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# KUREPA-TREES AND NAMBA FORCING

BERNHARD KÖNIG AND YASUO YOSHINOBU

ABSTRACT. We show that compact cardinals and MM are sensitive to  $\lambda$ -closed forcings for arbitrarily large  $\lambda$ . This is done by adding ‘regressive’  $\lambda$ -Kurepa-trees in either case. We argue that the destruction of regressive Kurepa-trees with MM requires the use of Namba forcing.

## 1. INTRODUCTION

Say that a tree  $T$  of height  $\lambda$  is  $\gamma$ -regressive if for all limit ordinals  $\alpha < \lambda$  with  $\text{cf}(\alpha) < \gamma$  there is a function  $f_\alpha : T_\alpha \rightarrow T_{<\alpha}$  which is regressive, i.e.  $f_\alpha(x) <_T x$  for all  $x \in T_\alpha$  and if  $x, y \in T_\alpha$  are distinct then  $f_\alpha(x)$  or  $f_\alpha(y)$  is strictly above the meet of  $x$  and  $y$ . We give a summary of the main results of this paper:

**5 Theorem.** *For all uncountable regular  $\lambda$  there is a  $\lambda$ -closed forcing  $\mathcal{K}_{\text{reg}}^\lambda$  that adds a  $\lambda$ -regressive  $\lambda$ -Kurepa-tree.*

This is contrasted in Section 4:

**7 Theorem.** *Assume that  $\kappa$  is a compact cardinal and  $\lambda \geq \kappa$  is regular. Then there are no  $\kappa$ -regressive  $\lambda$ -Kurepa-trees.*

Theorems 5 and 7 establish that compact cardinals are sensitive to  $\lambda$ -closed forcings for arbitrarily large  $\lambda$ . This should be compared with the well-known result that a supercompact cardinal  $\kappa$  can be made indestructible by  $\kappa$ -directed-closed forcings [10]. These results drive a major wedge between the notions of  $\lambda$ -closed and  $\lambda$ -directed-closed. Another contrasting known result is that a strong cardinal  $\kappa$  can be made indestructible by  $\kappa^+$ -closed forcings [3]. In Section 7 we prove

**13 Theorem.** *Under MM, there are no  $\omega_1$ -regressive  $\lambda$ -Kurepa-trees for any uncountable regular  $\lambda$ .*

This shows that MM is sensitive to  $\lambda$ -closed forcings for arbitrarily large  $\lambda$ , thus answering a question from both [7] and [8]. Note that MM is indestructible by  $\omega_2$ -directed-closed forcings [8], so again we

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find a remarkable gap between the notions of  $\omega_2$ -closed and  $\omega_2$ -directed-closed. Interestingly enough though,  $\omega_2$ -closed forcings can only violate a very small fragment of MM. To see this, let us denote by  $\Gamma_{\text{cov}}$  the class of posets that preserve stationary subsets of  $\omega_1$  and have the *covering property*, i.e. every countable set of ordinals in the extension can be covered by a countable set in the ground model. Then we have the following result from [7, p.302]:

**1 Theorem.** *The axioms PFA,  $\text{MA}(\Gamma_{\text{cov}})$  and  $\text{MA}^+(\Gamma_{\text{cov}})$  are all indestructible by  $\omega_2$ -closed forcings respectively.<sup>1</sup>*

So Theorem 5 gives

**2 Corollary.** *If  $\lambda \geq \omega_2$  is regular, then  $\text{MA}^+(\Gamma_{\text{cov}})$  is consistent with the existence of a  $\lambda$ -regressive  $\lambda$ -Kurepa-tree.*

Again, compare this with Theorem 13. It is interesting to add that  $\text{MA}^+(\Gamma_{\text{cov}})$  in particular implies the axioms  $\text{PFA}^+$  and SRP. The typical example of a forcing that preserves stationary subsets of  $\omega_1$  but does not have the covering property is Namba forcing and the proofs confirm that Namba forcing plays a crucial role in this context. It has already been established in [9] and [11] that  $\text{MA}(\Gamma_{\text{cov}})$  can be preserved in an  $(\omega_1, \infty)$ -distributive forcing extension in which the Namba-fragment of MM fails. In our case though, the failure of MM is obtained with a considerably milder forcing, i.e.  $\lambda$ -closed for arbitrarily large  $\lambda$ .

