 \leq j \leq N_\theta-1}, \\
 235 &
 \end{cases} \quad (5)$$

236 with  $F_j^0 = f^0(\theta_j)$ .

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### 2.2. Grazing Collisions

241 In the case of grazing collisions, the scattering cross section  $B$  is  
 242 localized and depends on a small parameter  $\varepsilon$ . In the nonCoulombian  
 243 case,  $B$  has the following expression (see Buet et al. (2001) and  
 244 Desvilletes (1992)):

$$B^\varepsilon(z) = \frac{1}{\varepsilon^3} \bar{B}(z/\varepsilon) \chi_{S^1}(z/\varepsilon)$$

248 where  $\chi$  denotes the characteristic function (i.e.,  $\chi_{S^1}(z) = 1$  if  $z \in S^1$  and  
 249 0 otherwise). We will here consider a simplified model for which the  
 250 scattering cross section is given by:

$$B^\varepsilon(z) = \begin{cases} B_0 & \text{if } |z| < \varepsilon\pi, \\ 0 & \text{else} \end{cases}$$

253 with  $B_0 = 1/\varepsilon^3$ . We then have:

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255 **Proposition 2.1.** *Setting  $\alpha = \varepsilon\pi/h$  and assuming that  $\alpha \leq 1$ , the grazing*  
 256 *collision matrix  $Q$  is given by:*

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$$Q = \frac{\pi^3}{3h} \begin{pmatrix} -2 + \frac{3}{4}\alpha & 1 - \frac{1}{2}\alpha & \frac{1}{8}\alpha & 0 & \dots & 0 & \frac{1}{8}\alpha & 1 - \frac{1}{2}\alpha \\ 1 - \frac{1}{2}\alpha & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{1}{8}\alpha \\ \frac{1}{8}\alpha & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{1}{8}\alpha \\ \frac{1}{8}\alpha & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 - \frac{1}{2}\alpha \\ 1 - \frac{1}{2}\alpha & \frac{1}{8}\alpha & 0 & \dots & 0 & \frac{1}{8}\alpha & 1 - \frac{1}{2}\alpha & -2 + \frac{3}{4}\alpha \end{pmatrix}. \quad (6)$$

270 **Proof.** The proof is detailed in the Appendix A1.

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By means of the same arguments applied in order to compute the matrix  $Q$  (see Appendix A1), we have that the mass matrix  $A$  reads:

$$A = \frac{h}{3} \begin{pmatrix} 2 & \frac{1}{2} & 0 & \dots & 0 & \frac{1}{2} \\ \frac{1}{2} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \frac{1}{2} \\ \frac{1}{2} & 0 & \dots & 0 & \frac{1}{2} & 2 \end{pmatrix}. \quad (7)$$

We remark that if we let  $\varepsilon \rightarrow 0$  then  $Q_{ij} \rightarrow (h\pi^3)/(3D_{ij})$ , where  $D_{ij}$  is the classical finite difference discretization of the Laplace-Beltrami operator  $\Delta f = \partial_{\theta\theta} f$ :

$$D = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & 0 & 1 & -2 \end{pmatrix}.$$

295 Moreover, the coefficient  $\pi^3/3$  is strictly related to the second order  
 296 moment of the scattering cross section, which in our case is equal to  $\pi^3/3$   
 297 (since  $\bar{B} = 1$ ).  
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### 2.3. Isotropic Collisions

302 In the following simulations, in order to compare the evolution of  
 303 the system of particles with grazing collisions and with isotropic  
 304 collisions, we will not compute the exact solution for the isotropic  
 305 Boltzmann-Lorentz operator (although it is possible), but we will approx-  
 306 imate the isotropic operator following the same discretization as before.

307 Recalling that for isotropic collisions the scattering cross section  $B$  is  
 308 a constant (which we will assume equal to one), we get the following finite  
 309 element discretization of the isotropic Lorentz operator. The gain matrix  
 310  $G$  is given by:  $G = (G_{ij})_{0 \leq i, j \leq N_\theta - 1}$ , with  $G_{ij} = h^2$ , whereas, the loss matrix  
 311  $L$  is such that:  $L = 2\pi A$ . Then we can apply the implicit finite element  
 312 scheme (5), where now the matrix  $Q = G - L$  represents the  $P^1$  finite  
 313 element approximation of the isotropic Lorentz operator.

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## 3. THE TEST PROBLEM

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We illustrate the finite element approximation for the Boltzmann-Lorentz operator with grazing collisions on the following transport equation:

$$\partial_t f + \cos \theta \partial_x f + \sigma f = \frac{1}{\tau} Q(f). \quad (8)$$

This transport type equation describes the evolution of a system of particles moving between two parallel plates (slab), subject to absorption by the host medium and to elastic collisions with other particles present in the slab. We recall that  $f = f(x, \theta, t)$  represents the number of particles which at time  $t \geq 0$ , are in the position  $x \in [-L, +L]$  with velocity  $v = (\cos \theta, \sin \theta)$ ,  $\theta \in [-\pi, \pi]$ . Moreover,  $Q(f)$  is the Lorentz collision operator defined by Eq. (1),  $\sigma \geq 0$  is the absorption cross section, and  $\tau$  is the relaxation time; this one is assumed to be equal to one, except in the last numerical test concerning the defocusing of a beam (see Sec. 4.4 below and Cordier et al. (2000)).

The interactions of the particles with the two plates of the slab may be described by means of a linear, bounded, and positive operator

337  $\Lambda : \Gamma^{\text{out}} \rightarrow \Gamma^{\text{in}}$  which relates the incoming and outgoing flux of particles  
338 as follows:

$$339 \quad f^{\text{in}} = \Lambda f^{\text{out}}. \quad (9)$$

341 In Eq. (9),  $f^{\text{in}}$  and  $f^{\text{out}}$  are the traces of the distribution function  
342 respectively to the incoming and outgoing boundary sets:

$$343 \quad \Gamma^{\text{in}} = \{(-L, \theta), \cos \theta > 0\} \times \{(L, \theta), \cos \theta < 0\},$$

$$344 \quad \Gamma^{\text{out}} = \{(L, \theta), \cos \theta > 0\} \times \{(-L, \theta), \cos \theta < 0\}.$$

346 We define the norms on the incoming and outgoing sets as usual:

$$347 \quad \|f\|_{\text{out}} = \int_{(x, \theta) \in \Gamma^{\text{out}}} f(x, \theta) |\cos \theta| d\theta, \quad \|f\|_{\text{in}} = \int_{(x, \theta) \in \Gamma^{\text{in}}} f(x, \theta) |\cos \theta| d\theta,$$

350 and we consider the norm for the operator  $\Lambda$  to be given by:

$$351 \quad \|\Lambda\| = \max_{\|f^{\text{out}}\|_{\text{out}}=1} \|\Lambda f^{\text{out}}\|_{\text{in}}.$$

