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VORTICES AND MAGNETIZATION IN KAC'S MODEL

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Abstract. We consider a 2-dimensional planar rotator on a large, but finite lattice with a ferromagnetic Kac potential $J_\gamma(i) = \gamma^2 J(\gamma i)$, J with compact support. The system is subject to boundary conditions with vorticity. Using a Glauber like dynamics, we compute minimizers of the free energy functional at low temperature, i.e. in the regime of phase transition. We have the numerical evidence of a vortex structure for minimizers, which present many common features with those of the Ginzburg-Landau functional.

0. Introduction.

Vector spin model with an internal continuous symmetry group, such classical $O^+(q)$ models (XY or “planar rotator” for $q = 2$, and Heisenberg model for $q = 3$,) play an important rôle in Statistical Physics. In one or two dimensions, if the range of the interaction decays at infinity fast enough, there is no breaking of the internal symmetry, and for the planar rotator with short range interactions, uniqueness of the Gibbs state holds in 1 or 2 space dimensions. Despite of this, a particular form for phase transition exists, which can be characterized by the change of behavior in the correlation functions. In the low temperature phase they have power law decay, showing that the system is in a long range order state (exhibiting in particular the so-called “spin waves”,) but they decay exponentially fast at high temperatures, breaking the long range order, even though thermodynamic quantities remain smooth across the transition. For the XY system, these transitions were described by Kosterlitz and Thouless in term of topological excitations called vortices : while these vortices are organized into dipoles at low temperature, a disordered state emerges at the transition, and correlation functions give information about dipole unbinding. But the observation of the spatial distribution of defects shows that it is not uniform ; rather, defects tend to cluster at temperatures slightly larger than the transition temperature, and there are still large ordered domains where the spins are almost parallel (see e.g. [LeVeRu], and references therein)

Here we consider a Kac version of the classical XY model on a “large” lattice $\Lambda \subset \mathbf{Z}^2$. The hamiltonian (except for the interaction with the boundary) is of the form

$$H_\gamma(\sigma_\Lambda) = -\frac{1}{2} \sum_{i,j \in \Lambda} \gamma^2 J(\gamma(i-j)) \langle \sigma_\Lambda(i), \sigma_\Lambda(j) \rangle$$

where γ is a small coupling constant and J denotes a cutoff function. Kac models are finite but long range, so that they share some features with the mean field model, and still exhibit

better mechanisms of phase transitions, which depend in particular on the dimension, as for the short range case. For the mean field model with $O^+(q)$ symmetry, we know that there is no phase transition for inverse temperature $\beta \leq 2$ (Gibbs measure is supported at the absolute minimum of the free energy functional,) while there is a phase transition for $\beta > 2$, with internal symmetry group $O^+(q)$.

When the model possesses internal symmetry and common features with the mean field, it is hard to expect vortices at low temperature, unless the symmetry is somehow broken, for instance if the system is subject to boundary conditions. This situation is met in other domains of condensed matter Physics, as in superconductivity, where vorticity is created by an exterior magnetic flux, or for superfluid. In that case, phase transitions of matter are well described by critical points of free energy (Ginzburg-Landau) functionals ([BeBrHe], [OvSi], etc...)

The free energy (or excess free energy) functional $F_{\beta,\gamma}(m_{\Lambda^*})$ at inverse temperature β in case of Kac models with internal symmetry, can be simply derived from a suitable renormalization of H_γ making use of the entropy for the free field (see Sect.1) In particular, we have replaced the spins σ_Λ by the magnetization m_{Λ^*} on another “coarser” or “mesoscopic” lattice Λ^* . To understand the significance of $F_{\beta,\gamma}$, one should think also of the formal “stationary phase” argument, as $\Lambda \rightarrow \infty$, which suggests that an important role in the averaging with respect to Gibbs measure, is played by configurations close to those which produce the local critical points of $F_{\beta,\gamma}$. This occurs in computing correlations functions (see e.g. [Z].) These critical points consist in ground states, or metastable states.

They will be determined as the attractors of a certain dynamics, similar to this given by the “heat operator”, but known in that context as the Glauber dynamics [DeMOrPrTr], [DeM], [Pr] Thus, we expect convergence of this dynamics toward a Gibbsian equilibrium, though this will not be formally established here.

Our main observation is the existence of vortices below the temperature of transition of phase for the free field model, induced by the vorticity at the boundary of the lattice Λ , together with large ordered domains where the magnetizations m_{Λ^*} become parallel.

We also have some numerical evidence that, as in the case of Ginzburg-Landau functional, Kirchoff-Onsager hamiltonian for the system of vortices gives a fairly good approximation of the minimizing free energy, despite of the non-local interactions.

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1. Description of Kac’s Hamiltonian.

Consider the lattice \mathbf{Z}^2 , consisting in a bounded, connected domain Λ (the interior

region), and its complement (the exterior region) Λ^c . In practice, we think of Λ as a large rectangle with sides parallel to the axis of \mathbf{Z}^2 , of length of the form $L = 2^n$, $n \in \mathbf{N}$. Physical objects make sense in the thermodynamical limit $\Lambda \rightarrow \mathbf{Z}^2$, but in this paper we work in large, but finite domains.

To each site $i \in \mathbf{Z}^2$ is attached a classical spin variable $\sigma_i \in \mathbf{S}^{q-1}$, $q = 2, 3$. The configuration space $\mathcal{X}(\mathbf{Z}^2) = (\mathbf{S}^{q-1})^{\mathbf{Z}^2}$ is the set of all such classical states of spin ; it has the natural internal symmetry group $O^+(q)$ acting on \mathbf{S}^{q-1} . The state $\sigma \in \mathcal{X}(\mathbf{Z}^2)$ will denote the map $\sigma : \mathbf{Z}^2 \rightarrow \mathbf{S}^{q-1}$, $i \mapsto \sigma(i)$. Given the partition $\mathbf{Z}^2 = \Lambda \cup \Lambda^c$, we define by restriction the interior and exterior configuration spaces $\mathcal{X}(\Lambda)$ and $\mathcal{X}(\Lambda^c)$, and the restricted configurations by σ_Λ and σ_{Λ^c} . The Hamiltonian in \mathbf{Z}^2 describes the interaction between different sites through Kac's potential defined as follows.

Let $0 \leq J \leq 1$ be a function on \mathbf{R}^2 with compact support and normalized by $\int_{\mathbf{R}^2} J = 1$. We can think of J also as a function on the lattice. There is a lot of freedom concerning the choice of J , but for numerical purposes, we take J as 1/2 the indicator function \tilde{J} of the unit rhombus with center at the origin, in other words $J(x) = J(|x|_1)$ where $|\cdot|_1$ is the ℓ^1 norm in \mathbf{R}^2 . Thus the support of \tilde{J} is thought of as a chip of unit area, and considered as a function on the lattice, \tilde{J} takes the value 1 at the center, and 1/4 at each vertex, so that $\sum_{i \in \Lambda} J(i) = 1$. For γ of the form 2^{-m} , we set $J_\gamma(x) = \gamma^2 J(\gamma x)$, and extend the definition above in the discrete case so that J_γ enjoys good scaling properties, namely the stratum of full dimension (i.e. the set of points interior to the chip) has weight 1, the strata of dimension 1 (the points on the sides on the chip) have weight 1/2, and those of dimension 0 (the vertices of the chip) have weight 1/4. Thus, again $\sum_{i \in \mathbf{Z}^2} J_\gamma(i) = 1$. The discrete convolution on Λ is defined as usual. For instance, $(J_\gamma * \sigma)(i) = \sum_{j \in \mathbf{Z}^2} J_\gamma(i-j)\sigma(j)$ represents, with conventions as above, the mean value of σ over the chip of size γ^{-1} and center i , with a weight that depends on the stratum containing j .

Note that we could replace the lattice \mathbf{Z}^2 by the torus $(\mathbf{Z}/L\mathbf{Z})^2$ or the cylinder $(\mathbf{Z}/L\mathbf{Z}) \times \mathbf{Z}$, which amounts to specify periodic boundary conditions in one or both directions. Thermodynamic limit is obtained as $L \rightarrow \infty$.

The coupling between spin at site i and spin at site j is given by $J_\gamma(i-j)$; this is known as Kac's potential. From Statistical Physics point of view, Kac's potential, for small γ , shares locally the main properties of the mean field, i.e. long range $\approx \gamma^{-1}$, large connectivity $\approx \gamma^{-2}$ of each site, small coupling constant $\approx \gamma^2$ of the bonds, and total strength of each site equal to 1.

Given the exterior configuration $\sigma_{\Lambda^c} \in \mathcal{X}(\Lambda^c)$, we define the Hamiltonian on \mathbf{Z}^2 as

$$(1.1) \quad H_\gamma(\sigma_\Lambda | \sigma_{\Lambda^c}) = -\frac{1}{2} \sum_{i,j \in \Lambda} J_\gamma(i-j) \langle \sigma(i), \sigma(j) \rangle - \sum_{(i,j) \in \Lambda \times \Lambda^c} J_\gamma(i-j) \langle \sigma(i), \sigma(j) \rangle$$

where $\sigma(i)$, for simplicity, stands for $\sigma_\Lambda(i)$ or $\sigma_{\Lambda^c}(i)$, and $\langle \cdot, \cdot \rangle$ is the standard scalar product in \mathbf{R}^q . We note that as $J \geq 0$, the interaction is ferromagnetic, i.e. energy decreases as spins align.