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The reader requires a strong background in set-theoretic forcing, a good prerequisite would be [4]. We give some definitions that might not be in this last reference or because we defined them in a slightly different fashion. If  $\Gamma$  is a class of posets then  $\text{MA}(\Gamma)$  denotes the statement that whenever  $\mathcal{P} \in \Gamma$  and  $D_\xi$  ( $\xi < \omega_1$ ) is a collection of dense subsets of  $\mathcal{P}$  then there exists a filter  $G$  on  $\mathcal{P}$  such that  $D_\xi \cap G \neq \emptyset$  for all  $\xi < \omega_1$ . The stronger  $\text{MA}^+(\Gamma)$  denotes the statement that whenever  $\mathcal{P} \in \Gamma$ ,  $D_\xi$  ( $\xi < \omega_1$ ) are dense subsets of  $\mathcal{P}$ , and  $\dot{S}$  is a  $\mathcal{P}$ -name such that

$$\Vdash_{\mathcal{P}} \dot{S} \text{ is stationary in } \omega_1$$

then there exists a filter  $G$  on  $\mathcal{P}$  such that  $D_\xi \cap G \neq \emptyset$  for all  $\xi < \omega_1$ , and

$$\dot{S}[G] = \{\gamma < \omega_1 : \exists q \in G (q \Vdash_{\mathcal{P}} \check{\gamma} \in \dot{S})\}$$

is stationary in  $\omega_1$ . In particular, PFA is  $\text{MA}(\text{proper})$  and MM is  $\text{MA}(\text{preserving stationary subsets of } \omega_1)$ . The interested reader is referred to [1] and [2] for the history of these *forcing axioms*.

<sup>1</sup>See below for a definition of the axioms  $\text{MA}(\Gamma)$  and  $\text{MA}^+(\Gamma)$ .

A partial order is  $\lambda$ -closed if it is closed under descending chains of length less than  $\lambda$ . It is  $\lambda$ -directed-closed if it is closed under directed subsets of size less than  $\lambda$ . [7] proves that PFA is preserved by  $\omega_2$ -closed forcings and [8] that MM is preserved by  $\omega_2$ -directed-closed forcings.

*Namba forcing* is denoted by Nm: conditions are trees  $t \subseteq \omega_2^{<\omega}$  with a *trunk*  $\text{tr}(t)$  such that  $t$  is linear below  $\text{tr}(t)$  and has splitting  $\aleph_2$  everywhere above the trunk. Smaller trees contain more information. It is known that Namba forcing preserves stationary subsets of  $\omega_1$ . If  $t \in \text{Nm}$  and  $x \in t$  then the last element of  $x$  is also called the *tag* of  $x$ , denoted as  $\text{tag}(x)$ , and we define  $\text{Suc}_t(x)$  to be the set of tags of all immediate successors of  $x$  in  $t$ . So  $\text{Suc}_t(x)$  is an unbounded subset of  $\omega_2$ . In an abuse of notation, a sequence is sometimes confused with its tag. We write  $[t]$  for the set of infinite branches through  $t$ .

## 2. STATIONARY LIMITS

For a tree  $T$  and an ordinal  $\alpha$ , let  $T_\alpha$  denote the  $\alpha$ th level of  $T$  and  $T_{<\alpha} = \bigcup_{\xi < \alpha} T_\xi$ . If  $X$  is a set of ordinals, we write  $T \upharpoonright X$  for the subtree  $\bigcup_{\xi \in X} T_\xi$ . The expression  $\text{ht}(T)$  denotes the *height* of  $T$ . We only consider trees of functions. If  $T$  is a tree and  $\mathcal{B}$  a collection of cofinal branches through  $T$  then we call  $\mathcal{B}$  *non-stationary over  $T$*  if there is a function  $f : \mathcal{B} \rightarrow T$  which is *regressive*, i.e.  $f(b) \in b$  for all  $b \in \mathcal{B}$  and if  $b, b' \in \mathcal{B}$  are distinct then  $f(b)$  or  $f(b')$  is strictly above  $b \cap b'$ . Otherwise we call  $\mathcal{B}$  *stationary over  $T$* . A tree  $T$  of height  $\kappa$  is called  $\gamma$ -*regressive* if  $T_\alpha$  is non-stationary over  $T_{<\alpha}$  for every limit ordinal  $\alpha < \kappa$  of cofinality less than  $\gamma$ . The following is easy to check:

**3 Remark.** *Assume that  $A \subseteq \alpha$  is cofinal in  $\alpha$ . Then  $T_\alpha$  is stationary over  $T_{<\alpha}$  iff  $T_\alpha$  is stationary over  $T \upharpoonright A$ .*

