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► **To cite this version:**

Baohua Fu. Extremal contractions, stratified Mukai flops and Springer maps. *Advances in Mathematics*, 2007, 213 (1), pp.165–182. hal-00069128

HAL Id: hal-00069128

<https://hal.science/hal-00069128>

Submitted on 16 May 2006

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Extremal contractions, stratified Mukai flops and Springer maps

Baohua Fu

May 16, 2006

Abstract

We prove that two Springer maps of the same degree over a nilpotent orbit closure are connected by stratified Mukai flops, and the latter is obtained by contractions of extremal rays of a natural resolution of the nilpotent orbit closure.

1 Introduction

Let \mathcal{O} be a nilpotent orbit in a simple complex Lie algebra \mathfrak{g} (with G its adjoint group). The closure $\overline{\mathcal{O}}$ in \mathfrak{g} is a singular variety whose smooth part admits a holomorphic symplectic Kostant-Kirillov form ω . A resolution $f : Z \rightarrow \overline{\mathcal{O}}$ is said *symplectic* if $f^*\omega$ extends to a symplectic form on the whole of Z , or equivalently if f is crepant. Note that there exist nilpotent orbit closures which admit no symplectic resolutions ([Fu1]).

A natural G -equivariant projective resolution of $\overline{\mathcal{O}}$ is given by $\mu : G \times^P \mathfrak{n} \rightarrow \overline{\mathcal{O}}$, where P is a parabolic subgroup associated to the Jacobson-Morozov sub-algebra of \mathcal{O} and \mathfrak{n} is a nilpotent ideal in $\mathfrak{p} = Lie(P)$ (see Section 2). This resolution plays an important role in the study of singularities of $\overline{\mathcal{O}}$ (see for example [Pan]). The resolution μ is symplectic if and only if \mathcal{O} is an even orbit ([Fu3]). The closure $\overline{Amp}(\mu)$ of the ample cone of μ is a simplicial polyhedral cone and a face of $\overline{Amp}(\mu)$ corresponds to the Stein factorization of the natural birational map $p : G \times^P \mathfrak{n} \rightarrow G \times^Q (Q \cdot \mathfrak{n})$, for a suitable parabolic subgroup Q of G containing P . Notice that μ factorizes through p , which gives a birational map $\pi : G \times^Q (Q \cdot \mathfrak{n}) \rightarrow \overline{\mathcal{O}}$. A natural question is if we can choose Q such that π is a symplectic resolution. If it is the

case and \mathcal{O} is not an even orbit, then p becomes an extremal contraction of μ , which is an important class of morphisms in Mori theory. We prove (Theorem 3.3, Theorem 4.2, Corollary 5.11 and Example 5.12) that except for some particular orbits in D_n and the orbit $\mathcal{O}_{D_7(a_2)}$ in E_8 , one can always obtain a symplectic resolution of the nilpotent orbit closure $\overline{\mathcal{O}}$ either by μ itself or by extremal contractions of μ , provided that $\overline{\mathcal{O}}$ admits a symplectic resolution. The proof is based on Proposition 2.3 and some combinatorial calculations. Here and throughout the paper, the notations of nilpotent orbits in exceptional Lie algebras are those in [CM].

Then we turn to study the birational geometry of Springer maps. Recall that ([Ric]) for any parabolic sub-group Q of G , the image of the moment map $T^*(G/Q) \rightarrow \mathfrak{g} \simeq \mathfrak{g}^*$ is a nilpotent orbit closure $\overline{\mathcal{O}}$. The orbit \mathcal{O} will be called a *Richardson orbit* and Q its *polarization*. The morphism $T^*(G/Q) \rightarrow \overline{\mathcal{O}}$ will be called the *Springer map* associated to Q , which is a generically finite surjective projective map. For two Springer maps $T^*(G/Q_i) \rightarrow \overline{\mathcal{O}}, i = 1, 2$ having the same degree, we prove (Corollary 5.9) that there exists a birational map $T^*(G/Q_1) \dashrightarrow T^*(G/Q_2)$ over $\overline{\mathcal{O}}$ which is decomposed into a finite sequence of (analytically) locally trivial families of stratified Mukai flops. Recall that ([Fu1]) any symplectic resolution of $\overline{\mathcal{O}}$ is given by a (degree one) Springer map, so this implies that two symplectic resolutions of $\overline{\mathcal{O}}$ are connected by stratified Mukai flops, which has been previously proved by Namikawa ([Na2]) using an ingenious argument. Our proof here is different from [Na2] and does not make use of the Springer correspondence and Mori theory, instead we will use a theorem of Hiraï ([Hir]) on polarizations of nilpotent orbits. Here are three features of this result:

(i) Nilpotent orbits usually have G -equivariant coverings, which turn to be important in representation theory (e.g. [BK]). An immediate corollary of our result is that Springer resolutions of coverings of nilpotent orbits are related by stratified Mukai flops.

(ii) As a special case of Kawamata's conjecture that K -equivalence implies D -equivalence, one conjectures the derived equivalence of the birational map $T^*(G/Q_1) \dashrightarrow T^*(G/Q_2)$. As remarked in [Na2], our result here reduces this conjecture to the cases of stratified Mukai flops.

(iii) This result provides evidence to the following:

Speculation: Any two (projective) symplectic resolutions of a symplectic singularity are related by stratified Mukai flops. Any birational map between two projective hyperKähler manifolds is decomposed into a sequence

of stratified Mukai flops.

Finally we turn to study stratified Mukai flops. Let $\phi : T^*(G/Q) \dashrightarrow T^*(G/Q')$ be such a flop. We show (Theorem 6.1) that the graph closure of ϕ is isomorphic to the variety $G \times^P \mathfrak{n}$ which gives the natural resolution of $\overline{\mathcal{O}}$ and the two graph projections $T^*(G/Q) \leftarrow G \times^P \mathfrak{n} \rightarrow T^*(G/Q')$ are given by the two contractions of extremal rays of μ . This gives a conceptual and explicit description of the flop ϕ .

Some interesting by-products are obtained by our methods here. We prove (Theorem 4.2 and Corollary 5.11) that for the orbits $D_4(a_1) + A_1$ in E_7 , $D_6(a_1)$ and $D_7(a_2)$ in E_8 , their closures admit a symplectic resolution, while for the four orbits $A_4 + A_1$, $D_5(a_1)$ in E_7 , $E_6(a_1) + A_1$ and $E_7(a_3)$ in E_8 , their closures do not admit a symplectic resolution (Corollary 5.11). Together with [Fu1] (see also Proposition 3.1, Proposition 4.1), this completes the classification of nilpotent orbits in a simple Lie algebra whose closure admits a symplectic resolution.

As remarked in [BM], the degree of the Springer map $T^*(G/Q) \rightarrow \overline{\mathcal{O}}$ plays an important role in several different contexts (e.g. [BB], Theorem 5.5, 5.6, 5.8). Another by-product of this paper is that we can determine (Remark 7.3) the degree of the Springer map associated to any parabolic sub-group Q in a very explicit and practical way. When \mathfrak{g} is classical, this is due to Hesselink ([Hes]). When \mathfrak{g} is exceptional, this result seems to be new.

Notations: We fix a Cartan sub-algebra \mathfrak{h} , a Borel sub-algebra \mathfrak{b} , a system of positive roots Φ^+ and simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$. The labels of roots in the Dynkin diagram are the same as those in [Bou]. For a subset $\Gamma \subset \Delta$, we denote by \mathfrak{p}_Γ the standard parabolic sub-algebra $\mathfrak{b} \oplus_{\beta \in \langle \Delta - \Gamma \rangle} \mathfrak{g}_\beta$ and P_Γ the standard parabolic subgroup in G with Lie algebra P_Γ . Note that in the literature, our \mathfrak{p}_Γ is usually denoted by $\mathfrak{p}_{\Delta - \Gamma}$. The marked Dynkin diagram of P_Γ is obtained by marking the nodes in Γ . We will denote by $\mathfrak{u}(P_\Gamma)$ or $\mathfrak{u}(\mathfrak{p}_\Gamma)$ the nil-radical of \mathfrak{p}_Γ and by $\mathfrak{l}(P_\Gamma)$ or $\mathfrak{l}(\mathfrak{p}_\Gamma)$ the Levi factor of \mathfrak{p}_Γ . More precisely, $\mathfrak{u}(\mathfrak{p}_\Gamma) = \oplus_{\beta \in -\langle \Delta - \Gamma \rangle} \mathfrak{g}_\beta$ and $\mathfrak{l}(\mathfrak{p}_\Gamma) = \mathfrak{h} \oplus_{\beta \in \langle \Delta - \Gamma \rangle} \mathfrak{g}_\beta$.

Acknowledgements: Part of this work has been done during my visit to I.M.S. at the Chinese University of Hong Kong and M.S.R.I. at Berkeley. It's my pleasure to thank C. Leung and Y. Ruan for the kind invitations and the two institutes for their hospitality. I want to thank L. Manivel for the references [Dem] [NSZ], and D. Alvis for explanations on [Alv].

