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**Some New Estimates on the Spectral Shift Function  
Associated with Random Schrödinger Operators**

**Jean-Michel Combes**<sup>1</sup>

Département de Mathématiques  
Université du Sud, Toulon-Var  
83130 La Garde, FRANCE

**Peter D. Hislop**<sup>2</sup>

Department of Mathematics  
University of Kentucky  
Lexington, KY 40506-0027 USA

**Frédéric Klopp**

L.A.G.A, Institut Galilée  
Université Paris-Nord  
F-93430 Villetaneuse, FRANCE  
et  
Institut Universitaire de France

**Abstract**

We prove some new pointwise-in-energy bounds on the expectations of various spectral shift functions (SSF) associated with random Schrödinger operators in the continuum having Anderson-type random potentials in both finite-volume and infinite-volume. These estimates are a consequence of our new Wegner estimate for finite-volume random Schrödinger operators [5]. For lattice models, we also obtain a representation of the infinite-volume density of states in terms of the expectation of a SSF for a single-site perturbation. For continuum models, the corresponding measure, whose density is given by this SSF, is absolutely continuous with respect to the density of states and agrees with it in certain cases. As an application of one-parameter spectral averaging, we give a short proof of the classical pointwise upper bound on the SSF for finite-rank perturbations.

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<sup>1</sup>Centre de Physique Théorique, CNRS Marseille, France

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# 1 Introduction: The Wegner Estimate and the Spectral Shift Function

Some recent analyses of random Schrödinger operators have involved three related concepts: the Wegner estimate for the finite-volume Hamiltonians, the spectral shift function (SSF), and the integrated density of states (IDS). In this note, we prove some new pointwise bounds on the expectation of some SSFs that occur in the theory of random Schrödinger operators in the continuum. These bounds result from an improved version of the Wegner estimate [5]. In earlier work [4, 8], we used  $L^p$ -bounds on the SSF in order to obtain better estimates on the IDS. In our most recent work, we obtain an optimal Wegner estimate directly without using the SSF and found, as a consequence, new pointwise bounds on the expectation of the SSF. It has often been conjectured that in the case of ergodic, random, Schrödinger operators of the form considered here the SSF for a local single-site perturbation should be in  $L_{loc}^\infty(\mathbb{R})$  once it is averaged over the random variables on which the disordered potential depends. We prove this in this note. We mention that these types of bounds on the SSF also play a motivating role in the fractional moment method for proving localization in the continuum [1]. For lattice models, the pointwise bounds on the SSF are a simple consequence of the fact that the corresponding perturbations are finite-rank (cf. [2, 20] and section 4.3).

We first recall a special case of the Wegner estimate proved in [5] that will be used for the bounds on the SSF. We refer to [5] for the general statement, valid for arbitrary bounded processes on  $\mathbb{Z}^d$ , and the proofs. The family of Schrödinger operators  $H_\omega = H_0 + V_\omega$ , on  $L^2(\mathbb{R}^d)$ , is constructed from a deterministic,  $\mathbb{Z}^d$ -periodic, background operator  $H_0 = (-i\nabla - A_0)^2 + V_0$ . We consider an Anderson-type random potential  $V_\omega$  constructed from the single-site potential  $u$  as

$$V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \omega_j u(x - j). \quad (1)$$

The family of random variables is assumed to be independent, and identically distributed (iid). The results are independent of the disorder provided it is nonzero.

We define local versions of the Hamiltonians and potentials associated with bounded regions in  $\mathbb{R}^d$ . By  $\Lambda_L(x)$ , we mean the open cube of side

length  $L$  centered at  $x \in \mathbb{R}^d$ . For  $\Lambda \subset \mathbb{R}^d$ , we denote the lattice points in  $\Lambda$  by  $\tilde{\Lambda} = \Lambda \cap \mathbb{Z}^d$ . For a cube  $\Lambda$ , we take  $H_0^\Lambda$  and  $H_\Lambda$  (omitting the index  $\omega$ ) to be the restrictions of  $H_0$  and  $H_\omega$ , respectively, to the cube  $\Lambda$ , with periodic boundary conditions on the boundary  $\partial\Lambda$  of  $\Lambda$ . We denote by  $E_0^\Lambda(\cdot)$  and  $E_\Lambda(\cdot)$  the spectral families for  $H_0^\Lambda$  and  $H_\Lambda$ , respectively. Furthermore, for  $\Lambda \subset \mathbb{R}^d$ , let  $\chi_\Lambda$  be the characteristic function for  $\Lambda$ . The local potential  $V_\Lambda$  is defined by