The  $\omega$ -cofinal limits will figure prominently when dealing with  $\omega_1$ -regressive trees, so we prove a useful Lemma about these. For simplicity we only consider trees of height  $\omega$ . The reader will notice that the following observations are applicable in Section 6. If  $T$  is of height  $\omega$  and  $\mathcal{B}$  a collection of infinite branches then for any subset  $S \subseteq T$  we let

$$\overline{S} = \{b \in \mathcal{B} : b \cap S \text{ is infinite}\}.$$

If  $S$  is countable and  $\overline{S}$  uncountable then we call  $S$  a *Cantor-subtree* of  $T$ . The class  $\mathcal{N}(T, \mathcal{B}) \subseteq [H_\theta]^{\aleph_0}$  (for some large enough regular  $\theta$ ) is defined by letting  $N \in \mathcal{N}(T, \mathcal{B})$  if and only if there is  $b \in \mathcal{B}$  such that  $b \subseteq N$  but  $b \notin N$ . We have the following

**4 Lemma.** *Assume that  $T$  has height  $\omega$  and size  $\aleph_1$  and that  $\mathcal{B}$  is a collection of infinite branches. Then the following are equivalent:*

- (1)  $\mathcal{B}$  is stationary over  $T$ .
- (2) (a) Either there is a Cantor-subtree  $S \subseteq T$  or  
 (b) if we identify  $T$  with  $\omega_1$  by any enumeration then

$$E_{\mathcal{B}} = \{\alpha < \omega_1 : \sup(b) = \alpha \text{ for some } b \in \mathcal{B}\}$$

is stationary in  $\omega_1$ .

- (3)  $\mathcal{N}(T, \mathcal{B})$  is stationary in  $[H_{\theta}]^{\aleph_0}$ .

*Proof.* The equivalence of (1) and (3) can be found in [6, p.112] and the implication (2)  $\implies$  (1) is easy.

For (3)  $\implies$  (2), assume  $\neg(2)$  and show  $\neg(3)$ : pick an enumeration  $e : \omega_1 \rightarrow T$  such that  $E_{\mathcal{B}}$  is nonstationary if we identify nodes with countable ordinals via the enumeration  $e$ . Pick a structure  $N \prec H_{\theta}$  such that  $e, T, \mathcal{B} \in N$  and set  $\gamma = N \cap \omega_1$ , so we have  $\gamma \notin E_{\mathcal{B}}$ . Let  $b \in \mathcal{B}$  be such that  $b \subseteq N$ . Then  $\sup(b) < \gamma$  holds. Now define

$$\mathcal{A} = \{c \in \mathcal{B} : \sup(c) = \sup(b)\}.$$

Note that  $\mathcal{A} \in N$  and  $\mathcal{A}$  is countable since we know by  $\neg(2)(a)$  that  $\overline{\sup(b)}$  is countable. So  $\mathcal{A} \subseteq N$ , therefore  $b \in N$ . This shows that  $N \notin \mathcal{N}(T, \mathcal{B})$  and  $\mathcal{N}(T, \mathcal{B})$  is non-stationary.  $\square$

Note that the equivalence of (1) and (3) is to some extent already in [1, p.955] but our result differs slightly from this last reference as we have a stronger notion of non-stationarity. See also [6] for variations of Lemma 4 in uncountable heights.

### 3. CREATING REGRESSIVE KUREPA-TREES

Let  $\lambda$  be a regular uncountable cardinal throughout this section. We describe the natural forcing  $\mathcal{K}_{\text{reg}}^{\lambda}$  to add a  $\lambda$ -regressive  $\lambda$ -Kurepa-tree and show that this forcing is  $\lambda$ -closed. We may assume the cardinal arithmetic  $2^{<\lambda} = \lambda$ , otherwise a preliminary Cohen-subset of  $\lambda$  could be added. Conditions of  $\mathcal{K}_{\text{reg}}^{\lambda}$  are pairs  $(T, h)$ , where

- (1)  $T$  is a tree of height  $\alpha + 1$  for some  $\alpha < \lambda$  and each level has size  $< \lambda$ .
- (2)  $T$  is  $\lambda$ -regressive, i.e. if  $\xi \leq \alpha$  then  $T_{\xi}$  is non-stationary over  $T_{<\xi}$ .
- (3)  $h : T_{\alpha} \rightarrow \lambda^{+}$  is 1-1.

The condition  $(T, h)$  is stronger than  $(S, g)$  if

- $S = T \upharpoonright \text{ht}(S)$ .
- $\text{rng}(g) \subseteq \text{rng}(h)$ .
- $g^{-1}(\nu) \leq_T h^{-1}(\nu)$  for all  $\nu \in \text{rng}(g)$ .