354 We will consider multiplying boundary conditions (see Mancini and  
355 Totaro (1998)), i.e., the operator  $\Lambda$  is such that  $\|\Lambda f^{\text{out}}\|_{\text{in}} \geq c \|f^{\text{out}}\|_{\text{out}}$ ,  
356 with  $c > 1$ . In other words, a multiplication of particles takes place on the  
357 boundaries. For instance, the boundary operator  $\Lambda$  may represent reflection,  
358 or diffusion boundary conditions, as well as a linear combination of  
359 them. In the following simulations, we will consider multiplying reflective  
360 boundary conditions:

$$361 \quad \begin{cases} f(-L, \theta) = \alpha_L f(-L, \theta'), & \text{for } \cos \theta > 0, \\ f(L, \theta) = \alpha_R f(L, \theta'), & \text{for } \cos \theta < 0, \end{cases} \quad (10)$$

364 with  $\theta' = \pi - \theta$ , where  $\alpha_L$  and  $\alpha_R$  respectively denote the reflection coefficients  
365 on the left and right boundaries of the slab (i.e., on  $x = -L$  and  
366  $x = L$  respectively). As the region where the evolution takes place is  
367 bounded, it is natural to consider also an absorption term  $\sigma f$  in order  
368 to avoid the growth of the number of particles (see the results of Fig. 7  
369 below). Summarizing, the problem which we approximate numerically  
370 reads as follows:

$$371 \quad \begin{cases} \partial_t f + \cos \theta \partial_x f + \sigma f = \tau^{-1} Q(f), \\ f^{\text{in}} = \Lambda f^{\text{out}}, \\ f(x, \theta, 0) = f_0(x, \theta). \end{cases} \quad (11)$$

376 It is clear that the efficiency of the absorption coefficient  $\sigma$  will  
377 depend on the “power” of the multiplication coefficient (i.e., the norm  
378 of the operator  $\Lambda$ ), on the modulus of the particle velocities  $|v|$  (here

379 equal to 1) and on the dimension of the considered region (here the  
 380 distance  $2L$  between the two plates). It has been proven in Mancini  
 381 and Totaro (1998), by means of the semi-groups theory and for a fixed  
 382  $\varepsilon$ , that if  $\|\Lambda\| \leq 1$ , then problem (11) admits a unique solution whatever is  
 383  $\sigma \geq 0$ ; whereas, if  $\|\Lambda\| > 1$ , then problem (11) admits a unique solution  
 384 provided that:

$$385 \quad \sigma > \frac{|v| \ln \|\Lambda\|}{2L} = \sigma_i; \quad (12)$$

388  $\sigma_i$  is called the theoretical critical absorption cross section. In the  
 389 particular case of multiplying reflective boundary conditions (10), the  
 390 theoretical value for the critical absorption cross section is given by  
 391 (see Belleni-Morante and Totaro (1996)):

$$393 \quad \sigma_i = \frac{|v| \ln(\max\{\alpha_L, \alpha_R\})}{2L}. \quad (13)$$

396 We will numerically compute a critical value  $\sigma_n$  for the absorption  
 397 cross section and we will precise the behavior of the solution of problem  
 398 (11) in terms of the absorption coefficient  $\sigma$ : we will see that, if  $\sigma < \sigma_n$  the  
 399 particle density significantly grows in a finite time, for  $\sigma > \sigma_n$  the particle  
 400 density rapidly goes to zero in a finite time, and for  $\sigma = \sigma_n$  it remains  
 401 bounded without vanishing (see Sec. 4.3 below). This numerical critical  
 402 value  $\sigma_n$  does not depend on  $\varepsilon$ ; moreover, it is smaller than the theoretical  
 403 one  $\sigma_i$ . This fact does not contradict at all the results proven in  
 404 Belleni-Morante and Totaro (1996) or Mancini and Totaro (1998); in  
 405 fact, it just shows the existence of a solution also for absorption cross  
 406 sections  $\sigma \in ]\sigma_n, \sigma_i[$  (since  $\sigma_n < \sigma_i$ ), which was not possible to prove by  
 407 means of the semigroup theory.

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#### 411 4. NUMERICAL SIMULATIONS

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413 In order to simulate the evolution problem (11), we apply a first  
 414 order splitting in time method. First, we approximate by means of an  
 415 upwind explicit finite difference method the following transport problem  
 (see paragraph 4.1 below):

416

$$417 \quad \begin{cases} \frac{\partial f}{\partial t} + \cos \theta \frac{\partial f}{\partial x} + \sigma f = 0, \\ f^{\text{in}} = \Lambda f^{\text{out}}, \\ f(x, \theta, 0) = f_0(x, \theta). \end{cases} \quad (14)$$

420

421       Second, we apply the implicit finite element scheme of Sec. 2.1 to the  
422 pure collision problem:

$$423 \quad \begin{cases} \frac{\partial f}{\partial t} = \tau^{-1} Qf \\ 425 \quad f(x, \theta, 0) = f_0(x, \theta). \end{cases} \quad (15)$$

426  
427       At each iteration, the program solves separately the transport and  
428 collision parts. Our whole numerical scheme is first order in all variables.  
429 We have done this choice in order to simplify the presentation of  
430 the method, but it would be naturally possible to improve the accuracy  
431 of the method by using appropriate second order schemes for each  
432 step (convection, collision) coupled with a second order (of Strang  
433 type) splitting algorithm.

434       We will here perform essentially three different numerical tests. In the  
435 first two ones, developed in paragraphs 4.2 and 4.3 below, the relaxation  
436 time  $\tau$  is equal to one. In paragraph 4.2, we essentially focalize on the  
437 discrete collision operator itself: the boundary conditions are classical  
438 specular reflection, and there is no absorption; the grazing collision  
439 operator is then compared to the Laplace-Beltrami operator (i.e., to its  
440 limit when  $\varepsilon$  goes to zero), to the isotropic operator and also to the  
441 collisionless case. The dependence of the solution on the grazing  
442 parameter  $\varepsilon$  is also numerically investigated, first for  $\varepsilon \leq h/\pi$ , but also  
443 for bigger values of  $\varepsilon$ . The influence of multiplicative boundary condi-  
444 tions is then studied in paragraph 4.3: in particular, the existence of a  
445 numerical critical absorption cross section, which does not depend on  $\varepsilon$   
446 (for sufficiently small values of  $\varepsilon$ ) nor on the initial data is shown.  
447 Moreover, we underline the existence of a unique profile of equilibrium  
448 (independent on  $\varepsilon$  and on the initial data) for the particle density. Finally,  
449 in paragraph 4.4, we study the influence of the time relaxation  $\tau$  for a  
450 photonic type problem already considered in a previous work (see Cordier  
451 et al. (2000)). This last test is further investigated in Cordier et al. (2002).  
452 Let us firstly precise the numerical scheme used to approximate the  
453 convective part of the equation, i.e., problem (14).

