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***Abstract*— The equality of the energy and group velocities of bulk acoustic waves in a lossless piezoelectric medium is demonstrated, with the energy velocity defined from the generalized energy density and the generalized Poynting vector.**

***Index Terms*— Bulk acoustic waves; Piezoelectric media; Energy velocity; Group velocity.**

I. INTRODUCTION

It is well known that the energy and the group velocities of bulk acoustic waves (BAW) in elastic media are equal [1], [2]. The energy velocity, v_e, is defined as the ratio of the Poynting vector to the energy density, with the help of the Poynting theorem, while the group velocity, v_g, is defined as the derivative of the phase velocity with respect to the direction of propagation. The group velocity is thus defined on a purely geometrical ground and is by construction normal to the slowness surface. Further properties are the equality of the BAW kinetic and the potential (or strain) energies, and the orthogonality of the slowness vector to the wavefront surface.

In the case of piezoelectric media, we are not aware of a general demonstration of equivalent properties, though generalized expressions have long been known for the energy density and the Poynting vector [1]. Nevertheless, the collinearity of the generalized Poynting vector and the group velocity is routinely used to predict the beam-steering angle, or direction of energy transport. However, Zaitsev and Kuznetsova [3] recently identified a possible discrepancy that can occur if the generalized Poynting vector is used to predict the direction of energy transport in strong piezoelectrics such as lithium niobate. Their argument relies on a dissymmetry of the mechanoelectrical and electromechanical contributions to the generalized Poynting vector. Their work has motivated us to establish the equality of the energy and group velocities of bulk acoustic waves in an arbitrary lossless piezoelectric medium, with the energy velocity obtained from the generalized forms of the Poynting vector and the energy density. This property settles the apparent discrepancy.

II. BASIC RELATIONS

We summarize in this section some well known energetic relations for bulk acoustic waves propagating in linear piezoelectric media. Let us first consider a perturbance of an arbitrary linear piezoelectric medium, characterized by the strain S_{ij}, the stress T_{ij}, the electric field strength E_k and the electric displacement D_k. These fields are related by the constitutive relations [1], [4]

$$T_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k, \quad (1)$$
$$D_j = e_{jkl}S_{kl} + \epsilon_{jk}E_k, \quad (2)$$

where c_{ijkl}, e_{kij} and ϵ_{jk} are the elastic, piezoelectric and dielectric tensors, respectively. The repeated summation index convention is used throughout this letter. The time-averaged generalized energy density is given by [1]

$$W = \frac{1}{2}\mathrm{Re}(T_{ij}S_{ij}^* + D_kE_k^*), \quad (3)$$

and the time-averaged generalized Poynting vector is

$$P_j = \frac{1}{2}\mathrm{Re}(-T_{ij}v_i^* + (\mathbf{E}\times\mathbf{H}^*)_j), \quad (4)$$

where v_i is the particle velocity, H_i is the magnetic field strength and $*$ denotes complex conjugation.

We then consider specifically a time-harmonic plane wave with angular frequency ω and slowness vector s_j. The displacements u_i are then of the form

$$u_i(\mathbf{x}, t) = u_i \exp(\jmath\omega(t - s_jx_j)), \quad (5)$$

and similar expressions hold for all fields. We specifically demand that the slowness vector be real-valued, i.e. the medium is lossless and evanescent or inhomogeneous bulk acoustic waves are not considered.

Considering further the quasi-static approximation [1], Maxwell's equations give

$$E_i = \jmath\omega s_i\Phi, \quad (6)$$
$$(\mathbf{s}\times\mathbf{H})_j = -D_j, \quad (7)$$
$$s_jD_j = 0, \quad (8)$$

where Φ is the electric potential. The constitutive relations become

$$T_{ij} = -\jmath\omega(c_{ijkl}s_ku_l + e_{kij}s_k\Phi), \quad (9)$$
$$D_j = -\jmath\omega(e_{jkl}s_ku_l - \epsilon_{jk}s_k\Phi). \quad (10)$$

The generalized energy density simplifies to

$$W = \frac{1}{2}\mathrm{Re}(\jmath\omega T_{ij}s_ju_i^*). \quad (11)$$

It can be seen that the electromagnetic part of the energy density vanishes. The generalized Poynting vector becomes

$$P_j = \frac{1}{2}\mathrm{Re}(\jmath\omega(T_{ij}u_i^* + D_j\Phi^*)), \quad (12)$$

where $v_i = \jmath\omega u_i$ has been used in Eq. (4).