A generic filter  $G$  for  $\mathcal{K}_{\text{reg}}^\lambda$  will produce a  $\lambda$ -regressive  $\lambda$ -tree  $T_G$  in the first coordinate and the sets

$$b_\nu = \{x \in T_G : \text{there is } (T, h) \in G \text{ such that } h(x) = \nu\}$$

for  $\nu < \lambda^+$  form a collection of  $\lambda^+$ -many mutually different  $\lambda$ -branches through the tree  $T_G$ . Notice also that the standard arguments for  $\lambda^+$ -cc go through here as we assumed  $2^{<\lambda} = \lambda$ .

So we are done once we show that  $\mathcal{K}_{\text{reg}}^\lambda$  is  $\lambda$ -closed. To this end, let  $(T^\xi, h^\xi)$  ( $\xi < \gamma$ ) be a descending chain of conditions of length less than  $\lambda$ . We can obviously assume that  $\gamma$  is a limit ordinal. If the height of  $T^\xi$  is  $\alpha^\xi + 1$ , let  $\alpha^\gamma = \sup_{\xi < \gamma} \alpha^\xi$ . We want to extend the tree

$$T^* = \bigcup_{\xi < \gamma} T^\xi,$$

so we have to define the  $\alpha^\gamma$ th level: whenever  $\nu \in \text{rng}(h^\xi)$  for some  $\xi < \gamma$ , then there is exactly one  $\alpha^\gamma$ -branch  $c_\nu$  that has color  $\nu$  on a final segment. Now define

$$T_{\alpha^\gamma}^\gamma = \{c_\nu : \nu \in \text{rng}(h^\xi) \text{ for some } \xi < \gamma\}$$

and let  $T^\gamma$  be the tree  $T^*$  with the level  $T_{\alpha^\gamma}^\gamma$  on top. The 1-1 function  $h^\gamma : T_{\alpha^\gamma}^\gamma \rightarrow \lambda^+$  is defined by letting

$$h^\gamma(c_\nu) = \nu.$$

We claim that  $(T^\gamma, h^\gamma)$  is a condition: the only thing left to check is that  $T_{\alpha^\gamma}^\gamma$  is non-stationary over  $T^*$ . But this is witnessed by the function

$$f(c_\nu) = \text{the } <_T\text{-least } x \in c_\nu \text{ such that there is } \xi < \gamma \text{ with } h^\xi(x) = \nu.$$

Notice that  $f$  is regressive: if

$$f(c_\nu) \leq_T f(c_\mu) \leq_T c_\nu \cap c_\mu,$$

let  $\xi$  witness that  $f(c_\mu) = x$ , i.e.  $h^\xi(x) = \mu$ . Then  $h^\xi(x)$  must be color  $\nu$  as well since  $f(c_\nu) \leq_T x$  has color  $\nu$ . Thus,  $\nu = h^\xi(x) = \mu$ .

But  $(T^\gamma, h^\gamma)$  extends the chain  $(T^\xi, h^\xi)$  ( $\xi < \gamma$ ), so we just showed

**5 Theorem.**  $\mathcal{K}_{\text{reg}}^\lambda$  is a  $\lambda$ -closed forcing that adds a  $\lambda$ -regressive  $\lambda$ -Kurepa-tree.

We emphasize again that the forcing  $\mathcal{K}_{\text{reg}}^\lambda$  is *not*  $\omega_2$ -directed-closed but the reader can check that the usual forcing to add a plain  $\lambda$ -Kurepa-tree (see e.g. [4]) actually *is*  $\lambda$ -directed-closed.

#### 4. DESTROYING REGRESSIVE KUREPA-TREES ABOVE A COMPACT CARDINAL

If  $\lambda$  is a regular uncountable cardinal then a tree  $T$  is called a *weak  $\lambda$ -Kurepa-tree* if

- $T$  has height  $\lambda$ ,
- each level has size  $\leq \lambda$  and
- $T$  has  $\lambda^+$ -many cofinal branches.

**6 Lemma.** *Suppose that  $\lambda$  is a regular uncountable cardinal and there is an elementary embedding  $j : V \longrightarrow M$  such that  $\eta = \sup(j''\lambda) < j(\lambda)$  and  $\text{cf}^M(\eta) < j(\kappa)$ . Then there are no  $\kappa$ -regressive weak  $\lambda$ -Kurepa-trees.*

*Proof.* Suppose that  $T$  is a  $\kappa$ -regressive weak  $\lambda$ -Kurepa-tree and  $j$  as above. Then there is a regressive function  $f_\eta$  defined on the level  $(jT)_\eta$ . If  $b$  is a cofinal branch through  $T$ , then we find  $\alpha_b < \lambda$  such that

$$f_\eta(jb \upharpoonright \eta) \leq_{jT} jb \upharpoonright j(\alpha_b) = j(b \upharpoonright \alpha_b).$$

