



HAL
open science

Some new estimates on the spectral shift function associated with random Schrödinger operators

Frédéric Klopp, Jean-Michel Combes, Peter Hislop

► **To cite this version:**

Frédéric Klopp, Jean-Michel Combes, Peter Hislop. Some new estimates on the spectral shift function associated with random Schrödinger operators. 2006. hal-00067801v1

HAL Id: hal-00067801

<https://hal.science/hal-00067801v1>

Preprint submitted on 9 May 2006 (v1), last revised 13 Oct 2006 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

**Some New Estimates on the Spectral Shift Function
Associated with Random Schrödinger Operators**

Jean-Michel Combes¹

Département de Mathématiques
Université du Sud, Toulon-Var
83130 La Garde, FRANCE

Peter D. Hislop²

Department of Mathematics
University of Kentucky
Lexington, KY 40506-0027 USA

Frédéric Klopp

L.A.G.A, Institut Galilée
Université Paris-Nord
F-93430 Villetaneuse, FRANCE
et
Institut Universitaire de France

Abstract

We prove some new pointwise-in-energy bounds on the expectations of various spectral shift functions associated with random Schrödinger operators in the continuum having Anderson-type random potentials in both finite-volume and infinite-volume. These estimates are a consequence of our new Wegner estimate for finite-volume random Schrödinger operators [4]. For lattice models, we also obtain a representation of the infinite-volume density of states in terms of a spectral shift function. For continuum models, the corresponding measure is absolutely continuous with respect to the density of states and agrees with it in certain cases. We present a variant of the spectral averaging result of [4] and use it to prove a pointwise upper bound on the SSF for finite-rank perturbations.

¹Centre de Physique Théorique, CNRS Marseille, France

²Supported in part by NSF grant DMS-0503784.

1 Introduction: The Wegner Estimate and the Spectral Shift Function

Some recent analyses of random Schrödinger operators have involved three related concepts: the Wegner estimate for the finite-volume Hamiltonians, the spectral shift function (SSF), and the integrated density of states (IDS). In this note, we prove some new pointwise bounds on the expectation of some SSFs that occur in the theory of random Schrödinger operators in the continuum. These bounds result from an improved version of the Wegner estimate [4]. In earlier work [3, 7], we used L^p -bounds on the SSF in order to obtain better estimates on the IDS. In our most recent work, we obtain an optimal Wegner estimate directly without using the SSF and found, as a consequence, new pointwise bounds on the expectation of the SSF. It has often been conjectured that in the case of ergodic, random, Schrödinger operators of the form considered here the SSF for a local single-site perturbation should be in $L_{loc}^\infty(\mathbb{R})$ once it is averaged over the random variables on which the disordered potential depends. We prove this in this note. We mention that these types of bounds on the SSF also play a motivating role in the fractional moment method for proving localization in the continuum [1]. For lattice models, the pointwise bounds on the SSF are a simple consequence of the fact that the corresponding perturbations are finite-rank (cf. [2, 21] and section 4.3).

We first recall a special case of the Wegner estimate proved in [4] that will be used for the bounds on the SSF. We refer to [4] for the general statement, valid for arbitrary bounded processes on \mathbb{Z}^d , and the proofs. The family of Schrödinger operators $H_\omega = H_0 + V_\omega$, on $L^2(\mathbb{R}^d)$, is constructed from a deterministic, \mathbb{Z}^d -periodic, background operator $H_0 = (-i\nabla - A_0)^2 + V_0$. We consider an Anderson-type random potential V_ω constructed from the single-site potential u as

$$V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \omega_j u(x - j). \quad (1)$$

The family of random variables is assumed to be independent, and identically distributed (iid). The results are independent of the disorder provided it is nonzero.

We define local versions of the Hamiltonians and potentials associated with bounded regions in \mathbb{R}^d . By $\Lambda_L(x)$, we mean the open cube of side

length L centered at $x \in \mathbb{R}^d$. For $\Lambda \subset \mathbb{R}^d$, we denote the lattice points in Λ by $\tilde{\Lambda} = \Lambda \cap \mathbb{Z}^d$. For a cube Λ , we take H_0^Λ and H_Λ (omitting the index ω) to be the restrictions of H_0 and H_ω , respectively, to the cube Λ , with periodic boundary conditions on the boundary $\partial\Lambda$ of Λ . We denote by $E_0^\Lambda(\cdot)$ and $E_\Lambda(\cdot)$ the spectral families for H_0^Λ and H_Λ , respectively. Furthermore, for $\Lambda \subset \mathbb{R}^d$, let χ_Λ be the characteristic function for Λ . The local potential V_Λ is defined by

$$V_\Lambda(x) = V_\omega(x)\chi_\Lambda(x), \quad (2)$$

and we assume this can be written as

$$V_\Lambda(x) = \sum_{j \in \tilde{\Lambda}} \omega_j u(x - j). \quad (3)$$

For example, if the support of u is contained in a single unit cube, the formula (3) holds. We refer to the discussion in [3] when the support of u is compact, but not necessarily contained inside one cube. In this case, V_Λ can be written as in (3) plus a boundary term of order $|\partial\Lambda|$ and hence it does not contribute to the large $|\Lambda|$ limit. Hence, we may assume (3) without any loss of generality. We will also use the local potential obtained from (3) by setting all the random variables to one, that is,

$$\tilde{V}_\Lambda(x) = \sum_{j \in \tilde{\Lambda}} u_j(x), \quad (4)$$

where we will write $u_j(x) = u(x - j)$.

