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**Circle Formation of Weak Robots
and
Lyndon Words**

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Circle Formation of Weak Robots and Lyndon Words

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Abstract

A Lyndon word is a non-empty word strictly smaller in the lexicographic order than any of its suffixes, except itself and the empty word. In this paper, we show how Lyndon words can be used in the distributed control of a set of n weak mobile robots. By weak, we mean that the robots are anonymous, memoryless, without any common sense of direction, and unable to communicate in an other way than observation. An efficient and simple deterministic protocol to form a regular n -gon is presented and proven for n prime.

1 Introduction

A Lyndon word is a non-empty word strictly smaller in the lexicographic order than any of its suffixes, except itself and the empty word. Lyndon words have been widely studied in the combinatorics of words area [Lot83]. However, only a few papers consider Lyndon words addressing issues in other areas than word algebra, e.g., [Che04, DR04, SM90].

In this paper, we address the class of distributed systems where computing units are *autonomous mobile* robots (or agents), i.e., devices equipped with sensors which do not depend on a central scheduler and designed to move in a two-dimensional plane. Also, we assume that the robots cannot remember any previous observation nor computation performed in any previous step. Such robots are said to be *oblivious* (or *memoryless*). The robots are also *uniform* and *anonymous*, i.e, they all have the same program using no local parameter (such that an identity) allowing to differentiate any of them. Moreover, none of them share any kind of common coordinate mechanism or common sense of direction, and they communicate only by observing the position of the others.

The motivation behind such a weak and unrealistic model is the study of the minimal level of ability the robots are required to have in the accomplishment of some basic cooperative tasks in a deterministic way [SS90, SY99, FPSW99, Pre02]. Among them, the *Circle Formation Problem* (CFP) has received a particular attention. The CFP consists in the design of a protocol insuring that starting from an initial arbitrary configuration, all n robots eventually form a circle with equal spacing between any two adjacent robots. In other words, a *regular n -gon* must be formed when the protocol terminated.

An informal CFP algorithm is presented in [Deb95] to show the relationship between the class of pattern formation algorithms and the concept of self-stabilization in distributed systems [Dol00]. In [SS96], an algorithm based on heuristics is proposed for the formation of a cycle approximation. A CFP protocol is given in [SY99] for non-oblivious robots with an unbounded memory. Two deterministic algorithms are provided in [DK02, CMN04]. In the former work, the robots asymptotically

converge toward a configuration in which they are uniformly distributed on the boundary of a circle. This solution is based on an elegant Voronoi Diagram construction. The latter work avoid this construction by making an extra assumption on the initial position of robots. All the above solutions work in a semi-asynchronous model. The solution in [Kat05] works on a fully asynchronous model, but when n is even, the robots may only achieve a biangular circle—the distance between two adjacent robots is alternatively either α or β .

In this paper, we show a straight application of the properties of Lyndon words. They are used to build and to prove an efficient and simple deterministic protocol solving the CFP for a prime number of robots.

2 Preliminaries

In this section, we define the distributed system and the problem considered in this paper. Next, the Lyndon words are introduced.

Distributed Model. We adopt the model of [SY96]. The *distributed system* considered in this paper consists of n robots r_1, r_2, \dots, r_n , where n is prime—the subscripts $1, \dots, n$ are used for notational purpose only. Each robot r_i , viewed as a point in the Euclidean plane, move on this two-dimensional space unbounded and devoid of any landmark. When no ambiguity arises, r_i also denotes the point in the plane occupied by that robot. It is assumed that the robots never collide and that two or more robots may simultaneously occupy the same physical location. Any robot can observe, compute and move with infinite decimal precision. The robots are equipped with sensors allowing to detect the instantaneous position of the other robots in the plane. Each robot has its own local coordinate system and unit measure. The robots do not agree on the orientation of the axes of their local coordinate system, nor on the unit measure. They are *uniform* and *anonymous*, i.e, they all have the same program using no local parameter (such that an identity) allowing to differentiate any of them. They communicate only by observing the position of the others and they are *oblivious*, i.e., none of them can remember any previous observation nor computation performed in any previous step.