We give here some heuristic derivation of the model we will consider, starting from principles of Statistical Physics. A thermodynamical system at equilibrium is described by Gibbs measure at inverse temperature β . We assume an a priori probability distribution ν for the states of spin, and because of the internal continuous symmetry of $\mathcal{X}(\Lambda)$, we take ν as the normalized surface measure on \mathbf{S}^{q-1} , i.e. $\nu(d\sigma_i) = \omega_q^{-1} \delta(|\sigma_i| - 1) d\sigma_i$, where ω_q is the volume of \mathbf{S}^{q-1} . Then Gibbs measure on $\mathcal{X}(\Lambda)$ with prescribed boundary condition σ_{Λ^c} is given by

$$\mu_{\beta,\gamma}(d\sigma_\Lambda | \sigma_{\Lambda^c}) = \frac{1}{Z_{\beta,\gamma}^\Lambda(\sigma_{\Lambda^c})} \exp[-\beta H_\gamma(\sigma_\Lambda | \sigma_{\Lambda^c})] \prod_{i \in \Lambda} \nu(d\sigma_\Lambda(i))$$

where $Z_{\beta,\gamma}^\Lambda(\sigma_{\Lambda^c})$, the partition function, is a normalization factor which makes of $\mu_{\beta,\gamma}$ a probability measure on $\mathcal{X}(\Lambda)$, conditioned by $\sigma_{\Lambda^c} \in \mathcal{X}(\Lambda^c)$. It is obtained by integration of $\mu_{\beta,\gamma}(d\sigma_\Lambda | \sigma_{\Lambda^c})$ over \mathbf{S}^{q-1} .

Since we are working on \mathbf{Z}^2 , there exists, for any $\beta > 0, \gamma > 0$, an infinite volume Gibbs state $\mu_{\beta,\gamma}$, i.e. a (unique) probability distribution $\mu_{\beta,\gamma}$ on the space \mathcal{X} of all configurations obtained by taking the thermodynamic limit $\Lambda \rightarrow \mathbf{Z}^2$. This measure satisfies suitable coherence conditions, i.e. DRL equations.

Nevertheless, we are faced with various difficulties, indicating that the microcanonical ensemble is not a suitable frame in this context ; it is known that to understand thermodynamical properties for Kac's model, one should instead average spins over mesoscopic regions and consider the image of Gibbs measure through this transformation, the so called "block-spin transformation". Since the model shares some features with the mean field, the basic idea is to approximate the local entropy density of the system i.e., the entropy in some intermediate boxes $\tilde{\Lambda}(x) \subset \Lambda$ or, roughly, the log of the number of configurations $\sigma_{\Lambda_{\text{int}}} \in \mathcal{X}(\Lambda_{\text{int}})$ contributing to the same energy of Hamiltonian H_γ restricted to $\tilde{\Lambda}(x)$, by this that would be given by the mean field theory.

Here $\tilde{\Lambda}(x)$ will be typically a square "centered" at a variable $x \in \mathbf{Z}^2$, with sides of length $\frac{\delta}{\gamma}$, δ of the form 2^{-p} , $p \in \mathbf{N}$, $\frac{\delta}{\gamma}$ typically much smaller than the diameter of Λ , but still containing many sites.

More generally we introduce the empirical magnetization in the finite box $\Delta \subset \mathbf{Z}^2$

$$(1.2) \quad m_\Delta(\sigma) = \frac{1}{|\Delta|} \sum_{i \in \Delta} \sigma(i)$$

and for any $m \in \mathbf{R}^q$, $|m| \leq 1$, we define the canonical partition function

$$(1.3) \quad Z_{\beta, \gamma}^{\Delta, \sigma_{\Delta^c}}(m) = \int_{\mathbf{S}^{q-1}} \exp[-\beta H_\gamma(\sigma_\Delta | \sigma_{\Delta^c})] \prod_{i \in \Delta} \nu(d\sigma(i)) \delta(m_\Delta(\sigma) - m)$$

Then it is shown, taking the thermodynamical limit $\Delta \rightarrow \mathbf{Z}^2$, that the quantity

$$(1.4) \quad F_\gamma(\beta, m) = \lim_{\Delta \rightarrow \mathbf{Z}^2} -\frac{1}{\beta|\Delta|} \log Z_{\beta, \gamma}^{\Delta, \sigma_{\Delta^c}}(m)$$

is well defined, and doesn't depend on the boundary condition on Δ^c ; it will be interpreted as the Gibbs free energy of the mesoscopic system. So far, parameter γ was kept small but constant; the limit $\gamma \rightarrow 0$ is called Lebowitz-Penrose limit. Lebowitz-Penrose theorem (in this simplified context) states that

$$(1.5) \quad \lim_{\gamma \rightarrow 0} F_\gamma(\beta, m) = \text{CE}(f_\beta(m))$$

where $f_\beta(m) = -\frac{1}{2}|m|^2 + \frac{1}{\beta}I(m)$ is the free energy for the mean field, ($I(m)$ denotes the entropy) and will be discussed below. See [BuPi] for the case of a 1-d lattice, the proof can be carried over to \mathbf{Z}^2 . CE denotes the convex envelope, to account for Maxwell correction law.

From this we sketch the renormalization procedure that leads to Lebowitz-Penrose theorem, as stated e.g. in [Pr, Thm. 3.2.1] (actually, this is the ‘‘pressure’’ version of Lebowitz-Penrose theorem, but the argument can easily be adapted to free energy.) We set $\Delta = \tilde{\Lambda}(x)$, and $m_\sigma(x) = m_{\tilde{\Lambda}(x)}(\sigma)$. The set of all such magnetizations $m_\sigma \in \mathbf{R}^q$ is the image of $\mathcal{X}(\mathbf{Z}^2)$ by the block-spin transformation (1.2), and will be denoted by $\tilde{\mathcal{X}}(\mathbf{Z}^2)$. It has again the continuous symmetry group $O^+(q)$, and this is a subset of the convex set \mathcal{M} of all functions $m : \mathbf{Z}^2 \rightarrow \mathbf{R}^q$ such that $|m(x)| \leq 1$ for all x . When considering microcanonical interior and exterior regions as above, the partition $\mathbf{Z}^2 = \Lambda \cup \Lambda^c$ induces of course restricted configuration spaces $\tilde{\mathcal{X}}(\Lambda^*)$ and $\tilde{\mathcal{X}}(\Lambda^{*c})$, where $\Lambda^* = \{x \in \mathbf{Z}^2 : \tilde{\Lambda}(x) \subset \Lambda\}$ and $\Lambda^{*c} = \{x \in \mathbf{Z}^2 : \tilde{\Lambda}(x) \subset \Lambda^c\}$. So let $m \in \mathcal{M}$. Formula (1.3) extends in this context to define the partition function

$$(1.6) \quad Z_{\beta, \gamma}(\{m_\sigma = m\}; \sigma_{\Lambda^c}) = \int_{\mathbf{S}^{q-1}} \exp[-\beta H_\gamma(\sigma_\Lambda | \sigma_{\Lambda^c})] \prod_{i \in \tilde{\Lambda}(\cdot)} \nu(d\sigma(i)) \delta(m_\sigma - m)$$

constrained to the configurations σ_Λ whose block spin are equal to m . Assuming that the diameter of all block spins equals $\gamma^{-1/2}$, one can show that $Z_{\beta,\gamma}(\{m_\sigma = m\}; \sigma_{\Lambda^c})$ is well approximated by a continuous free energy functional of the form

$$(1.7) \quad F_{\beta,\gamma}(m_{\Lambda^*} | m_{\Lambda^{*c}}) = \int_{\Lambda^*} \left[-\frac{1}{\beta} I((m_{\Lambda^*}(r))) + U_\gamma(m_{\Lambda^*} | m_{\Lambda^{*c}}) \right] dr$$

with the entropy function I as in (1.5) and $U_\gamma(m_{\Lambda^*} | m_{\Lambda^{*c}})$ is a suitable energy term. This is essentially the content of [Pr,Thm. 3.2.3], and one of the main ingredients for the proof of [Pr,Thm. 3.2.1], the other one being the computation of the limit of U_γ , as $\gamma \rightarrow 0$.

Having this construction in mind, we shall proceed the other way around, and make a simple renormalization of H_γ (see Proposition 1.1 below). Actually our sole purpose is to give a discrete analogue for the excess free energy functional as in (1.7), most adapted to numerical experiments on the lattice.

2. Mean field approximation and renormalized hamiltonian.

The free energy for the mean field given in (1.5) is given by

$$(1.9) \quad f_\beta(m) = -\frac{1}{2}|m|^2 + \frac{1}{\beta}I(m)$$

where $I(m)$ is the entropy function of the a priori measure ν , which can be computed following [BuPi]. We introduce the moment generating function

$$\phi(h) = \int_{S^{q-1}} e^{\langle h, \sigma \rangle} d\nu(\sigma)$$

and define $I(m)$ as Legendre transformation

$$(1.10) \quad I(m) = \widehat{I}(|m|) = \sup_{h \in \mathbf{R}^q} (\langle h, m \rangle - \log \phi(h))$$

For $q = 2$, we have $\phi(h) = \widehat{\phi}(|h|) = J_0(i|h|)$ (Bessel function of order 0.) Function $\rho \mapsto \widehat{I}(\rho)$ is convex, strictly increasing on $[0, 1]$, $\widehat{I}(\rho) \sim \rho^2$ as $\rho \rightarrow 0$, $\widehat{I}(\rho) \sim -\frac{1}{2} \log(1 - \rho)$, as $\rho \rightarrow 1$, and these relations can be differentiated. We have also $\widehat{I}' = ((\log \widehat{\phi})')^{-1}$ and $(\log \widehat{\phi})'(t) = -iJ_1(it)/J_0(it)$, this is of course a real valued function. The phase transition of mean field type is given by the critical point of the free energy f_β , i.e. the positive root of equation $\beta m_\beta = \widehat{I}'(m_\beta)$, which exists iff $\beta > \widehat{I}''(0) = 2$. So the critical manifold has again $O^+(2)$ invariance.