We will always make the following four assumptions:

- (H1).** The background operator $H_0 = (-i\nabla - A_0)^2 + V_0$ is a lower semi-bounded, \mathbb{Z}^d -periodic Schrödinger operator with a real-valued, \mathbb{Z}^d -periodic, potential V_0 , and a \mathbb{Z}^d -periodic vector potential A_0 . We assume that V_0 and A_0 are sufficiently regular so that H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$.
- (H2).** The periodic operator H_0 has the unique continuation property, that is, for any $E \in \mathbb{R}$ and for any function $\phi \in H_{loc}^2(\mathbb{R}^d)$, if $(H_0 - E)\phi = 0$, and if ϕ vanishes on an open set, then $\phi \equiv 0$.
- (H3).** The nonzero, non negative, compactly supported, single-site potential $u \in L_0^\infty(\mathbb{R}^d)$, and it is strictly positive on a nonempty open set. We assume that $\|u\|_\infty \leq 1$.

(H4). The random coupling constants $\{\omega_j \mid j \in \mathbb{Z}^d\}$, are independent and identically distributed. The probability distribution μ_0 of ω_0 is compactly supported with a bounded density $h_0 \in L_0^\infty(\mathbb{R})$.

These imply that the infinite-volume random Schrödinger operator H_ω is ergodic with respect to the group of \mathbb{Z}^d -translations.

Our results also apply to the randomly perturbed Landau Hamiltonian $H_\omega(\lambda) = H_L(B) + \lambda V_\omega$, for $\lambda \neq 0$, where V_ω is an Anderson-type potential as in (1). The Landau Hamiltonian $H_L(B)$ on $L^2(\mathbb{R}^2)$ is given by

$$H_L(B) = (-i\nabla - A_0)^2, \quad \text{with } A_0(x, y) = \frac{B}{2}(-y, x). \quad (5)$$

The constant $B \neq 0$ is the magnetic field strength.

Under these assumptions, the Wegner estimate necessary for our purposes has the following form.

Theorem 1.1 *We assume that the family of random Schrödinger operators $H_\omega = H_0 + V_\omega$ on $L^2(\mathbb{R}^d)$ satisfies hypotheses (H1)-(H4). Then, there exists a locally uniform constant $C_W > 0$ such that for any $E_0 \in \mathbb{R}$, and $\epsilon \in (0, 1]$, the local Hamiltonians H_Λ satisfy the following Wegner estimate*

$$\begin{aligned} \mathbb{P}\{\text{dist}(\sigma(H_\Lambda), E_0) < \epsilon\} &\leq \mathbb{E}\{\text{Tr} E_\Lambda([E_0 - \epsilon, E_0 + \epsilon])\} \\ &\leq C_W \epsilon |\Lambda|. \end{aligned} \quad (6)$$

A similar estimate holds for randomly perturbed Landau Hamiltonians.

This theorem immediately implies the Lipschitz continuity of the integrated density of states [4]. As a consequence, the density of states (DOS) exists and is a locally bounded function. In this note, we use Theorem 1.1 to prove new pointwise bounds on the expectation of the SSF for both finite-volume and infinite-volume random Schrödinger operators. We comment on the relation of these results to various results concerning the SSF for random Schrödinger operators in section 4.

2 Bounds on the Spectral Shift Function for Finite-Volume Hamiltonians

We use the result of Theorem 1.1 to bound the expectation of the SSF for a single-site perturbation of a finite-volume Hamiltonian H_Λ . Since the size of

the support of the perturbation u is of order one relative to $|\Lambda|$, we expect the SSF to be of order one also. For a discussion of the relation between the IDS and the SSF, we refer the reader to [7] and references therein. A nice review of results concerning the SSF may be found in [2]. We recall that for a pair of self-adjoint operators $(H(1), H(0))$, such that $f(H(1)) - f(H(0))$ is trace-class, the SSF $\xi(E; H(1), H(0))$ is defined through the trace formula. For example, for any $f \in C_0^1(\mathbb{R})$, we have

$$\text{Tr}[f(H(1)) - f(H(0))] = \int_{\mathbb{R}} f'(E) \xi(E; H(1), H(0)) dE. \quad (7)$$

We first consider a one-parameter family of self-adjoint operators $H(\lambda) = H_0 + \lambda V$, with $V \geq 0$, and λ uniformly distributed on $[0, 1]$. Birman and Solomyak proved a relation (cf. [18]) between the averaged, weighted, trace of the spectral projector $E_\lambda(\cdot)$ of $H(\lambda)$, and the SSF for the pair $H(1) \equiv H(\lambda = 1)$ and $H(0) \equiv H(\lambda = 0) = H_0$. For any measurable $\Delta \subset \mathbb{R}$, this formula has the form

$$\int_0^1 d\lambda \text{Tr} V^{1/2} E_\lambda(\Delta) V^{1/2} = \int_{\Delta} dE \xi(E; H_0 + V, H_0), \quad (8)$$

whenever all the terms exist. For example, if V is relatively H_0 -trace class, then all the terms are well-defined.