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#### 4.1. Transport Problem Approximation

457

458       We consider  $N_x (= 50$  in simulations, unless otherwise specified)  
459 discretization points  $x_i$ ,  $i \in \{1, \dots, N_x\}$ , for the space variable  
460  $x \in [-L, L]$  (with  $L = 1$ ) and  $N_\theta (= 50$  in simulations, unless otherwise  
461 specified) discretization points for the angle variable  $\theta \in S^1$ . We introduce  
462 the following symmetric discretization of the circle  $S^1$ :  $\theta_j = -\pi + hj$  with

463  $h = 2\pi/N_\theta$  and  $j = 0, \dots, N_\theta - 1$ . We have that if  $\cos \theta_j$  is a discretization  
 464 point, then also  $-\cos \theta_j$  is one. Moreover, if  $j$  and  $j'$  are such that  $j' = s(j)$ ,  
 465 where  $s(j) = (N_\theta/2 - j) \bmod(N)$  (see Fig. 1), then:  $\cos \theta_{j'} = -\cos \theta_j$ .

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466 Considering the above discretization, we approximate the transport  
 467 problem (14) by means of an upwind explicit finite difference scheme as  
 468 follows:

469

470 • If  $\cos \theta_j > 0$  then,

471

$$472 \quad f_{1,j}^{n+1} = \alpha_L f_{1,s(j)}^{n+1}$$

473

474 and for  $i = 2 \dots N_x$

475

$$476 \quad f_{ij}^{n+1} = (1 - (\Delta t/\Delta x) \cos \theta_j - \sigma \Delta t) f_{ij}^n + (\Delta t/\Delta x) \cos \theta_j f_{i-1,j}^n. \quad (16)$$

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• If  $\cos \theta_j < 0$  then

$$481 \quad f_{N_x,j}^{n+1} = \alpha_R f_{N_x,s(j)}^{n+1}$$

483 and for  $i = 1 \dots N_x - 1$

$$484 \quad f_{ij}^{n+1} = (1 + (\Delta t/\Delta x) \cos \theta_j - \sigma \Delta t) f_{ij}^n - (\Delta t/\Delta x) \cos \theta_j f_{i+1,j}^n. \quad (17)$$

We recall that  $f_{ij}^n = f(x_i, \cos \theta_j, t_n)$  is the value of the distribution  
 function  $f$  at the position  $x_i = -L + \Delta x(i - 1)$ ,  $i = 1 \dots N_x$ , and  
 $\Delta x = 2L/(N_x - 1)$ , with velocity  $\cos \theta_j$ ,  $j = 0 \dots N_\theta - 1$ , and at time  
 $t_n = n \Delta t$ ,  $n = 1 \dots N_{\text{iter}}$ .

Setting  $\lambda = \Delta t/\Delta x$ , the CFL condition writes (for any  $\sigma \geq 0$ ):  $\lambda \leq 1$ .  
 In the simulations below,  $\lambda$  is taken equal to 0.8 (but  $\lambda = 1$  would give  
 exactly the same results). Moreover, we choose  $\alpha_L = 2\alpha_R = 4$  and  
 $\varepsilon = 0.001$ , unless otherwise specified.

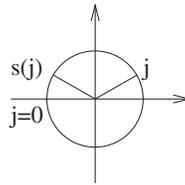


Figure 1. Discretization on the sphere.

#### 4.2. Grazing Collisions and Comparison with Other Collision Terms

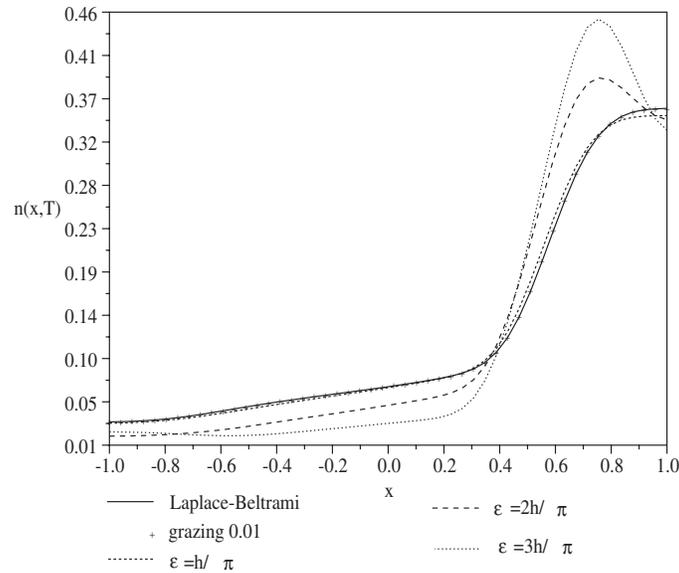
In this section we present numerical results concerning the grazing collision limit at the discrete level and also a comparison with different types of collisions. We take into account only specular reflection boundary conditions (i.e.,  $\alpha_L = \alpha_R = 1$ ), the absorption cross section is given by  $\sigma = 0$  (i.e., the number of particles is constant along all the evolution), and the relaxation time is  $\tau = 1$ . We introduce the density function:

$$n(x, t) = \int f(x, \theta, t) d\theta$$

and the number of particles inside the slab at time  $t$ :

$$N(t) = \int_{-L}^{+L} n(x, t) dx. \quad (18)$$

In Fig. 2, we show how the behavior of the density  $n(x, t)$  changes when changing the value of the grazing parameter  $\varepsilon$ . We have first considered values of  $\varepsilon$  such that  $\varepsilon \leq h/\pi$ , but also larger values of  $\varepsilon$ : in this last case,



**Figure 2.** The density  $n(x, T)$ , with  $\sigma = 0$ ,  $\alpha_R = \alpha_L = 1$ , for different values of  $\varepsilon$ :  $\varepsilon = 10^{-2}$ ,  $h/\pi$ ,  $2h/\pi$ ,  $3h/\pi$  and for the Laplace-Beltrami operator, with initial data given by a Dirac,  $t = \Delta t N_{\text{iter}}$ , with  $N_{\text{iter}} = 100$ ,  $N_x = 50$ ,  $N_\theta = 10$ .

547 the collision terms have been computed by use of standard first  
 548 order quadrature formulas. In particular, we note that for  $\varepsilon < 10^{-2}$  the  
 549 difference between the grazing collision case and the Laplace-Beltrami  
 550 operator is too small to be observed. In fact, the difference is of order  $\varepsilon h$ ,  
 551 so that it is rapidly very small for small values of  $\varepsilon$  (and for small  $h$  as  
 552 well). In order to precise this behavior, we have computed in Fig. 3 the  
 553  $L^\infty$  norm of the relative error between the density  $n_\varepsilon$  of the grazing  
 554 collision problem and that of the limiting one, denoted by  $n_{LB}$ , i.e.,

$$\max_i \left( \frac{n_\varepsilon(i, T) - n_{LB}(i, T)}{n_{LB}(i, T)} \right),$$

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 558 in terms of  $\varepsilon$  and for different values of  $h$  (with  $T = 50\Delta t$ ). We first  
 559 remark that this error is very small for  $\varepsilon = 10^{-3}$ . But as expected, we  
 560 observe a difference between the case  $\varepsilon = h/\pi$  and  $\varepsilon = 10^{-2}$  for large  
 561 values of  $h$  ( $h = \pi/2$ ), while this difference decreases for smaller values  
 562 of  $h$ . Figure 2 is related to the “medium” case  $h = \pi/5$  (i.e.,  $N_\theta = 10$ ).