2 Extremal contractions

Let \mathfrak{g} be a simple Lie algebra and G its adjoint group. For a nilpotent element $x \in \mathfrak{g}$, the Jacobson-Morozov theorem gives an \mathfrak{sl}_2 -triplet (x, y, h) , i.e. $[h, x] = 2x, [h, y] = -2y, [x, y] = h$. Up to replacing this triplet by a conjugate one, we can assume that $h \in \mathfrak{h}$ and h is Δ -dominant. This triplet makes \mathfrak{g} an \mathfrak{sl}_2 -module, so we have a decomposition $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, where $\mathfrak{g}_i = \{z \in \mathfrak{g} \mid [h, z] = iz\}$. The Jacobson-Morozov parabolic sub-algebra of this triplet is $\mathfrak{p} := \bigoplus_{i \geq 0} \mathfrak{g}_i$. Its conjugacy class is uniquely determined by the nilpotent orbit $\mathcal{O} = G \cdot x$.

Recall that a nilpotent orbit is uniquely determined by its weighted Dynkin diagram, which is obtained by assigning $\alpha(h)$ to the node α . Under our assumption, $\alpha(h) \in \{0, 1, 2\}$.

Proposition 2.1. *The marked Dynkin diagram of \mathfrak{p} is obtained from the weighted Dynkin diagram of \mathcal{O} by marking the nodes with non-zero weights.*

Proof. Let Γ be the set of marked nodes in the marked Dynkin diagram of \mathfrak{p} , then the set $\Delta - \Gamma$ consists of simple roots α such that $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ are contained in \mathfrak{p} . Notice that $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\alpha(h)}, \mathfrak{g}_{-\alpha} \subset \mathfrak{g}_{-\alpha(h)}$, which gives that $\alpha(h) = 0$. \square

The closure $\overline{\mathcal{O}}$ of \mathcal{O} in \mathfrak{g} is singular. A natural resolution of $\overline{\mathcal{O}}$ is given by $\mu : G \times^P \mathfrak{n} \rightarrow \overline{\mathcal{O}}$, where P is a connected subgroup of G with Lie algebra \mathfrak{p} and $\mathfrak{n} := \bigoplus_{i \geq 2} \mathfrak{g}_i$ is a nilpotent ideal of \mathfrak{p} . Note that the variety $G \times^P \mathfrak{n}$ and the map μ is independent (up to isomorphisms) of the choice of the element $x \in \mathcal{O}$ and the standard \mathfrak{sl}_2 -triplet. This resolution is symplectic if and only if \mathcal{O} is an *even orbit*, i. e. $\mathfrak{g}_1 = 0$, or equivalently the weights in the weighted Dynkin diagram of \mathcal{O} are only 0 and 2 ([Fu3]).

If we denote by $\tilde{\mu}$ the Stein factorization of μ , then $\overline{Am\tilde{p}}(\tilde{\mu})$ and $\overline{NE}(\tilde{\mu})$ are both simplicial polyhedral cones. The contraction of a face in $\overline{NE}(\tilde{\mu})$ is given by the Stein factorization of the morphism

$$p : G \times^P \mathfrak{n} \rightarrow G \times^Q (Q \cdot \mathfrak{n}),$$

for a suitable parabolic sub-group Q in G containing P . The map μ factorizes through p , which gives a birational map $\pi : G \times^Q (Q \cdot \mathfrak{n}) \rightarrow \overline{\mathcal{O}}$. An interesting question is if we can choose Q such that π becomes a symplectic resolution.

Remark 2.2. When \mathcal{O} is even, then μ is already a symplectic resolution. In this case, any contraction as above will produce a singular variety $G \times^Q (Q \cdot \mathfrak{n})$.

Proposition 2.3. *Let Q be a parabolic subgroup containing P and $\mathfrak{u}(Q)$ the nil-radical of $\mathfrak{q} := \text{Lie}(Q)$. If $\mathfrak{n} \subset \mathfrak{u}(Q)$ and $2 \dim(\mathfrak{u}(Q)) = \dim \mathcal{O}$, then*

(1) $G \times^Q (Q \cdot \mathfrak{n}) \simeq T^*(G/Q)$ and π is a symplectic resolution,

(2) if \mathcal{O} is not an even orbit, then the map p is a contraction of an extremal face of $\overline{NE}(\tilde{\mu})$.

Proof. By assumption, we have $Q \cdot \mathfrak{n} \subset Q \cdot \mathfrak{u}(Q) = \mathfrak{u}(Q)$. Note that $Q \cdot \mathfrak{n}$ is the image of $Q \times^P \mathfrak{n}$ under the projective map $Q/P \times \mathfrak{g} \rightarrow \mathfrak{g}$, so it is closed in \mathfrak{g} . Since π is birational, we have $\dim(Q \cdot \mathfrak{n}) = \dim(\mathcal{O}) - \dim(G/Q) = \dim(\mathcal{O}) - \dim(\mathfrak{u}(Q)) = \dim(\mathfrak{u}(Q))$, which gives $Q \cdot \mathfrak{n} = \mathfrak{u}(Q)$ since $\mathfrak{u}(Q)$ is closed and irreducible. Now assertion (1) follows immediately.

Assume that \mathcal{O} is not even, then p is not an isomorphism. The exceptional set E of p has pure codimension 1 since $G \times^Q (Q \cdot \mathfrak{n})$ is smooth. Let $E = \cup_i E_i$ be the decomposition into irreducible components, then $K := K_{G \times^P \mathfrak{n}} = \sum_i a_i E_i$ with $a_i > 0$. Now it is easy to see that $K \cdot C < 0$ for any curve C contracted to a point by p . For any such a curve C , its class lies in an extremal face of $\overline{NE}(\tilde{\mu})$, the one dual to the face in $\overline{Amp}(\tilde{\mu})$ determined by the map p . \square

Remark 2.4. The advantage of using extremal contractions is that the degree of the map $T^*(G/Q) \rightarrow \overline{\mathcal{O}}$ is automatically 1. This turns out to be helpful when \mathfrak{g} is exceptional, since in this case, it is not easy to calculate the degrees of Springer maps.

One should bear in mind that even when $T^*(G/Q) \rightarrow \overline{\mathcal{O}}$ is a symplectic resolution, in general the birational map $G \times^Q (Q \cdot \mathfrak{n}) \rightarrow \overline{\mathcal{O}}$ is not a symplectic resolution. In fact, the variety $G \times^Q (Q \cdot \mathfrak{n})$ can even be singular if we drop the conditions in the precedent proposition, as shown by the following example.

Example 2.5. In \mathfrak{sl}_5 , let \mathcal{O} be the nilpotent orbit with Jordan type $[4, 1]$. Then the Jacobson-Morozov parabolic subgroup of \mathcal{O} is a Borel subgroup. Let Q be the standard parabolic subgroup with flag type $[2, 1, 1, 1]$, then $T^*(G/Q) \rightarrow \overline{\mathcal{O}}$ is a symplectic resolution, so $\dim \mathfrak{u}(Q) = 1/2 \dim \mathcal{O}$, but \mathfrak{n} is not contained in $\mathfrak{u}(Q)$. A direct calculus shows that $Q \cdot \mathfrak{n}$ is defined by some quadric equations, and it is singular in codimension 1.

3 Classical types

For a nilpotent orbit \mathcal{O} in a classical simple Lie algebra \mathfrak{g} , we denote by Θ_i the set of nodes in Δ with weight i , for $i = 0, 1$ or 2 . The standard Jacobson-

Morozov parabolic sub-algebra of \mathcal{O} will be denoted by \mathfrak{p} , which is obtained by marking $\Theta_1 \cup \Theta_2$ (Prop. 2.1). We denote by P the connected subgroup of G with Lie algebra \mathfrak{p} . Let $\mathbf{d} = [d_1, \dots, d_k]$ be the Jordan type of \mathcal{O} . Recall the following classification theorem from [Fu1].

Proposition 3.1. *Assume \mathfrak{g} is simple classical. Then the closure $\overline{\mathcal{O}}$ admits a symplectic resolution if and only if \mathcal{O} is in the following list:*

- (i) \mathcal{O} is in $\mathfrak{g} = \mathfrak{sl}_n$;
- (ii) $\mathfrak{g} = \mathfrak{so}_{2n+1}$ (resp. \mathfrak{sp}_{2n}) and there exists an odd (resp. even) number $q \geq 0$ such that d_1, \dots, d_q are odd and d_{q+1}, \dots, d_k are even;
- (iii) $\mathfrak{g} = \mathfrak{so}_{2n}$ and either
 - (iii-a) there exists an even number $q \geq 4$ such that d_1, \dots, d_q are odd and d_{q+1}, \dots, d_k are even; or
 - (iii-b) there exist exactly two odd parts in \mathbf{d} at positions $2t - 1$ and $2t$ for some number $t \geq 1$.