We apply this formula as follows. First, we must also make a stronger hypothesis on the probability distribution than (H4). We will assume:

(H4'). The random coupling constants $\{\omega_j \mid j \in \mathbb{Z}^d\}$, are independent and identically distributed. The probability distribution μ_0 of ω_0 is the uniform distribution on $[0, 1]$.

Second, without loss of generality, we normalize the single-site potential $u \geq 0$ so that $0 \leq \sum_j u_j \leq 1$ and it follows trivially that

$$\sum_{j \in \tilde{\Lambda}} \text{Tr} u_j^{1/2} E_\Lambda(\Delta) u_j^{1/2} \leq \text{Tr} E_\Lambda(\Delta). \quad (9)$$

We now consider the effect of the variation of one random variable ω_j , for $j \in \tilde{\Lambda}$, on the local Hamiltonian. In formula (8), we take $H(0) = H_\Lambda(\omega_j = 0)$,

$H(1) = H_\Lambda(\omega_j = 1)$, so that $\lambda = \omega_j$, and $V = u_j \geq 0$. We write $H_{j^\perp}^\Lambda$ for H_Λ with $\omega_j = 0$. The Birman-Solomyak formula (8) then has the form

$$\int_0^1 d\omega_j \operatorname{Tr} u_j^{1/2} E_\Lambda(\Delta) u_j^{1/2} = \int_\Delta dE \xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda). \quad (10)$$

Taking the expectation of (9) and using formula (10), we obtain

$$\begin{aligned} \mathbb{E}\left\{\sum_{j \in \tilde{\Lambda}} \operatorname{Tr} u_j^{1/2} E_\Lambda(\Delta) u_j^{1/2}\right\} &= \sum_{j \in \tilde{\Lambda}} \mathbb{E}\left\{\int_\Delta dE \xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda)\right\} \\ &\leq \mathbb{E}\{\operatorname{Tr} E_\Lambda(\Delta)\} \\ &\leq C_0 |\Delta| |\Lambda|, \end{aligned} \quad (11)$$

where we used the result of the proof of Theorem 1.1 on the last line. We conclude from (11) that

$$\frac{1}{|\Delta|} \int_\Delta dE \left\{ \frac{1}{|\Lambda|} \sum_{j \in \tilde{\Lambda}} \mathbb{E}\{\xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda)\} \right\} \leq C_0. \quad (12)$$

If the spatially averaged expectation of the SSF is $L^1_{loc}(\mathbb{R})$ in E , we can conclude a pointwise bound from (12), for Lebesgue almost every energy E , of the form

$$\mathbb{E}\left\{\frac{1}{|\Lambda|} \sum_{j \in \tilde{\Lambda}} \xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda)\right\} \leq C_0. \quad (13)$$

In [7], we proved that the SSF for local Schrödinger operators with compactly-supported perturbations is locally- L^1 , so this pointwise bound (13) holds. Finally, we observe that due to the periodic boundary conditions on $\partial\Lambda$ and the \mathbb{Z}^d -periodicity of H_0 , we have that for any $j, k \in \tilde{\Lambda}$

$$\mathbb{E}\{\xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda)\} = \mathbb{E}\{\xi(E; H_{k^\perp}^\Lambda + u_k, H_{k^\perp}^\Lambda)\}, \quad (14)$$

and consequently it follows from (13) that for any $j \in \tilde{\Lambda}$,

$$\mathbb{E}\{\xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda)\} \leq C_0. \quad (15)$$

Theorem 2.1 *Under the hypotheses (H1)-(H4'), the expectation of the spectral shift function, corresponding to the variation of a single site of the finite-volume Hamiltonian, is uniformly locally bounded in energy. That is, for any*

bounded energy interval, there is a constant $C_I > 0$, independent of Λ , so that for Lebesgue almost every $E \in I$, we have

$$\mathbb{E}\{\xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda)\} \leq C_I, \quad (16)$$

for any $j \in \tilde{\Lambda}$.

In the lattice case, the perturbation u_j is rank-one, so by the general theory (cf. [21, 2], or see section 4.3), we have the bound

$$\xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda) \leq 1, \quad (17)$$

for any $j \in \tilde{\Lambda}$, uniformly in $E \in \mathbb{R}$.

3 Bounds on the Spectral Shift Function for Infinite-Volume Hamiltonians

We consider the thermodynamic limit of the SSF in (13). The Birkhoff Ergodic Theorem implies that the limit of the right side of (13) is the expectation of the SSF corresponding to the pair of infinite-volume Hamiltonians $(H_{0^\perp}, H_{0^\perp} + u_0)$ if we replace $\xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda)$ by $\xi(E; H_{j^\perp} + u_j, H_{j^\perp})$, where H_{j^\perp} is the infinite-volume Hamiltonian with $\omega_j = 0$. This is the content of the next proposition.