563 In Fig. 4, we compare four types of collisions: the collisionless case  
 564 ( $Q = 0$ ), the case of isotropic collisions (with  $B = 1/4\pi$  in order to fit the  
 565 first nonzero eigenvalues), the grazing collisions case (with  $\varepsilon = 10^{-3}$ ) and  
 566 its limit when  $\varepsilon \rightarrow 0$ , i.e., the Laplace-Beltrami operator. We trace the  
 567 evolution of the density  $n(\cdot, T)$  for four different times of computation  
 568  $T = \Delta t N_{\text{iter}}$ , with  $N_{\text{iter}} = 50, 100, 150, 200$ . We observe that the grazing  
 569 collisions case and the Laplace-Beltrami operator give the same results  
 570 (up to a difference of order  $10^{-4}$  which is not noticeable on the figure).

571 Naturally, in all cases, we have checked numerically that the total  
 572 density  $N(t)$  defined by Eq. (18) was constant with respect to time.

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#### 4.3. Absorption Cross Section and Multiplying Boundary Conditions

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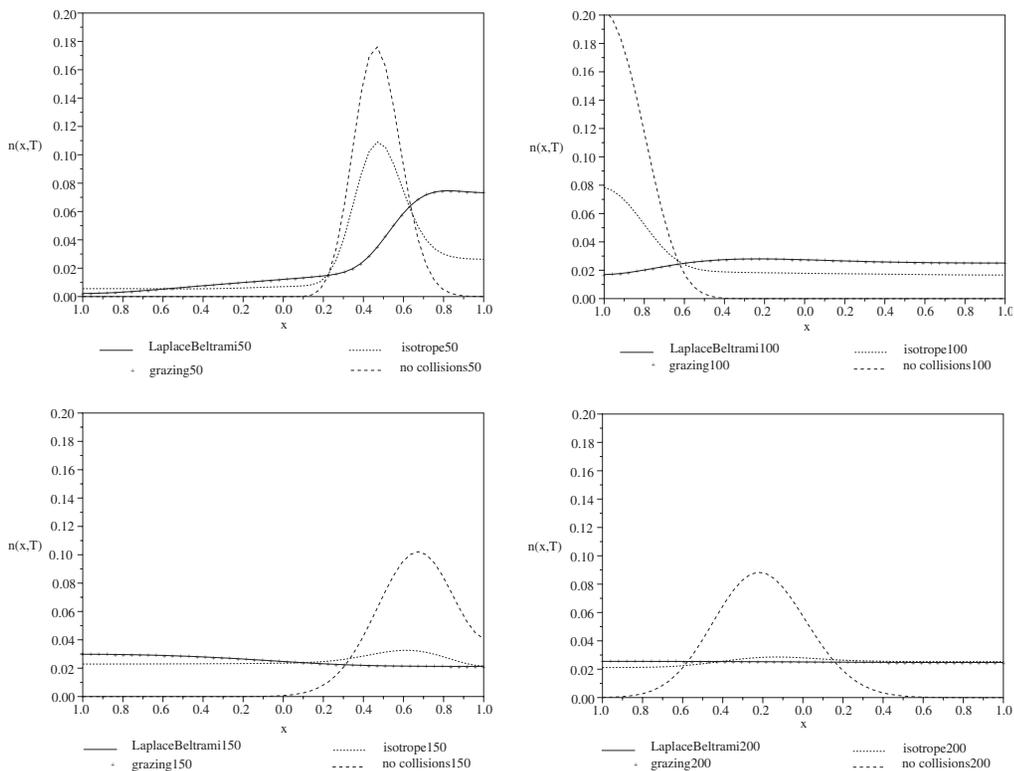
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	$N_\theta = 4$	$N_\theta = 10$	$N_\theta = 20$
$\varepsilon = h/\pi$	0.15639	0.03291	0.01392
$\varepsilon = 10^{-2}$	0.00269	0.00158	0.00136
$\varepsilon = 10^{-3}$	0.00027	0.00016	0.00013

**Figure 3.** Influence of parameters  $\varepsilon$  and  $h$ .

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**Figure 4.** The density  $n(\cdot, T)$ , with  $\sigma = 0$ ,  $\alpha_R = \alpha_L = 1$ , for the grazing collisions with  $\varepsilon = 0.001$ , the Laplace-Beltrami operator, the isotropic collisions case, and the collisionless case, with initial data given by a Dirac,  $T = \Delta t N_{\text{iter}}$ , with  $N_{\text{iter}} = 50$  (top on the left), 100 (top on the right), 150 (bottom on the left), 200 (bottom on the right).

589 data: first, a constant function both in space and velocity; second, a Dirac  
590 mass both in space and velocity.

591 In Fig. 5, we show the convergence of the sequence of critical absorp-  
592 tion cross sections towards a finite value  $\sigma_n$  for  $\varepsilon=0.001$ . The critical  
593 absorption value we compute is independent (for large times) on the  
594 initial data: in fact we observe that both the constant initial data and  
595 the Dirac mass give the same value. Moreover, this value is smaller than  
596 the theoretical one  $\sigma_l$  given by Eq. (13). The way the computation is  
597 performed is the following one:

- 598
- 599 • We compute the total densities  $N(t_i, \sigma)$  for  $t_i = (i + 1)50\Delta t$ , with  
600  $i = 0, \dots, 9$ , and for different values of  $\sigma$ .
  - 601 • We determine the intersection point of two successive curves  
602  $N(t_i, \cdot)$  and  $N(t_{i+1}, \cdot)$  describing the evolution of the density in  
603 terms of  $\sigma$ ; the respective value of  $\sigma$  for the intersection points  
604 are called critical and denoted by  $\sigma_c(i)$ . In fact, for each  $\sigma_c(i)$ , the  
605 total density is constant for the two successive times of iteration  $t_i$   
606 and  $t_{i+1}$ .