From now on, we will assume that \mathcal{O} is not an even orbit, i. e. the Jordan type of \mathcal{O} has parts with different parities. We have the following lemma, whose proof is left to the reader.

Lemma 3.2. *For cases (i), (ii) and (iii-a) in Proposition 3.1, the set Θ_1 has even number of elements. For case (iii-b) in Proposition 3.1, there are the following cases:*

- (iii-b-1) if $t = 1$, then $\alpha_n \in \Theta_1$ and the number of elements in Θ_1 is odd;
- (iii-b-2) if $k = 2t \geq 4$, then the number of elements in Θ_1 is even and $\alpha_{n-1}, \alpha_n \in \Theta_1$;
- (iii-b-3) if $k > 2t \geq 4$, then the number of elements in Θ_1 is even.

For cases (i), (ii) and (iii-a), we decompose the set $\Theta_1 =: \{\alpha_{m_i} | i = 1, \dots, 2l, m_i < m_{i+1}, \forall i\}$ as the disjoint union $\Theta_1^I \cup \Theta_1^{II}$, where $\Theta_1^I = \{\alpha_{m_{2j-1}} | j = 1, \dots, l\}$ and $\Theta_1^{II} = \{\alpha_{m_{2j}} | j = 1, \dots, l\}$. For case (iii-b-1), we obtain a decomposition in a similar way $\Theta_1 = \Theta_1^I \cup \Theta_1^{II}$, with the extra element α_n in Θ_1^{II} . For case (iii-b-2), we have two distinct decompositions (except when $d_{2t-1} = d_{2t} = 1$) of Θ_1 as $\Theta_1^I \cup \Theta_1^{II} = \Theta_1^I \cup \Theta_1^{\prime II}$, where $\Theta_1^I, \Theta_1^{II}$ are defined as above and $\Theta_1^{\prime I}$ (resp. $\Theta_1^{\prime II}$) is obtained from Θ_1^I by replacing α_{n-1} (resp. α_n) by α_n (resp. α_{n-1}).

Let \mathfrak{q}_1 (resp. $\mathfrak{q}_2, \mathfrak{q}'_1, \mathfrak{q}'_2$) be the standard parabolic sub-algebra obtained by marking the nodes in $\Theta_1^I \cup \Theta_2$ (resp. $\Theta_1^{II} \cup \Theta_2, \Theta_1^{\prime I} \cup \Theta_2, \Theta_1^{\prime II} \cup \Theta_2$). Let Q_1, Q_2, Q'_1, Q'_2 be the parabolic subgroups with Lie algebras $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}'_1, \mathfrak{q}'_2$

respectively. One remarks that the standard Jacobson-Morozov parabolic sub-algebra \mathfrak{p} is contained in these sub-algebras. Let $\pi_i : G \times^{Q_i} (Q_i \cdot \mathfrak{n}) \rightarrow \overline{\mathcal{O}}, i = 1, 2$, be two maps obtained from contractions of μ . Similarly one has π'_i for case (iii-b-2).

Theorem 3.3. *For case (i), the two maps π_1, π_2 are both symplectic resolutions. The rational map $\pi_2^{-1} \circ \pi_1$ is resolved by $G \times^P \mathfrak{n}$;*

For cases (ii), (iii-a) and (iii-b-1) the map π_2 is a symplectic resolution;

For case (iii-b-2), π_i, π'_i are all symplectic resolutions, which are all dominated by μ .

For case (iii-b-3), none of the maps arising from extremal contractions of μ is a symplectic resolution.

Proof. The strategy is to apply Prop. 2.3, so we need to check the conditions $\mathfrak{n} \subset \mathfrak{u}(\mathfrak{q})$ and $\dim(\mathfrak{u}(\mathfrak{q})) = 1/2 \dim \mathcal{O}$. The first condition is easily checked by our choice of the decomposition of Θ_1 . The second condition is equivalent to $\dim \mathfrak{u}(\mathfrak{p}) - \dim \mathfrak{u}(\mathfrak{q}) = \dim \mathfrak{u}(\mathfrak{q}) - \dim \mathfrak{n}$. We will check this condition case by case.

For case (i), $\dim \mathfrak{u}(\mathfrak{p}) - \dim \mathfrak{u}(\mathfrak{q}_1)$ is the number of positive roots $\beta = \sum_{i \leq k \leq j} \alpha_k$ such that there exists a unique k_0 such that $\alpha_{k_0} \in \Theta_1^I$ and for $i \leq k \neq k_0 \leq j$, we have $\alpha_k \in \Theta_0$. This is also the sum $\sum_{k=1}^l N(\alpha_{m_{2k}})$, where $N(\alpha_{m_{2k}})$ is the number of connected subgraphs containing the node $\alpha_{m_{2k}}$ and the other nodes are in Θ_0 . On the other hand, $\dim \mathfrak{u}(\mathfrak{q}_1) - \dim \mathfrak{n}$ is the number of positive roots $\beta = \sum_{i \leq k \leq j} \alpha_k$ such that the sum of weights of nodes in β is 1 and there exists some $i \leq k_0 \leq j$ such that $\alpha_{k_0} \in \Theta_1^I$. This number is the sum $\sum_{k=1}^l N(\alpha_{m_{2k-1}})$. Note that the weighted Dynkin diagram of any nilpotent orbit in \mathfrak{sl}_n is invariant under the non-trivial graph automorphism (Lemma 3.6.5 [CM]), so we have $N(\alpha_{m_{2k-1}}) = N(\alpha_{m_{2l+2-2k}})$, which gives the equality $\dim \mathfrak{u}(\mathfrak{p}) - \dim \mathfrak{u}(\mathfrak{q}_1) = \dim \mathfrak{u}(\mathfrak{q}_1) - \dim \mathfrak{n}$. Similar arguments apply to \mathfrak{q}_2 . Thus π_1, π_2 are both symplectic and dominated by μ .

For case (ii), first consider $\mathfrak{g} = \mathfrak{so}_{2n+1}$. As easily seen, the weighted Dynkin diagram of \mathcal{O} has the following form (where nodes are replaced by their weights):

$$\dots - 2 - 0^{2a} - 1 - 0^{r_1} - 1 - 0^{2a} - 1 - \dots - 0^{2a} - 1 - 0^{r_l} - 1 - 0^{a-1} \Rightarrow 0$$

where r_1, \dots, r_l are non-negative integers, $2a + 1 = q$ and 0^m means the consecutive m nodes have weights 0. Note that the weights of nodes on

the left-hand side of the node with weight 2 can be only 0 or 2. Like case (i), one has

$$\dim \mathfrak{u}(\mathfrak{p}) - \dim \mathfrak{u}(\mathfrak{q}_2) = \sum_{i=1}^l N(\alpha_{m_{2i-1}}) = \sum_{i=1}^l (2a+1)(r_i+1) = q\left(\sum_{i=1}^l r_i + l\right).$$

On the other hand

$$\dim \mathfrak{u}(\mathfrak{q}_2) - \dim \mathfrak{n} = \sum_{i=1}^{l-1} N(\alpha_{m_{2i}}) + N = q\left(\sum_{i=1}^l r_i + l\right) + (N - q(r_l + 1)),$$

where N is the number of positive roots such that one (with multiplicity) of whose summands is α_{n-a} (the rightmost node with weight 1) and the others are in $\{\alpha_j | n-a-r_l \leq j \neq n-a\}$. Using the table for positive roots ([Bou]), we find $N = q(r_l + 1)$, which gives the assertion.

The proof for $\mathfrak{g} = \mathfrak{sp}_{2n}$ is similar to the case of \mathfrak{so}_{2n+1} . The key point is to notice that the weighted Dynkin diagram of \mathcal{O} is of the following form:

$$\dots - 2 - 0^{q-1} - 1 - 0^{r_1} - \dots - 0^{q-1} - 1 - 0^{r_l} - 1 - 0^{q/2-1} \Leftarrow 0,$$

where r_1, \dots, r_l are non-negative integers. For case (iii-a), the argument is the same, by noticing that the weighted Dynkin diagram has the following form:

$$\dots - 2 - 0^{q-1} - 1 - 0^{r_1} - 1 - \dots - 0^{r_l} - 0^{q/2},$$

where $0^{q/2}$ means the last $q/2$ nodes have weights 0. The other two cases (iii-b-1) and (iii-b-2) are similar.

For case (iii-b-3), we notice that the Levi type of any (degree 1) polarization Q of \mathcal{O} is $\mathbf{d}' = [d_1, \dots, d_{2t-2}, d_{2t-1} + 1, d_{2t} - 1, d_{2t+1}, \dots, d_k]$. The dual partition of \mathbf{d}' has the form $\text{ord}(\mathbf{d}') := [k^{2s}, q_1, \dots, q_l]$, with $q_j \leq k - 2s, \forall j$ and $s \geq 1$, since even parts appear with even multiplicity in the Jordan type of any nilpotent orbit in \mathfrak{so}_{2n} . By our assumption, $q_l \geq 2$ since d_1 is even. By [Hes], this implies that every flag type determined by $\text{ord}(\mathbf{d}')$ corresponds to two marked Dynkin diagrams, i.e. either α_{n-1} or α_n is marked, but not both.