Theorem 3.1 *Let H_{0^\perp} be the infinite-volume random Hamiltonian H_ω with $\omega_0 = 0$ and assume hypotheses (H1)-(H4'). Then the SSF $\xi(E; H_{0^\perp} + u_0, H_{0^\perp})$ is well-defined and $\mathbb{E}\{\xi(E; H_{0^\perp} + u_0, H_{0^\perp})\} \in L_{loc}^\infty(\mathbb{R})$.*

Proof: 1. We begin with the integrated expression (12) and write

$$\begin{aligned} & \frac{1}{|\Delta|} \int_\Delta dE \mathbb{E} \left\{ \frac{1}{|\tilde{\Lambda}|} \sum_{j \in \tilde{\Lambda}} \xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda) \right\} \\ &= \frac{1}{|\Delta|} \int_\Delta dE \mathbb{E} \left\{ \frac{1}{|\tilde{\Lambda}|} \sum_{j \in \tilde{\Lambda}} \xi(E; H_{j^\perp} + u_j, H_{j^\perp}) \right\} + \frac{\mathcal{E}_\Lambda(\Delta)}{|\Delta|}, \quad (18) \end{aligned}$$

where the error term is

$$\mathcal{E}_\Lambda(\Delta) \equiv \int_\Delta dE \mathbb{E} \left\{ \frac{1}{|\Lambda|} \sum_{j \in \tilde{\Lambda}} (\xi(E; H_{j^\perp}^\Lambda + u_j, H_{j^\perp}^\Lambda) - \xi(E; H_{j^\perp} + u_j, H_{j^\perp})) \right\}. \quad (19)$$

We will prove below that $\mathcal{E}_\Lambda(\Delta) \rightarrow 0$ as $|\Lambda| \rightarrow \infty$. Assuming this for the moment, it follows from the Birkhoff Ergodic Theorem and (18) that

$$\begin{aligned} & \lim_{|\Lambda| \rightarrow \infty} \frac{1}{|\Delta|} \int_\Delta dE \mathbb{E} \left\{ \frac{1}{|\Lambda|} \sum_{j \in \tilde{\Lambda}} \xi(E; H_{j^\perp} + u_j, H_{j^\perp}) \right\} \\ &= \frac{1}{|\Delta|} \int_\Delta dE \mathbb{E} \{ \xi(E; H_{0^\perp} + u_0, H_{0^\perp}) \} \leq C_I < \infty. \end{aligned} \quad (20)$$

In order to justify the interchange of the expectation and the infinite-volume limit, we note that the nonnegative series in brackets on the first line of (20) converges pointwise a. e. to the integrand on the second line of (20). As the SSF $\xi(E; H_{j^\perp} + u_j, H_{j^\perp}) \in L^1_{loc}(\mathbb{R})$, the term in the brackets on the right of the first line of (20) is uniformly bounded, so the exchange is justified by the Lebesgue Dominated Convergence Theorem. We apply the Lebesgue Differentiation Theorem to the second line of (20), since the SSF is in $L^1_{loc}(\mathbb{R})$, and obtain the pointwise bound in Theorem 3.1.

2. It remains to prove the vanishing of the error term in (19) in the infinite-volume limit. Using the identity on the first line of (11), we obtain

$$\mathcal{E}_\Lambda(\Delta) = \mathbb{E} \left\{ \frac{1}{|\Lambda|} \sum_{j \in \tilde{\Lambda}} \left[\text{Tr} u_j^{1/2} E_\Lambda(\Delta) u_j^{1/2} - \text{Tr} u_j^{1/2} E(\Delta) u_j^{1/2} \right] \right\}. \quad (21)$$

We define a local nonnegative measure κ_Λ by

$$\kappa_\Lambda(\Delta) \equiv \frac{1}{|\Lambda|} \mathbb{E} \left\{ \sum_{j \in \tilde{\Lambda}} \text{Tr} u_j^{1/2} E_\Lambda(\Delta) u_j^{1/2} \right\}, \quad (22)$$

and the nonnegative measure $\tilde{\kappa}_\Lambda$, defined similarly but with the spectral projection $E(\cdot)$ for the infinite-volume Hamiltonian H_ω . In terms of these local measures, we can express the right side of (21) as

$$\mathcal{E}_\Lambda(\Delta) = \frac{1}{|\Lambda|} [\kappa_\Lambda(\Delta) - \tilde{\kappa}_\Lambda(\Delta)]. \quad (23)$$

We first prove that the measure $\mathcal{E}_\Lambda(\cdot)$ converges vaguely to zero by computing the Laplace transform of the measure. The Laplace transform $\mathcal{L}(\mathcal{E}_\Lambda)(t)$ is easily seen to be given by

$$\mathcal{L}(\mathcal{E}_\Lambda)(t) = \frac{1}{|\Lambda|} \mathbb{E} \{ \text{Tr} \tilde{V}_\Lambda (e^{-tH_\omega} - e^{-tH_\Lambda}) \}. \quad (24)$$

Using the Feynman-Kac formula for the heat semigroups, for example, one easily shows, as in [11], that

$$\lim_{|\Lambda| \rightarrow \infty} \mathcal{L}(\mathcal{E}_\Lambda)(t) = 0, \quad (25)$$

for $t > 0$ pointwise, for a reasonable expanding family of regions Λ . This implies the measure $\mathcal{E}_\Lambda(\cdot)$ converges vaguely to zero which, in turn, implies that the right side of (23) converges to zero. \square

A consequence of this result is an apparently new relationship between the infinite-volume SSF and the DOS for lattice models. The analogous relation for continuum models defines a new measure absolutely continuous with respect to Lebesgue measure and to the DOS measure. These results follow easily from the proof of Proposition 3.1.