Now we specify the choice of mesoscopic boxes $\widetilde{\Lambda}(x)$ and construct the excess free energy functional by the procedure sketched above. When $q = 2$, it is convenient to use the underlying complex structure of $\mathcal{X}(\mathbf{Z}^2)$, so we shall write (1.1), with obvious notations, as

$$(1.20) \quad H_\gamma(\sigma_\Lambda | \sigma_{\Lambda^c}) = -\frac{1}{2} \sum_{i,j \in \Lambda} J_\gamma(i-j) \sigma(i) \overline{\sigma(j)} - \text{Re} \sum_{(i,j) \in \Lambda \times \Lambda^c} J_\gamma(i-j) \sigma(i) \overline{\sigma(j)}$$

We introduce in detail the mesoscopic ensemble averages, or coarse graining approximation to renormalize H_γ . Let $\delta > 0$ be small, but still much larger than γ , we take again $\delta = 2^{-m}$, for some $m \in \mathbf{N}$. We take for $\tilde{\Lambda}(x)$, $x \in \mathbf{Z}^2$, a square “centered” at x , of diameter $\frac{\delta}{\gamma}$, and of the form $\tilde{\Lambda}_\delta(x) = \{i = (i_1, i_2) \in \mathbf{Z}^2 : i_k \in \frac{\delta}{\gamma}[x_k, x_k + 1]\}$ where we define as in (1.2), $m_\delta(x) = (\frac{\gamma}{\delta})^2 \sum_{i \in \tilde{\Lambda}(x)} \sigma(i)$. Thus we magnify by a factor δ/γ the “coarse graining” (or mesoscopic ensemble) labelled by $x \in \mathbf{Z}^2$, to the “smooth graining” (or microcanonical ensemble) labelled by $i \in \mathbf{Z}^2$. We have :

Proposition 1.1: There is $0 < \alpha < \frac{1}{4}$ such that

$$(1.22) \quad \left(\frac{\gamma}{\delta}\right)^2 H_\gamma(\sigma_\Lambda | \sigma_{\Lambda^c}) + U_{\text{ext}}(m_\delta) + U_{\text{int}}(m_\delta) - |\Lambda| f_\beta(m_\beta) = \mathcal{F}(m_\delta) + W(m_\delta) + |\Lambda| \mathcal{O}(\delta^{2\alpha})$$

where

$$(1.23) \quad \begin{aligned} \mathcal{F}(m_\delta) &= \frac{1}{4} \sum_{x, y \in \Lambda^*} J_\delta(x - y) |m_\delta(x) - m_\delta(y)|^2 + \sum_{x \in \Lambda^*} f_\beta(m_\delta(x)) - f_\beta(m_\beta) \\ W(m_\delta) &= \frac{1}{2} \sum_{(x, y) \in \Lambda^* \times \Lambda^{*c}} J_\delta(x - y) |m_\delta(x) - m_\delta(y)|^2 \\ U_{\text{ext}}(m_\delta) &= \frac{1}{2} \sum_{(x, y) \in \Lambda^* \times \Lambda^{*c}} J_\delta(x - y) |m_\delta(y)|^2 \\ U_{\text{int}}(m_\delta) &= \frac{1}{\beta} \sum_{x \in \Lambda^*} I(m_\delta(x)) \end{aligned}$$

Proof: To start with, consider the first term in (1.1)

$$(1.25) \quad \begin{aligned} &\left(\frac{\gamma}{\delta}\right)^2 \sum_{i, j \in \Lambda} J_\gamma(i - j) \sigma(i) \overline{\sigma(j)} = \sum_{x, y \in \Lambda^*} J_\delta(x - y) m_\delta(x) \overline{m_\delta(y)} \\ &+ \gamma^2 \sum_{x, y \in \Lambda^*} \sum_{(i, j) \in \tilde{\Lambda}_\delta(x) \times \tilde{\Lambda}_\delta(y)} (J(\gamma(i - j)) - J(\delta(x - y))) \sigma(i) \overline{\sigma(j)} \end{aligned}$$

and denote by $R(\Lambda^*)$ the second sum in the RHS of (1.25). Let $C_0 = B_1(0, \frac{1}{\delta})$ be the rhombus (or ℓ^1 -ball in \mathbf{R}^2) of center 0 and radius $\frac{1}{\delta}$, corresponding to the shape of the interaction J , and for $x' \in \mathbf{Z}^2$, its translate $C_{x'} = \frac{1}{\delta}x' + C_0$, we denote also by $C_{x'}^* \subset \Lambda^*$ the corresponding lattice obtained from $C_{x'}$ by deleting 2 of its sides, so that $\Lambda^* = \bigcup_{x' \in \mathbf{Z}^2} C_{x'}^*$ (disjoint union), and Λ^* is covered by those $C_{x'}^*$, with $x' = (x'_1, x'_2)$, $x'_j \in \{\pm 1, \dots, \pm \gamma L\}$. Let also $E(x, y) = \{(i, j) \in \tilde{\Lambda}_\delta(x) \times \tilde{\Lambda}_\delta(y) : J(\gamma(i - j)) - J(\delta(x - y)) \neq 0\}$. We can consider $E(x, y)$ as a symmetric relation $E : \Lambda^* \rightarrow \Lambda^*$, $E(x) = \{y \in \Lambda^* : E(x, y) \neq \emptyset\}$. By translation invariance of J , for any $x' \in \mathbf{Z}^2$, we have $|E(x, y)| = |E(x - \frac{1}{\delta}x', y - \frac{1}{\delta}x')|$, so that

$$(1.26) \quad \sum_{x, y \in \Lambda^*} |E(x, y)| \leq 4 \left(\frac{\gamma L}{\delta}\right)^2 \sum_{x, y \in C_0^*} |E(x, y)|$$

With the choice of ℓ^1 norm, we have $E(x, y) \neq \emptyset$ for all $x, y \in C_0^*$, and $\max_{x \in C_0^*} |E(x)| = (1 + \frac{1}{2\delta})^2$, while $\min_{x \in C_0^*} |E(x)|$ is of order unity. In any case, $|E(x)|$ depends on x and δ , but not on γ , and it is easy to see that for some $0 < \alpha < \frac{1}{4}$, $\sum_{x \in C_0^*} |E(x)| = \mathcal{O}(\delta^{-2(1-\alpha)})$, $\delta \rightarrow 0$. [Actually, this kind of estimate is well-known, see e.g. [BlLe] and references therein for related results, and applies whenever the support of J is a convex set.]

On the other hand, we have the rough estimate $|E(x, y)| \leq |\tilde{\Lambda}_\delta(x) \times \tilde{\Lambda}_\delta(y)| = (\frac{\delta}{\gamma})^4$, and since $|\sigma(i)| = 1$,

$$\begin{aligned} & \left| \sum_{x, y \in C_0^*} \sum_{(i, j) \in \tilde{\Lambda}_\delta(x) \times \tilde{\Lambda}_\delta(y)} (J(\gamma(i-j)) - J(\delta(x-y))) \sigma(i) \overline{\sigma(j)} \right| \\ & \leq \left(\frac{\delta}{\gamma}\right)^4 \sum_{x \in C_0^*} |E(x)| = \left(\frac{\delta}{\gamma}\right)^4 \mathcal{O}(\delta^{-2(1-\alpha)}) \end{aligned}$$

This, together with (1.26), shows that $R(\Lambda^*) \leq \text{Const.} \delta^{2\alpha} L^2$. A similar argument gives an estimate on the remainder $R(\Lambda^* | \Lambda^{*c})$ for the second term in (1.1). Once we have replaced $(\frac{\gamma}{\delta})^2 \sum_{i, j} J_\gamma(i-j) \sigma(i) \overline{\sigma(j)}$ by $\sum_{x, y} J_\delta(x-y) m_\delta(x) \overline{m_\delta(y)}$ modulo $R(\Lambda_{\text{int}}^*)$ and $R(\Lambda^* | \Lambda^{*c})$, which verify the estimate given in (1.22), we use the identity

$$-2 \operatorname{Re} m_\delta(x) \overline{m_\delta(y)} = |m_\delta(x) - m_\delta(y)|^2 - |m_\delta(x)|^2 - |m_\delta(y)|^2$$

and express the “density” term $\frac{1}{2}|m|^2$ in term of the mean field free energy $f_\beta(m)$ as in (1.9). Summing over (x, y) and making use of the fact that J_δ is normalized in $\ell^1(\mathbf{Z}^2)$ eventually gives the Proposition. ♣

Remarks: 1) In homogenization problems, one usually associates the discrete configuration $\sigma \in \mathcal{X}(\Lambda)$ with the function σ_γ on \mathbf{R}^2 taking the constant value $\sigma(i)$ on the square “centered” at γi , $i = (i_1, i_2)$, i.e. on $[\gamma i_1, \gamma(i_1+1)] \times [\gamma i_2, \gamma(i_2+1)]$. Furthermore the size of the domain Λ is normalized, so that taking the thermodynamic limit $\Lambda \rightarrow \infty$ is a problem of convergence for piecewise constant functions (or discrete measures) in some suitable functional space. In Kac’s model it is then convenient to take a smooth interaction J . Thus a version of Proposition 1.1 was obtained in [BuPi] by replacing the discrete average $m_\delta(x)$ around $x \in \Lambda$ by an integral, or in [DeMOrPrTr], [DeM], [Pr], ... by averaging J_γ over boxes of type $C_{x'}$ as above. (For short we refer henceforth to the review article [Pr]). Since our ultimate purpose here consists in numerical simulations on a lattice, we chose instead to give a discrete renormalization for H_γ .

2) Our renormalized Hamiltonian is now given by $\mathcal{F}(m_\delta) + W(m_\delta)$, the quantities we have subtracted are $-U_{\text{ext}}(m_\delta)$, attached to the configuration space $\mathcal{X}(\Lambda_{\text{ext}})$, and $-U_{\text{int}}(m_\delta)$

that can be interpreted as β^{-1} times the entropy of the system in Λ_{int} . Note we have also included self-energy terms $i = j$ in the original Hamiltonian. Of course, relevance of this free energy to Gibbs measure (or rather its image through the block-spin transformation) after taking the thermodynamic limit, is a rather subtle question which will not be discussed here, since we content to finite lattices.