Note that if  $b$  and  $b'$  are two distinct branches through  $T$  then  $jb$  and  $jb'$  must disagree below  $\eta$ . Moreover,  $j(b \upharpoonright \alpha_b) \neq j(b' \upharpoonright \alpha_{b'})$  holds because  $f_\eta$  is regressive. Then the assignment  $b \longmapsto b \upharpoonright \alpha_b$  must be 1-1, which is a contradiction to the fact that  $T$  has  $\lambda^+$ -many branches.  $\square$

Recall that a cardinal  $\kappa$  is  *$\lambda$ -compact* if there is a fine ultrafilter on  $\mathcal{P}_\kappa\lambda$ . If  $\lambda$  is regular, the elementary embedding  $j : V \longrightarrow M$  with respect to such a fine ultrafilter has the following properties:

- the critical point of  $j$  is  $\kappa$ ,
- there is a discontinuity at  $\lambda$ , i.e.  $\eta = \sup(j''\lambda) < j(\lambda)$  and
- $\text{cf}^M(\eta) < j(\kappa)$ .

(see [5, §22] for more details). A cardinal  $\kappa$  is said to be *compact* if it is  $\lambda$ -compact for all  $\lambda$ , so it follows from Lemma 6 and the above definition:

**7 Theorem.** *Assume that  $\kappa$  is a compact cardinal and  $\lambda \geq \kappa$  is regular. Then there are no  $\kappa$ -regressive weak  $\lambda$ -Kurepa-trees.*

Using Theorem 5, we have

**8 Corollary.** *Compact cardinals are sensitive to  $\lambda$ -closed forcings for arbitrarily large  $\lambda$ .*

It was known before that adding a slim<sup>2</sup>  $\kappa$ -Kurepa-tree destroys the ineffability of  $\kappa$  and that slim  $\kappa$ -Kurepa-trees can be added with  $\kappa$ -closed forcing. But note that our notion of regressive is more universal: slim Kurepa-trees can exist above compact or even supercompact cardinals.

## 5. OSCILLATING BRANCHES

Now assume that  $T$  is an  $\omega_2$ -tree: we enumerate each level by letting

$$(5.1) \quad T_\alpha = \{\tau(\alpha, \xi) : \xi < \omega_1\} \text{ for all } \alpha < \omega_2.$$

In this situation we identify branches with functions from  $\omega_2$  to  $\omega_1$  that are induced by the enumerations of the levels. If  $A \subseteq \omega_2$  is unbounded and  $b : \omega_2 \rightarrow \omega_1$  is an  $\omega_2$ -branch through  $T$  then we say that  $b$  *oscillates on*  $A$  if for all  $\alpha < \omega_2$  and all  $\zeta < \omega_1$  there is  $\beta > \alpha$  in  $A$  and  $\xi > \zeta$  such that  $b(\beta) = \xi$ .

**9 Lemma.** *Assume that  $T$  is an  $\omega_2$ -Kurepa-tree with an enumeration  $\tau(\alpha, \xi)$  ( $\alpha < \omega_2, \xi < \omega_1$ ) as in (5.1) and  $A_\iota$  ( $\iota < \omega_2$ ) are  $\aleph_2$ -many unbounded subsets of  $\omega_2$ . Then there is an  $\omega_2$ -branch  $b$  through  $T$  that oscillates on every  $A_\iota$  ( $\iota < \omega_2$ ).*

*Proof.* Assume not, then for every  $\omega_2$ -branch  $b$  there is  $\iota_b < \omega_2$  and there are  $\alpha_b < \omega_2, \zeta_b < \omega_1$  such that

$$b \upharpoonright (A_{\iota_b} \setminus \alpha_b) \subseteq \{\tau(\alpha, \xi) : \alpha \in A_{\iota_b} \setminus \alpha_b, \xi < \zeta_b\}.$$

By a cardinality argument we can find  $\aleph_3$ -many branches  $b$  such that  $\iota_0 = \iota_b, \alpha_0 = \alpha_b$  and  $\zeta_0 = \zeta_b$ . But then each of these branches is a different branch through the tree

$$T_0 = \{\tau(\alpha, \xi) : \alpha \in A_{\iota_0} \setminus \alpha_0, \xi < \zeta_0\}.$$

$T_0$  has countable levels but  $\aleph_3$ -many branches, a contradiction.  $\square$

## 6. DESTROYING REGRESSIVE KUREPA-TREES WITH MM

We introduce a simplified notation for the following arguments: if  $f : t \rightarrow \omega_1$  for some  $t \in \text{Nm}$  and  $\pi \in [t]$  then we let

$$\sup^{(f)}(\pi) = \sup_{n < \omega} f(\pi \upharpoonright n).$$

If  $b : \omega_2 \rightarrow \omega_1$  is an  $\omega_2$ -branch and  $x \in \omega_2^{<\omega}$  then  $b(x)$  really denotes the countable ordinal  $b(\text{tag}(x))$ .