$$V_\Lambda(x) = V_\omega(x)\chi_\Lambda(x), \quad (2)$$

and we assume this can be written as

$$V_\Lambda(x) = \sum_{j \in \tilde{\Lambda}} \omega_j u(x - j). \quad (3)$$

For example, if the support of  $u$  is contained in a single unit cube, the formula (3) holds. We refer to the discussion in [4] when the support of  $u$  is compact, but not necessarily contained inside one cube. In this case,  $V_\Lambda$  can be written as in (3) plus a boundary term of order  $|\partial\Lambda|$  and hence it does not contribute to the large  $|\Lambda|$  limit. Hence, we may assume (3) without any loss of generality. We will also use the local potential obtained from (3) by setting all the random variables to one, that is,

$$\tilde{V}_\Lambda(x) = \sum_{j \in \tilde{\Lambda}} u_j(x), \quad (4)$$

where we will write  $u_j(x) = u(x - j)$ .

We will always make the following four assumptions:

- (H1). The background operator  $H_0 = (-i\nabla - A_0)^2 + V_0$  is a lower semi-bounded,  $\mathbb{Z}^d$ -periodic Schrödinger operator with a real-valued,  $\mathbb{Z}^d$ -periodic, potential  $V_0$ , and a  $\mathbb{Z}^d$ -periodic vector potential  $A_0$ . We assume that  $V_0$  and  $A_0$  are sufficiently regular so that  $H_0$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R}^d)$ .
- (H2). The periodic operator  $H_0$  has the unique continuation property, that is, for any  $E \in \mathbb{R}$  and for any function  $\phi \in H_{loc}^2(\mathbb{R}^d)$ , if  $(H_0 - E)\phi = 0$ , and if  $\phi$  vanishes on an open set, then  $\phi \equiv 0$ .
- (H3). The nonzero, non negative, compactly supported, bounded single-site potential  $u \in L_0^\infty(\mathbb{R}^d)$ , and it is strictly positive on a nonempty open set.

(H4). The random coupling constants  $\{\omega_j \mid j \in \mathbb{Z}^d\}$ , are independent and identically distributed. The probability distribution  $\mu_0$  of  $\omega_0$  is compactly supported with a bounded density  $h_0 \in L^\infty(\mathbb{R})$ .

These imply that the infinite-volume random Schrödinger operator  $H_\omega$  is ergodic with respect to the group of  $\mathbb{Z}^d$ -translations.

Our results also apply to the randomly perturbed Landau Hamiltonian  $H_\omega(\lambda) = H_L(B) + \lambda V_\omega$ , for  $\lambda \neq 0$ , where  $V_\omega$  is an Anderson-type potential as in (1). The Landau Hamiltonian  $H_L(B)$  on  $L^2(\mathbb{R}^2)$  is given by

$$H_L(B) = (-i\nabla - A_0)^2, \quad \text{with } A_0(x, y) = \frac{B}{2}(-y, x). \quad (5)$$

The constant  $B \neq 0$  is the magnetic field strength.

Under these assumptions, the Wegner estimate necessary for our purposes has the following form.

**Theorem 1.1** *We assume that the family of random Schrödinger operators  $H_\omega = H_0 + V_\omega$  on  $L^2(\mathbb{R}^d)$  satisfies hypotheses (H1)-(H4). Then, there exists a locally uniform constant  $C_W > 0$  such that for any  $E_0 \in \mathbb{R}$ , and  $\epsilon \in (0, 1]$ , the local Hamiltonians  $H_\Lambda$  satisfy the following Wegner estimate*

$$\begin{aligned} \mathbb{P}\{\text{dist}(\sigma(H_\Lambda), E_0) < \epsilon\} &\leq \mathbb{E}\{\text{Tr}E_\Lambda([E_0 - \epsilon, E_0 + \epsilon])\} \\ &\leq C_W \epsilon |\Lambda|. \end{aligned} \quad (6)$$

*A similar estimate holds for randomly perturbed Landau Hamiltonians.*

This theorem immediately implies the Lipschitz continuity of the integrated density of states [5]. As a consequence, the density of states (DOS) exists and is a locally bounded function. In this note, we use Theorem 1.1 to prove new pointwise bounds on the expectation of the SSF for both finite-volume and infinite-volume random Schrödinger operators. We comment on the relation of these results to various results concerning the SSF for random Schrödinger operators in section 4. We also show that one-parameter spectral averaging can be used to recover the classical pointwise bound on the SSF for finite-rank perturbations.