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III. Energy velocity

We now introduce the generalized notation

$$\bar{c}_{ijkl} = \begin{cases} c_{ijkl} & i,l = 1,2,3 \\ e_{kij} & l = 4, i = 1,2,3 \\ e_{jkl} & i = 4, l = 1,2,3 \\ -\epsilon_{jk} & i,l = 4 \end{cases} . \tag{13}$$

This notation should not be confused with the piezoelectrically stiffened elastic constants [4]. With the generalized displacements, $\bar{u}_i$, and stresses, $\bar{T}_{ij}$, defined as

$$\bar{u}_i = u_i, i = 1,2,3, \tag{14}$$

$$\bar{u}_4 = \Phi, \tag{15}$$

$$\bar{T}_{ij} = T_{ij}, i = 1,2,3, \tag{16}$$

$$\bar{T}_{4j} = D_j, \tag{17}$$

the constitutive relations can be written in a compact way as

$$\bar{T}_{ij} = -\jmath\omega\bar{c}_{ijkl}s_k\bar{u}_l, \tag{18}$$

and the Christoffel equation [1] governing the wave dynamics assumes the form

$$\bar{c}_{ijkl}s_js_k\bar{u}_l = \rho_{il}\bar{u}_l, \tag{19}$$

with

$$\rho_{il} = \rho \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{20}$$

Contracting the Christoffel Eq. (19) with $\bar{u}_i^*$ yields the relation

$$\bar{c}_{ijkl}s_js_k\bar{u}_l\bar{u}_i^* = \rho_{il}\bar{u}_l\bar{u}_i^* = \rho u_i^* u_i. \tag{21}$$

Inserting Eq. (9) into Eq. (11) we obtain

$$W = \frac{\omega^2}{2}\mathrm{Re}(c_{ijkl}s_js_ku_lu_i^* + e_{kij}s_js_k\Phi u_i^*), \tag{22}$$

or by inspection

$$W = \frac{\omega^2}{2}\mathrm{Re}(\bar{c}_{ijkl}s_js_k\bar{u}_l\bar{u}_i^*). \tag{23}$$

Using Eq. (21), we observe that this quantity is real and thus

$$W = \frac{\omega^2}{2}\bar{c}_{ijkl}s_js_k\bar{u}_l\bar{u}_i^* = \frac{\omega^2}{2}\rho u_i^* u_i. \tag{24}$$

This relation proves the equality of potential and kinetic energies for bulk acoustic waves as in the case of elastic media [2].

The expression for the Poynting vector can similarly be transformed by inserting Eqs. (9-10) into Eq. (12), yielding

$$P_j = \frac{\omega^2}{2}\mathrm{Re}(c_{ijkl}s_ku_lu_i^* + e_{kij}s_k\Phi u_i^* + e_{jkl}s_ku_l\Phi^* - \epsilon_{jk}s_k\Phi\Phi^*), \tag{25}$$

or simply

$$P_j = \frac{\omega^2}{2}\mathrm{Re}(\bar{c}_{ijkl}s_k\bar{u}_l\bar{u}_i^*). \tag{26}$$

Defining the energy velocity as for bulk acoustic waves in elastic media by the ratio of the Poynting vector to the energy density, we obtain at once

$$(V_e)_j = \frac{P_j}{W} = \frac{\mathrm{Re}(\bar{c}_{ijkl}s_k\bar{u}_l\bar{u}_i^*)}{\rho u_i^* u_i}. \tag{27}$$

Furthermore, contracting this expression with s_j, the following useful relation is obtained

$$s_j(V_e)_j = 1, \tag{28}$$

as in the case of elastic media [1], [2].