Corollary 3.1 *Let ν be the DOS measure for the random Hamiltonian H_ω . For lattice models, for any Borel set $A \subset \mathbb{R}$, we have*

$$\nu(A) = \int_A dE \mathbb{E} \{ \xi(E; H_{0^\perp} + u_0, H_{0^\perp}) \}. \quad (26)$$

For continuum models, there is a nonnegative measure κ , absolutely continuous with respect to the DOS measure and Lebesgue measure, with distribution given in (30), so that

$$\kappa(A) = \int_A dE \mathbb{E} \{ \xi(E; H_{0^\perp} + u_0, H_{0^\perp}) \}. \quad (27)$$

For any closed bounded interval $I \subset \mathbb{R}$, there are constants $0 < c_I \leq C_I < \infty$, so that for any Borel set $A \subset I$, we have

$$0 \leq \kappa(A) \leq c_I |A|, \text{ and } 0 \leq \nu(A) \leq C_I \kappa(A). \quad (28)$$

Proof: From the Birkhoff Ergodic Theorem, and expression (10), we can express the integral on the right in (20) in terms of a positive measure κ as follows

$$\begin{aligned} \int_{\Delta} dE \mathbb{E}\{\xi(E; H_{0^\perp} + u_0, H_{0^\perp})\} &= \mathbb{E}\left(\lim_{|\Lambda| \rightarrow \infty} \frac{1}{|\Lambda|} \sum_{j \in \tilde{\Lambda}} \text{Tr} u_j^{1/2} E(\Delta) u_j^{1/2}\right) \\ &= \mathbb{E}\{\text{Tr} u_0^{1/2} E(\Delta) u_0^{1/2}\} \\ &\equiv \kappa(\Delta), \end{aligned} \quad (29)$$

where $\kappa(\cdot)$ is the nonnegative measure with distribution function given by

$$K(E) \equiv \mathbb{E}\{\text{Tr} u_0^{1/2} P(E) u_0^{1/2}\}, \quad (30)$$

where $P(E)$ is the spectral family for H_ω . For the lattice case, this measure is just the DOS measure, since $u_0 = \delta_0$, so that $\mathbb{E}\{\xi(E; H_{0^\perp} + u_0, H_{0^\perp})\}$ is a representation of the DOS. It follows immediately from (29) and Theorem 3.1 that for any closed bounded interval $I \subset \mathbb{R}$, there exists a finite constant $0 < C_I < \infty$, so that for any Lebesgue measurable set $A \subset I$, we have

$$0 \leq \kappa(A) = \int_A dE \mathbb{E}\{\xi(E; H_{0^\perp} + u_0, H_{0^\perp})\} \leq C_I |A|. \quad (31)$$

Lebesgue measure, It remains to prove that κ is bounded above by the DOS measure. This implies the absolute continuity with respect to ν . We simply note that there exists a constant $0 < C_0 < \infty$, depending only on u , so that

$$0 \leq \sum_{j \in \tilde{\Lambda}} \text{Tr} u_j^{1/2} E_\Lambda(\Delta) u_j^{1/2} \leq C_0 \text{Tr} E_\Lambda(\Delta), \quad (32)$$

and recall the definition of the DOS measure. This implies that $0 \leq \kappa(A) \leq C_I \nu(A)$, for $A \subset I \subset \mathbb{R}$. \square .

This measure κ is similar to the DOS measure for continuum models. The distribution function for the DOS for continuum models is given by $N(E) = \mathbb{E}\{\text{Tr} \chi_{\Lambda_1(0)} P(E) \chi_{\Lambda_1(0)}\}$. The measure κ is equivalent to the DOS measure ν if the single-site potential satisfies $c_0 \chi_{\Lambda_1(0)} \leq u$, for some $c_0 > 0$, and it is equal to ν in the special case that $u = \chi_{\Lambda_1(0)}$. The equivalence of measures means that there are constants $C_0, c_0 > 0$ so that

$$c_0 \nu(A) \leq \kappa(A) \leq C_0 \nu(A), \quad (33)$$

for all Borel subsets $A \subset \mathbb{R}$.

4 Comments

We make three comments on various other results concerning the SSF associated with random Schrödinger operators that have recently occurred in the literature related to random Schrödinger operators. For deterministic Schrödinger operators, *pointwise bounds* are known only in a few specific cases, such as finite-rank perturbations (cf. [21, 2] and Theorem 4.3 below) or perturbations of the Laplacian on $L^2(\mathbb{R}^d)$ by sufficiently smooth potentials [19].

4.1 Related Results on the Averaged SSF

Bounds on the L^p -norm of the SSF, for $0 < p \leq 1$, were proved in [7] and improved in [10]. More recently, Hundertmark, et. al. [9], obtained some new integral bounds on the SSF that indicate that one cannot expect that, in general, the SSF is locally bounded. Examples of potential perturbations of Laplacians for which the SSF diverges at certain energies were constructed by Kirsch [12, 13]. Raikov and Warzel [17] considered the SSF for the Landau Hamiltonian (5) and a perturbation by a compactly-supported potential. They showed that the SSF diverges at the Landau energies.