The weighted Dynkin diagram of \mathcal{O} has the following form:

$$\dots - 1 - 0 - 1 - 0^{k-3} - 1^2,$$

where 1^2 means that the nodes α_{n-1}, α_n have weights 1. Suppose that Q contains P , then in the flag type of Q , there is either a part equal to $k - 1$ or a part equal to $q' \geq k + 1$, which is a contradiction. \square

Remark 3.4. (1). Notice that for \mathfrak{so}_{2n+1} , although one has $\mathfrak{n} \subset \mathfrak{u}(\mathfrak{q}_1)$, but $\dim \mathfrak{u}(\mathfrak{p}) - \dim \mathfrak{u}(\mathfrak{q}_1) > \dim \mathfrak{u}(\mathfrak{q}_1) - \dim \mathfrak{n}$, thus π_1 is not a symplectic resolution. Similar remark applies to other cases.

(2). For case (iii-b-3), as we know the Levi type of a degree one polarization, we can also obtain the marked Dynkin diagram of a symplectic resolution. Later on, we will find all symplectic resolutions of a nilpotent orbit closure by starting from any given one.

4 Exceptional cases

Let us recall the following classification result from [Fu1]. The notations of orbits are those in [CM] (p. 128-134).

Proposition 4.1. (i) For the following Richardson orbits, we do not know if their closures admit a symplectic resolution or not:

$D_4(a_1) + A_1, A_4 + A_1, D_5(a_1)$ in E_7 and
 $D_6(a_1), D_7(a_2), E_6(a_1) + A_1, E_7(a_3)$ in E_8 .

(ii) For other orbits in a simple exceptional Lie algebra \mathfrak{g} , its closure admits a symplectic resolution if and only if it is a Richardson orbit. The following is the complete list of such orbits:

(ii-a) even orbits;

(ii-b) C_3 in F_4 , $2A_1, A_2 + 2A_1, A_3, A_4 + A_1, D_5(a_1)$ in E_6 , $D_5 + A_1, D_6(a_1)$ in E_7 and $A_4 + A_2 + A_1, A_6 + A_1, E_7(a_1)$ in E_8 .

Theorem 4.2. For the orbits in (ii-b) of Proposition 4.1 and orbits $D_4(a_1) + A_1$ in E_7 , $D_6(a_1)$ in E_8 , one can always obtain a symplectic resolution by an extremal contraction of the natural resolution $\mu : G \times^P \mathfrak{n} \rightarrow \overline{\mathcal{O}}$. In particular, the closures of the orbit $D_4(a_1) + A_1$ in E_7 and the orbit $D_6(a_1)$ in E_8 admit a symplectic resolution.

Proof. We will verify the conditions in Proposition 2.3, and then apply it to conclude. The notations of roots are those in [Bou]. We just give the corresponding polarization Q to each orbit in the statement. The condition $\mathfrak{n} \subset \mathfrak{u}(\mathfrak{q})$ and the dimension check can be done by using the tables of root systems in [Bou]. For example for $\mathfrak{g} = F_4$ and $\mathcal{O} = \mathcal{O}_{C_3}$, the Jacobson-Morozov standard parabolic is $P = P_{\alpha_1\alpha_2\alpha_3}$ while $Q = P_{\alpha_3\alpha_4}$. One checks that $\dim \mathfrak{u}(P) - \dim \mathfrak{u}(Q) = \dim \mathfrak{u}(Q) - \dim \mathfrak{n} = 2$. Notice that if we take $Q' = P_{\alpha_1\alpha_4}$, then \mathfrak{n} is not contained in $\mathfrak{u}(Q')$, since $\alpha_2 + 2\alpha_3$ is still a positive

root, so Proposition 2.3 is not applicable to Q' . In the tables below, the first row gives the Lie algebras, the second row lists the nilpotent orbits and the third row gives the corresponding (degree one) polarizations.

F_4	E_6				
C_3	$2A_1$	$A_2 + 2A_1$	A_3	$A_4 + A_1$	$D_5(a_1)$
$P_{\alpha_3\alpha_4}$	$P_{\alpha_1}, P_{\alpha_6}$	$P_{\alpha_3}, P_{\alpha_5}$	$P_{\alpha_1\alpha_2}, P_{\alpha_2\alpha_6}$	$P_{\alpha_3\alpha_5}$	$P_{\alpha_2\alpha_3\alpha_6}, P_{\alpha_1\alpha_2\alpha_5}$
E_7			E_8		
$D_4(a_1) + A_1$	$D_5 + A_1$	$A_4 + A_2 + A_1$	$A_6 + A_1$	$A_6 + A_1$	$D_6(a_1)$
$P_{\alpha_2\alpha_7}$	$P_{\alpha_1\alpha_3\alpha_5}$	$P_{\alpha_1\alpha_2\alpha_3\alpha_7}$	P_{α_3}	P_{α_4}	$P_{\alpha_1\alpha_2\alpha_3}$

Finally for the orbit $E_7(a_1)$ in E_8 , we take $Q = P_{\alpha_1\alpha_2\alpha_3\alpha_7\alpha_8}$.

□

Remark 4.3. *The proof also gives another way to show that these orbits are Richardson.*

5 Birational geometry

The precedent sections give a particular symplectic resolution of a nilpotent orbit closure $\overline{\mathcal{O}}$ provided we know the existence of such a resolution. In this section, we will describe a way to find all symplectic resolutions of $\overline{\mathcal{O}}$ starting from any given one. This procedure has been previously described in [Na2].

For two standard parabolic subgroups P_Γ and $P_{\Gamma'}$, we define $P_\Gamma \sim_R P_{\Gamma'}$ (or $\Delta - \Gamma \sim_R \Delta - \Gamma'$) if the Richardson orbits corresponding to P_Γ and $P_{\Gamma'}$ are the same, say \mathcal{O} . We say that P_Γ and $P_{\Gamma'}$ are *equivalent* (write $P_\Gamma \sim P_{\Gamma'}$ or $\Delta - \Gamma \sim \Delta - \Gamma'$) if furthermore the degrees of the two Springer maps $T^*(G/P_\Gamma) \rightarrow \overline{\mathcal{O}} \leftarrow T^*(G/P_{\Gamma'})$ are the same.

Theorem 5.1 (Hirai [Hir]). *Assume \mathfrak{g} is simple. The equivalence relation \sim_R is generated by the following fundamental ones:*

- (1) In B_n or C_n with $n = 3k - 1, k \geq 1$, $P_{\alpha_{2k-1}} \sim_R P_{\alpha_{2k}}$.
- (2) In D_4 , $P_{\alpha_2} \sim_R P_{\alpha_3\alpha_4}$.
- (3) In D_n with $n = 3k + 1, k \geq 2$, $P_{\alpha_{2k}} \sim_R P_{\alpha_{2k+1}}$.
- (4) In G_2 , $P_{\alpha_1} \sim_R P_{\alpha_2}$.
- (5) In F_4 , $P_{\alpha_2} \sim_R P_{\alpha_3} \sim_R P_{\alpha_1\alpha_4}$.
- (6) In E_6 , $P_{\alpha_4} \sim_R P_{\alpha_2\alpha_5}$.
- (7) In E_8 , $P_{\alpha_5} \sim_R P_{\alpha_2\alpha_3}$.
- (8) In A_n , $P_{\alpha_i} \sim_R P_{\alpha_{n+1-i}}, \forall i$.

(9) In D_{2k+1} ($k \geq 2$), $P_{\alpha_{2k}} \sim_R P_{\alpha_{2k+1}}$.

(10) In E_6 , $P_{\alpha_1} \sim_R P_{\alpha_6}$, and $P_{\alpha_3} \sim_R P_{\alpha_5}$.

(GP)[General principle] If Δ_1, Δ_2 are two subsets of Δ orthogonal to each other. Let $\Gamma_i \subset \Delta_i, i = 1, 2$ be two subsets and $\Gamma'_1 \subset \Delta_1$ a subset such that $\Delta_1 - \Gamma_1 \sim_R \Delta_1 - \Gamma'_1$ in the root system $\langle \Delta_1 \rangle$, then $P_{\Gamma_1 \cup \Gamma_2} \sim_R P_{\Gamma'_1 \cup \Gamma_2}$. Here Δ_i can be empty.

The proof of this theorem is essentially a type-by-type check, since one can determine the Richardson orbit of any parabolic subgroup (in classical cases, this is given by the Spaltenstein map, while in most exceptional cases, it suffices to do just a dimension counting. Some particular attention should be paid to a few cases, for details see [Hir]).

Our result is to give a list of fundamental relations for the equivalence \sim .

Theorem 5.2. *Suppose that \mathfrak{g} is simple. Then the equivalence \sim is generated by the relations (8), (9), (10) and (GP) in Theorem 5.1.*

Remark 5.3. As we can see in the proof, the theorem is not true if \mathfrak{g} is not simple.