## 2 Bounds on the Spectral Shift Function for Finite-Volume Hamiltonians

We use the result of Theorem 1.1 to bound the expectation of the SSF for a single-site perturbation of a finite-volume Hamiltonian  $H_\Lambda$ . Since the size of the support of the perturbation  $u$  is of order one relative to  $|\Lambda|$ , we expect the SSF to be of order one also. For a discussion of the relation between the IDS and the SSF, we refer the reader to [8] and references therein. A nice review of results concerning the SSF may be found in [2]. We recall that for a pair of self-adjoint operators  $(H(1), H(0))$ , such that  $f(H(1)) - f(H(0))$  is trace-class, the SSF  $\xi(E; H(1), H(0))$  is defined through the trace formula. For example, for any  $f \in C_0^1(\mathbb{R})$ , we have

$$\text{Tr}[f(H(1)) - f(H(0))] = \int_{\mathbb{R}} f'(E) \xi(E; H(1), H(0)) dE. \quad (7)$$

We first consider a one-parameter family of self-adjoint operators  $H(\lambda) = H_0 + \lambda V$ , with  $V \geq 0$ , and  $\lambda$  uniformly distributed on  $[0, 1]$ . Birman and Solomyak proved a relation (cf. [18]) between the averaged, weighted, trace of the spectral projector  $E_\lambda(\cdot)$  of  $H(\lambda)$ , and the SSF for the pair  $H(1) \equiv H(\lambda = 1)$  and  $H(0) \equiv H(\lambda = 0) = H_0$ . For any measurable  $\Delta \subset \mathbb{R}$ , this formula has the form

$$\int_0^1 d\lambda \text{Tr} V^{1/2} E_\lambda(\Delta) V^{1/2} = \int_{\Delta} dE \xi(E; H_0 + V, H_0), \quad (8)$$

whenever all the terms exist. For example, if  $V$  is relatively  $H_0$ -trace class, then all the terms are well-defined.

We apply this formula as follows. First, we must also make a stronger hypothesis on the probability distribution than (H4). We will assume:

**(H4')**. The random coupling constants  $\{\omega_j \mid j \in \mathbb{Z}^d\}$ , are independent and identically distributed. The probability distribution  $\mu_0$  of  $\omega_0$  is the uniform distribution on  $[0, 1]$ .

As, for some  $C > 0$ , one has  $0 \leq \sum_j u_j \leq C$ , it follows trivially that

$$\sum_{j \in \bar{\Lambda}} \text{Tr} u_j^{1/2} E_\Lambda(\Delta) u_j^{1/2} \leq C \text{Tr} E_\Lambda(\Delta). \quad (9)$$

We now consider the effect of the variation of one random variable  $\omega_j$ , for  $j \in \tilde{\Lambda}$ , on the local Hamiltonian. In formula (8), we take  $H(0) = H_\Lambda(\omega_j = 0)$ ,  $H(1) = H_\Lambda(\omega_j = 1)$ , so that  $\lambda = \omega_j$ , and  $V = u_j \geq 0$ . We write  $H_{j^\perp}^\Lambda$  for  $H_\Lambda$  with  $\omega_j = 0$ . The Birman-Solomyak formula (8) then has the form

$$\int_0^1 d\omega_j \operatorname{Tr} u_j^{1/2} E_\Lambda(\Delta) u_j^{1/2} = \int_\Delta dE \xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda). \quad (10)$$