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<sup>2</sup>A  $\kappa$ -Kurepa-tree  $T$  is called *slim* if  $|T_\alpha| \leq |\alpha|$  for all  $\alpha < \kappa$ .



**10 Lemma.** *Assume that  $T$  is an  $\omega_2$ -Kurepa-tree and  $\mathcal{B}$  is the set of branches. Let  $\tau(\alpha, \xi)$  ( $\alpha < \omega_2, \xi < \omega_1$ ) be an enumeration as in (5.1). Then in the Namba extension  $V^{\text{Nm}}$  there is a sequence*

$$\Delta_G = \langle \delta_n^G : n < \omega \rangle$$

*cofinal in  $\omega_2^V$  such that*

$$\dot{E}_{\mathcal{B}} = \{\text{sup}^{(b)}(\Delta_G) : b \in \mathcal{B}\}$$

*is stationary relative to every stationary  $S \subseteq \omega_1$  in  $V$ , i.e.  $\dot{E}_{\mathcal{B}} \cap S$  is stationary for all stationary  $S \subseteq \omega_1$  in the ground model.*

*Proof.* Assume that  $\dot{C}$  is an Nm-name for a club in  $\omega_1$ ,  $S \subseteq \omega_1$  is a stationary set in  $V$  and  $t_0$  a condition in Nm. Our goal is to find a condition  $t_3 \leq t_0$  and an ordinal  $\xi_0 \in S$  such that  $t_3 \Vdash \xi_0 \in \dot{C} \cap \dot{E}_{\mathcal{B}}$ . By a fusion argument similar to the ones in [11, p.188], we construct a condition  $t_1 \leq t_0$  and a coloring  $f : t_1 \rightarrow \omega_1$  such that

- (1)  $f$  increases on chains, i.e. if  $v \subsetneq x$  are elements of  $t_1$  then  $f(v) < f(x)$ .
- (2) if the height of  $x$  in  $t_1$  is odd and  $x$  is above the trunk then there is  $\zeta < \omega_1$  such that

$$|\{x \hat{\ } \beta \in t_1 \mid f(x \hat{\ } \beta) = \xi\}| = \aleph_2 \text{ for all } \xi > \zeta,$$

i.e. each ordinal in a final segment of  $\omega_1$  has  $\aleph_2$ -many preimages in the set  $\text{Suc}_{t_1}(x)$ .

- (3) if  $G \subseteq \text{Nm}$  is generic with  $t_1 \in G$  and  $\pi : \omega \rightarrow \omega_2^V$  is the corresponding Namba-sequence then  $\text{sup}^{(f)}(\pi) \in \dot{C}[G]$ .

Given the condition  $t_1$ , we apply Lemma 9 to find a branch  $b$  that oscillates on all sets  $\text{Suc}_{t_1}(x)$  ( $x \in t_1$ ). Using (1) and (2), we thin out again to get a condition  $t_2 \leq t_1$  with the following property:

- (4) if  $v \subsetneq x \subsetneq y$  is a chain in  $t_2$  above the trunk and the height of  $x$  is odd then  $f(v) < b(x) < f(y)$ .

Note that in particular (2) can be preserved by passing to the condition  $t_2$ , so we may assume that  $t_2$  has properties (1)-(4). Let us also assume for notational simplicity that the height of  $\text{tr}(t_2)$  is even. The next step is to find  $t_3 \leq t_2$  and  $\xi_0 \in S$  such that

- (5)  $\text{sup}^{(f)}(\pi) = \xi_0$  for all branches  $\pi$  in  $[t_3]$ .

To find  $t_3$  and  $\xi_0$ , we define a game  $\mathbb{G}(\gamma)$  for every limit  $\gamma < \omega_1$ . Fix a ladder sequence  $l(\gamma) = (\gamma_n : n < \omega)$  for each such  $\gamma$ . The game  $\mathbb{G}(\gamma)$  is played as follows:

I	$\alpha_0$	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\dots$
II	$\beta_0$	$\beta_1$	$\beta_2$	$\beta_3$	$\dots$

where for all  $n < \omega$

- $\alpha_n < \beta_n < \omega_2$ ,
- $s_n = \text{tr}(t_2) \frown (\beta_i : i \leq n) \in t_2$  and
- $f(s_n) \in (\gamma_n, \gamma)$  whenever  $n$  is even.