3. Euler-Lagrange equations and non local dynamics.

We are interested in the critical points of $\mathcal{F}(m_\delta) + W(m_\delta)$. Denote as usual resp. by ∂_m and $\bar{\partial}_m$ the holomorphic and anti-holomorphic derivatives, we have for $m = m_\delta$ (for short), and any tangent vector of type (1,0) in the holomorphic sense, $\delta m \in T_m^{(1,0)} \tilde{\mathcal{X}}(\mathbf{Z}^2)$:

$$\begin{aligned} \langle \partial_m W(m), \delta m \rangle &= \frac{1}{2} \sum_{(x,y) \in \Lambda_{\text{int}}^* \times \Lambda_{\text{ext}}^*} J_\delta(x-y) (\bar{m}(x) - \bar{m}(y)) \delta m(x) \\ \langle \partial_m \mathcal{F}(m), \delta m \rangle &= \frac{1}{2} \sum_{x,y \in \Lambda_{\text{int}}^*} J_\delta(x-y) (\bar{m}(x) - \bar{m}(y)) \delta m(x) \\ &+ \sum_{x \in \Lambda_{\text{int}}^*} \left(-\frac{1}{2} \bar{m}(x) + \frac{1}{\beta} \frac{\partial I(m)}{\partial m}(x) \right) \delta m(x) \end{aligned}$$

Using again the normalization of J_δ in $\ell^1(\mathbf{Z}^2)$, the relation $I(m) = \widehat{I}(|m|)$, and setting as before $J_\delta * m(x) = \sum_{y \in \mathbf{Z}^2} J_\delta(x-y) m(y)$, we obtain

$$(1.28) \quad \langle \partial_m W(m) + \partial_m \mathcal{F}(m), \delta m \rangle = \frac{1}{2} \sum_{x \in \Lambda^*} \left(-J_\delta * \bar{m}(x) + \frac{1}{\beta} \frac{\widehat{I}'(|m|)}{|m|} \bar{m}(x) \right) \delta m(x)$$

Since $\mathcal{F} + W$ is real, this gives Euler-Lagrange equation :

$$(1.29) \quad -J_\delta * m(x) + \frac{1}{\beta} \frac{\widehat{I}'(|m|)}{|m|} m(x) = 0$$

Let $f = (\widehat{I}')^{-1} = \frac{\widehat{\phi}}{\phi}$ denote the inverse of the function \widehat{I}' . Thus $f : [0, +\infty[\rightarrow [0, 1[$ is strictly concave, $f(0) = 0$, $f'(0) = 1/2$, and $f(\rho) \rightarrow 1$ as $\rho \rightarrow +\infty$. Since the inverse of $m \mapsto \widehat{I}'(|m|) \frac{m}{|m|}$ defined on the unit disk is given by $n \mapsto f(|n|) \frac{n}{|n|}$, $n \in \mathbf{C}$, (1.29) takes the form

$$(1.30) \quad -m + f(\beta |J_\delta * m|) \frac{J_\delta * m}{|J_\delta * m|} = 0$$

Following [Pr], to find the critical points minimizing the excess free energy functional $\mathcal{F} + W$ we solve the ‘‘heat equation’’

$$(1.31) \quad \frac{dm}{dt} = -m + f(\beta |J_\delta * m|) \frac{J_\delta * m}{|J_\delta * m|} \text{ in } \Lambda^*$$

with prescribed (time independent) boundary condition on Λ^{*c} , and initial condition $m|_{\Lambda^*} = m_0$. By Cauchy-Lipschitz theorem, equation (1.31) has a unique solution, defined for all $t > 0$, valued in $\tilde{\mathcal{X}}(\Lambda^*)$. Monotonicity of $\mathcal{F} + W$ is given in the following :

Proposition 1.2: There exists a Lyapunov function for equation (1.31), i.e. $\mathcal{I} : \tilde{\mathcal{X}}(\Lambda^*) \rightarrow \mathbf{R}^+$, $\mathcal{I}(m) = 0$ iff m solves (1.30), and

$$\frac{d}{dt}(\mathcal{F} + W)(m(\cdot, t)) = -\mathcal{I}(m(\cdot, t))$$

along the integral curves of (1.31).

Proof: We have, using (1.28) and (1.31)

(1.33)

$$\begin{aligned} \mathcal{I}(m(\cdot, t)) &= -\frac{d}{dt}(\mathcal{F} + W) = -\langle \partial_m(\mathcal{F} + W), \frac{\partial m}{\partial t} \rangle - \langle \bar{\partial}_m(\mathcal{F} + W), \frac{\partial \bar{m}}{\partial t} \rangle \\ &= \frac{1}{\beta} \operatorname{Re} \sum_{x \in \Lambda^*} \left(-\beta J_\delta * \bar{m}(x) + \frac{\widehat{I}'(|m|)}{|m|} \bar{m}(x) \right) (m(x) - f(\beta |J_\delta * m|) \frac{\beta J_\delta * m}{|\beta J_\delta * m|}(x)) \end{aligned}$$

Let $m = \rho e^{i\theta}$, $\beta J_\delta * m = \rho' e^{i\theta'}$, $\mathcal{I}(m(\cdot, t))$ equals a sum of terms of the form

$$R = \frac{2}{\beta} (\rho' f(\rho') + \rho \widehat{I}'(\rho) - (\rho \rho' + f(\rho') \widehat{I}'(\rho)) \cos(\theta - \theta'))$$

then using $(\rho - f(\rho'))(\widehat{I}'(\rho) - \rho') \geq 0$ for any ρ, ρ' since \widehat{I}' is increasing, we obtain the lower bound $R \geq \frac{2}{\beta} (1 - \cos(\theta - \theta')) (\rho \rho' + f(\rho') \widehat{I}'(\rho)) \geq 0$. And because $\rho \rho' + f(\rho') \widehat{I}'(\rho) = 0$ iff $\rho = 0$ or $\rho' = 0$, this estimate easily implies the Proposition. ♣

From Proposition 1.2 and a compactness argument as in [Pr], follow that in the closure of each orbit of equation (1.31) there is a solution of Euler-Lagrange equation (1.30) or (1.29), i.e. a critical point for $\mathcal{F} + W$. As suggested by numerical simulations, this critical point is not unique, and depends on initial conditions (except of course when $\beta \leq 2$.) We expect however some uniqueness in the thermodynamical limit $\Lambda^* \rightarrow \infty$, modulo the symmetry group.

Now we give estimates on solutions of (1.31) or (1.30), borrowing some ideas to [Pr]. Eq. (1.31) can be rewritten in the integrated form :

$$(1.35) \quad m(x, t) = e^{-t} m(x, 0) + \int_0^t dt_1 e^{t_1 - t} f(\beta |J_\delta * m|) \frac{J_\delta * m}{|J_\delta * m|}(x, t_1)$$

An effective construction of the solution is given by the “time-delayed” approximations. It will also be used, discretizing time, in the numerical simulations below. We define inductively

$m_h(x, t)$, $h > 0$, on the intervals $[hk, h(k+1)[$, $k \in \mathbf{N}$, by $m_h(x, t) = m_0(x)$ for $0 \leq t < h$, and for $hk \leq t < h(k+1)$, $k \geq 2$:

$$(1.36) \quad m_h(x, t) = e^{kh-t} m_h(x, kh) + \int_{hk}^t dt_1 e^{t_1-t} f(\beta e^{-h} |J_\delta * m_h|) \frac{J_\delta * m_h}{|J_\delta * m_h|}(x, t_1 - h)$$

while for $k = 1$, just replace the first term $e^{h-t} m_h(x, h)$ on the RHS of (1.36) by $e^{-t} m_0(x)$. Using Lipschitz properties of the coefficients, it is easy to see that, as $h \rightarrow 0$, $m_h(x, t)$ tends to the solution $m(x, t)$ of (1.31) uniformly for $x \in \Lambda^*$ and t in compact sets of \mathbf{R}_+ . We prove estimates on $m(x, t)$ using sub- and supersolutions of (1.31). We start with :

Lemma 1.3: Assume $\beta > 2$, and let $\lambda(t)$, $t > 0$ be the solution of

$$(1.37) \quad \frac{d\lambda}{dt}(t) + \lambda(t) - f(\beta\lambda(t)) = 0, \quad \lambda(0) = \lambda \in [0, 1]$$

If $\lambda > m_\beta$, then $\lambda(t) \leq \lambda$ for all $t > 0$.

Proof: Write (1.37) in the integrated form as in (1.35) and consider the approximating sequence $\lambda_h(t)$. Since $\lambda_h(t)$ tends to $\lambda(t)$ uniformly on compact sets of \mathbf{R}_+ , it suffices to show the property stated in the Lemma for λ_h , and $h > 0$ small enough. For $0 \leq t < h$, $\lambda_h(t) = \lambda$, so the property holds, while for $h \leq t < 2h$, performing the integration in (1.36), and taking in account the modification for $k = 1$, we get $\lambda_h(t) = e^{h-t}(\lambda' - f(\beta\lambda')) + f(\beta\lambda')$, with $\lambda' = e^{-h}\lambda$. So by the discussion after (1.10), if $\lambda > m_\beta$, and $h > 0$ small enough, then $\lambda' - f(\beta\lambda') \geq 0$, and $\lambda_h(t) \leq \lambda' < \lambda$. By induction, using also that f is increasing, but without changing $h > 0$ anymore, it is easy to see that this property carries over for all $t > 0$.

♣.

Then we claim that the modulus of the magnetization doesn't increase in time. More precisely we have :

Proposition 1.4: Assume $\beta > 2$, and let $m(x, t)$ be the solution of (1.31) such that $m_0(x) = m(x, 0)$ satisfies $|m_0(x)| \leq \lambda < 1$, for some $\lambda > m_\beta$, and all $x \in \mathbf{Z}^2$ (so including the exterior region .) Then $|m(x, t)| \leq \lambda$ for all $x \in \mathbf{Z}^2$, and all $t > 0$.