The averaged SSF is expected to be better behaved. In addition to the pointwise bounds of Theorems 2.1 and 3.1, Aizenman, et. al. [1] proved an interesting bound on a spectral shift function related to the ones treated here. They consider the SSF $\xi(t, E)$ for a pair of Hamiltonians $H_t = H_0 + tV$ and $H_t + U$, where V and U are nonnegative bounded potentials such that V is strictly positive on a neighborhood of the support of U . Specifically, for any $\delta > 0$, we define the set $Q_\delta \equiv \{x \in \mathbb{R}^d \mid \text{dist}(x, \text{supp}(U)) < \delta\}$. We then require that V be strictly positive on Q_δ .

Theorem 4.1 *For any $0 < s < \min(2/d, 1/2)$, there is a finite positive constant $C_{s,\delta} > 0$ so that the SSF $\xi(t, E)$ satisfies the bound*

$$\int_0^1 |\xi(t, E)|^s dt \leq C_{s,\delta} \|U\|_\infty (1 + |E - E_0| + \|V\|_\infty)^{2s(d+1)}, \quad (34)$$

where $E \geq E_0 \equiv \inf \sigma(H_0)$.

4.2 Spectral Shift Density

Kostrykin and Schrader [14, 15] introduced the *spectral shift density* (SSD) that is closely related to the integrated density of states. The SSD is the density of a measure Ξ obtained by the thermodynamic limit

$$\int_{\mathbb{R}} g(\lambda) d\Xi(\lambda) = \lim_{|\Lambda| \rightarrow \infty} \int_{\mathbb{R}} g(\lambda) \frac{\xi(E; H_0 + \chi_{\Lambda} V_{\omega})}{|\Lambda|}. \quad (35)$$

Note that the size of the perturbation $\chi_{\Lambda} V_{\omega}$ is of order $|\Lambda|$. They prove that the SSD $\tilde{\xi}(E)$ is given as

$$\tilde{\xi}(E) = N_0(E) - N(E), \text{ a. e. } E \in \mathbb{R}, \quad (36)$$

where $N_0(E)$ is the IDS of H_0 and $N(E)$ is the IDS of H_{ω} .

4.3 Maximally Dissipative Operators and Finite Rank Perturbations

J. Schenker (private communication) indicated to us that we can obtain the following version of the main spectral averaging result of [4]. Let us recall a main result in the theory of maximally dissipative operators (cf. [16, 20]). A closed operator A is maximally dissipative if $\Im A \geq 0$ and A has no dissipative extension.

Proposition 4.1 *Suppose A is a maximally dissipative operator on a separable Hilbert space \mathcal{H} . Then, there exists a Hilbert space $\tilde{\mathcal{H}}$, containing \mathcal{H} as a subspace, an orthogonal projection $P : \tilde{\mathcal{H}} \rightarrow \mathcal{H}$, and a self-adjoint dilation L so that for $z \in \mathbb{C}$ with $\Im z < 0$,*

$$(A - z)^{-1} = P(L - z)^{-1}P^*. \quad (37)$$

Note that the signs of the imaginary parts in the denominator of the left side of (37) are the same. Using this result, we are able to replace the weight function B used in [4] by $B^{1/2}$. This allows us to give a new proof of a well-known result on the pointwise bound of the SSF when the perturbation is finite-rank. The abstract result, corresponding to Theorem 3.1 of [4], is the following theorem.

Theorem 4.2 *Let A be a maximally dissipative operator and let $B \geq 0$ be a bounded, nonnegative self-adjoint operator on a separable Hilbert space \mathcal{H} . Then, for any $\phi \in \mathcal{H}$, we have the bound*

$$\sum_{n \in \mathbb{Z}} \sup_{y \in [0,1]} \Im \langle B^{1/2} \phi, \frac{1}{A + (n+y)B + iB} B^{1/2} \phi \rangle \leq 2\pi \|B\| \|\phi\|^2. \quad (38)$$

Proof: Let $\delta > 0$ be a small parameter and set $B_\delta \equiv B + \delta > \delta$, since $B \geq 0$. As B_δ is bounded and invertible, we can write

$$\langle B_\delta^{1/2} \phi, \frac{1}{A + (n+y)B_\delta + iB_\delta} B_\delta^{1/2} \phi \rangle = \langle \phi, \frac{1}{B_\delta^{-1/2} A B_\delta^{-1/2} + (n+y) + i} \phi \rangle. \quad (39)$$

Since $B \geq 0$, and A is maximally dissipative, so is $B_\delta^{-1/2} A B_\delta^{-1/2}$. Let P and L be the orthogonal projector and self-adjoint dilation associated with A as in Proposition 4.1. Let μ_L^ψ be the spectral measure for L and the vector ψ . We can write the matrix element in (39) as

$$\langle P^* \phi, \frac{1}{L + (n+y) + i} P^* \phi \rangle = \int_{\mathbb{R}} d\mu_L^{P^* \phi}(s) \frac{1}{s + (n+y) + i}. \quad (40)$$