Taking the expectation of (9) and using formula (10), we obtain

$$\begin{aligned} \mathbb{E} \left\{ \sum_{j \in \tilde{\Lambda}} \operatorname{Tr} u_j^{1/2} E_\Lambda(\Delta) u_j^{1/2} \right\} &= \sum_{j \in \tilde{\Lambda}} \mathbb{E} \left\{ \int_\Delta dE \xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda) \right\} \\ &\leq C \mathbb{E} \{ \operatorname{Tr} E_\Lambda(\Delta) \} \\ &\leq C_0 |\Delta| |\Lambda|, \end{aligned} \quad (11)$$

where we used the result of the proof of Theorem 1.1 on the last line. We conclude from (11) that

$$\frac{1}{|\Delta|} \int_\Delta dE \left\{ \frac{1}{|\Lambda|} \sum_{j \in \tilde{\Lambda}} \mathbb{E} \{ \xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda) \} \right\} \leq C_0. \quad (12)$$

If the spatially averaged expectation of the SSF is  $L_{loc}^1(\mathbb{R})$  in  $E$ , we can conclude a pointwise bound from (12), for Lebesgue almost every energy  $E$ , of the form

$$\mathbb{E} \left\{ \frac{1}{|\Lambda|} \sum_{j \in \tilde{\Lambda}} \xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda) \right\} \leq C_0. \quad (13)$$

In [8], we proved that the SSF for local Schrödinger operators with compactly-supported perturbations is locally- $L^1$ , so this pointwise bound (13) holds. Finally, we observe that due to the periodic boundary conditions on  $\partial\Lambda$  and the  $\mathbb{Z}^d$ -periodicity of  $H_0$ , we have that for any  $j, k \in \tilde{\Lambda}$

$$\mathbb{E} \{ \xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda) \} = \mathbb{E} \{ \xi(E; H_{k^\perp}^\Lambda + u_k, H_{k^\perp}^\Lambda) \}, \quad (14)$$

and consequently it follows from (13) that for any  $j \in \tilde{\Lambda}$ ,

$$\mathbb{E} \{ \xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda) \} \leq C_0. \quad (15)$$

**Theorem 2.1** *Under the hypotheses (H1)-(H4'), the expectation of the spectral shift function, corresponding to the variation of a single site of the finite-volume Hamiltonian, is uniformly locally bounded in energy. That is, for any bounded energy interval, there is a constant  $C_I > 0$ , independent of  $\Lambda$ , so that for Lebesgue almost every  $E \in I$ , we have*

$$\mathbb{E}\{\xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda)\} \leq C_I, \quad (16)$$

for any  $j \in \tilde{\Lambda}$ .

In the lattice case, the perturbation  $u_j$  is rank-one, so by the general theory (cf. [20, 2], or see section 4.3), we have the bound

$$\xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda) \leq 1, \quad (17)$$

for any  $j \in \tilde{\Lambda}$ , uniformly in  $E \in \mathbb{R}$ .

### 3 Bounds on the Spectral Shift Function for Infinite-Volume Hamiltonians

We consider the thermodynamic limit of the SSF in (13). The Birkhoff Ergodic Theorem implies that the limit of the right side of (13) is the expectation of the SSF corresponding to the pair of infinite-volume Hamiltonians  $(H_{0^\perp}, H_{0^\perp} + u_0)$  if we replace  $\xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda)$  by  $\xi(E; H_{j^\perp} + u_j, H_{j^\perp})$ , where  $H_{j^\perp}$  is the infinite-volume Hamiltonian with  $\omega_j = 0$ . This is the content of the next proposition.

**Theorem 3.1** *Let  $H_{0^\perp}$  be the infinite-volume random Hamiltonian  $H_\omega$  with  $\omega_0 = 0$  and assume hypotheses (H1)-(H4'). Then the SSF  $\xi(E; H_{0^\perp} + u_0, H_{0^\perp})$  is well-defined and  $\mathbb{E}\{\xi(E; H_{0^\perp} + u_0, H_{0^\perp})\} \in L_{loc}^\infty(\mathbb{R})$ .*