We begin the proof by some lemmas.

Lemma 5.4. (i) *For the parabolic subgroups appeared in each equivalence relation of (1) - (7) in Theorem 5.1, there is only one such that the associated Springer map is birational.*

(ii) *For any parabolic subgroup appeared in the relations (8) - (10) in Theorem 5.1, the associated Springer map is birational.*

Proof. For the case of B_{3k-1} (resp. C_{3k-1}), the Richardson orbit has Jordan type $[3^{2k-1}, 1^2]$ (resp. $[3^{2k-2}, 2^2]$). The Springer map associated to $P_{\alpha_{2k-1}}$ (resp. $P_{\alpha_{2k}}$) is of degree 1, while that of $P_{\alpha_{2k}}$ (resp. $P_{\alpha_{2k-1}}$) is of degree 2. Here we used Hesselink's formula for the degrees of Springer maps (Theorem 7.1 [Hes]) in classical Lie algebras.

In D_4 , the Richardson orbit of P_{α_2} and $P_{\alpha_3\alpha_4}$ has Jordan type $[3^2, 1^2]$. One calculates that the degrees of the Springer maps associated to the two polarizations are respectively 1 and 2.

In D_{3k+1} , $k \geq 2$, the Richardson orbit for $P_{\alpha_{2k}}$ and $P_{\alpha_{2k+1}}$ has Jordan type $[3^{2k}, 1^2]$. The degrees of the Springer maps are respectively 1 and 2.

In G_2 , the Richardson orbit of P_{α_1} and P_{α_2} is the sub-regular orbit $\mathcal{O} := \mathcal{O}_{G_2(a_1)}$. It is an even orbit with weighted Dynkin diagram $2 \equiv > 0$, so π_2 is

a symplectic resolution, where $\pi_i : T^*(G/P_{\alpha_i}) \rightarrow \overline{\mathcal{O}}, i = 1, 2$ are the Springer maps. The closure $\overline{\mathcal{O}}$ is normal since \mathcal{O} is the sub-regular orbit, whose singular part is the closure of the codimension 2 orbit $\mathcal{O}' := \mathcal{O}_{\tilde{A}_1}$. A slice transversal to \mathcal{O}' has an isolated normal surface singularity, which admits a unique crepant resolution. Suppose that π_1 is birational, then the birational map $\phi := \pi_2^{-1} \circ \pi_1$ is an isomorphism over the pre-images of \mathcal{O}' . Let \mathcal{L} be a π_2 -ample line bundle and C' an irreducible component of the π_1 -fiber of a point in \mathcal{O}' , then the line bundle $\phi^*(\mathcal{L})$ satisfies $\phi^*(\mathcal{L}) \cdot C' = \mathcal{L} \cdot \phi(C') > 0$. But the Picard group of $T^*(G/P_{\alpha_1})$ is \mathbb{Z} and π_1 is projective, so $\phi^*(\mathcal{L})$ is π_1 -ample, which implies that ϕ is in fact an isomorphism. Note that π_1, π_2 are both G -equivariant, so is the isomorphism ϕ . This implies that G/P_{α_1} and G/P_{α_2} are isomorphic as G -varieties, which is absurd since P_{α_1} and P_{α_2} are not G -conjugate. In conclusion, π_1 is not birational.

Now consider (5). The Richardson orbit is given by $\mathcal{O} = \mathcal{O}_{F_4(a_3)}$, whose weighted Dynkin diagram is given by $0-2 \Rightarrow 0-0$. This is an even orbit, thus the Jacobson-Morozov parabolic subgroup P_{α_2} gives a symplectic resolution. Notice that the Picard group of $G/P_{\alpha_1\alpha_4}$ is \mathbb{Z}^2 which is different from that of G/P_{α_2} , so the Springer map associated to $P_{\alpha_1\alpha_4}$ is not birational. By [Bro], the orbit closure $\overline{\mathcal{O}}$ is normal, whose singular part contains a codimension 2 orbit $C_3(a_1)$. Now a similar argument as that for the relation (4) shows that the Springer map of P_{α_3} is not birational.

For case (6), the Richardson orbit is $D_4(a_1)$, which is an even orbit. The Jacobson-Morozov parabolic sub-group is P_{α_4} , so it gives a symplectic resolution, while $P_{\alpha_2\alpha_5}$ does not, for the reason of different Picard groups.

For case (7), the Richardson orbit is $E_8(a_7)$, which is again an even orbit with the Jacobson-Morozov parabolic sub-group P_{α_5} . The situation is similar to (6).

For case (8), (9), the Springer map associated to each parabolic sub-group is birational, by Theorem 3.3 (cases (i) and (iii-b-2)). For case (10), the associated Springer map is birational by the proof of Theorem 4.2. \square

Remark 5.5. (i) In the appendix, we will calculate explicitly the degree of the Springer map associated to each parabolic sub-group appeared in (4)–(7) of Theorem 5.1 by using a formula of Borho-MacPherson ([BM]). However, we prefer to give the more geometric proof here.

(ii) The two varieties G_2/P_{α_1} and G_2/P_{α_2} are not isomorphic even as algebraic varieties, since they have different automorphism groups (see [Dem], also [BK]). The variety F_4/P_{α_2} is not isomorphic to F_4/P_{α_3} as algebraic

varieties since their Chow groups are different (see for example [NSZ]).

Let $P \subset Q$ be two standard parabolic subgroups in G with Lie algebras $\mathfrak{p}, \mathfrak{q}$ and L a Levi subgroup of Q . The projection to the first factor of the direct sum $\mathfrak{u}(\mathfrak{p}) = \mathfrak{u}(\mathfrak{l}(\mathfrak{q}) \cap \mathfrak{p}) \oplus \mathfrak{u}(\mathfrak{q})$ gives an L -equivariant map $f : L \cdot \mathfrak{u}(\mathfrak{p}) \rightarrow L \cdot \mathfrak{u}(\mathfrak{l}(\mathfrak{q}) \cap \mathfrak{p})$. Let $g : L \times^{(L \cap P)} \mathfrak{u}(\mathfrak{l}(\mathfrak{q}) \cap \mathfrak{p}) \rightarrow L \cdot \mathfrak{u}(\mathfrak{l}(\mathfrak{q}) \cap \mathfrak{p})$ and $g' : Q \times^P \mathfrak{u}(\mathfrak{p}) \rightarrow Q \cdot \mathfrak{u}(\mathfrak{p}) = L \cdot \mathfrak{u}(\mathfrak{p})$ be the natural morphisms. Note that g is a product of isomorphisms with Springer maps (in some simple Lie sub-algebras of $\mathfrak{l}(\mathfrak{q})$), so it is generically finite.

Lemma 5.6. *The morphism g' is the pull-back via f of the map g , i.e. the following diagram is Cartesian. In particular, $\deg(g') = \deg(g)$.*

$$\begin{array}{ccc} Q \times^P \mathfrak{u}(\mathfrak{p}) & \xrightarrow{g'} & Q \cdot \mathfrak{u}(\mathfrak{p}) = L \cdot \mathfrak{u}(\mathfrak{p}) \\ \downarrow & & \downarrow f \\ L \times^{L \cap P} \mathfrak{u}(\mathfrak{l}(\mathfrak{q}) \cap \mathfrak{p}) & \xrightarrow{g} & L \cdot \mathfrak{u}(\mathfrak{l}(\mathfrak{q}) \cap \mathfrak{p}). \end{array}$$

Proof. Let Z be the fiber product of f and g , then we have an L -equivariant map $\eta : Z \rightarrow L/(L \cap P)$. The fiber of η over the identity is isomorphic to $\mathfrak{u}(\mathfrak{p})$. If we denote by U the unipotent subgroup of Q , then $U \subset P$ and $Q/P = LU/P \simeq L/(L \cap P)$. This shows that Z is isomorphic to $Q \times^P \mathfrak{u}(\mathfrak{p})$. Under this isomorphism, the projection from Z to $Q \cdot \mathfrak{u}(\mathfrak{p}) = L \cdot \mathfrak{u}(\mathfrak{p})$ is identified to the morphism g' , since it is Q -equivariant. \square

We use notations in Theorem 5.1 (GP) in the following. Assume furthermore that $\Delta_1 - \Gamma_1 \sim_R \Delta_1 - \Gamma'_1$ is one of the fundamental relations in Theorem 5.1. Let ν, ν' be the Springer maps associated to P_{Γ_1} and $P_{\Gamma'_1}$ in the root system $\langle \Delta_1 \rangle$. Denote $P = P_{\Gamma_1 \cup \Gamma_2}$, $P' = P_{\Gamma'_1 \cup \Gamma_2}$ and $Q = P_{\Gamma_2}$. Let π, π' be the Springer maps associated to P, P' and \mathcal{O} their Richardson orbit.