We recall a special case of Lemma 3.1 of [4]:

$$\sup_{\kappa \in \mathbb{R}} \sum_{n \in \mathbb{Z}} \sup_{y \in [0,1]} \frac{1}{(y + n + \kappa)^2 + 1} \leq 2\pi. \quad (41)$$

Taking the imaginary part, we obtain from (40) and (41),

$$\begin{aligned} & \sum_{n \in \mathbb{Z}} \sup_{y \in [0,1]} \Im \langle B_\delta^{1/2} \phi, \frac{1}{A + (n+y)B_\delta + iB_\delta} B_\delta^{1/2} \phi \rangle \\ & \leq \int_{\mathbb{R}} d\mu_L^{P^* \phi}(s) \sup_{\kappa \in \mathbb{R}} \sum_{n \in \mathbb{Z}} \sup_{y \in [0,1]} \frac{1}{(y + n + \kappa)^2 + 1} \\ & \leq 2\pi. \end{aligned} \quad (42)$$

We now take $\delta \rightarrow 0$ and obtain the result. \square

We apply this result to the following situation. Let $B \geq 0$ be a nonnegative finite-rank operator with rank N . Let $H_s = H_0 + sB$ be a perturbation of a self-adjoint, lower-semibounded operator H_0 . We consider the SSF $\xi(E; H_1, H_0)$ and recover the well-known result (cf. [21, 2]) that the SSF is pointwise bounded above by N (we obtain a constant 2π that is not optimal).

Theorem 4.3 *The spectral shift function for the pair of self-adjoint operators (H_1, H_0) , where $0 \leq B \equiv H_1 - H_0$ is a finite-rank operator of rank $N < \infty$ and $\|B\| < 1$, satisfies the bound*

$$0 \leq \xi(E; H_1, H_0) \leq 2\pi N. \quad (43)$$

Proof: Let $f \in C_0^1(\mathbb{R})$ and consider the formula for the SSF:

$$\begin{aligned} \text{Tr}(f(H_1) - f(H_0)) &= - \int_{\mathbb{R}} f'(E) \xi(E; H_1, H_0) dE \\ &= \int_0^1 \frac{d}{ds} \text{Tr} f(H_s) ds \\ &= \int_0^1 ds \text{Tr} B^{1/2} f'(H_s) B^{1/2} \\ &= \sum_{j=1}^N \int_0^1 ds \langle \phi_j, B^{1/2} f'(H_s) B^{1/2} \phi_j \rangle \end{aligned} \quad (44)$$

Let $E_s(\cdot)$ be the spectral family for H_s . The matrix element in (44) is written as

$$\begin{aligned} \langle \phi_j, B^{1/2} f'(H_s) B^{1/2} \phi_j \rangle &= \int_{\mathbb{R}} f'(\lambda) d\langle B^{1/2} \phi_j, E_s(\lambda) B^{1/2} \phi_j \rangle \\ &= \int_{\mathbb{R}} f'(\lambda) d\mu_{H_s}^{\psi_j}(\lambda), \end{aligned} \quad (45)$$

where $\psi_j \equiv B^{1/2} \phi_j$ and $\mu_{H_s}^{\psi_j}$ is the corresponding spectral measure for H_s and ψ_j . We divide the support of f' into p subintervals Δ_k and bound the absolute value of the integral over λ in (45) from above as

$$\left| \int_{\mathbb{R}} f'(\lambda) d\mu_{H_s}^{\psi_j}(\lambda) \right| \leq \sum_{k=1}^p |f'(x_k)| \mu_{H_s}^{\psi_j}(\Delta_k), \quad x_k \in \Delta_k, \quad (46)$$

so it remains to estimate

$$\mu_{H_s}^{\psi_j}(\Delta_k) = \langle B^{1/2} \phi_j, E_s(\Delta_k) B^{1/2} \phi_j \rangle. \quad (47)$$

As in [3], we have the inequality,

$$\langle \phi, E_s(\Delta) \phi \rangle \leq (4/\pi) \int_{\Delta} dt \langle \phi, \Im(H_s - t - i|\Delta|)^{-1} \phi \rangle. \quad (48)$$

Inserting this into (47), we have

$$\int_{\Delta_k} dt \langle B^{1/2} \phi_j, \Im(H_s - t - i|\Delta_k|)^{-1} B^{1/2} \phi_j \rangle. \quad (49)$$

Returning to the s integral in (44), and taking $|\Delta_k| = \epsilon \ll 1$, we divide the interval $[0, 1]$ into intervals of length ϵ and obtain

$$\begin{aligned} & \int_0^1 ds \langle B^{1/2} \phi_j, \Im(H_0 + sB - t - i\epsilon)^{-1} B^{1/2} \phi_j \rangle \\ & \leq \frac{1}{\epsilon} \sum_{m=0}^{N_\epsilon} \int_{m\epsilon}^{(m+1)\epsilon} ds \langle B^{1/2} \phi_j, \Im(K + (s/\epsilon)B - iB)^{-1} B^{1/2} \phi_j \rangle \\ & \leq \sum_{m=0}^{N_\epsilon} \int_0^1 dy \langle B^{1/2} \phi_j, \Im(K + (m+y)B - iB)^{-1} B^{1/2} \phi_j \rangle, \end{aligned} \quad (50)$$

where $K \equiv (1/\epsilon)(H_0 - t) - i(1 - B)$ and we let $s/\epsilon = m + y$, with $y \in [0, 1]$. Since we require $\|B\| < 1$, the operator $(1 - B) > 0$ so that $\Im K < 0$. Hence, $-K$ is a dissipative operator and $\Im(K + (m+y)B - iB) = \Im K - B < 0$.