**Proof:** 1. We begin with the integrated expression (12) and write

$$\begin{aligned} & \frac{1}{|\Delta|} \int_\Delta dE \mathbb{E} \left\{ \frac{1}{|\Lambda|} \sum_{j \in \tilde{\Lambda}} \xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda) \right\} \\ &= \frac{1}{|\Delta|} \int_\Delta dE \mathbb{E} \left\{ \frac{1}{|\Lambda|} \sum_{j \in \tilde{\Lambda}} \xi(E; H_{j^\perp} + u_j, H_{j^\perp}) \right\} + \frac{\mathcal{E}_\Lambda(\Delta)}{|\Delta|}, \quad (18) \end{aligned}$$



where the error term is

$$\mathcal{E}_\Lambda(\Delta) \equiv \int_\Delta dE \mathbb{E} \left\{ \frac{1}{|\Lambda|} \sum_{j \in \bar{\Lambda}} (\xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda) - \xi(E; H_{j^\perp} + u_j, H_{j^\perp})) \right\}. \quad (19)$$

We will prove below that  $\mathcal{E}_\Lambda(\Delta) \rightarrow 0$  as  $|\Lambda| \rightarrow \infty$ . Assuming this for the moment, it follows from the Birkhoff Ergodic Theorem and (18) that

$$\begin{aligned} & \lim_{|\Lambda| \rightarrow \infty} \frac{1}{|\Delta|} \int_\Delta dE \mathbb{E} \left\{ \frac{1}{|\Lambda|} \sum_{j \in \bar{\Lambda}} \xi(E; H_{j^\perp} + u_j, H_{j^\perp}) \right\} \\ &= \frac{1}{|\Delta|} \int_\Delta dE \mathbb{E} \{ \xi(E; H_{0^\perp} + u_0, H_{0^\perp}) \} \leq C_I < \infty. \end{aligned} \quad (20)$$

In order to justify the interchange of the expectation and the infinite-volume limit, we note that the nonnegative series in brackets on the first line of (20) converges pointwise a. e. to the integrand on the second line of (20). As the SSF  $\xi(E; H_{j^\perp} + u_j, H_{j^\perp}) \in L_{loc}^1(\mathbb{R})$ , the term in the brackets on the right of the first line of (20) is uniformly bounded, so the exchange is justified by the Lebesgue Dominated Convergence Theorem. We apply the Lebesgue Differentiation Theorem to the second line of (20), since the SSF is in  $L_{loc}^1(\mathbb{R})$ , and obtain the pointwise bound in Theorem 3.1.

2. It remains to prove the vanishing of the error term in (19) in the infinite-volume limit. Using the identity on the first line of (11), we obtain

$$\mathcal{E}_\Lambda(\Delta) = \mathbb{E} \left\{ \frac{1}{|\Lambda|} \sum_{j \in \bar{\Lambda}} \left[ \text{Tr} u_j^{1/2} E_\Lambda(\Delta) u_j^{1/2} - \text{Tr} u_j^{1/2} E(\Delta) u_j^{1/2} \right] \right\}. \quad (21)$$

We define a local nonnegative measure  $\kappa_\Lambda$  by

$$\kappa_\Lambda(\Delta) \equiv \frac{1}{|\Lambda|} \mathbb{E} \left\{ \sum_{j \in \bar{\Lambda}} \text{Tr} u_j^{1/2} E_\Lambda(\Delta) u_j^{1/2} \right\}, \quad (22)$$

and the nonnegative measure  $\tilde{\kappa}_\Lambda$ , defined similarly but with the spectral projection  $E(\cdot)$  for the infinite-volume Hamiltonian  $H_\omega$ . In terms of these local measures, we can express the right side of (21) as

$$\mathcal{E}_\Lambda(\Delta) = \frac{1}{|\Lambda|} [\kappa_\Lambda(\Delta) - \tilde{\kappa}_\Lambda(\Delta)]. \quad (23)$$

We first prove that the measure  $\mathcal{E}_\Lambda(\cdot)$  converges vaguely to zero by computing the Laplace transform of the measure. The Laplace transform  $\mathcal{L}(\mathcal{E}_\Lambda)(t)$  is easily seen to be given by

$$\mathcal{L}(\mathcal{E}_\Lambda)(t) = \frac{1}{|\Lambda|} \mathbb{E} \{ \text{Tr} \tilde{V}_\Lambda (e^{-tH_\omega} - e^{-tH_\Lambda}) \}. \quad (24)$$

Using the Feynman-Kac formula for the heat semigroups, for example, one easily shows, as in [12], that

$$\lim_{|\Lambda| \rightarrow \infty} \mathcal{L}(\mathcal{E}_\Lambda)(t) = 0, \quad (25)$$

for  $t > 0$  pointwise, for a reasonable expanding family of regions  $\Lambda$ . This implies the measure  $\mathcal{E}_\Lambda(\cdot)$  converges vaguely to zero which, in turn, implies that the right side of (23) converges to zero.  $\square$

A consequence of this result is an apparently new relationship between the infinite-volume SSF and the DOS for lattice models. The analogous relation for continuum models defines a new measure absolutely continuous with respect to Lebesgue measure and to the DOS measure. These results follow easily from the proof of Proposition 3.1.