II wins if he can make legal moves at each step, so the game is determined.

**10.1 Claim.** *II wins  $\mathbb{G}(\gamma)$  for club many  $\gamma$ 's.*

*Proof.* Assume not, then there is a stationary  $U \subseteq \omega_1$  such that player I wins  $\mathbb{G}(\gamma)$  for each  $\gamma \in U$  via the strategy  $\sigma_\gamma$ . Now pick a countable elementary  $N$  such that  $\xi = N \cap \omega_1 \in U$  and  $t_2, f, l, U \in N$ .

A ladder sequence  $l(\xi) = (\xi_n : n < \omega)$  converging to  $\xi$  is given and we define a sequence  $(\beta_n : n < \omega)$  inductively as follows: let  $\beta_n$  be the least

$$\beta > \sup_{\gamma \in U} \sigma_\gamma(\beta_i : i < n)$$

such that

- $s = \text{tr}(t_2) \frown (\beta_i : i < n) \frown \beta \in t_2$  and
- $f(s) \in (\xi_n, \xi)$  whenever  $n$  is even.

Such a  $\beta$  exists in  $N$  by (2) and elementarity. Note that  $(\beta_n : n < \omega)$  is a possible record of moves for player II if player I goes along with the strategy  $\sigma_\xi$ . But II obviously wins the game  $\mathbb{G}(\xi)$  if the sequence  $(\beta_n : n < \omega)$  is played, a contradiction. This proves the claim.  $\square$

Given the claim, pick  $\xi_0 \in S$  above all  $b(x)$  ( $x \subseteq \text{tr}(t_2)$ ) such that II wins the game  $\mathbb{G}(\xi_0)$ . Now we can easily find a condition  $t_3 \leq t_2$  with property (5).

If we fix a generic  $G \subseteq \text{Nm}$  with  $t_3 \in G$  and let  $\pi_G : \omega \longrightarrow \omega_2^V$  be the corresponding Namba-sequence, we can define  $\delta_n^G = \pi_G(2n + 1)$  and  $\Delta_G = \langle \delta_n^G : n < \omega \rangle$ . Then we have

- (6)  $\text{sup}^{(f)}(\Delta_G) \in \dot{C}[G]$  by (3),
- (7)  $\text{sup}^{(f)}(\Delta_G) = \xi_0$  by (5) and
- (8)  $\text{sup}^{(f)}(\Delta_G) = \text{sup}_{n < \omega} b(\delta_n^G) = \text{sup}^{(b)}(\Delta_G)$  by (4).

But this finishes the proof since

$$\xi_0 \in \dot{C}[G] \cap \dot{E}_{\mathcal{B}}[G] \cap S.$$

$\square$

**11 Corollary.** *Assume that  $T$  is an  $\omega_2$ -Kurepa-tree and  $\mathcal{B}$  the set of ground model branches through  $T$ . Then  $\mathcal{B}$  is stationary over  $T$  in the Namba extension.*

Finally we get the main result for  $\omega_2$ . We will prove a more general version of this in Theorem 13.

**12 Theorem.** *There are no  $\omega_1$ -regressive  $\omega_2$ -Kurepa-trees under MM.*

*Proof.* Assume that  $T$  is an  $\omega_1$ -regressive  $\omega_2$ -Kurepa-tree and that

$$\tau(\alpha, \xi) \ (\alpha < \omega_2, \xi < \omega_1)$$

is an enumeration as in (5.1). Look at the iteration  $\mathbb{P} = \text{Nm} * \text{CS}(\dot{E}_{\mathcal{B}})$ , where  $\text{CS}(\dot{E}_{\mathcal{B}})$  shoots a club through the set  $\dot{E}_{\mathcal{B}}$  from the statement of Lemma 10. The poset  $\mathbb{P}$  preserves stationary subsets of  $\omega_1$  by the fact that  $\dot{E}_{\mathcal{B}}$  is stationary relative to every stationary set in  $V$ . But we have that  $\dot{E}_{\mathcal{B}}$  is club in  $V^{\mathbb{P}}$ , so we can use MM to get a sequence  $\Delta = \langle \delta_n : n < \omega \rangle$  converging to  $\delta < \omega_2$  such that

$$\{\text{sup}^{(b)}(\Delta) : b \text{ is a } \delta\text{-sequence in } T_\delta\}$$

is club in  $\omega_1$ . Using Lemma 4, we see that  $T_\delta$  is definitely stationary over  $T \upharpoonright \Delta$ . So  $T_\delta$  is stationary over  $T_{<\delta}$  by Remark 3. Since  $\text{cf}(\delta) = \omega$ , this contradicts the fact that  $T$  is  $\omega_1$ -regressive.  $\square$

## 7. LARGER HEIGHTS

Starting from Theorem 12, we generalize the result to weak Kurepa-trees in all uncountable regular heights.