Proof: Eq. (1.35) shows that

$$(1.38) \quad |m(x, t)| \leq e^{-t} |m_0(x)| + \int_0^t dt_1 e^{t_1-t} f(\beta |J_\delta * m|)(x, t_1)$$

Now by the monotony properties of the convolution and the function f , we have $f(\beta |J_\delta * m|)(x, t_1) \leq f(\beta J_\delta * |m|)(x, t_1)$, so the solution $\lambda(t)$ of (1.37) with $\lambda(0) = \lambda$ is a supersolution for (1.38), and Lemma 1.3 easily implies the Proposition. ♣

We now look for lower bounds on $m(x, t)$. Since there are in general vortices, one cannot expect a global, positive lower bound on $|m(x, t)|$, unless there is no vorticity on initial and boundary values. On the other hand, we know (at least for a 1-d lattice, see [BuPi],) that the Gibbs measure of the configurations at equilibrium $m_\delta \in \tilde{\mathcal{X}}(\Lambda^* | \Lambda^{*c})$ with $|m_\delta(x)|$ arbitrarily close to m_β , has to be large. We have :

Proposition 1.5: Assume $\beta > 2$, and let $m(x, t)$ be the solution of (1.31) such that $m_0(x) = m(x, 0)$ as in Proposition 1.4 satisfies $\text{Re}(\nu m_0(x)) \geq \mu$, for some fixed $\nu \in \mathbf{S}^1$ and $\mu > 0$ and all $x \in \Lambda^*$. Assume furthermore that μ satisfies $(\mu^2 + \lambda^2)^{1/2} < \beta f(\beta\lambda)$, where λ is as in Proposition 1.4. Then $\text{Re}(\nu m(x, t)) \geq \mu$ for all $x \in \Lambda^*$, and all $t > 0$.

Proof: As in the proof of Proposition 1.4 we make use of a comparison function. So let $\mu(t)$ verify the differential equation

$$(1.40) \quad \frac{d\mu}{dt}(t) + \mu(t) - \beta f(\beta\lambda) \frac{\mu(t)}{(\mu(t)^2 + \lambda^2)^{1/2}} = 0, \quad \mu(0) = \mu$$

Write (1.40) in the integrated form as in (1.35) and consider the approximating sequence $\mu_h(t)$ as in (1.36), including the modification for $k = 1$. We shall show that $\mu_h(t) \geq \mu$ for all $t > 0$ provided $\mu(0) = \mu$ verifies the inequality given in the Proposition. Namely, this holds for $0 \leq t < h$, because then $\mu_h(t) = \mu$, while for $h \leq t < 2h$, performing the integration as in (1.36), we get $\mu_h(t) = e^{h-t} \mu e^{-h} (1 - M_h) + \mu e^{-h} M_h$, where $M_h = \beta f(\beta\lambda) (e^{-2h} \mu^2 + \lambda^2)^{-1/2}$, so by assumption $\mu_h(t) > e^{-h} \mu$ and $\mu_h(t) \geq \mu$ for $h > 0$ small enough. By induction, using that the function $\rho \mapsto \rho(\rho^2 + \lambda^2)^{-1/2}$ is increasing on \mathbf{R}_+ , and possibly decreasing $h > 0$ once more, it is easy to see that $\mu_h(t) \geq \mu$ holds for all $t > 0$. Because the coefficients of (1.40) are uniformly Lipschitz, $\mu_h(t)$ tends to $\mu(t)$ uniformly on compact sets in \mathbf{R}_+ , and this property holds again for $\mu(t)$.

Now we turn to the equation for $m(x, t)$. Possibly after rotating the coordinates, we may assume $\nu = 1$, i.e. $\text{Re} m_0(x) \geq \mu$ and all $x \in \Lambda^*$ (again, we have included the boundary condition in the initial configuration.) Write $m(x, t) = u(x, t) + iv(x, t)$, u, v real and take real part of (1.31). The integrating form of the resulting equation writes :

$$(1.41) \quad u(x, t) = e^{-t} u(x, 0) + \int_0^t dt_1 e^{t_1-t} f(\beta |J_\delta * m|) \frac{\beta J_\delta * u}{\beta |J_\delta * m|}(x, t_1)$$

As $\rho' \mapsto \frac{f(\rho')}{\rho'}$ is decreasing on \mathbf{R}_+ , and by Proposition 1.4, $|J_\delta * m| \leq (|J_\delta * u|^2 + \lambda^2)^{1/2}$, we have

$$\frac{f(\beta |J_\delta * m|)}{\beta |J_\delta * m|} \geq \frac{f(\beta (|J_\delta * u|^2 + \lambda^2)^{1/2})}{\beta (|J_\delta * u|^2 + \lambda^2)^{1/2}} \geq \frac{f(\beta\lambda)}{\beta (|J_\delta * u|^2 + \lambda^2)^{1/2}}$$

the last inequality because f is increasing. Since $u(x, 0) \geq \mu$, by continuity we have $u(x, 0) > 0$ at least for small $t > 0$, and (1.41) gives

$$(1.42) \quad u(x, t) \geq e^{-t}u(x, 0) + \beta f(\beta\lambda) \int_0^t dt_1 e^{t_1-t} (J_\delta * u) ((J_\delta * u)^2 + \lambda^2)^{-1/2}(x, t_1)$$

Now, using the monotony of the convolution, and again the fact that the function $\rho \mapsto \rho(\rho^2 + \lambda^2)^{-1/2}$ is increasing on \mathbf{R}_+ , we can easily show that the solution $\mu(t)$ of (1.40) with $\mu(0) = \mu$ is actually a subsolution for (1.42), for all $t > 0$; the properties proved already for $\mu(t)$ then imply the Proposition. ♣

Of course, by continuity, Propositions 1.4 and 1.5 imply the corresponding estimates for the solution of Euler-Lagrange equation (1.29) or (1.31). Our last result states that if $\beta \leq 2$, then $m(x, t)$ tends to 0 $t \rightarrow \infty$, which is consistent with the absence of phase transition (or spontaneous magnetization) at high temperature.

Proposition 1.6: Assume $\beta \leq 2$, and let $m(x, t)$ be the solution of (1.31). Then $m(x, t) \rightarrow 0$ on Λ^* as $t \rightarrow +\infty$.

Proof: Using that $f(\rho') \leq \frac{1}{2}\rho'$, all $\rho' > 0$, (1.38) shows that

$$|m(x, t)| \leq e^{-t}|m_0(x)| + \frac{\beta}{2} \int_0^t dt_1 e^{t_1-t} J_\delta * |m|(x, t_1)$$

So by taking convolution

$$J_\delta * |m|(x, t_1) \leq e^{-t_1} J_\delta * |m_0|(x) + \frac{\beta}{2} \int_0^{t_1} dt_2 e^{t_2-t_1} J_\delta^{*(2)} * |m|(x, t_2)$$

and integrating the resulting inequality :

$$|m(x, t)| \leq e^{-t} [|m_0(x)| + \frac{\beta t}{2} J_\delta * |m_0|(x) + \left(\frac{\beta}{2}\right)^2 T^{(2)}(e^{(\cdot)} J_\delta^{*2} * |m|(x, \cdot))(t)]$$

where $T^{(k)}u(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{k-1}} dt_k u(t_k)$ denotes the k -fold integral of u , and J_δ^{*k} the k -fold convolution product of J_δ with itself. By induction, we get :

$$\begin{aligned} |m(x, t)| \leq e^{-t} [& |m_0(x)| + \frac{\beta}{2} t J_\delta * |m_0|(x) + \cdots + \left(\frac{\beta}{2}\right)^k \frac{t^k}{k!} J_\delta^{*k} * |m_0|(x) \\ & + T^{(k+1)}(e^{(\cdot)} J_\delta^{*(k+1)} * |m|(x, \cdot))(t)] \end{aligned}$$

The series is uniformly convergent for t in compact sets so we can write

$$|m(x, t)| \leq e^{-t} \sum_{k=0}^{+\infty} \left(\frac{\beta}{2}\right)^k \frac{t^k}{k!} J_\delta^{*k} * |m_0|(x)$$

When $\beta < 2$, using $J_\delta^{*k} * |m|(x, 0) \leq |m_0(x)| \leq 1$, it follows that $m(x, t) \rightarrow 0$ for all $x \in \Lambda^*$ as $t \rightarrow \infty$. This holds again for $\beta = 2$ since we may assume that m_0 has compact support, and we know (see [Hö, Lemma 1.3.6]) that $J_\delta^{*k} \rightarrow 0$ uniformly on \mathbf{R}^2 (or on \mathbf{Z}^2 in the discrete case,) as $k \rightarrow \infty$. ♣

4. Vortices.

We consider here the problem of finding numerically the critical points of Euler-Lagrange equation (1.30) by solving (1.31) subject to a boundary condition on Λ^{*c} presenting vorticity. Let $m : \mathbf{R}^2 \rightarrow \mathbf{C}$ be a differentiable function, considered as a vector field on \mathbf{R}^2 , and subject to the condition $|m(x)| \rightarrow \ell > 0$ as $|x| \rightarrow \infty$ uniformly in $\hat{x} = x/|x|$. Then the integer

$$\deg_R m = \frac{1}{2\pi} \int_{|x|=R} d(\arg m) = \frac{1}{2i\pi} \int_{|x|=R} \frac{dm}{m}$$

is independent of R when $R > 0$ is large enough, is called the (topological) degree of m at infinity, and denoted by $\deg_\infty m$.

We define in the same way the local degree (or topological defect) $\deg_{x_0} m$ of m near x_0 , provided $m(x) \neq 0$, $x \neq x_0$, by integrating on a small loop around x_0 . The local degree takes values $d_j \in \mathbf{Z}$. When m has finitely many zeros x_j inside the disc of radius R , its total degree (or vorticity) is defined again as the sum of all local degrees near the x_j 's. In many boundary value problems, (or generalized boundary value problems, in the sense that the boundary is at infinity,) such as Ginzburg-Landau equations, the total vorticity is conserved, i.e. $\deg_\infty m = \sum_j \deg_{x_j} m$. Generically $d_j = \pm 1$ ("simple poles".) Our aim is to check this conservation principle in the present situation.

So in the discrete case, we define analogously the degree of $m(x) = \rho e^{i\theta}$ at infinity to be the degree restricted to the lattice Λ^{*c} , e.g. by $d = \deg_{\Lambda^{*c}} m = \frac{1}{2\pi} \sum_j (\theta_{j+1} - \theta_j)$ along some closed loop $\Gamma_i \subset \Lambda^{*c}$ encircling Λ^* , the sites along Γ_i being labelled by j , assuming that this integer takes the same value on each Γ_i .

The local degree near x_0 , where $m(x_0) = 0$, is identified again by computing the angle circulation on a loop encircling x_0 . Local degrees are also expected to take, generically, values ± 1 .