**Corollary 3.1** *Let  $\nu$  be the DOS measure for the random Hamiltonian  $H_\omega$ . For lattice models, for any Borel set  $A \subset \mathbb{R}$ , we have*

$$\nu(A) = \int_A dE \mathbb{E} \{ \xi(E; H_{0^\perp} + u_0, H_{0^\perp}) \}. \quad (26)$$

*For continuum models, there is a nonnegative measure  $\kappa$ , absolutely continuous with respect to the DOS measure and Lebesgue measure, with distribution given in (30), so that*

$$\kappa(A) = \int_A dE \mathbb{E} \{ \xi(E; H_{0^\perp} + u_0, H_{0^\perp}) \}. \quad (27)$$

*For any closed bounded interval  $I \subset \mathbb{R}$ , there are constants  $0 < c_I \leq C_I < \infty$ , so that for any Borel set  $A \subset I$ , we have*

$$0 \leq \kappa(A) \leq c_I |A|, \text{ and } 0 \leq \nu(A) \leq C_I \kappa(A). \quad (28)$$

**Proof:** From the Birkhoff Ergodic Theorem, and expression (10), we can express the integral on the right in (20) in terms of a positive measure  $\kappa$  as follows

$$\begin{aligned} \int_{\Delta} dE \mathbb{E}\{\xi(E; H_{0^\perp} + u_0, H_{0^\perp})\} &= \mathbb{E} \left( \lim_{|\Lambda| \rightarrow \infty} \frac{1}{|\Lambda|} \sum_{j \in \bar{\Lambda}} \text{Tr} u_j^{1/2} E(\Delta) u_j^{1/2} \right) \\ &= \mathbb{E}\{\text{Tr} u_0^{1/2} E(\Delta) u_0^{1/2}\} \\ &\equiv \kappa(\Delta), \end{aligned} \quad (29)$$

where  $\kappa(\cdot)$  is the nonnegative measure with distribution function given by

$$K(E) \equiv \mathbb{E}\{\text{Tr} u_0^{1/2} P(E) u_0^{1/2}\}, \quad (30)$$

where  $P(E)$  is the spectral family for  $H_u$ . For the lattice case, this measure is just the DOS measure, since  $u_0 = \delta_0$ , so that  $\mathbb{E}\{\xi(E; H_{0^\perp} + u_0, H_{0^\perp})\}$  is a representation of the DOS. It follows immediately from (29) and Theorem 3.1 that for any closed bounded interval  $I \subset \mathbb{R}$ , there exists a finite constant  $0 < C_I < \infty$ , so that for any Lebesgue measurable set  $A \subset I$ , we have

$$0 \leq \kappa(A) = \int_A dE \mathbb{E}\{\xi(E; H_{0^\perp} + u_0, H_{0^\perp})\} \leq C_I |A|. \quad (31)$$

Lebesgue measure, It remains to prove that  $\kappa$  is bounded above by the DOS measure. This implies the absolute continuity with respect to  $\nu$ . We simply note that there exists a constant  $0 < C_0 < \infty$ , depending only on  $u$ , so that

$$0 \leq \sum_{j \in \bar{\Lambda}} \text{Tr} u_j^{1/2} E_\Lambda(\Delta) u_j^{1/2} \leq C_0 \text{Tr} E_\Lambda(\Delta), \quad (32)$$

and recall the definition of the DOS measure. This implies that  $0 \leq \kappa(A) \leq C_I \nu(A)$ , for  $A \subset I \subset \mathbb{R}$ .  $\square$ .