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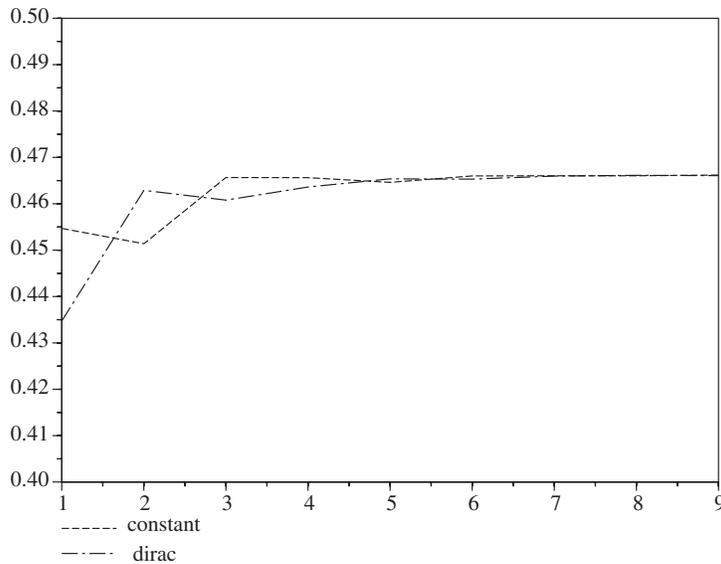
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627 **Figure 5.** The interpolation curve of  $\sigma_c(i)$ . Each point is given by the intersection  
628 of the total density curves for two successive iteration times, i.e. 1 corresponds to  
629 the intersection point of 50 and 100 iteration curves. We increase of 50 iterations  
630 at every ordinate point, finishing by 500 iterations.

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631 We observe in Fig. 5 that these values of critical absorption cross  
 632 sections numerically converge to a finite value  $\sigma_n$ , which is the searched  
 633 value. Moreover, additional numerical simulations show that this value  
 634  $\sigma_n$  does not depend on  $\varepsilon$  (for sufficiently small values of  $\varepsilon$ ).

635 In Fig. 6, we show the limit profile of the curve  $n(\cdot, t)$  for the computed critical absorption  $\sigma = \sigma_n = 0.466$  and  $t = 500\Delta t$ . Again the initial  
 636 condition is given by a constant function or a Dirac mass. We underline  
 637 the fact that for both initial data, we end up with the same (up to a  
 638 normalization) profile which does not depend on  $\varepsilon$ , for  $\varepsilon$  small enough.

639 In Fig. 7, we show the influence of multiplying reflection boundary  
 640 conditions with respect to different values of the absorption cross section.  
 641 When time evolves, we clearly see the significant growth of the number  
 642 of particles  $N(t)$  when  $\sigma < \sigma_n$ , and its fast shoot to zero for  $\sigma > \sigma_n$ .  
 643 For  $\sigma = \sigma_n$ , this number remains bounded without “exploding” both in  
 644 the case of the initial condition given by a constant function or a Dirac  
 645 mass. Moreover, it seems to reach an almost constant value. Hence, this  
 646 numerical approximation  $\sigma_n$  of the critical cross-section  $\sigma_c$  done in Fig. 5  
 647 seems to be satisfactory.

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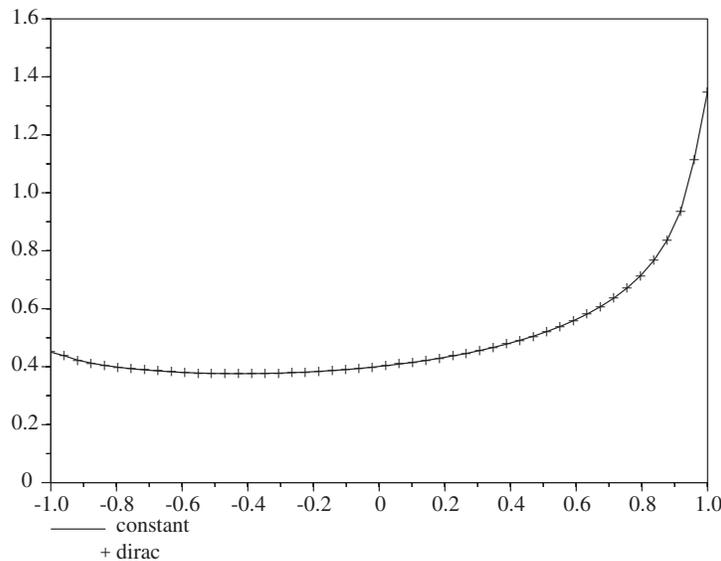
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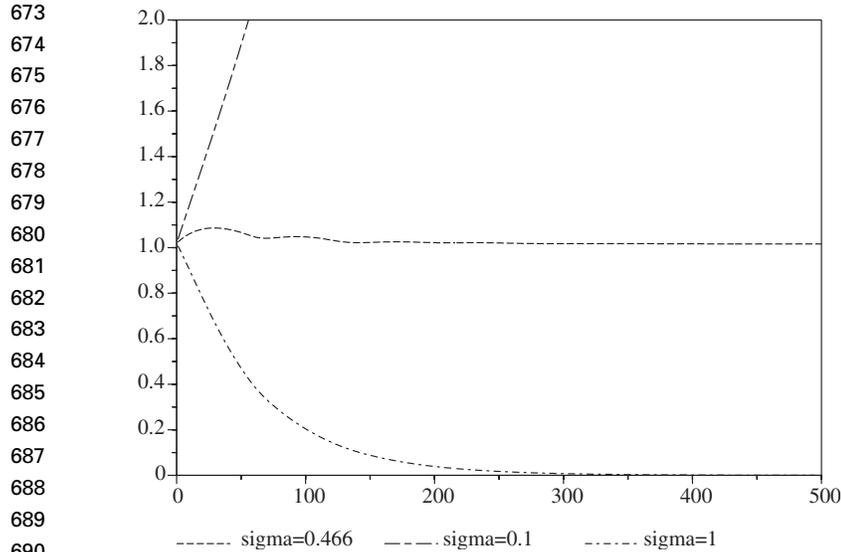
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670 **Figure 6.** The final profile of  $n(\cdot, t)$  with initial data a Dirac mass or a constant  
 671 function, both in position and velocity;  $\sigma = 0.466$ ,  $\varepsilon = 0.001$ ,  $\alpha_R = 2\alpha_L = 4$ ,  
 672  $t = 500\Delta t$ .



691 **Figure 7.** The total density  $N(t)$  for different values of the absorption ( $\sigma > \sigma_n$ ,  
 692  $\sigma < \sigma_n$ , and  $\sigma = \sigma_n = 0.466$ ) with initial data a constant function,  $\varepsilon = 0.001$  and  
 693  $t = 500\Delta t$ .