Time is represented as an infinite sequence of time instant $t_0, t_1, \dots, t_j, \dots$. The set of positions in the plane occupied by the n robots at a given time instant t_j ($j \geq 0$) is called a *configuration* of the distributed system. At each time instant t_j ($j \geq 0$), each robot r_i is either *active* or *inactive*. The former means that, during the computation step (t_j, t_{j+1}) , using a given algorithm, r_i computes in its local coordinate system a position $p_i(t_{j+1})$ depending only on the system configuration at t_j , and moves towards $p_i(t_{j+1})$ — $p_i(t_{j+1})$ can be equal to $p_i(t_j)$, making the location of r_i unchanged. In the latter case, r_i does not perform any local computation and remains at the same position.

The concurrent activation of robots is modeled by the interleaving model in which the robot activations are driven by a *fair scheduler*. At each instant t_j ($j \geq 0$), the scheduler arbitrarily activates a (non empty) set of robots. Fairness means that every robot is infinitely often activated by the scheduler.

The Circle Formation Problem. Consider a configuration at time t_k ($k \geq 0$) in which the positions of the n robots are located at distinct positions on the circumference of a non degenerate circle C —the radius of C is greater than *zero*. At time t_k , the *successor* r_j of any robot r_i , $i, j \in 1 \dots n$

and $i \neq j$, is the unique robot such that no robot exists between r_i and r_j on C in the clockwise direction. Given r_i and its successor r_j on C centered in O , $\widehat{r_i O r_j}$ denotes the angle between r_i and r_j , i.e., the angle centered in O and with sides the half-lines $[O, r_i)$ and $[O, r_j)$.

The problem considered in this paper consists in the design of a distributed protocol which arranges a group of n ($n \geq 2$) mobile robots with initial distinct positions into a *non degenerate regular n -gon* in finite time, i.e., the robots eventually take place in a non degenerate circle C centered in O such that for every pair r_i, r_j of robots, if r_j is the successor of r_i on C , then $\widehat{r_i O r_j} = \alpha$, where $\alpha = \frac{2\pi}{n}$. Angle α is called the *characteristic angle* of the n -gon.

Lyndon Word. Let an ordered alphabet A be a finite set of letters. Denote \prec an order on A . A non empty *word* w over A is a finite sequence of letters $a_1, \dots, a_i, \dots, a_l$, $l > 0$. The *concatenation* of two words u and v , denoted $u \circ v$ or simply uv , is equal to the word $a_1, \dots, a_i, \dots, a_k, b_1, \dots, b_j, \dots, b_l$ such that $u = a_1, \dots, a_i, \dots, a_k$ and $v = b_1, \dots, b_j, \dots, b_l$. Let ϵ be the *empty word* such that for every word w , $w\epsilon = \epsilon w = w$. The *length* of a word w , denoted by $|w|$, is equal to the number of letters of w — $|\epsilon| = 0$.

A word u is *lexicographically* smaller than or equal to a word v , denoted $u \preceq v$, iff there exists either a word w such that $v = uw$ or three words r, s, t and two letters a, b such that $u = ras$, $v = rbt$, and $a \prec b$.

Let k and j be two positive integers. The k^{th} *power* of a word w is the word denoted s^k such that $s^0 = \epsilon$, and $s^k = s^{k-1}s$. A word u is said to be *primitive* if and only if $u = v^k \Rightarrow k = 1$. The j^{th} *rotation* of a word w , notation $R_j(w)$, is defined by:

$$R_j(w) \stackrel{\text{def}}{=} \begin{cases} \epsilon & \text{if } w = \epsilon \\ a_j, \dots, a_l, a_1, \dots, a_{j-1} & \text{otherwise } (w = a_1, \dots, a_l, l \geq 1) \end{cases}$$

Note that $R_1(w) = w$. A word w is said to be *minimal* iff $\forall j \in 1, \dots, l$, $w \preceq R_j(w)$.