Proposition 5.7. *Under the above hypothesis, we have :*

(i) $Q \cdot \mathfrak{u}(P) = Q \cdot \mathfrak{u}(P')$.

(ii) $\deg(\pi) \deg(\nu') = \deg(\pi') \deg(\nu)$.

(iii) *The diagram $T^*(G/P) \xrightarrow{\phi} G \times^Q (Q \cdot \mathfrak{u}(P)) = G \times^Q (Q \cdot \mathfrak{u}(P')) \xleftarrow{\phi'}$ $T^*(G/P')$ is an analytically locally trivial family of the diagram given by ν and ν' .*

Proof. Let $Q = LU$ be a Levi decomposition of Q , then $U \subset P \cap P'$ since P and P' are contained in Q . Notice that $\mathfrak{p} \cap \mathfrak{l}(\mathfrak{q})$ and $\mathfrak{p}' \cap \mathfrak{l}(\mathfrak{q})$ are parabolic

sub-algebras in $\mathfrak{l}(\mathfrak{q})$ corresponding to two polarizations of a same orbit, so $L \cdot \mathfrak{u}(\mathfrak{p} \cap \mathfrak{l}(\mathfrak{q})) = L \cdot \mathfrak{u}(\mathfrak{p}' \cap \mathfrak{l}(\mathfrak{q}))$. Now claim (i) follows from the fact $Q \cdot \mathfrak{u}(P) = L \cdot \mathfrak{u}(P)$ and $\mathfrak{u}(P) = \mathfrak{u}(\mathfrak{p} \cap \mathfrak{l}(\mathfrak{q})) \oplus \mathfrak{u}(\mathfrak{q})$.

Let $\psi : G \times^Q (Q \cdot \mathfrak{u}(P)) \rightarrow \overline{\mathcal{O}}$ be the natural map. Then $\pi = \psi \circ \phi$ and $\pi' = \psi \circ \phi'$. Note that ϕ is the composition of the following maps:

$$T^*(G/P) \simeq G \times^P \mathfrak{u}(P) \simeq G \times^Q (Q \times^P \mathfrak{u}(P)) \rightarrow G \times^Q (Q \cdot \mathfrak{u}(P)),$$

so the degree of ϕ is the same as that of the map $Q \times^P \mathfrak{u}(P) \rightarrow Q \cdot \mathfrak{u}(P)$, which is equal to the degree of the map $L \times^{(L \cap P)} \mathfrak{u}(\mathfrak{l}(\mathfrak{q}) \cap \mathfrak{p}) \rightarrow L \cdot (\mathfrak{u}(\mathfrak{l}(\mathfrak{q}) \cap \mathfrak{p}))$ by Lemma 5.6. The latter is in fact a trivial family of the morphism ν , which gives $\deg(\phi) = \deg(\nu)$. A similar argument shows that $\deg(\phi') = \deg(\nu')$, which gives (ii). Note that the morphism f in Lemma 5.6 is an affine bundle, so it is analytically locally trivial, which proves claim (iii). \square

Remark 5.8. *One can prove directly assertion (ii) by using a formula of Borho-MacPherson (see Proposition 7.1).*

This proposition is analogue to Proposition 4.4 in [Na2], but the proof is different here. Now we turn to the proof of Theorem 5.2. We will argue case-by-case for the simple Lie algebra \mathfrak{g} .

If \mathfrak{g} is A_n, G_2 or F_4 , then the relation \sim is generated by (8) and (GP), by Lemma 5.4 and the degree formula in Proposition 5.7.

Assume \mathfrak{g} is B_n and P_Γ a polarization of \mathcal{O} , where

$$\Gamma = \{\alpha_{p_1}, \alpha_{p_1+p_2}, \dots, \alpha_{p_1+\dots+p_s}\}, p_i > 0, \forall i.$$

To simplify the notations, we encode Γ by the sequence of ordered numbers $[p_1, \dots, p_s]$. If we want to perform the relation \sim_R in (1) of Theorem 5.1 for some B_{3k-1} , one should have $p_1 + \dots + p_{s-1} = n + 1 - 3k$ and $p_s = 2k - 1$ or $2k$. We consider the case $p_s = 2k - 1$, since the other one can be done similarly. Then $P_{\Gamma^{(1)}} \sim_R P_\Gamma$, where $\Gamma^{(1)} = [p_1^{(1)}, \dots, p_s^{(1)}]$ with $p_i^{(1)} = p_i$ for $i \neq s$ and $p_s^{(1)} = 2k$. Now any $P_{\Gamma^{(2)}}$ obtained from $P_{\Gamma^{(1)}}$ by performing (8) and (GP) in Theorem 5.1 has the following form: $\Gamma^{(2)} = [p_{\sigma(1)}^{(1)}, \dots, p_{\sigma(s)}^{(1)}]$ for some element σ in the symmetric group \mathfrak{S}_s . For simplicity, we will denote by $\deg(\Gamma)$ the degree of the Springer map associated to P_Γ . Then we have $\deg(\Gamma^{(2)}) = \deg(\Gamma^{(1)}) = 2 \deg(\Gamma)$.

If we want to change the degree, we need to perform once again the operation in (1) of Theorem 5.1. There are only two possibilities: (i) perform the operation in (1) for B_{3k-1} ; (ii) perform the operation in (1) for B_{3k-4} .

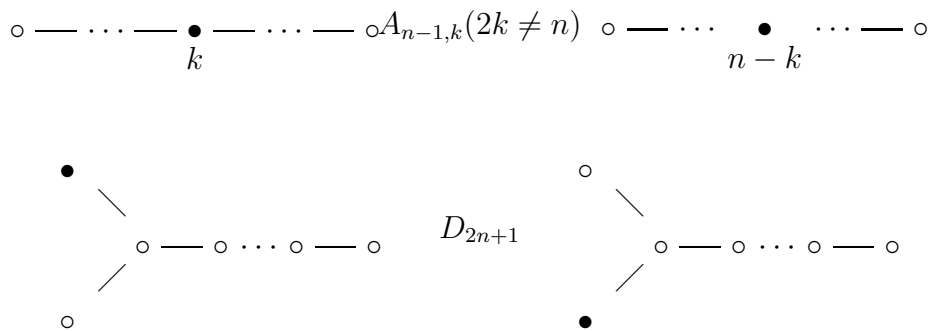
For case (i), after the operation, the rightmost marked node goes back to its original position and the degree remains the same. For case (ii), after the operation, one obtains $\Gamma^{(3)}$, but the degree goes higher: $\deg(\Gamma^{(3)}) = 2 \deg(\Gamma^{(2)}) = 4 \deg(\Gamma)$. By this way, we see that to obtain the same degree as $\deg(\Gamma)$ for $\Gamma^0 = [q_1, \dots, q_s]$, one should have $\sum_{j=1}^s q_j = \sum_{i=1}^s p_i$. In other words, the rightmost marked node should stay at the same position. Now it follows that Γ^0 can be obtained from Γ by just performing operations in (8) and (GP) of Theorem 5.1.

Similar arguments can be done to the case $\mathfrak{g} = C_n$. When $\mathfrak{g} = D_n$, we have two possible operations (2) and (3) in Theorem 5.1 which do not preserve degrees. The key point is that if we have performed one of them, then we can not perform the other one, so the situation is similar to the B_n case.

If \mathfrak{g} is E_6 , let P_Γ be a parabolic subgroup. The only possible operation not preserving the degree is (2) of Theorem 5.1, since (6) is settled by Lemma 5.4. But then one should have $\alpha_1, \alpha_6 \in \Gamma$. Now it is easy to see that for any $P_{\Gamma'}$ equivalent to P_Γ , one can arrive Γ' from Γ by just performing operations (8) and (GP) of Theorem 5.1.

If \mathfrak{g} is E_7 , we can perform either (2) or (6) of Theorem 5.1. For (2), one should have $\alpha_1, \alpha_6 \in \Gamma$. For (6), one has $\alpha_7 \in \Gamma$. The argument is similar to the case of E_6 . We can do the similar to the case of E_8 , noticing that (7) is already done by Lemma 5.4. This finishes the proof of Theorem 5.2. Q.E.D.

Following Namikawa [Na2], the diagrams given by the Springer maps of dual parabolic subgroups in (8), (9), (10) of Theorem 5.1 will be called *stratified Mukai flops* of type $A, D, E_{6,I}$ and $E_{6,II}$ respectively. The following is the list of the dual marked Dynkin diagrams.



Corollary 5.10. *Let \mathcal{O} be a nilpotent orbit in a simple exceptional Lie algebra \mathfrak{g} . Then the degrees of Springer maps associated to polarizations of \mathcal{O} can take at most two values.*

This follows directly from the proof of Theorem 5.2. An immediately consequence is that $\overline{\mathcal{O}}$ has at most two G -coverings which admit a Springer resolution. Note that this corollary is not true if \mathfrak{g} is classical.