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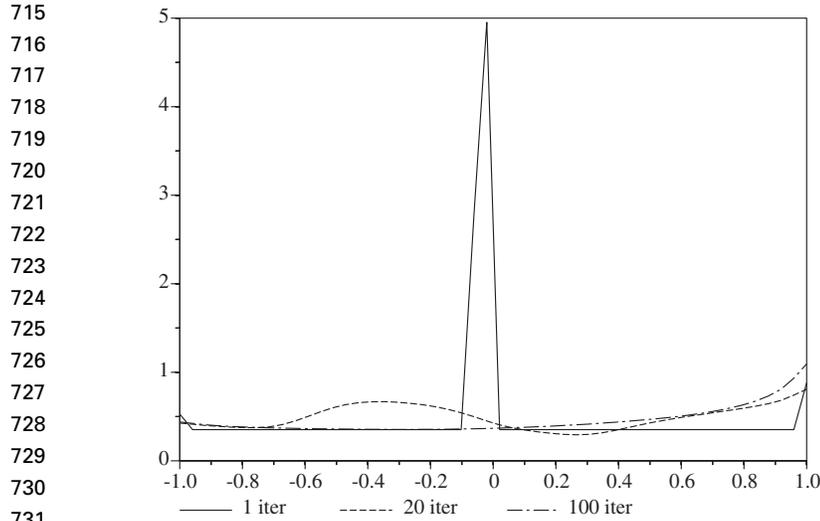
696 Finally, in Fig. 8, we show the behavior of the density function  $n(\cdot, t)$   
 697 for three different iteration times. The initial data which we consider is a  
 698 Dirac mass both in space and velocity. The absorption cross section  $\sigma$  is  
 699 the numerical critical coefficient  $\sigma_n = 0.466$ . It is clearly seen that the  
 700 Dirac mass is convected and diffused respectively by means of the trans-  
 701 port equation and of the Boltzmann-Lorentz operator for grazing collisions  
 702 (with  $\varepsilon = 0.001$ ). Finally, the density is also multiplied and reflected  
 703 at the boundaries, and we can see the beginning of boundary layers.

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#### 4.4. Defocusing of a Beam

708 Following the test performed in Cordier et al. (2000) for a photonic  
 709 type model, we now assume that between the two plates there is vacuum.  
 710 Moreover, a focussed beam enters in the slab on the left side (i.e., at  
 711  $x = -L$ ) and scatters. The absorption cross section is fixed to  $\sigma = 0$ ,  
 712 the grazing coefficient to  $\varepsilon = 0.001$  and the discretization points in  $x$   
 713 and  $\theta$  respectively to  $N_x = 50$  and  $N_\theta = 50$ . We now consider the influ-  
 714 ence of the relaxation time  $\tau$ . The boundary condition corresponding to



**Figure 8.** The evolution of  $n(\cdot, t)$  with initial data a Dirac mass (normalized) in position and velocity.

736 the injection of a mono-kinetic beam in the direction perpendicular to the  
737 planes of the slab is given by:

$$738 \quad f(-L, \cos \theta > 0, t) = \delta,$$

739 where  $\delta$  denotes the delta measure with respect to the angle variable  $\theta$ ; on  
740 the right boundary we have:

$$742 \quad f(L, \cos \theta < 0, t) = 0,$$

743 which means that the particles leaving the slab on the right side never  
744 return.

745 In Fig. 9, we draw the focalization coefficient, defined as the ratio  
746 between the particles leaving the slab on the right side with a velocity  
747 perpendicular to the plane, and the injected particles:

$$748 \quad \mathcal{F}(\tau, t) = f(L, \theta = 0, t).$$

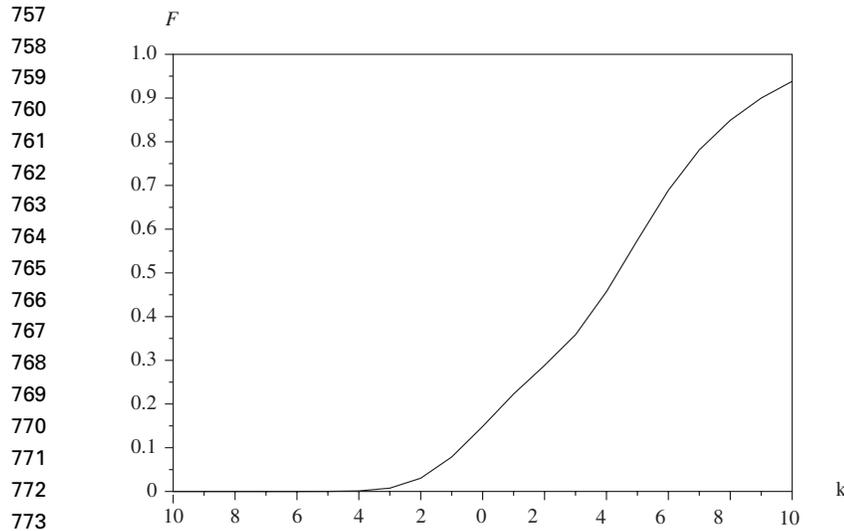
749 The computations are done for  $t$  sufficiently large ( $t = 150\Delta t$ ) so that  
750 the density has reached a stationary state.

752 Finally, in Fig. 10 we show for three values of the relaxation time,  
753  $\tau = 2^{-5}, 1, 2^5$ , and for four iteration times  $N_{\text{iter}} = 25, 50, 100, 200$  the  
754 particle density  $n(\cdot, t)$  evolution (with  $t = \Delta t N_{\text{iter}}$ ).

755 Let us note that the numerical results we have obtained for this  
756 model are in perfect agreement with those performed in Cordier et al. (2000)

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774 **Figure 9.** The focalization  $\mathcal{F}(\tau, t = 150\Delta t)$ .  $\tau = 2^k$ ,  $k = -10, \dots, 10$ .

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777 for a 1D in space and 2D in velocity (i.e.,  $R \times S^2$ ) analogous problem,  
778 where the collision operator involved was the Laplace-Beltrami operator  
779 on the unit sphere.  
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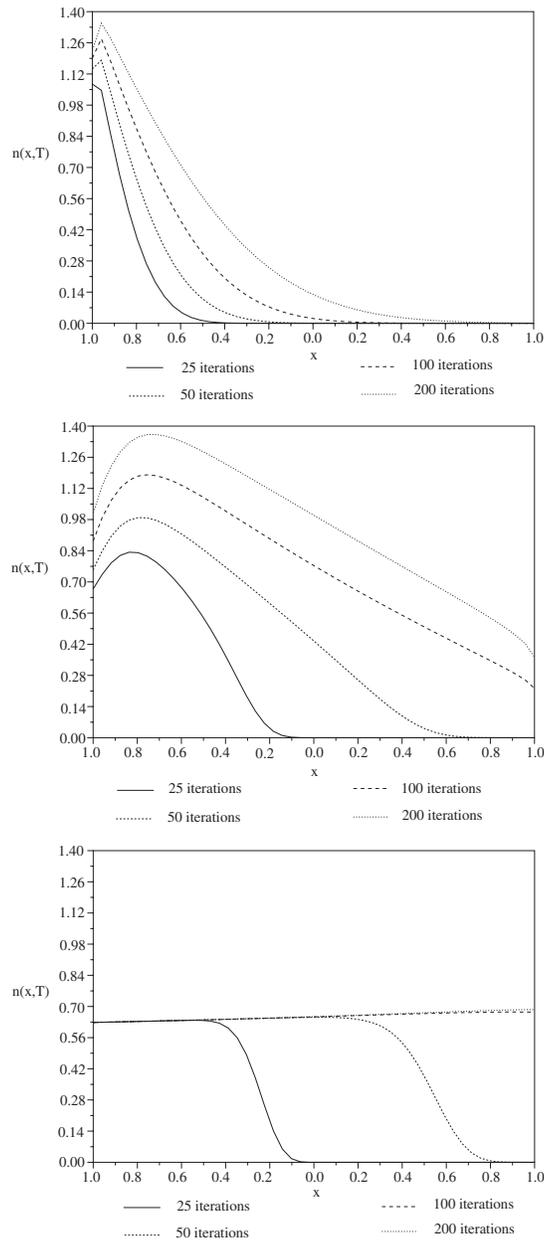
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## 5. CONCLUDING REMARKS