Definition 1 (Lyndon Word) A word w ($|w| > 0$) is a *Lyndon word* iff w is nonempty, primitive and minimal, i.e., $w \neq \epsilon$ and $\forall j \in 2, \dots, |w|$, $w \prec R_j(w)$.

For instance, if $A = \{a, b\}$, then a, b, ab, aab, abb are Lyndon words, whereas aba , and $abab$ are not— aba is not minimal ($aab \preceq aba$) and $abab$ is not primitive ($abab = (ab)^2$).

3 Algorithm

In this section, we present our main result based on Lyndon words for n robots, n being prime and greater than or equal to 5. Note that if the system contains two robots only, then CFP is trivially solved because they always form a 2-gon. If $n = 3$, then the three robots always form a triangle, which must be equilateral to form a regular 3-gon. If the triangle is not equilateral, then either (1) the three robots belongs to the same line, (2) they form an isosceles triangle, or (3) they form an ordinary triangle. In all cases, it is always possible to elect a unique robot as the leader which moves to the unique position making the triangle equilateral—a formal algorithm for $n = 3$ is given and proved in the appendix.

The technique developed in this paper is based on a Leader Election using properties of Lyndon words. The words we consider are made over the alphabet such that the letters are the angles between neighboring robots located on the boundary of a unique non degenerate circle. So, we focus

only on configurations where the robots are either all or all but one on the boundary of the circle. Thus, we integrate the first algorithm proposed in [DK02] in our solution, in the sequel referred to Algorithm ϕ_{circle} .

Theorem 2 ([DK02]) *Starting from an arbitrary configuration where n robots are located at distinct positions, Algorithm ϕ_{circle} leads the robots into a configuration where the robots are located on the boundary of a non degenerate circle.*

In the rest of this section, we first present how Lyndon words are used to make a leader election among n robots located on the boundary of a unique circle. Next, we define a particular type of configurations called *oriented* configurations. We then give an algorithm to arrange the robots in a regular n -gon starting from an oriented configuration. Finally, we provide our general scheme to solve the Circle Formation Problem.

3.1 Leader Election

In this subsection, we use the subscript i in the notation of a robot r_i , $i \in 1 \dots n$, to denote the order of the robots in an arbitrary clockwise direction on C . We proceed as follows: A robot is arbitrarily chosen as r_1 on C . Next, for any $i \in 1 \dots n-1$, r_{i+1} denotes the successor of r_i on C (in the clockwise direction). Finally, the successor of r_n is r_1 .

Let the alphabet A be the set of k ($k \leq n$) strictly positive reals x_1, x_2, \dots, x_k such that $\forall i \in 1 \dots n$, there exists $j \in 1 \dots k$ such that $x_j = r_i \widehat{Or}_{i+1}$, where O is the center of C . An example of such an alphabet is shown in Figure 1— $A = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$.

The order on A is the natural order ($<$) on the reals. So, the lexicographic order \preceq on the words made over A is defined as follows:

$$u \preceq v \stackrel{\text{def}}{\equiv} (\exists w | v = uw) \vee (\exists r, s, t, \exists a, b \in A | (u = ras) \wedge (v = rbt) \wedge (a < b))$$

For instance, if $A = \{\frac{\pi}{3}, \frac{\pi}{2}\}$, then $(\frac{\pi}{3}) \preceq (\frac{\pi}{3}\frac{\pi}{3}) \preceq (\frac{\pi}{3}\frac{\pi}{2}) \preceq (\frac{\pi}{2}\frac{\pi}{2}) \preceq (\frac{\pi}{2})$.