We chose our parameters as follows. We start with prescribing the degree of the spin variable σ on Λ^c , and take on Γ_i , the i :th loop away from Λ , containing N_i sites, ($N_i = 4i + P$, where P is the perimeter of Λ , we take enough i 's to cover the range of interaction,) with a uniform distribution $\sigma_j = \exp i(2\pi dj/N_i + \phi_0)$, $1 \leq j \leq N_i$; here ϕ_0 is a constant (e.g. $\phi_0 = 1$) that "breaks" the symmetry of the rectangle Λ . We can also take a random distribution

$\sigma_j = \exp(2i\pi d(j/N_i + \varepsilon_{ij}))$, where ε_{ij} are uniform random variables with $\sum_{j=1}^{N_i} \varepsilon_{ij} = 0$, and $(\varepsilon_{ij})_{i,1 \leq j \leq N_i-1}$, independent, identically distributed, with a variance small enough.

To this spin distribution on Λ^c , we apply the block spin transformation (1.2), so to have a distribution of magnetization on Λ^{*c} , then we prescribe arbitrary initial conditions inside Λ^* , which we choose again to be random numbers, with absolute value less than, or comparable to m_β . We can also take zero initial values, which gives a particular symmetry to the solution (see below.)

We usually fix the inverse temperature $\beta = 5$; the results do not depend on β in an essential way, we just observe that magnetization tends to 0 as $\beta \rightarrow 2^+$. The diameter N of the lattice Λ ranges from 2^6 to 2^{10} , the size δ/γ of the diameter of the block-spin $\Delta(x)$ is set to 4 (most of the time) so the diameter L^* of the lattice Λ^* ranges from 2^4 to 2^8 . The lattice is either a square, or a rectangle.

The size $1/\delta$ of the length of interaction in Λ^* ranges from 2 to 32, thus the corresponding interaction in Λ has length $1/\gamma = 4/\delta$ between 8 to 128.

Equation (1.31) is solved by “time-delayed” approximations as in (1.36), implemented by the second order trapezoidal method to compute the integrals.

These experiments lead to the following observations, vortices display in a different way, according to the initial configuration on Λ^* , but always obey the conservation of total vorticity.

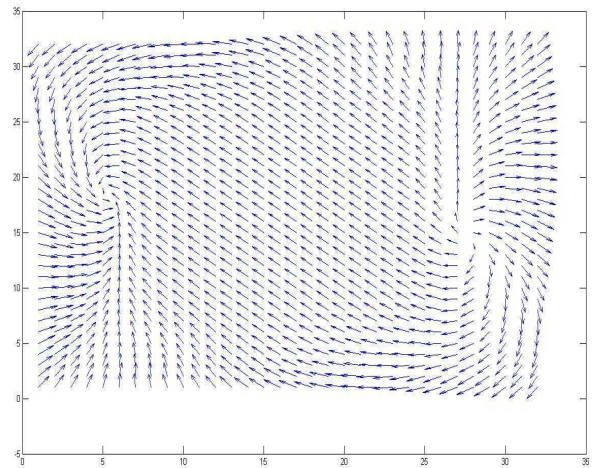
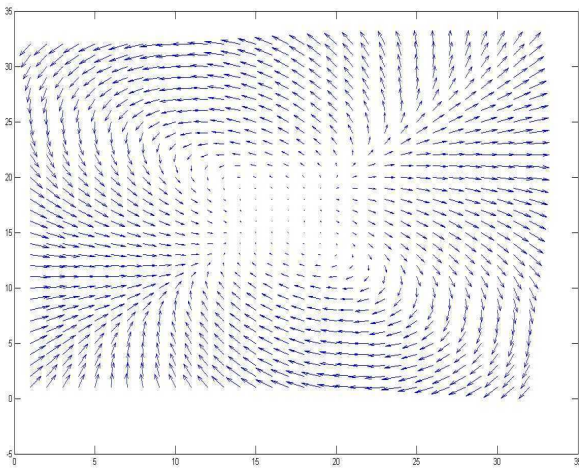


Fig 1.a: $L^* = 128, d = 2$, zero initial condition

Fig 1.b: $L^* = 128, d = 2$, random initial condition

1. To start with, we assume uniform distribution of spins on the boundary.

The particular case of zero initial values and a square lattice, gives raise to interesting symmetries (or degeneracies) in the picture : namely, vortices tend to occupy most of Λ^* so

to cope with the symmetry of the square. So for $d = 1$ there is a single vortex in the center, for $d = 2$ (cf Fig 1.a) a vortex of multiplicity 2, (unless the degeneracy is lifted and turns into 2 nearby vortices,) for $d = 4 - 1$, (cf Fig 2.a) one vortex of degree -1 surrounded by 4 vortices of degree +1 near the corners, for $d = 4$, 4 vortices of degree +1 near the corners, for $d = 4 + 1$, same configuration as for $d = 3$, for $d = 4 + 2$ the picture looks alike, with a double vortex at the center, for $d = 2 \times 4 - 1$, 4 new vortices appear near the center (cf Fig 3.a), etc... So the configuration depends essentially of the residue of d modulo 4 : new vortices show up from the middle towards the corners along the diagonals of Λ^* .

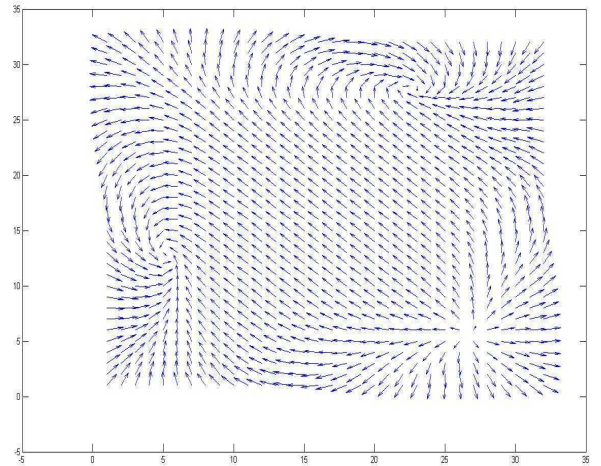
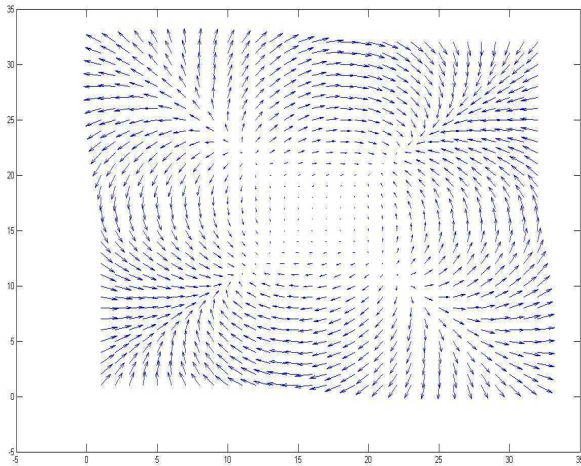


Fig 2.a: $L^* = 128, d = 3$, zero initial condition

Fig 2.b: $L^* = 128, d = 3$, random initial condition

Still for a square lattice, but random initial conditions inside Λ^* , vortices are simple and tend to display at the periphery of Λ^* , in a pretty regular way, leaving some large ordered domain near the center.

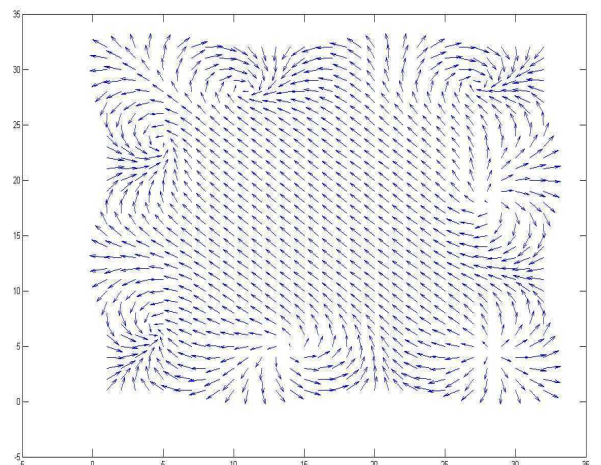
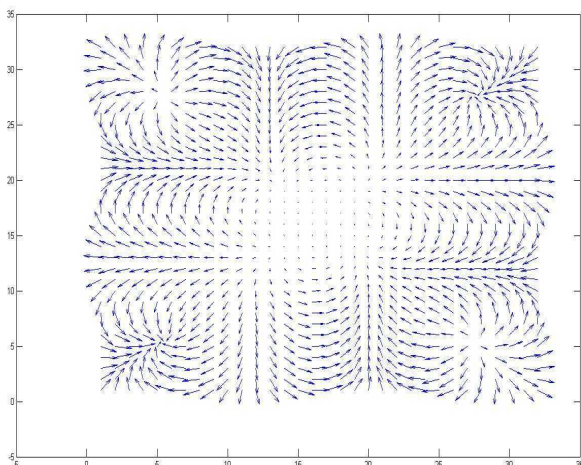


Fig 3.a: $L^* = 128, d = 7$, zero initial condition

Fig 3.b: $L^* = 128, d = 7$, random initial condition

Thus, these configurations maximize the area of the lattice where the magnetizations are aligned, with an absolute value close to m_β , (in accordance with the fact that energy H_γ of the microcanonical ensemble decreases as the spins align.) Their direction in general, points out along one of the diagonals of Λ^* . This is illustrated in Fig.1,2,3.b above, for a vorticity $d = 2, 3, 7$ resp. In particular, Fig.2 shows the topological bifurcation from $d=4-1$ to $d=3$. These simulations also suggest that the equilibrium configurations shouldn't depend on the initial conditions, but for an exceptional set.

Now we vary the shape of the lattice, changing the square into a rectangle, keeping in mind that thermodynamic limit, most of the time, should be taken in the sense of Fisher, i.e. the length of the rectangle Λ^* doesn't exceed a constant times $|\Lambda^*|^{1/2}$. As expected, vortices tend to align along the largest dimension, but again, limiting configurations depend on whether the initial condition inside Λ^* is set to zero or not.