IV. Group velocity

The components of the slowness vector can be written $s_i = sn_i$ where the n_i are the components of a unit vector, i.e. $n_in_i = 1$. With this notation, the Christoffel equation (19) becomes a generalized eigenvalue equation for the square of the phase velocity, $V = s^{-1}$,

$$\bar{c}_{ijkl}n_jn_k\bar{u}_l = V^2\rho_{il}\bar{u}_l. \tag{29}$$

The group velocity is defined as [2]

$$(V_g)_j = \frac{\partial V}{\partial n_j}, \tag{30}$$

which implies that the group velocity vector is normal to the slowness surface by construction. The group velocity vector can be obtained by differentiating with respect to n_j the identity formed by contracting Eq. (29) with $\bar{u}_i^*$, or

$$\bar{c}_{ijkl}n_jn_k\bar{u}_l\bar{u}_i^* = V^2\rho_{il}\bar{u}_l\bar{u}_i^*, \tag{31}$$

which is merely a restatement of Eq. (21). In this equation, $\bar{u}_l$ is a function of the unit vector $\mathbf{n}$ since it is the eigenvector associated with the eigenvalue V^2 of the Christoffel equation (29) for a fixed propagation direction. The differentiation of Eq. (31) with respect to n_j results in

$$2\bar{c}_{ijkl}n_k\bar{u}_l\bar{u}_i^* + \bar{c}_{i\beta kl}n_\beta n_k\frac{\partial\bar{u}_l}{\partial n_j}\bar{u}_i^* + \bar{c}_{i\beta kl}n_\beta n_k\bar{u}_l\frac{\partial\bar{u}_i^*}{\partial n_j}$$
$$= 2V(V_g)_j\rho_{il}\bar{u}_l\bar{u}_i^* + V^2\rho_{il}\frac{\partial\bar{u}_l}{\partial n_j}\bar{u}_i^* + V^2\rho_{il}\bar{u}_l\frac{\partial\bar{u}_i^*}{\partial n_j}. \tag{32}$$

This equation can be simplified by considering two different contractions of the Christoffel equation (29). First, the contraction by $\frac{\partial\bar{u}_i^*}{\partial n_j}$ yields

$$\bar{c}_{i\beta kl}n_\beta n_k\bar{u}_l\frac{\partial\bar{u}_i^*}{\partial n_j} = V^2\rho_{il}\bar{u}_l\frac{\partial\bar{u}_i^*}{\partial n_j}. \tag{33}$$

Second, complex conjugation of Eq. (29) and subsequent contraction by $\frac{\partial\bar{u}_i}{\partial n_j}$ results in

$$\bar{c}_{i\beta kl}n_\beta n_k\bar{u}_l^*\frac{\partial\bar{u}_i}{\partial n_j} = V^2\rho_{il}\bar{u}_l^*\frac{\partial\bar{u}_i}{\partial n_j}. \tag{34}$$

Permutation of indices i and l, and β and k, respectively, and consideration of the symmetries $\bar{c}_{i\beta kl} = \bar{c}_{lk\beta i}$ and $\rho_{il} = \rho_{li}$ lead to

$$\bar{c}_{i\beta kl}n_\beta n_k\bar{u}_i^*\frac{\partial\bar{u}_l}{\partial n_j} = V^2\rho_{il}\bar{u}_i^*\frac{\partial\bar{u}_l}{\partial n_j}. \tag{35}$$

Eventually, Eq. (32) simplifies to

$$(V_g)_j = \frac{\bar{c}_{ijkl}n_k\bar{u}_l\bar{u}_i^*}{V\rho u_iu_i^*} = \frac{\bar{c}_{ijkl}s_k\bar{u}_l\bar{u}_i^*}{\rho u_iu_i^*}. \tag{36}$$

This is identical with the expression (27) for the energy velocity and incidentally shows that the $Re(.)$ operators in expressions (26) and (27) can be dropped. We have thus shown the equality of energy and group velocities for bulk acoustic waves in a lossless piezoelectric medium.

V. CONCLUSION

We have demonstrated that the energy and group velocities for bulk acoustic waves are equal in a lossless piezoelectric medium. Our derivation closely parallels and generalizes for piezoelectric media the derivation of Ref. [2], that was limited to purely elastic media. The possible discrepancy pointed out in Ref. [3], that the mechanoelectrical and electromechanical power flows calculated from the generalized form of the Poynting vector for piezoelectric media do not compensate each other, has then no significance for the estimation of the power flow direction. This observation is moreover supported by the experimental results of Havlice *et al.* [5], who demonstrated that the generalized Poynting vector indeed predicts the correct beam-steering angle of longitudinal waves along the y axis of lithium niobate. We finally observe that attempting to separate the generalized energy density or the generalized Poynting vector into their purely electrical, purely mechanical, mechanoelectrical and electromechanical parts can be confusing, since the wave displacements are obtained as the solution of an eigenvalue problem involving the mixed elastic, electrical and piezoelectric properties of the medium.

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