For each robots r_i , let us define the word $SA(r_i)$ (respectively, $\overline{SA(r_i)}$) over A (SA stands for “string of angles” [CP02]) as follows:

$$\begin{aligned} SA(r_i) &= x_i x_{i+1} \dots x_n x_1 \dots x_{i-1} \\ (\text{resp. } \overline{SA(r_i)}) &= x_{i-1} \dots x_1 x_n x_{n-1} \dots x_i \end{aligned}$$

An example showing a string of angle is given in Figure 1. Note that for every robot r_i , $|SA(r_i)| = |\overline{SA(r_i)}| = n$. Moreover, if the configuration is a regular n -gon, then for every robot r_i , $SA(r_i) = \overline{SA(r_i)} = \alpha^n$, where $\alpha = \frac{2\pi}{n}$ is the characteristic angle of the n -gon.

Lemma 3 *If all the robots are at distinct positions on the boundary of a same circle C without forming a regular n -gon, then there exists exactly one robot r_i such that $SA(r_i)$ is a Lyndon Word.*

Proof. We prove this in two steps. First, we show that r_i always exists. Next, we show the uniqueness of r_i .

Existence. Let $minSA$ be the string of angle such that $\forall i \in 1 \dots n$, $minSA \preceq SA(r_i)$. Let us show that $minSA$ is a Lyndon word. By definition, $minSA$ is minimal. Assume by contradiction that $minSA$ is not primitive. So, by definition, there exists a word u such as $minSA = u^k$ with $k > 1$. By definition of the k^{th} power of a word u , $|minSA| = k * |u|$, $|u|$ being a divisor of n . Since $|minSA| = n$ and n is prime, there are only two cases to consider:

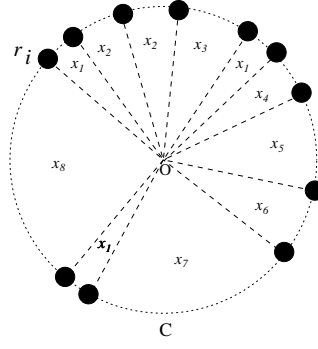


Figure 1: $SA(r_i) = x_1(x_2)^2x_3x_1x_4x_5x_6x_7x_1x_8$.

1. $|u| = |\min SA|$. Since $|\min SA| = k * |u|$, this implies that $k = 1$, which contradicts $k > 1$.
2. $|u| = 1$. In this case, u is a letter ($\in A$), and $\min SA = u^n$. Since the letters are angles, the angle between every pair of successive robots is equal to the value u . So, the robots form a regular n -gon and $u = \alpha$, the characteristic angle of the n -gon. This contradicts that the configuration is not a regular n -gon.

Unicity. Assume by contradiction that there exists two different robots r_i, r_j ($r_i \neq r_j$) such that $SA(r_i)$ and $SA(r_j)$ are Lyndon words. Since $SA(r_j)$ (respectively, $\overline{SA(r_j)}$) is a Lyndon word, by Definition 1, $\forall k \in 2 \dots n$, $SA(r_i) \prec R_k(SA(r_i))$ (resp. $\forall k' \in 2 \dots n$, $\overline{SA(r_j)} \prec R_{k'}(\overline{SA(r_j)})$). Since $r_i \neq r_j$, there exists $k \in 2 \dots n$ such that $SA(r_j) = R_k(SA(r_i))$ (resp. there exists $k' \in 2 \dots n$ such that $\overline{SA(r_i)} = R_{k'}(\overline{SA(r_j)})$). Hence, $SA(r_i) \prec SA(r_j)$ and $\overline{SA(r_j)} \prec \overline{SA(r_i)}$. A contradiction. \square

Clearly, following the reasoning as for Lemma 3:

Lemma 4 *If all the robots are at distinct positions on the boundary of a same circle C without forming a regular n -gon, then there exists exactly one robot r_i such that $\overline{SA(r_i)}$ is a Lyndon Word.*

Let LWS be the set of robots r_i such that $SA(r_i)$ or $\overline{SA(r_i)}$ is a Lyndon word.