This measure  $\kappa$  is similar to the DOS measure for continuum models. The distribution function for the DOS for continuum models is given by  $N(E) = \mathbb{E}\{\text{Tr} \chi_{\Lambda_1(0)} P(E) \chi_{\Lambda_1(0)}\}$ . The measure  $\kappa$  is equivalent to the DOS measure  $\nu$  if the single-site potential satisfies  $c_0 \chi_{\Lambda_1(0)} \leq u$ , for some  $c_0 > 0$ , and it is equal to  $\nu$  in the special case that  $u = \chi_{\Lambda_1(0)}$ . The equivalence of measures means that there are constants  $C_0, c_0 > 0$  so that

$$c_0 \nu(A) \leq \kappa(A) \leq C_0 \nu(A), \quad (33)$$

for all Borel subsets  $A \subset \mathbb{R}$ .

## 4 Comments

We make three comments on various other results concerning the SSF associated with random Schrödinger operators that have recently occurred in the literature related to random Schrödinger operators. For deterministic Schrödinger operators, *pointwise bounds* are known only in a few specific cases, such as finite-rank perturbations (cf. [20, 2] and Theorem 4.2 below) or perturbations of the Laplacian on  $L^2(\mathbb{R}^d)$  by sufficiently smooth potentials [19].

### 4.1 Related Results on the Averaged SSF

Bounds on the  $L^p$ -norm of the SSF, for  $0 < p \leq 1$ , were proved in [8] and improved in [11]. More recently, Hundertmark, et. al. [10], obtained some new integral bounds on the SSF that indicate that one cannot expect that, in general, the SSF is locally bounded. Indeed, Kirsch [13, 14] proved that if the Dirichlet Laplacian in  $\Lambda_L$ , a cube of side length  $L$  centered at the origin, is perturbed by a nonnegative, bounded potential supported inside the unit cube  $\Lambda_1$ , then the corresponding finite-volume SSF, at any positive energy, diverges as  $L \rightarrow \infty$ . Raikov and Warzel [17] considered the SSF for the Landau Hamiltonian (5) and a perturbation by a compactly-supported potential. They showed that the SSF diverges at the Landau energies.

The averaged SSF is expected to be better behaved. In addition to the pointwise bounds of Theorems 2.1 and 3.1, Aizenman, et. al. [1] proved an interesting bound on a spectral shift function related to the ones treated here. They consider the SSF  $\xi(t, E)$  for a pair of Hamiltonians  $H_t = H_0 + tV$  and  $H_t + U$ , where  $V$  and  $U$  are nonnegative bounded potentials such that  $V$  is strictly positive on a neighborhood of the support of  $U$ . Specifically, for any  $\delta > 0$ , we define the set  $Q_\delta \equiv \{x \in \mathbb{R}^d \mid \text{dist}(x, \text{supp}(U)) < \delta\}$ . We then require that  $V$  be strictly positive on  $Q_\delta$ .

**Theorem 4.1** *For any  $0 < s < \min(2/d, 1/2)$ , there is a finite positive constant  $C_{s,\delta} > 0$  so that the SSF  $\xi(t, E)$  satisfies the bound*

$$\int_0^1 |\xi(t, E)|^s dt \leq C_{s,\delta} \|U\|_\infty (1 + |E - E_0| + \|V\|_\infty)^{2s(d+1)}, \quad (34)$$

where  $E \geq E_0 \equiv \inf \sigma(H_0)$ .

## 4.2 Spectral Shift Density

Kostrykin and Schrader [15, 16] introduced the *spectral shift density* (SSD) that is closely related to the integrated density of states. The SSD is the density of a measure  $\Xi$  obtained by the thermodynamic limit

$$\int_{\mathbb{R}} g(\lambda) d\Xi(\lambda) = \lim_{|\Lambda| \rightarrow \infty} \int_{\mathbb{R}} g(\lambda) \frac{\xi(E; H_0 + \chi_{\Lambda} V_{\omega})}{|\Lambda|}. \quad (35)$$

Note that the size of the perturbation  $\chi_{\Lambda} V_{\omega}$  is of order  $|\Lambda|$ . They prove that the SSD  $\tilde{\xi}(E)$  is given as

$$\tilde{\xi}(E) = N_0(E) - N(E), \text{ a. e. } E \in \mathbb{R}, \quad (36)$$

where  $N_0(E)$  is the IDS of  $H_0$  and  $N(E)$  is the IDS of  $H_{\omega}$ .