**13 Theorem.** *Under MM, there are no  $\omega_1$ -regressive weak  $\lambda$ -Kurepa-trees for any uncountable regular  $\lambda$ .*

*Proof.* Since PFA destroys weak  $\omega_1$ -Kurepa-trees (see [1]), we may assume that  $\lambda$  is at least  $\omega_2$ . Now assume that  $T$  is an  $\omega_1$ -regressive weak  $\lambda$ -Kurepa-tree and let  $\mathcal{P} = \text{Col}(\omega_2, \lambda)$  be the usual  $\omega_2$ -directed collapse. Note that  $\mathcal{P}$  has the  $\lambda^+$ -cc, because  $\lambda^{\omega_1} = \lambda$  holds under MM (see [2]). So the tree  $T$  has a cofinal subtree  $T^*$  in  $V^{\mathcal{P}}$  that is an  $\omega_1$ -regressive weak  $\omega_2$ -Kurepa-tree. By throwing away some nodes if necessary, we may assume that  $T^*$  has the property that

$$(7.1) \quad T_x^* = \{y \in T^* : x \leq_T y\} \text{ has } \aleph_3\text{-many branches for all } x \in T^*.$$

Now we define an  $\omega_2$ -directed forcing  $\mathcal{Q}$  in  $V^{\mathcal{P}}$  that shoots an actual  $\omega_2$ -Kurepa-subtree through the tree  $T^*$ : conditions of  $\mathcal{Q}$  are pairs of the form  $(S, B)$ , where

- (1)  $S$  is a downward-closed subtree of  $T^*$  of height  $\alpha + 1$  for some ordinal  $\alpha < \omega_2$ ,
- (2)  $|S| \leq \omega_1$ ,
- (3)  $B$  is a nonempty set of branches cofinal in  $T^*$  and  $|B| \leq \aleph_1$ ,
- (4)  $b \upharpoonright (\alpha + 1) \subseteq S$  for all  $b \in B$ .

We let  $(S_0, B_0) \geq_{\mathcal{Q}} (S_1, B_1)$  if  $S_0 = S_1 \upharpoonright \text{ht}(S_0)$  and  $B_0 \subseteq B_1$ .

If  $X \subseteq \mathcal{Q}$  is a set of mutually compatible conditions of size  $\leq \aleph_1$  then we let  $S_X$  and  $B_X$  be the unions over the first respectively second coordinates of  $X$ . Now  $S_X$  can be end-extended to a tree  $\bar{S}_X$  of successor height by extending at least the cofinal branches in the non-empty set  $B_X$ . But then  $(\bar{S}_X, B_X)$  is a condition stronger than every condition in  $X$ , hence  $\mathcal{Q}$  is  $\omega_2$ -directed-closed. An easy cardinality argument shows that  $\mathcal{Q}$  has the  $\aleph_3$ -cc because  $2^{\aleph_1} = \aleph_2$  holds in  $V^{\mathcal{P}}$ . It is now straightforward that a generic filter  $H \subseteq \mathcal{Q}$  will produce an  $\omega_2$ -tree in the first coordinate which is  $\omega_1$ -regressive since it is a subtree of the original tree  $T$  and notice that  $\mathcal{P}$  and  $\mathcal{Q}$  both preserve uncountable cofinalities. On the other hand, a density argument using (7.1) shows that the set

$$\mathfrak{B} = \bigcup \{B : \text{there is } S \text{ such that } (S, B) \in H\}$$

has cardinality  $\aleph_3$ , so  $H$  induces an  $\omega_1$ -regressive  $\omega_2$ -Kurepa-tree. The composition of two  $\omega_2$ -directed-closed forcings is again  $\omega_2$ -directed-closed and it was mentioned in the introduction that  $\omega_2$ -directed-closed forcings preserve MM, so we have the situation:

- $V^{\mathcal{P}*\mathcal{Q}} \models \text{MM}$
- $V^{\mathcal{P}*\mathcal{Q}} \models \text{“there is an } \omega_1\text{-regressive } \omega_2\text{-Kurepa-tree.”}$

But this contradicts Theorem 12. □

Using Theorem 5, we have

**14 Corollary.** *MM is sensitive to  $\lambda$ -closed forcing algebras for arbitrarily large  $\lambda$ .*

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