Lemma 5 *If all the robots are at distinct positions on the boundary of a same circle C without forming a regular n -gon, then $|LWS| = 2$.*

Proof. From Lemmas 3 and 4, there exists exactly one robot r_a such that $SA(r_a)$ is a Lyndon word, and exactly one robot r_b such that $\overline{SA(r_b)}$ is a Lyndon word. However, r_a can be the same robot as r_b . So, $1 \leq |LWS| \leq 2$.

Let assume that $|LWS| = 1$, i.e., there exists a unique robot r such that both $SA(r) = a_1a_2 \dots a_n$ and $\overline{SA(r)} = a_na_{n-1} \dots a_1$ are Lyndon words. Since the robots do not form a regular n -gon, $SA(r) \neq a_1^n$. So, $SA(r)$ contains at least two different letters. There are three cases to consider:

1. $a_n \prec a_1$. So, $a_na_1a_2 \dots a_{n-1} \prec a_1 \dots a_{n-1}a_n$, i.e., $R_n(SA(r)) \prec SA(r)$. This contradicts that $SA(r)$ is a Lyndon word—by Definition 1, $SA(r) \prec R_n(SA(r))$.
2. $a_1 \prec a_n$. So, $a_1a_na_{n-1} \dots a_2 \prec a_na_{n-1} \dots a_2a_1$, i.e., $R_n(\overline{SA(r)}) \prec \overline{SA(r)}$. This contradicts that $\overline{SA(r)}$ is a Lyndon word—by Definition 1, $\overline{SA(r)} \prec R_n(\overline{SA(r)})$.

3. $a_1 = a_n$. Since $SA(r)$ is a Lyndon word, by Definition 1, we have $a_1a_2 \dots a_{n-1}a_n \prec a_na_1a_2 \dots a_{n-1}$. Since $a_1 = a_n$, we have $a_2 \dots a_{n-1}a_n \prec a_1a_2 \dots a_{n-1}$. Since $a_1 = a_n$ again, we have also $a_2 \dots a_{n-1}a_na_1 \prec a_1a_2 \dots a_{n-1}a_n$, i.e., $R_2(SA(r)) \prec SA(r)$. This contradicts Definition 1, $SA(r) \prec R_2(SA(r))$.

□

Now, consider the distributed computation of string of angles. We borrow Algorithm 1 from [CP02] which describes a function called Function *ComputeSA*. Each robot r_i arbitrarily determines its own clockwise direction of C in its local coordinate system. Note that since the robots are uniform, all of them apply the same algorithm to determine the clockwise direction. However, the robots do not share a common coordinate system. So, for any pair r_i, r_j , the clockwise direction of r_i (resp., r_j) may be the counterclockwise of r_j (resp., r_i).

function *ComputeSA*(r_j): **word**

$SA := \epsilon$; $r := r_j$;

for ($k = 1$; $k \leq n$; $k := k + 1$) **do**

$r' := Succ(r)$; $SA := SA \circ Angle(r, r')$; $r := r'$;

done

return SA ;

Algorithm 1: Function *ComputeSA* for any robot r_i

Algorithm 1 uses two functions, $Succ(r)$ and $Angle(r, r')$. The former returns the successor of r in the local coordinate system of the robot executing $Succ(r)$. The latter returns the absolute value of $\widehat{rOr'}$, where O is the center of C . Using this algorithm, each robot r_i computes $SA(r_j)$, $\forall j \in 1 \dots n$. Similarly, for each robot r_j , $\overline{SA(r_j)}$ can also be easily computed by any r_i by either computing the mirroring word of $SA(r_j)$ or following Algorithm 1 and replacing Function $Succ(r)$ by a function $Pred(r)$, which returns the unique predecessor of r in the counterclockwise direction.

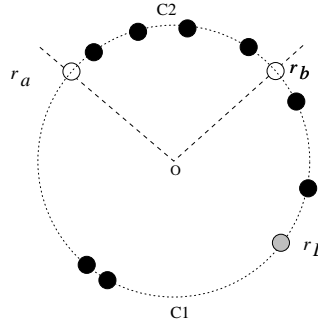


Figure 2: An example showing how the leader robot r_L is computed ($n = 11$).