## 4.3 A Pointwise Bound on the SSF for Finite Rank Perturbations

We consider the SSF for a finite-rank perturbation. Let  $B \geq 0$  be a non-negative finite-rank operator with rank  $N$ . Let  $H_s = H_0 + sB$  be the one-parameter perturbation of a self-adjoint, lower-semibounded operator  $H_0$ . The variable  $s \in [0, 1]$  is uniformly distributed. We consider the SSF  $\xi(E; H_1, H_0)$  and recover the classical pointwise upper bound  $N$  usually obtained by other methods cf. [2, 20].

**Theorem 4.2** *The spectral shift function for the pair of self-adjoint operators  $(H_1, H_0)$ , where  $0 \leq B \equiv H_1 - H_0$  is a finite-rank operator of rank  $N < \infty$ , satisfies the bound*

$$0 \leq \xi(E; H_1, H_0) \leq N. \quad (37)$$

**Proof:** Let  $f \in C_0^1(\mathbb{R})$  and consider the formula for the SSF:

$$\begin{aligned} \text{Tr}(f(H_1) - f(H_0)) &= - \int_{\mathbb{R}} f'(E) \xi(E; H_1, H_0) dE \\ &= \int_0^1 \frac{d}{ds} \text{Tr} f(H_s) ds \\ &= \int_0^1 ds \text{Tr} B^{1/2} f'(H_s) B^{1/2} \\ &= \sum_{j=1}^N \int_0^1 ds \langle \phi_j, B^{1/2} f'(H_s) B^{1/2} \phi_j \rangle \end{aligned} \quad (38)$$

Let  $E_s(\cdot)$  be the spectral family for  $H_s$ . The matrix element in (38) is written as

$$\begin{aligned} \langle \phi_j, B^{1/2} f'(H_s) B^{1/2} \phi_j \rangle &= \int_{\mathbb{R}} f'(\lambda) d\langle B^{1/2} \phi_j, E_s(\lambda) B^{1/2} \phi_j \rangle \\ &= \int_{\mathbb{R}} f'(\lambda) d\mu_{H_s}^{\psi_j}(\lambda), \end{aligned} \quad (39)$$

where  $\psi_j \equiv B^{1/2} \phi_j$  and  $\mu_{H_s}^{\psi_j}$  is the corresponding spectral measure for  $H_s$  and  $\psi_j$ . We divide the support of  $f'$  into  $p$  subintervals  $\Delta_k$  and bound the absolute value of the integral over  $\lambda$  in (39) from above as

$$\left| \int_{\mathbb{R}} f'(\lambda) d\mu_{H_s}^{\psi_j}(\lambda) \right| \leq \sum_{k=1}^p |f'(x_k)| \mu_{H_s}^{\psi_j}(\Delta_k), \quad (40)$$

where  $x_k \in \Delta_k$  is such that  $|f'(x_k)| = \sup_{x \in \Delta_k} |f'(x)|$ . Inserting this into the integral over  $s$  in (38), we see that it remains to estimate

$$\int_0^1 \mu_{H_s}^{\psi_j}(\Delta_k) ds = \int_0^1 \langle B^{1/2} \phi_j, E_s(\Delta_k) B^{1/2} \phi_j \rangle ds. \quad (41)$$

For bounded probability distributions with compact support, the integral on the right in (41) was estimated in [3]. The result is

$$\int_0^1 \langle B^{1/2} \phi_j, E_s(\Delta_k) B^{1/2} \phi_j \rangle ds \leq |\Delta_k|, \quad (42)$$

since  $\|\psi_j\| \leq 1$ . Combining this bound with (38)–(39) and recalling the approximation (40), we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}} f'(E) \xi(E; H_1, H_0) dE \right| &\leq \sum_{j=1}^N \sum_{k=1}^p |f'(x_k)| |\Delta_k| \\ &\leq N \|f'\|_1, \end{aligned} \quad (43)$$

which, extending the estimate to any  $f \in L^1(\mathbb{R})$ , we conclude that

$$|\xi(E; H_1, H_0)| \leq N, \quad (44)$$

proving the result.  $\square$

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