785 We have described an implicit finite element approximation for the  
786 grazing collision Boltzmann-Lorentz operator. This approximation  
787 allows to pass to the limit  $\varepsilon \rightarrow 0$  for a fixed discrete scheme (with respect  
788 to the velocity variable). This fact is due essentially to the possibility of  
789 carrying out the exact computations of both the mass matrix and the  
790 collision matrix with no need of quadrature formulas. Other methods  
791 would have been possible: for example, a finite volume method based  
792 on a piecewise parabolic operator or a spectral method (see Buet et al.  
793 (2001), Pareschi et al. (2002)). Our choice of a FEM method has been  
794 essentially motivated by the nature of the limiting problem and also by  
795 previous computations done for the Laplace-Beltrami operator in 3D.

796 The generalization to the three-dimensional case seems to be possi-  
797 ble: we can in fact compute the matrices  $A$  and  $Q$ , but the coefficients  
798 we find are much more complicated. According to Buet et al. (2001),

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**Figure 10.** The density  $n(\cdot, T)$ ,  $\tau = 2^{-5}$  (left), 1 (center),  $2^5$  (right),  $T = N_{\text{iter}}\Delta t$  with  $N_{\text{iter}} = 25, 50, 100, 200$ .

841 a spectral method should be a good idea for the numerical approximation  
842 of the 3D grazing collision Lorentz operator.

843 It should be also interesting to generalize the approximation  
844 presented in this article to nonconstant grazing collisions scattering  
845 cross sections and also to the Coulombian case (we refer to Cordier et  
846 al. (2002) for a numerical comparison between the Boltzmann-Lorentz  
847 operator and the Fokker-Planck one in the Coulombian case).

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### 851 A1. APPENDIX

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Here we perform the computation of the coefficients  $Q_{ij}$  of the graz-  
ing collision matrix  $Q$ . We first express the hat functions  $\varphi_i$  as follows:  
 $\varphi_i(\theta) = \varphi_h(\theta - \theta_i)$ , where:

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$$\varphi_h(z) = \begin{cases} h^{-1}(h - |z|), & \text{if } |z| < h, \\ 0 & \text{else.} \end{cases}$$

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Replacing in Eq. (3)  $\bar{\theta} = \theta - \theta_i$  and  $\bar{\theta}' = \theta' - \theta_j$  yields:

$$Q_{ij} = \int_{S^1} \int_{S^1} B^\varepsilon(\bar{\theta} - \bar{\theta}' + \Delta\theta) [\varphi_h(\bar{\theta}) \varphi_h(\bar{\theta}') - \varphi_h(\bar{\theta}) \varphi_h(\bar{\theta} + \Delta\theta)] d\bar{\theta} d\bar{\theta}',$$

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$$\varphi(z) = \begin{cases} (1 - |z|), & \text{if } |z| < 1, \\ 0 & \text{else} \end{cases}$$

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we have  $\varphi_h(z) = \varphi(z/h)$ . Now using the new variables  $\theta = \bar{\theta}/h$  and  
 $\theta' = \bar{\theta}'/h$ , we can write:

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$$\begin{aligned} Q_{ij} &= h^2 \int_{-1}^{+1} \int_{-1}^{+1} B^\varepsilon(h\theta - h\theta' + \Delta\theta) [\varphi(\theta) \varphi(\theta') \\ &\quad - \varphi(\theta) \varphi(\theta + (i - j))] d\theta d\theta' \\ &= h^2 \int_{-1}^{+1} \int_{-1}^{+1} B^\varepsilon(h(\theta - \theta' + (i - j))) [\varphi(\theta) \varphi(\theta') \\ &\quad - \varphi(\theta) \varphi(\theta + (i - j))] d\theta d\theta'. \end{aligned}$$

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Let us set  $B_h(z) = B(hz)$ , i.e., (with  $B_0 = 1/\varepsilon^3$ ):

$$B_h(z) = \begin{cases} B_0 & \text{if } |z| < \alpha = \varepsilon\pi/h, \\ 0 & \text{else;} \end{cases}$$

883 then the collision coefficient  $Q_{ij}$  reads:

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$$\begin{aligned}
 Q_{ij} &= h^2 \int_{-1}^{+1} \int_{-1}^{+1} B_h(\theta - \theta' + (i-j)) [\varphi(\theta) \varphi(\theta') \\
 &\quad - \varphi(\theta) \varphi(\theta + (i-j))] d\theta d\theta' \\
 &= h^2 B_0 \int \int_{|\theta - \theta' + (i-j)| < \alpha} [\varphi(\theta) \varphi(\theta') - \varphi(\theta) \varphi(\theta + (i-j))] d\theta d\theta'
 \end{aligned}$$

with  $\theta, \theta' \in [-1, +1]$ . We split  $Q_{ij}$  into a gain part  $G_{ij}$  and a loss part  $L_{ij}$ , i.e., we set  $Q_{ij} = G_{ij} - L_{ij}$ , where:

$$\begin{aligned}
 G_{ij} &= h^2 B_0 \int \int_{|\theta - \theta' + (i-j)| < \alpha} \varphi(\theta) \varphi(\theta') d\theta d\theta', \\
 L_{ij} &= h^2 B_0 \int \int_{|\theta - \theta' + (i-j)| < \alpha} \varphi(\theta) \varphi(\theta + (i-j)) d\theta d\theta'.
 \end{aligned}$$

We now have to compute separately these gain and loss terms. For this, we will assume that  $\alpha \leq 1$ , so that  $\varepsilon \leq h/\pi = 2/N_\theta$ : it is thus possible to pass to the limit first for  $\varepsilon \rightarrow 0$  and then for  $h \rightarrow 0$ .