Let us now describe how the above results can be used to elect a leader. An example showing our method is given in Figure 2 with $n = 11$. Following Lemmas 3, 4, and 5, when any robot computes the set LWS , then $LWS = \{r_a, r_b\}$ and r_a and r_b refer to the same robots for every robot. Consider both half-lines $[O, r_a)$ and $[O, r_b)$. These two half-lines divide the circle C into two sides,

C_1 and C_2 , where n_{C_1} (resp. n_{C_2}) represent the number of robots inside C_1 (resp C_2). Note that $n_{C_1} + n_{C_2} = n - 2$. Since $n - 2$ is odd (n is prime), there exists one side with an even number of robots, and one side with an odd number of robots. Without loss of generality, let us assume that n_{C_1} is odd. Let r_L be the unique robot which is the median robot on C_1 , i.e., the $(\lfloor \frac{n_{C_1}}{2} \rfloor + 1)^{\text{th}}$ robot starting indifferently from r_a or r_b .

Let us define Function $Elect()$ which returns the unique leader robot r_L for every robot r_i .

Lemma 6 *If all the robots are at distinct positions on the boundary of a same circle C without forming a regular n -gon, then for every robot r_i , Function $Elect()$ returns a unique leader robot r_L among the n robots.*

3.2 Oriented Configuration

A configuration is said to be *oriented* if the following conditions hold:

1. All the robots are at distinct positions on the same circle C_0 , except only one of them, called r_0 , located inside C_0 ;
2. r_0 is not located at the center O of C_0 ;
3. there is no robot on $C_0 \cap [O, r_0)$.

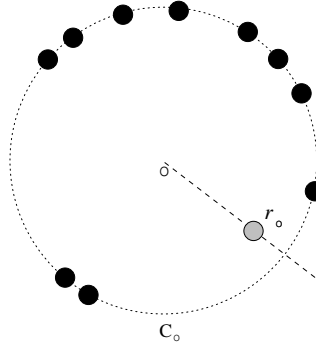


Figure 3: An oriented configuration ($n = 11$).

An example of an oriented configuration is shown in Figure 3 with $n = 11$. Let us denote an oriented configuration by the pair of its two main parameters, i.e., (C_0, r_0) . Two oriented configurations (C_0^α, r_0^α) and (C_0^β, r_0^β) are said to be *equivalent* if $C_0^\alpha = C_0^\beta$ and both r_0^α and r_0^β are located at the same position. In other words, the only possible difference between two equivalent oriented configurations (C_0^α, r_0^α) and (C_0^β, r_0^β) is different positions of robots between C_0^α and C_0^β .

We now describe Procedure ϕ_0 shown in Algorithm 2. This procedure assumes that the configuration is oriented. In such a configuration, we will build a partial order among the robots to eventually form an n -gon.

Let p_1, \dots, p_n be the final positions of the robots when the regular n -gon is formed. Let $p_1 = C_0 \cap [O, r_0)$. Then, for each $k \in 2 \dots n$, p_k is the point on C_0 such that $\widehat{p_1 O p_k} = \frac{2k\pi}{n}$. Clearly, while the distributed system remains in an equivalent oriented configuration, all the final positions remain unchanged for every robot. A position p_k , $k \in 2 \dots n$, is said to be *free* if no robot takes place at p_k .

```

procedure  $\phi_0$ 
   $C_0 :=$  circle where  $n - 1$  robots are located;
   $O :=$  center of  $C_0$ ;
   $r_0 :=$  robot inside  $C_0$ ;
   $p_1 := C_0 \cap [O, r_0)$ ;
   $PS := FindFinalPos(C_0, p_1)$ ;
   $FRS :=$  set of robots which are not located on a position in  $PS$ , except  $r_0$ ;
  if  $FRS = \emptyset$ 
  then // Every robot on  $C_0$  is located on a final position ( $\in PS$ ).
    if  $r_i = r_0$  then move to Position  $p_1$ ;
    endif
  else  $EFR := ElectFreeRobots(FRS)$ ;
    if  $r_i \in EFR$  then move to Position  $Associate(r_i)$ ;