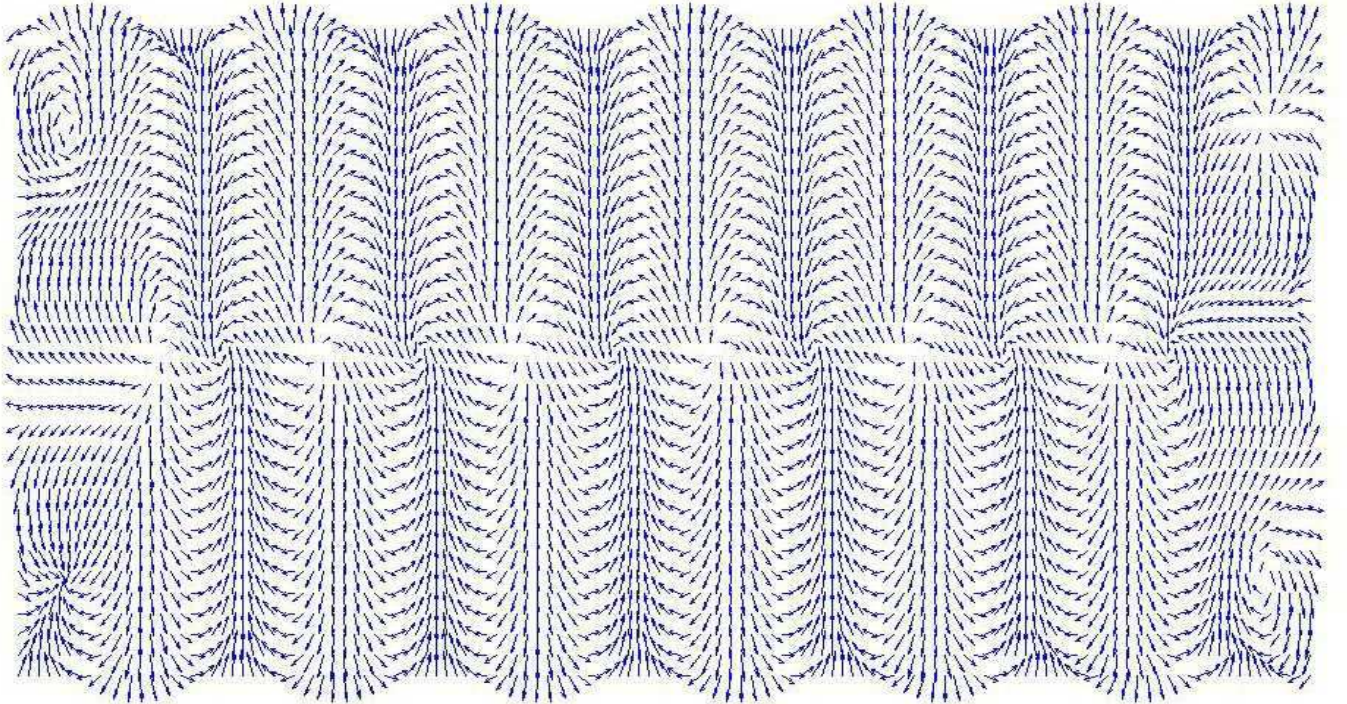


Fig 4.a: $L^* = 512, \ell^* = 128, d = 16$, zero initial condition

Thus, for zero initial condition, vortices display along the largest median of Λ^* , with possible extra vortices near the corners (inheriting the features of the square lattice.) Namely, they tend to repel each other so the energy cost in clustering is minimized by occupying the corners. Typically, such configurations occur if $d \geq 4$ and the length of Λ^* is only twice its width. But for sufficiently long lattices, or small degree, they just stand the median line. See Fig.4.a and 4.c.

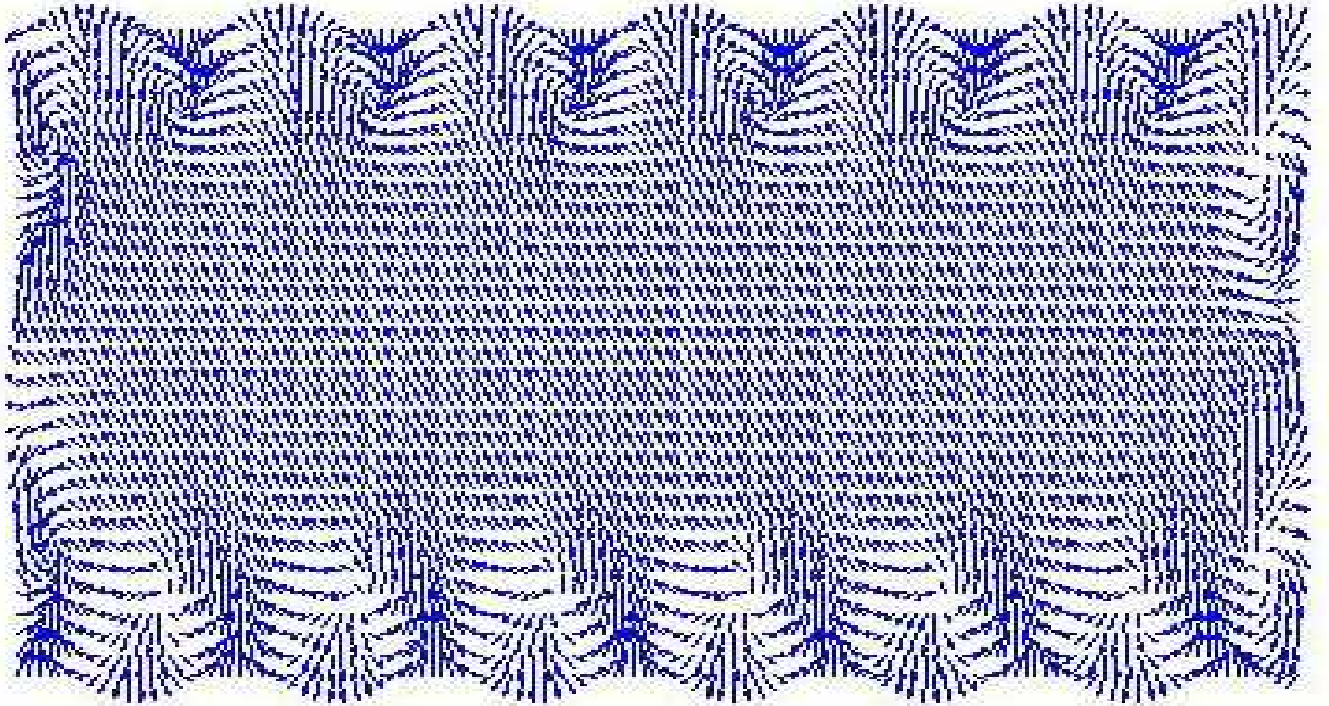


Fig 4.b: $L^* = 512, \ell^* = 128, d = 16$, random initial condition

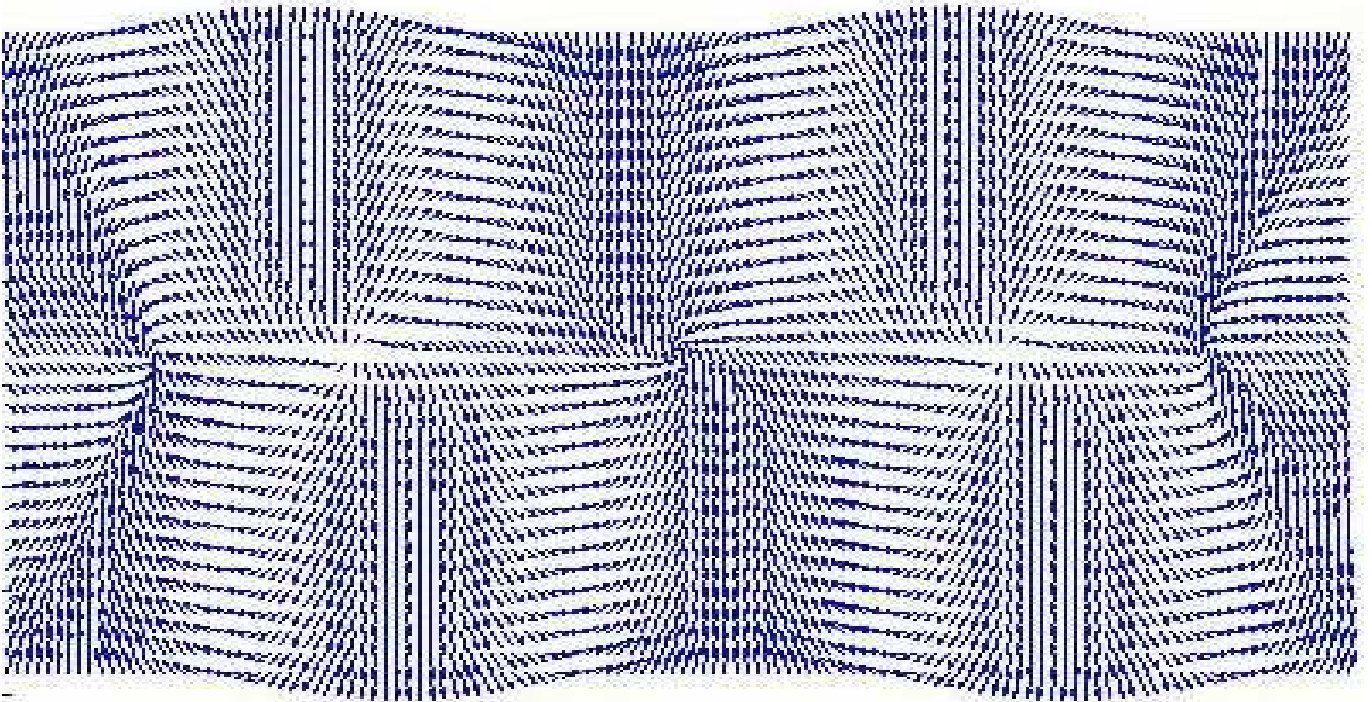


Fig 4.c: $L^* = 512, \ell^* = 128, d = 5$, zero initial condition

For random initial conditions (Fig.4.b), we recover the general picture of square lattices, i.e. vortices set along the boarder of Λ^* , leaving a large space in the middle with parallel

magnetizations. In any case, degeneracies are lifted, and all vortices have degree +1.