Let us first compute the gain coefficients  $G_{ij}$ . As the  $\varphi$  function is null outside a domain of length 2, it is clear that for  $|i-j| \geq 3$ ,  $G_{ij} = 0$  (see also Fig. 11). Thus, we have to compute  $G_{ij}$  for  $i=j$ ,  $|i-j|=1$ , and  $|i-j|=2$ . Considering also the symmetry of the collision operator, we only have to compute the integrals for  $i-j = -2$ ,  $i-j = -1$ , and  $i=j$ .

**F11**

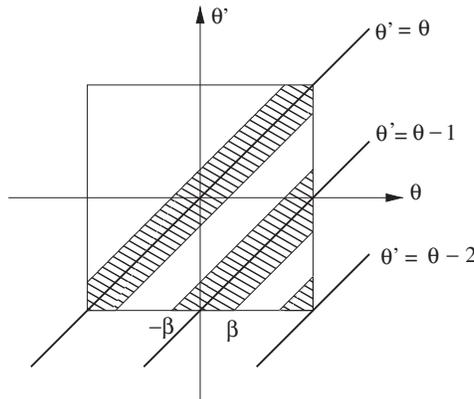


Figure 11. Set of definition of  $B_h(\theta - \theta' + i - j)$ .

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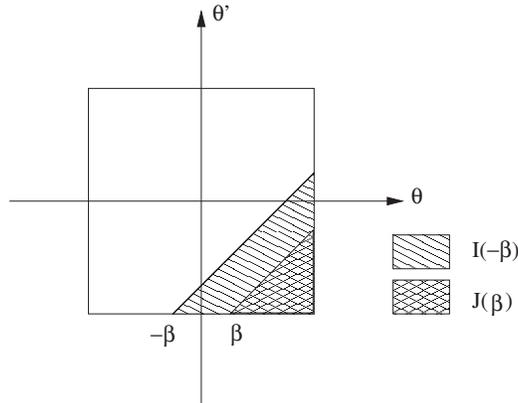


Figure 12. The integrals  $I(-\beta)$  and  $J(\beta)$ .

Looking at Fig. 12, we define the following integrals for a given **F12**  
 $\beta > 0$ :

$$\begin{aligned}
 I(-\beta) &= h^2 B_0 \int_{-\beta}^0 (1 + \theta) \left( \int_{-1}^{\theta+\beta-1} (1 + \theta') d\theta' \right) d\theta \\
 &\quad + h^2 B_0 \int_0^{1-\beta} (1 - \theta) \left( \int_{-1}^{\theta+\beta-1} (1 + \theta') d\theta' \right) d\theta \\
 &\quad + h^2 B_0 \int_{1-\beta}^1 (1 - \theta) \left( \int_{-1}^0 (1 + \theta') d\theta' \right) d\theta \\
 &\quad + h^2 B_0 \int_{1-\beta}^1 (1 - \theta) \left( \int_0^{\theta+\beta-1} (1 - \theta') d\theta' \right) d\theta \\
 &= h^2 B_0 \left( \frac{1}{12} \beta^3 (\beta - 4) + \frac{1}{24} (1 - \beta)(1 + \beta)(\beta^2 + 4\beta + 1) + \frac{1}{4} \beta^2 \right) \\
 &= h^2 B_0 \left( -\frac{1}{8} \beta^4 + \frac{1}{6} \beta^3 + \frac{1}{4} \beta^2 + \frac{1}{6} \beta + \frac{1}{24} \right)
 \end{aligned}$$

and

$$\begin{aligned}
 J(\beta) &= h^2 B_0 \int_{\beta}^1 (1 - \theta) \left( \int_{-1}^{\theta-\beta-1} (1 + \theta') d\theta' \right) d\theta \\
 &= h^2 B_0 \frac{1}{24} (\beta - 1)^4
 \end{aligned}$$

## Finite Element Approximation of Grazing Collisions

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967 In particular,  $I(-\beta)$  represents the integral with support on the  
 968 bigger triangle and  $J(\beta)$  represents the integral with support on the  
 969 small triangle (Fig. 12)

970 We get:

971

$$972 \quad G_{ij} = J(1 - \alpha) = \frac{1}{24} h^2 B_0 \alpha^4, \quad \text{for } i - j = -2 \quad (19)$$

973

$$974 \quad G_{ij} = I(-\alpha) - J(\alpha) = h^2 B_0 \left( -\frac{1}{6} \alpha^4 + \frac{1}{3} \alpha^3 + \frac{1}{3} \alpha \right), \quad \text{for } |i - j| = 1 \quad (20)$$

975

$$976 \quad G_{ij} = h^2 B_0 \left( \int_{-1}^{+1} (1 - |\theta|) d\theta \right) \left( \int_{-1}^{+1} (1 - |\theta'|) d\theta' \right) - 2I(-1 + \alpha)$$

$$977 \quad = h^2 B_0 \left( \frac{1}{4} \alpha^4 - \frac{2}{3} \alpha^3 + \frac{4}{3} \alpha \right), \quad \text{for } i = j \quad (21)$$

978

979 We now compute the loss coefficients  $L_{ij}$ . As before, we just have to  
 980 compute  $L_{ij}$  for  $|i - j| = 2, 1$ , and 0. We first remark that for  $|i - j| \geq 2$ ,  
 981 the product  $\varphi(\theta)\varphi(\theta + i - j)$  is null. In fact  $\varphi(\theta)$  differs from zero on the set  
 982  $[-1, +1]$  while  $\varphi(\theta + i - j)$  differs from zero on the set  $[1, 3]$ . Hence,  
 983  $L_{ij} = 0$  for  $|i - j| \geq 2$ . Moreover, for  $|i - j| = 1$  the product  
 984  $\varphi(\theta)\varphi(\theta + i - j)$  is not null only for  $\theta \in [0, 1]$ , and the dependence of  $L_{ij}$   
 985 on the variable  $\theta'$  appears only through the kernel  $B_h$  which is not null on  
 986 a set of amplitude  $2\alpha$ . Thus, we have:

987

$$988 \quad L_{ij} = h^2 B_0 2\alpha \int_0^1 \theta(1 - \theta) d\theta$$

$$989 \quad = h^2 B_0 2\alpha \frac{1}{6} = \frac{1}{3} h^2 B_0 \alpha, \quad \text{for } |i - j| = 1 \quad (22)$$

990

991 Analogously, for the principal diagonal, we have:

992

$$993 \quad L_{ii} = h^2 B_0 2\alpha \int_{-1}^{+1} (1 - |\theta|)^2 d\theta$$

$$994 \quad = h^2 B_0 2\alpha \frac{2}{3} = \frac{4}{3} h^2 B_0 \alpha. \quad (23)$$

995

996 Thus, collecting Eqs. (19)–(23) and recalling that  $B_0 = 1/\varepsilon^3$  and that  
 997  $\alpha = \varepsilon\pi/h$ , the computation of the coefficients  $Q_{ij}$  is completed.

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1009 We finally remark that the computation of the coefficients  $A_{ij}$  of the  
1010 mass matrix  $A$  may be done analogously.

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1014

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