Corollary 5.11. *Let \mathcal{O} be a nilpotent orbit in a simple exceptional Lie algebra. Then $\overline{\mathcal{O}}$ admits a symplectic resolution if and only if \mathcal{O} is Richardson and \mathcal{O} is not one of the following orbits:*

$$A_4 + A_1, D_5(a_1) \text{ in } E_7, \quad E_6(a_1) + A_1, E_7(a_3) \text{ in } E_8.$$

Proof. First consider the orbit $\mathcal{O} := \mathcal{O}_{D_7(a_2)}$ in E_8 . A polarization of \mathcal{O} is given by $P_{\alpha_1\alpha_4}$ ([Hir]). Let π be the corresponding Springer map. Remark that we can perform the operation (3) in Theorem 5.1 for D_7 to obtain another polarization $P_{\alpha_1\alpha_5}$. If we denote by π' the Springer map of $P_{\alpha_1\alpha_5}$, then by Proposition 5.7 and Lemma 5.4, we have $\deg(\pi) = 2 \deg(\pi')$. Notice that the fundamental group of \mathcal{O} is S_2 ([CM], p. 134), so the degree of any Springer map of \mathcal{O} is either 1 or 2, which gives $\deg(\pi') = 1$ and $\deg(\pi) = 2$. In particular, the closure of the orbit $\mathcal{O}_{D_7(a_2)}$ in E_8 admits a symplectic resolution.

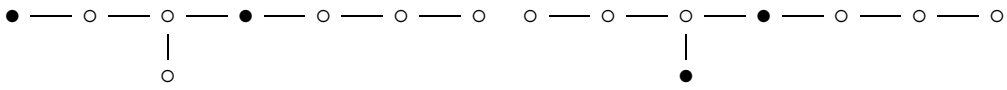
Then consider the orbits in the Corollary. The following is a list of a polarization for each orbit ([Hir]):

algebra	E_7		E_8	
orbit	$A_4 + A_1$	$D_5(a_1)$	$E_6(a_1) + A_1$	$E_7(a_3)$
polarization	$P_{\alpha_2\alpha_3}$	$P_{\alpha_1\alpha_2\alpha_3}$	$P_{\alpha_1\alpha_2\alpha_4}$	$P_{\alpha_1\alpha_2\alpha_3\alpha_4}$

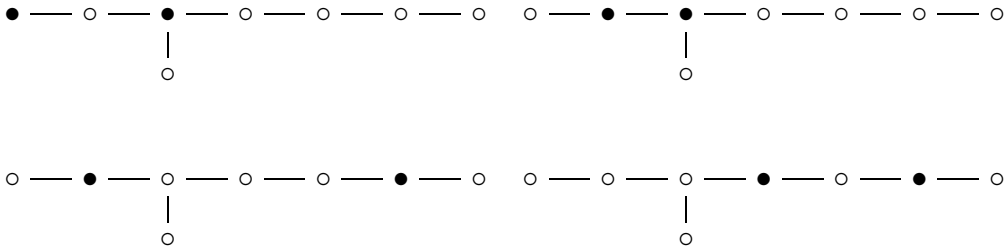
Now we do a case-by-case check to show that the relation \sim_R coincides with \sim , i. e. we can not perform the operations (2), (3) (for D_7), (6) and (7) in Theorem 5.1 to any polarization of the orbit. This implies that for any two polarizations Q_1, Q_2 of one of the four orbits, the degrees of the associated Springer maps are the same. In the appendix, we calculate the degrees of the Springer maps associated to the above polarizations by using a formula in [BM], which turn out to be 2. Thus the four orbit closures do not admit a symplectic resolution.

The proof is completed by applying Proposition 4.1 and Theorem 4.2. \square

Example 5.12. Let $\mathcal{O} = \mathcal{O}_{D_7(a_2)}$ in E_8 and \mathcal{O}_s its universal G -covering. Then $\overline{\mathcal{O}}$ admits exactly two symplectic resolutions, given by $P_{\alpha_1\alpha_5}$ and $P_{\alpha_2\alpha_5}$ (with marked Dynkin diagrams listed in the following), which is a locally trivial family of ordinary Mukai flops of $T^*\mathbb{P}^4$. Notice that neither of the two parabolic sub-groups contains the Jacobson-Morozov parabolic sub-group P of \mathcal{O} , so we cannot obtain a symplectic resolution of $\overline{\mathcal{O}}$ by extremal contractions of $\mu : G \times^P \mathfrak{n} \rightarrow \overline{\mathcal{O}}$.



The closure $\overline{\mathcal{O}}_s$ admits exactly four different Springer resolutions, given by $P_{\alpha_1\alpha_4}$, $P_{\alpha_3\alpha_4}$, $P_{\alpha_3\alpha_7}$ and $P_{\alpha_5\alpha_7}$ (their marked Dynkin diagrams are listed in the following), where the diagram of two consecutive symplectic resolutions is a locally trivial family of stratified Mukai flops of $T^*(\mathbb{P}^2)$, $T^*(Gr(2, 7))$ and of type $E_{6,II}$ respectively.



6 A description of stratified Mukai flops

By Corollary 5.9, to understand the birational geometry of Springer maps of the same degree, one is led to understand stratified Mukai flops of type $A, D, E_{6,I}$ and $E_{6,II}$. We will give a uniform resolution and explicit description of these flops. Note that some other descriptions of these flops were presented in [Cha].

Let $Q = P_{\alpha_i}, Q' = P_{\alpha_j}$ be a pair of the dual standard parabolic subgroups in (8), (9), (10) of Theorem 5.1 with Lie algebras $\mathfrak{q}, \mathfrak{q}'$ and \mathcal{O} their Richardson orbit. We denote by π and π' the Springer maps associated to Q and Q' . Write P the Jacobson-Morozov parabolic subgroup of \mathcal{O} with Lie algebra \mathfrak{p} and \mathfrak{n} the natural nilpotent ideal in $\mathfrak{u}(P)$.

Theorem 6.1. (i) The weighted Dynkin diagram of \mathcal{O} has weight 1 on nodes α_i, α_j and 0 on other nodes.

(ii) \mathfrak{q} and \mathfrak{q}' are the only (non-trivial) standard parabolic sub-algebras containing \mathfrak{p} and $\mathfrak{n} = \mathfrak{u}(\mathfrak{q}) \cap \mathfrak{u}(\mathfrak{q}')$.

(iii) The variety $G \times^P \mathfrak{n}$ is isomorphic to the graph closure of the flop $\phi : T^*(G/Q) \dashrightarrow T^*(G/Q')$.

(iv) We have the following commutative diagram:

$$\begin{array}{ccccc}
 & & G \times^P \mathfrak{n} & & \\
 & \swarrow \nu & \downarrow \mu & \searrow \nu' & \\
 G \times^Q (Q \cdot \mathfrak{n}) \simeq T^*(G/Q) & & & & T^*(G/Q') \simeq G \times^{Q'} (Q' \cdot \mathfrak{n}) \\
 & \searrow \pi & \downarrow & \swarrow \pi' & \\
 & & \overline{\mathcal{O}} & &
 \end{array}$$

The natural morphisms ν, ν' are contractions of extremal rays of μ and they are also the two graph projections under the isomorphism in (iii).

Proof. The following is a list of Richardson orbits appeared in the stratified Mukai flops.

type	$A_{n-1,k} (k < n/2)$	D_{2n+1}	$E_{6,I}$	$E_{6,II}$
orbit	$\mathcal{O}_{[2^k, 1^{n-2k}]}$	$\mathcal{O}_{[2^{2n}, 1^2]}$	\mathcal{O}_{2A_1}	$\mathcal{O}_{A_2+2A_1}$

Now claim (i) follows immediately (see [CM]). The first part of claim (ii) follows from (i). Claim (iv) and the inclusion $\mathfrak{n} \subseteq \mathfrak{u}(Q) \cap \mathfrak{u}(Q')$ have already been verified in Theorem 3.3 and Theorem 4.2 while the inclusion $\mathfrak{u}(Q) \cap \mathfrak{u}(Q') \subseteq \mathfrak{n}$ is obvious.

To show (iii), note that the two projections $G/P \rightarrow G/Q, G/P \rightarrow G/Q'$ embed G/P as a sub-variety in $G/Q \times G/Q'$ (the incident variety). The composition of morphisms

$$G \times^P \mathfrak{n} \rightarrow G/P \times \overline{\mathcal{O}} \xrightarrow{\eta} (G/Q \times \overline{\mathcal{O}}) \times (G/Q' \times \overline{\mathcal{O}})$$

embeds $G \times^P \mathfrak{n}$ into a closed sub-variety of $T^*(G/Q) \times T^*(G/Q')$ (here one uses (ii)), where η is given by $\eta([gP], x) = (([gQ], x), ([gQ'], x))$. The image is in fact the graph closure of the flop ϕ in $T^*(G/Q) \times_{\overline{\mathcal{O}}} T^*(G/Q')$, since it is irreducible closed and contains the diagonal embedding of \mathcal{O} into the fiber product. □

Remark 6.2. (i) It is possible to show that stratified Mukai flops are the only flops which appear in symplectic resolutions of nilpotent orbit closures with properties (i), (ii), (iv) in Theorem 6.1.