2. Now we examine the case of random distribution of spins on the boundary Λ^{*c} . Vortices change their place according to the initial random value, and tend again to gather inside Λ^* , but take always the value +1 (we always assume $d > 0$.) The sole effect of randomness in the boundary condition is to change the place of the vortices: namely they tend to get even closer to the boundary, so to leave larger ordered regions in the middle. See Fig 5.a, 5.b and compare with Fig 2.b, 3.b.

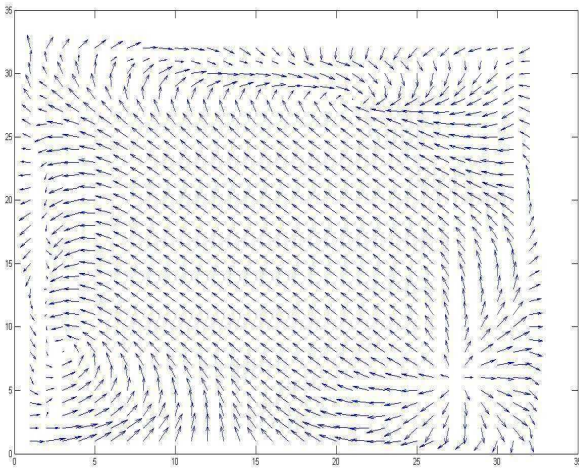


Fig 5.a: $L^* = 128, d = 3$

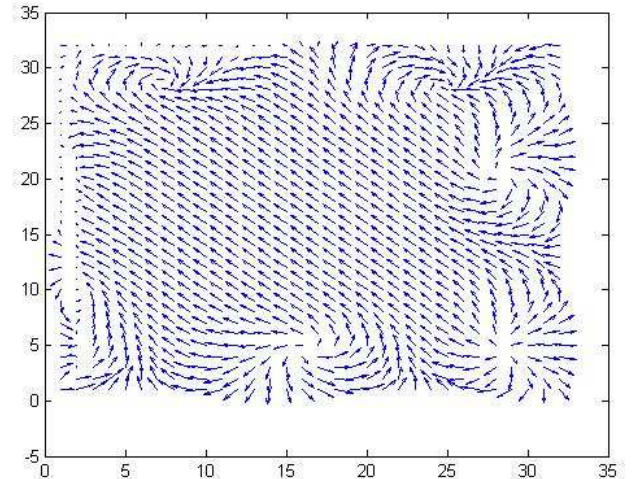


Fig 5.b: $L^* = 128, d = 7$

Another interesting result concerns the value of energy for the minimizing configurations. In case of Ginzburg-Landau equation, $-\Delta\psi + (|\psi|^2 - 1)\psi = 0$, where ψ is subject to a boundary condition with vorticity, it is known that energy of the minimizer vs. vorticity, has an asymptotic, as the n vortices x_j become distant from each other, the leading order term is given by a “proper energy”, proportional to $\sum_{i=1}^n d_i^2$, and the next correction is the inter-vortex energy given by so-called Kirchhoff-Onsager hamiltonian, of the form $W_0 = -\pi \sum_{i \neq j} d_i d_j \log |x_i - x_j|$ (see e.g. [BeBrHe] and [OvSi2] for precise statements.) It can be interpreted as the electrostatic energy for a system of charges d_j interacting through Coulomb forces. It turns out that, despite forces in action have no electrostatic character, Kirchhoff-Onsager correction holds with a good accuracy in our case, even for long range interactions (i.e. for small γ ,) but provided the inter-vortex distance is bounded below by the range of the interaction. We have listed below some graphs of $K = (\mathcal{F} + W) - W_0$, obtained with uniform boundary conditions, which show that K roughly grows linearly with d (cf Fig 6).

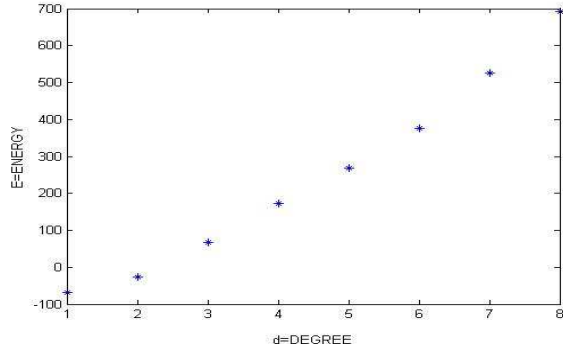


Fig 6.a: $L^* = 128$, zero initial condition

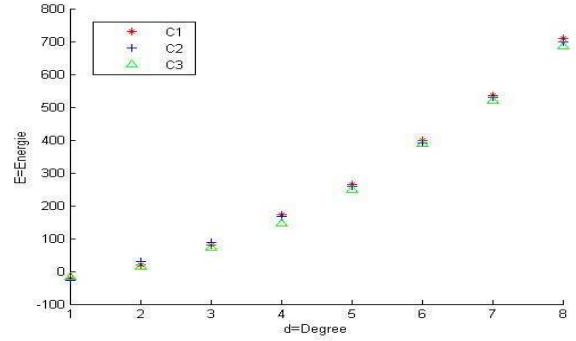


Fig 6.b: $L^* = 128$, random initial condition

Fig 6.b, shows that several random trials for initial conditions give approximately the same renormalized energy K .

We end up with studying minimizers of the free energy functional, when we set up totally random boundary values, i.e without prescribing any topological degree. The configuration, of course, have no particular structure, but show some ordered areas among more chaotic regions. See Fig 7.a, 7.b.

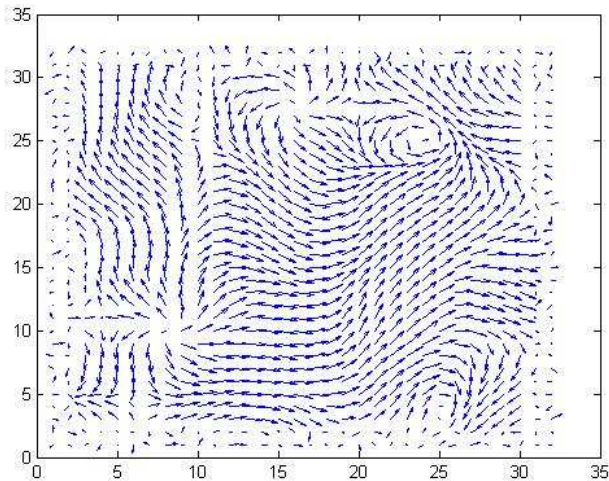


Fig 7.a: $L^* = 128$

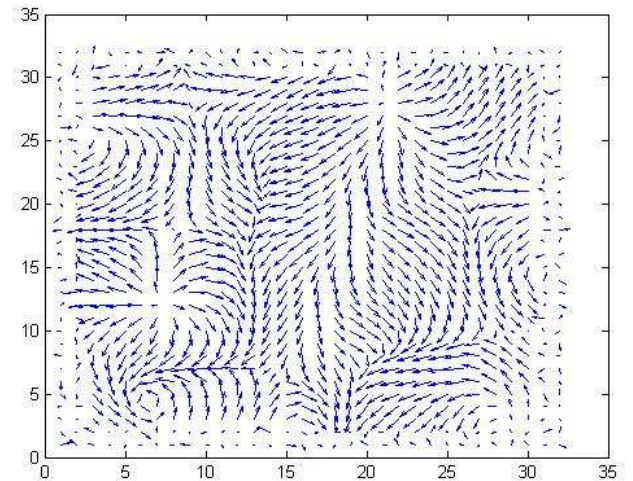


Fig 7.b: $L^* = 128$

References

- [Ba] M.Bauer. Introduction aux liens entre mécanique statistique et théorie des champs. Lecture notes, Rabat 1992.
- [BeBrHel] F.Bethuel, H.Brezis, F.Helein. Ginzburg-Landau vortices, Birkhäuser, Basel 1994
- [BlLe] P.Bleher, J.Lebowitz. Energy-level statistics of model quantum systems: universality and scaling in a lattice-point problem. J.Stat.Phys. 74 (1994), p.167-217.
- [BuPi] P.Butta, P.Picco. Large-deviation principle for one-dimensional vector spin models with Kac potentials. J.Stat.Phys. 92 (1998), p.101-150.

- [DeM] A.DeMasi. Spins systems with long range interactions. Progress in Probability, Birkhäuser, Vol 54, (2003), p.25-81.
- [DeMOrPrTr] A.DeMasi, E.Orlandi, E.Presutti, L.Triolo. Uniqueness and global stability of the instanton in non-local evolution equations. Rendiconti di Mat., Serie VII, 14, (1994), p.693-723
- [Hö] L.Hörmander. The Analysis of Partial Differential Operators I. Springer, 1983.
- [LeVeRu] X.Leoncini, A.Verga, S.Ruffo. Hamiltonian dynamics and the phase transition of the XY model. Phys. Rev. E 57, 6377-6389 (1998).
- [MiZh] R.Minlos, E.Zhizhina. Asymptotics of the decay of correlations for the Gibbs spin fields. Theoret.Math.Phys. 77(1) (1988), p.1003-1009.
- [OvSi] Y.Ovchinnikov, I.M.Sigal. Ginzburg-Landau Equation I. Static vortices. CRM Proceed. Vol 12, 1997, p.199-220
- [OvSi2] Y.Ovchinnikov, I.M.Sigal. The energy of Ginzburg-Landau vortex, European J. of Applied Mathematics 13 (2002), 153-178.
- [Pr] E.Presutti. From statistical mechanics towards continuum mechanics. Preprint M.Planck Institut, Leipzig, 1999
- [Ru] D.Ruelle. Statistical Mechanics. World Scientific, 1999.
- [S] Y.Sinai. Theory of phase transitions: rigorous results. Pergamon Press, 1982.
- [Z] J.Zinn-Justin. Transitions de Phase et Groupe de Renormalization. Editions du CNRS, 2004.