```

Algorithm 2: Procedure ϕ_0 for any robot r_i in an oriented configuration if $n \geq 5$

Similarly, a robot r_i on C_0 (i.e., $r_i \neq r_0$) is called a *free* robot if its current position does not belong to $\{p_2, \dots, p_n\}$. Define Function $FindFinalPos(C_0, p_1)$ which returns the set of final positions on C_0 with respect to p_1 . Clearly, all the robots compute the same set of final positions, stored in PS . Each robot also temporarily stores the set of free robots in the variable called FRS .

Basically, if $FRS = \emptyset$ while the n -gon is not formed, then it remains r_0 only to move from its current position inside C_0 to p_1 . Otherwise, the robots move in waves to the final positions following the order defined by Function $ElectFreeRobots()$. The elected robots are the closest free robots from p_1 . Clearly, the result of Function $ElectFreeRobots()$ return the same set of robots for every robot. Also, the number of elected robots is at most equal to 2, one for each direction on C_0 with respect to p_1 . Note that it can be equal to 1 when there is only one free robot, i.e., when only one robot on C_0 did not reach the last free position.

Function $Associate(r)$ assigns a unique free position to an elected robot as follows: If $ElectFreeRobots()$ returns only one robot r , then r is associated to the unique free remaining position p . This allows r to move to p . If $ElectFreeRobots()$ returns a pair of robots $\{r_a, r_b\}$ ($r_a \neq r_b$), then the closest robot to p_1 , in the clockwise (respectively, counterclockwise) is associated with the closest position to p_1 in the clockwise (resp., counterclockwise) direction. Note that, even if the robots may have opposite clockwise directions, r_a , r_b , and their associated positions are the same for every robot.

Lemma 7 *If the robots are in an oriented configuration at time t_j ($j \geq 0$), then at time t_{j+1} , either the robots are in an equivalent oriented configuration or they form a regular n -gon.*

Proof. By contradiction, assume that starting from an oriented configuration at time t_j , the robots are not in an equivalent oriented configuration and they do not form a regular n -gon at time t_{j+1} . By assumption, at each time instant t_j , at least one robot is active. So, by fairness, starting from an oriented configuration, at least one robot executes Procedure ϕ_0 . Assume first that no robot executing Procedure ϕ_0 moves from t_j to t_{j+1} . In that case, since the robots are located on the same positions at t_j and at t_{j+1} , the robots are in the same oriented configuration at t_{j+1} . An oriented configuration being equivalent to itself, this contradicts the assumption. So, at least one robot moves from t_j to t_{j+1} . There are two cases to consider:

1. No final position is free at time t . So, every robots on C_0 ($\neq r_0$) is located on a final position and no final position is free. Then, for every robot executing Procedure ϕ_0 , $FRS = \emptyset$. In that case, only r_0 is allowed to move. Since by assumption at least one robot moves from t_j to t_{j+1} , r_0 moves from its current position to p_1 . Therefore, the robots form a regular n -gon at time t_{j+1} . A contradiction.
2. Final positions are free at time t . Then, for every robot executing Procedure ϕ_0 , $FRS \neq \emptyset$. By construction, there exists either one or two robots which belongs to the set EFR and r_0 is not allowed to move. Therefore, at least one of them moves from its current position on C_0 to its associated free position. Obviously, the robots are again in distinct positions from t_j to t_{j+1} . Furthermore, since all the final positions are on C_0 and r_0 remains at the same position from t_j to t_{j+1} , the robots are in an equivalent oriented configuration at t_{j+1} . A contradiction.