(ii) A similar diagram holds for some other flops (see Theorem 3.3 and 4.2).

(iii) The variety $G \times^P \mathfrak{n}$ is in fact isomorphic to the conormal bundle of G/P in $G/Q \times G/Q'$.

Example 6.3. Consider the Mukai flop of type A_n with $i = 1, j = n$. Then $G \times^P \mathfrak{n}$ is isomorphic to the blowup of $T^*(\mathbb{P}^n)$ along the zero section and G/P is the incidence variety in $\mathbb{P}^n \times (\mathbb{P}^n)^*$. The two extremal contractions are blow-downs of G/P along two different directions.

Let $q : T^*(G/Q) \rightarrow G/Q$ and $q' : T^*(G/Q') \rightarrow G/Q'$ be the two projections and $\Phi = \nu'_* \nu^* : K(T^*(G/Q)) \rightarrow K(T^*(G/Q'))$ the natural morphism between two K -groups. It has been observed in [Na1] that Φ is not an isomorphism even when $n = 2$. In fact, for $-n + 1 \leq k \leq 0$, we have $\Phi(q^* \mathcal{O}(k)) = (q')^* \mathcal{O}(-k)$ and $\Phi(q^* \mathcal{O}(1)) = (q')^* \mathcal{O}(-1) \otimes \mathcal{I}_{G/Q'}$, where $\mathcal{I}_{G/Q'}$ is the ideal sheaf of G/Q' in $T^*(G/Q')$. Now using the Koszul resolution and the Euler exact sequence, we can obtain that $\Phi(q^* \mathcal{O}(1)) = -n(q')^* \mathcal{O}(-1) + \sum_{i=0}^{n-1} k_i (q')^* (\mathcal{O}(i))$ for some integers k_i , which shows that Φ is not an isomorphism as soon as $n \geq 2$. However, the morphism

$$\Phi \otimes_{\mathbb{Z}} \mathbb{Q} : K(T^*(G/Q)) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K(T^*(G/Q')) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. Similarly for the G -equivariant K -groups, we have also an isomorphism:

$$\Phi_G \otimes_{\mathbb{Z}} \mathbb{Q} : K^G(T^*(G/Q)) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K^G(T^*(G/Q')) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

It seems plausible that similar results hold for other stratified Mukai flops. More generally, for two birational K -equivalent varieties, it seems that the graph closure gives an isomorphism between the K -groups with rational coefficients (see [Wan] for further discussions and related references).

7 Appendix: The degrees of Springer maps

Let W be the Weyl group of G . The Springer correspondence ([Sp2]) assigns to any irreducible W -module a unique pair (\mathcal{O}, ϕ) consisting of a nilpotent orbit \mathcal{O} in \mathfrak{g} and an irreducible representation ϕ of the component group

$A(\mathcal{O}) := G^x/(G^x)^\circ$ of \mathcal{O} , where x is any point in \mathcal{O} and $(G^x)^\circ$ is the identity component of G^x . The corresponding irreducible W -module will be denoted by $\rho_{(\mathcal{O},\phi)}$. This correspondence is not surjective onto the set of all pairs (\mathcal{O}, ϕ) . A pair will be called *relevant* if it corresponds to an irreducible W -module, then the Springer correspondence establishes a bijection between irreducible W -modules and relevant pairs in \mathfrak{g} . For exceptional cases, the Springer correspondence has been completely worked out in [Sp1] for G_2 , in [Sho] for F_4 and in [AL] for E_n ($n = 6, 7, 8$).

Consider a parabolic sub-group Q in G . Let L be a Levi sub-group of Q and T a maximal torus in L . The Weyl group of L is $W(L) := N_L(T)/T$, where $N_L(T)$ is the normalizer of T in L . It is a sub-group of the Weyl group W of G . Let $\epsilon_{W(L)}$ be the sign representation of $W(L)$ and $\text{Ind}_{W(L)}^W(\epsilon_{W(L)})$ the induced representation of $\epsilon_{W(L)}$ to W .

Proposition 7.1 ([BM], Corollary 3.9). *Let $\pi_Q : T^*(G/Q) \rightarrow \overline{\mathcal{O}}_Q$ be the Springer map associated to the parabolic sub-group Q . Then*

$$\deg(\pi_Q) = \sum_{\phi} \text{mtp}(\rho_{(\mathcal{O}_Q, \phi)}, \text{Ind}_{W(L)}^W(\epsilon_{W(L)})) \deg \phi,$$

where the sum is over all irreducible representations ϕ of $A(\mathcal{O}_Q)$ such that (\mathcal{O}_Q, ϕ) is a relevant pair, $\text{mtp}(\rho_{(\mathcal{O}_Q, \phi)}, \text{Ind}_{W(L)}^W(\epsilon_{W(L)}))$ is the multiplicity of $\rho_{(\mathcal{O}_Q, \phi)}$ in $\text{Ind}_{W(L)}^W(\epsilon_{W(L)})$ and $\deg \phi$ is the dimension of the irreducible representation ϕ .

The multiplicity $\text{mtp}(\rho_{(\mathcal{O}_Q, \phi)}, \text{Ind}_{W_0}^W(\rho))$ has been worked out in [Alv], for any irreducible representation ρ of any maximal parabolic sub-group W_0 of W , where $\text{Ind}_{W_0}^W(\rho)$ is the induced representation of ρ to W . Note that $\text{Ind}_{W(L)}^W(\epsilon_{W(L)}) = \text{Ind}_{W_0}^W(\text{Ind}_{W(L)}^{W_0}(\epsilon_{W(L)}))$ for any maximal parabolic sub-group W_0 of W containing $W(L)$ and $\text{Ind}_{W(L)}^{W_0}(\epsilon_{W(L)})$ can be determined by the Littlewood-Richardson rules when W_0 is classical and by [Alv] when W_0 is exceptional. Using this, we can calculate the degrees of the Springer maps associated to the parabolic subgroups appeared in Theorem 5.1, and the result is as follows:

Lie algebra	G_2	F_4	F_4	E_6	E_8
parabolic subgroup	P_{α_1}	P_{α_3}	$P_{\alpha_1\alpha_4}$	$P_{\alpha_2\alpha_5}$	$P_{\alpha_2\alpha_3}$
nilpotent orbit	$G_2(a_1)$	$F_4(a_3)$	$F_4(a_3)$	$D_4(a_1)$	$E_8(a_7)$
component group	S_3	S_4	S_4	S_3	S_5
degree	2	4	6	3	10

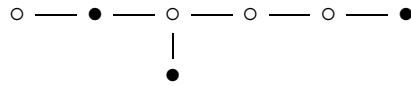
In a similar way, we obtain the degrees of the Springer maps for the orbits in the list of Corollary 5.11:

Lie algebra	E_7	E_7	E_8	E_8
parabolic subgroup	$P_{\alpha_2\alpha_3}$	$P_{\alpha_1\alpha_2\alpha_3}$	$P_{\alpha_1\alpha_2\alpha_4}$	$P_{\alpha_1\alpha_2\alpha_3\alpha_4}$
nilpotent orbit	$A_4 + A_1$	$D_5(a_1)$	$E_6(a_1) + A_1$	$E_7(a_3)$
degree	2	2	2	2

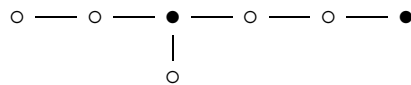
Remark 7.2. The correspondence between notations of irreducible characters of E_n ($n = 6, 7, 8$) in [Alv] and those in [AL] is given in [BL]. See also [GP] (Appendix C).

Remark 7.3. When \mathfrak{g} is classical, the degree of a Springer map is given by Hesselink's formula ([Hes]), in terms of the flag type of the parabolic subgroup. Our method here allows one to find the degree of the Springer map from the marked Dynkin diagram of the parabolic sub-group in exceptional Lie algebras. (This also works for Lie algebras of classical type). In fact, when \mathfrak{g} is exceptional, for any Richardson orbit, we have either given a degree one polarization or proved the degree of any polarization is 2 (Theorem 4.2 and Corollary 5.11). Now we can use Proposition 5.7, Lemma 5.4 and the above results to determine the degree associated to any other polarization.

Example 7.4. We will calculate the degree d of the Springer map associated to the following parabolic sub-group Q in E_7 :



We can perform the operation (6) in Theorem 5.1 to obtain the following parabolic Q' in E_7 :



Note that Q' is the Jacobson-Morozov parabolic sub-group of the even orbit $\mathcal{O} := E_7(a_5)$, so its associated Springer map is birational. By Proposition 5.7, we get that the degree of the Springer map associated to Q is $d = 3$. The component group $A(\mathcal{O})$ is isomorphic to S_3 , so the 2-fold, 6-fold G -coverings of $\overline{\mathcal{O}}$ have no Springer resolution by Corollary 5.10.

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