We now apply Theorem 4.2 to estimate the matrix element in (50). We obtain

$$\int_0^1 ds \langle B^{1/2} \phi_j, \Im(H_0 + sB - t - i\epsilon)^{-1} B^{1/2} \phi_j \rangle \leq 2\pi. \quad (51)$$

This implies that the expectation of (49) is bounded above by $2\pi|\Delta_k|$. Combining this bound with (44)–(45) and recalling the approximation (46), we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}} f'(E) \xi(E; H_1, H_0) dE \right| & \leq \sum_{j=1}^N \sum_{k=1}^p |f'(x_k)| \int_0^1 ds \mu_{H_s}^{\psi_j}(\Delta_k) \\ & \leq 2\pi N \|f'\|_1, \end{aligned} \quad (52)$$

which, extending the estimate to any $f \in L^1(\mathbb{R})$, we conclude that

$$|\xi(E; H_1, H_0)| \leq 2\pi N, \quad (53)$$

proving the result. \square

References

- [1] M. Aizenman, A. Elgart, S. Naboko, G. Stolz, J. Schenker: Moment Analysis for Localization in Random Schrödinger Operators, *Inventiones Mathematicae* **163**, 343–413 (2006).
- [2] M. S. Birman, D. R. Yafaev: The spectral shift function: The work of M. G. Krein and its further development, *St. Petersburg Mathematics Journal* **4**, 833–870 (1992).
- [3] J. M. Combes, P. D. Hislop, F. Klopp: Hölder continuity of the Integrated Density of States for some random operators at all energies, *IMRN* **2003** No. 4, 179–209.
- [4] J. M. Combes, P. D. Hislop, F. Klopp: An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators, preprint 2006.
- [5] J. M. Combes, P. D. Hislop, F. Klopp, G. Raikov: Global continuity of the integrated density of states for random Landau Hamiltonians, *Commun. Part. Diff. Eqns.* **29**, 1187–1214 (2004).
- [6] J. M. Combes, P. D. Hislop, E. Mourre: Spectral Averaging, Perturbation of Singular Spectrum, and Localization, *Trans. Amer. Math. Soc.* **348**, 4883–4894 (1996).
- [7] J. M. Combes, P. D. Hislop, S. Nakamura: The L^p -theory of the spectral shift function, the Wegner estimate, and the integrated density of states for some random operators, *Commun. Math. Phys.* **218**, 113–130 (2001).
- [8] P. D. Hislop, F. Klopp: The integrated density of states for some random operators with nonsign definite potentials, *J. Funct. anal.* **195**, 12–47 (2002).
- [9] D. Hundertmark, R. Killip, S. Nakamura, P. Stollmann, I. Veselic: Bounds on the spectral shift function and the density of states, *Commun. Math. Phys.* **262**, 489–503 (2005).
- [10] D. Hundertmark, B. Simon: An optimal L^p -bound on the Krein spectral shift function, *J. Anal. Math.* **87**, 199–208 (2002).

- [11] W. Kirsch: Random Schrödinger operators: A course, in *Schrödinger operators, Sonderborg DK 1988*, ed. H. Holden and A. Jensen, Lecture Notes in Physics **345**, Berlin: Springer 1989.
- [12] W. Kirsch: Small perturbations and the eigenvalues of the Laplacian on large bounded domains, *Proc. Amer. Math. Soc.* **101**, 509–512 (1987).
- [13] W. Kirsch: The stability of the density of states of Schrödinger operator under very small perturbations, *Integral Equations Operator Theory* **12**, 383–391 (1989).
- [14] V. Kostrykin, R. Schrader: Scattering theory approach to random Schrödinger operators in one dimension, *Rev. Math. Phys.* **11**, 187–242 (1999).
- [15] V. Kostrykin, R. Schrader: The density of states and the spectral shift density of random Schrödinger operators, *Rev. Math. Phys.* **12**, 807–847 (2000).
- [16] S. Naboko: The structure of singularities of operator functions with a positive imaginary part, *Funktsional. Anal. i Prilozhen* **25**, 1–13 (1991).
- [17] G. Raikov, S. Warzel: Quasi-classical versus non-classical spectral asymptotics for magnetic Schrödinger operators with decreasing electric potentials, *Rev. Math. Phys.* **14**, 1051–1072 (2002).
- [18] B. Simon: Spectral averaging and the Krein spectral shift, *Proc. Am. Math. Soc.* **126**, 1409–1413 (1998).
- [19] A. Sobolev: Efficient bounds for the spectral shift functions, *Ann. Inst. Poincaré* **58**, 55–83 (1993).
- [20] B. Sz.-Nagy, C. Foias: *Harmonic analysis of operators on Hilbert space*, North-Holland Publishing Co. Amsterdam, 1970.
- [21] D. R. Yafaev: *Mathematical Scattering Theory: General Theory*, Translations of Mathematical Monographs **105**, Providence, RI: American Mathematical Society, 1992.