□

Theorem 8 *Starting from an oriented configuration, Algorithm $\phi_{oriented}$ solves the Circle Formation Problem.*

Proof. It follows from the proof of Lemma 7 that in a oriented configuration, there exists at least one robot which can move to any final position. The theorem follows by fairness and Lemma 7. □

3.3 Main Algorithm

The main part of our solution is presented in Algorithm 4. Using Algorithm ϕ_{circle} [DK02], the robots are first placed in a circle C . Then, a robot is pointed out as the leader r_0 —using Function $Elect()$ described in Subsection 3.1. Next, Robot r_0 moves inside the circle on the segment $[O, r_0]$, at a position arbitrarily chosen at the middle of $[O, r_0]$. Finally, following the partial order provided by Procedure ϕ_0 , the robots eventually take place in a regular n -gon.

```

if the robots do not take place in a regular  $n$ -gon
then if the robots are in an oriented configuration
    then Execute Procedure  $\phi_0$ 
    else if the robots take place in a circle  $C$ 
        then if  $r_i = Elect()$  //Robot  $r_i$  is the leader robot
            then move to Position  $p$  such that  $p = \frac{Dist(r_i, O)}{2}$ ,  $O$  being the center of  $C$ ;
            endif
        else Execute Procedure  $\phi_{circle}$ ;

```

Algorithm 3: Procedure ϕ_0 for any robot r_i in an oriented configuration if $n \geq 5$

Algorithm 4: (ϕ_{n-gon}) Algorithm for any robot r_i if $n \geq 5$

Theorem 9 *Algorithm ϕ_{n-gon} solves the problem of circle formation for $n \geq 5$ robots.*

Proof. From Algorithm 4, if the robots form an n -gon, then no robot moves and the the robots form an n -gon forever. If the robots do not form an n -gon, then from Theorem 2, Algorithm 4 again, Lemma 6, and Theorem 8, the robots eventually form an n -gon. □

4 Conclusion

We showed how Lyndon words can be used in the distributed control of a set of n anonymous robots being memoryless, without any common sense of direction, and unable to communicate in an other way than observation. An efficient and simple deterministic protocol to form a regular n -gon was presented for a prime number of robots. We believe that the new idea presented in this paper should help in the design of the protocol for the Circle Formation Problem with any arbitrary number of robots. That would be the main goal of our future research.

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A Appendix : Circle Formation for 3 Robots

The aim of the algorithm, shown in Algorithm 5, is to lead the 3 robots to eventually take place in a regular 3-gon, i.e., an equilateral triangle. The correctness proof of Algorithm 5 is based on the following lemma:

Lemma 10 *If the three robots $\{r_1, r_2, r_3\}$ do not form an equilateral triangle, then it is possible to elect only one robot among them.*

Proof. If $\{r_1, r_2, r_3\}$ do not form an equilateral triangle, then either (Case 1) they belong to the same line, (Case 2) they form an isosceles triangle, or (Case 3) they form an ordinary triangle. In the first case, the elected robot is the median one, which is unique. In Case 2, the elected robot is the one placed at the unique angle different from the two others. In Case 3, there exists a unique angle which is strictly smallest than the others. The elected robot is the one placed at this unique angle. \square

Lemma 10 allows to define the Boolean function $Elect()$, which can be executed by any robots r_i . $Elect()$ returns the unique leader robot according to Lemma 10.

if the robots do not take place in a regular 3-gon

then if $r_i = Elect()$

then (r_j, r_k) denotes the two other robots than myself;

 move to Position p such that $\{p, r_j, r_k\}$ form a regular 3-gon;

Algorithm 5: (ϕ_3 -gon) Algorithm for any robot r_i if $n = 3$

Theorem 11 *Algorithm ϕ_3 -gon solves the problem of circle formation for three robots.*

Proof. From Lemma 10, we can distinguish a unique robot, called the leader. According to Algorithm 5, if the robots do not form an equilateral triangle at time t , then only the leader is allowed to move. Since by assumption at least one robot is activated at each instant time, at time $t + 1$ the robots form an equilateral triangle. \square